# IS THE GRAVITATIONAL QUANTIZATION ANOTHER CONSEQUENCE FROM GENERAL RELATIVITY?

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ABSTRACT. In this paper we show that the perturbation of the Newton's inverse square law that gives the Schwarzschild solution for the case of a punctual or spherical and homogeneous mass has a similar form as the development with respect to the delay of the retarded scalar potential proposed in [12, 13]. This observation suggests the possibility that the gravitational quantization is another consequence from General Relativity.

## 1. INTRODUCTION

The anomalous precession of the Mercury's perihelion was first noticed in 1859, when the French astronomer Le Verrier observed that the perihelion of the planet Mercury precesses at a slightly faster rate than the one that can be accounted by Newtonian mechanics with the distribution of masses of the solar system.

Einstein found that the extra precession unavoidably arises from the fundamental principles of General Relativity, see [4, 5]. The general problem of the integration of the Einstein equations, given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R$ is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor and $T_{\mu\nu}$ is the stress-energy tensor, is extremely difficult and the determination of the explicit solutions is only possible in a restricted number of cases. One of the most important is the Schwarzschild solution for the case of a punctual mass or spherical and homogeneous and with the assumption that the limit values in the infinite of the $g_{\mu\nu}$ are the galilean values. A way to determine the relativistic prediction

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for the advance of an elliptical orbit from the Schwarzschild solution in a very comprehensible and clear form is given in [17, 22]. We reproduce here the computations to obtain the perturbation of the Newton's inverse-square law.

To determine the motion of planets and light rays in a Schwartzschild spacetime we must first find the geodesic equations. This is best done by working from the Lagrangian

$$\mathcal{L} = \frac{1}{c}\left[-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^{1/2}.$$

Assuming that the orbits remain permanently in the equatorial plane (as in Newtonian theory ) i.e. $\Theta = \pi/2$, the Lagrangian is:

$$\mathcal{L} = \left\{\left(1-\frac{2GM}{c^2r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left[\left(1-\frac{2GM}{c^2r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2\right]\right\}^{1/2}$$

From Euler-Lagrange equations we obtain the energy conservation and the angular momentum conservation given by

$$\left(1-\frac{2GM}{c^2r}\right)\frac{dt}{d\tau} = k, \qquad \frac{d\theta}{d\tau} = \frac{h}{r^2},$$

where $k$ and $h$ are constants of integration, determined by the initial conditions of the orbit. Remember that $\mathcal{L} = \varepsilon$ with $\varepsilon = 1$ for time-like orbits and $\varepsilon = 0$ for null orbits. Hence, we can substitute these derivatives into the Lagrangian to obtain

$$\varepsilon^2 = \left(\frac{k^2 r}{r-2M}\right) - \left(\frac{r}{r-2M}\right)\left(\frac{dr}{d\tau}\right)^2 - \frac{h^2}{r^2},$$

taking $c = G = 1$. Isolating $(dr/d\tau)^2$ from the above expression, we have

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2Mh^2}{r^3} - \frac{h^2}{r^2} + \frac{2\varepsilon M}{r} + k^2 - \varepsilon^2. \tag{2}$$

If we differentiate equation (2) with respect to $\tau$ and we divide by $2(dr/d\tau)$, see [17], we obtain

$$\frac{d^2r}{d\tau^2} = -\frac{\varepsilon^2 M}{r^2} + \frac{h^2}{r^3} - \frac{3Mh^2}{r^4}. \tag{3}$$

Let $\omega = d\theta/d\tau$ be the proper angular speed, taking into account that $h = \omega r^2$, the above equation can be written as

$$\frac{d^2r}{d\tau^2} = -\frac{\varepsilon^2 M}{r^2} + \omega^2(r - 3M). \tag{4}$$

Obviously, if $\omega = 0$ we obtain the Newton's inverse-square law for radial gravitational acceleration. If $\omega$ is non–zero the term $r\omega^2$ corresponds to the Newtonian centripetal acceleration. Hence, defining the tangential velocity $v_t = \omega r$, this term would be equal to the classical $v_t^2/r$. Following the reasonings of [17], this term serves to offset the inward pull of gravity, but in the relativistic version we find not $w^2 r$ but $w^2(r - 3M)$. For values of $r$ much greater than $3M$ this difference can be neglected, but clearly if $r$ approaches $3M$ we can expect to see non–classical effects, and of course if $r$ becomes less than $3M$ we would expect a completely un–classical behavior. In fact, this corresponds to the cases when an orbiting particle spirals into the center, which never happens in classical theory.

In a similar way as in the resolution of the classical case, if we introduce the change of variable $u(\theta) = 1/r(\tau)$, equation (3) becomes

$$\left(1 + 6\left(\frac{M}{h}\right)^2\right)\frac{d^2u}{d\theta^2} + u = \left(\frac{M}{h^2} + \frac{3M^3}{h^4}\right) + 3M\left(\frac{d^2u}{d\theta^2}\right)^2 , \tag{5}$$

where we have taken $\varepsilon = 1$ for a timelike orbit. The value of $d^2u/d\theta^2$ in typical astronomical problems is numerically quite small (many orders of magnitude less than 1), so the quantity $3M(d^2u/d\theta^2)^2$ on the right hand side will be negligible for planetary motions, see [17].

## 2. post-Newtonian approximations of the General Relativity

Einstein's equations may be written in a very simple way, which leads straightly to the analogy with Maxwell equations, if we consider the so called weak field approximation. We can use this kind of linear approximation if we deal with a central mass whose gravitational field is weak and, if the central mass is rotating, its rotation is not relativistic. We are interested in the general linear solution of the gravitational field equations (1). We assume that a global background inertial frame with coordinates $x^\mu = (ct, \mathbf{x})$ and Minkowski metric $\eta_{\mu\nu}$ is perturbed by due to the presence of the central mass in such a way that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$, where $|h_{\mu,\nu}| \ll 1$ is a small perturbation of the Minkowski metric. Then we define $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, where $h$ is the trace of $h_{\mu\nu}$, i.e., $h = \eta^{\mu\nu}h_{\mu\nu}$. Hence, expanding the field equations (1) in powers of $\bar{h}_{\mu\nu}$ and taking the linear order terms, we obtain

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} , \tag{6}$$

where we have imposed the Lorentz gauge condition $\bar{h}^{\mu\nu}{}_{,\nu} = 0$. The linearized field equations (6) are similar to the Maxwell equations

$$\Box A^{\mu} = 4\pi j^{\mu} \,,$$

where the role of the electromagnetic four-vector potential $A^{\mu}$ is played by the tensor potential $\bar{h}_{\mu\nu}$, and the role of the four-current $j^{\mu}$ is played by the stress-energy tensor $T_{\mu\nu}$. The special retarded solution of equation (6) is given by

$$\bar{h}_{\mu\nu} = \frac{4G}{c^4} \int \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|/c \,, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d^3\mathbf{x}' \,.$$

Defining the matter density $\rho$ and the matter current $\mathbf{j} = \rho\mathbf{v}$ from $T_{00} = \rho c^2$ and $T_{0i} = c j_i$, and assuming that the central mass consists on a finite distribution of slowly moving matter with $|\mathbf{v}| \ll c$, we have that $|\bar{h}_{00}| \gg |\bar{h}_{ij}|$ and $|\bar{h}_{0j}| \gg |\bar{h}_{ij}|$. Moreover, $\bar{h}_{00} = 4\phi_g/c^2$ and $\bar{h}_{0j} = -2A_j/c^2$, where $\phi_g$ is the gravitoelectric potential and $\mathbf{A}_g$ is the gravitomagnetic vector potential given by

$$\phi_g \sim \frac{GM}{r}, \qquad \mathbf{A}_g \sim \frac{G}{c}\frac{\mathbf{J} \times \mathbf{x}}{r^3},$$

where $r = |\mathbf{x}|$, and $\mathbf{J}$ is the angular momentum of the central mass. Hence, under these conditions, the space-time metric has the form

$$ds^2 = -c^2(1 - 2\frac{\phi_g}{c^2})dt^2 - \frac{4}{c}(\mathbf{A}_g \cdot d\mathbf{x})dt + (1 + 2\frac{\phi_g}{c^2})\delta_{ij}dx^i dx^j. \tag{7}$$

This linear approximation of the gravitational field equations is known as Gravitoelectromagnetism (GEM), see [16].

If we consider the case of a non-rotating central mass, i.e., $\mathbf{J} \equiv \mathbf{0}$, and taking into account that $\phi_g \sim GM/r$ we have that the metric (7) takes the form

$$ds^2 = -c^2(1 - 2\frac{GM}{c^2 r})dt^2 + (1 + 2\frac{GM}{c^2 r})\delta_{ij}dx^i dx^j, \tag{8}$$

and is called the post-Newtonian metric, which is a particular case of the PPN metric, see for instance in [19] pag. 1097. The derivation of the perihelion shift in the PPN formalism can be checked in [19] pag. 1115. In the case of the post-Newtonian metric (8), it gives the result

$$\delta\phi_0 \equiv \frac{4}{3} \cdot \frac{6\pi M}{a(1 - e^2)},$$

where $a$ is the semi-major axis, $e$ is the eccentricity of the ellipse and the correct value of the General relativity is $6\pi M/(a(1-e^2))$. In the case of a rotating mass, $\mathbf{J} \not\equiv \mathbf{0}$, we have that the gravitomagnetic component is not zero and it is possible to derive the most famous gravitomagnetic effect which is the Lense-Thirring effect. At the beginning of the 20th century, J. Lense and H. Thiring [15], studied the effects of rotating masses within the Relativistic Theory of Gravitation. Their starting intention was to incorporate Mach's principle into General Relativity Theory.

Hence, the excess of motion of Mercury's perihelion is not explained only in terms of a relativistic gravitoelectric correction to the Newtonian gravitational potential of the Sun. In fact, we must take into account second orders in $M/r$ in the Schwarzchild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2(d\theta^2 + \sin^2\Theta d\Theta^2).$$

where we have taken $c = G = 1$. First we transform to isotropic coordinates, and then we expand the metric coefficients in powers of $M/r$ to obtain a more accurate post-Newtonian approximation

$$ds^2 = -\left[1 - 2\frac{M}{r} + 2\left(\frac{M}{r}\right)^2\right] dt^2 + \left[1 + 2\frac{M}{r}\right] [dr^2 + r^2(d\theta^2 + \sin^2\Theta d\Theta^2)].$$

With this approximation the correct value of the General Relativity is obtained, see [19] pag. 1110. Hence, including second order in $M/r$ the more accurate post-Newtonian approximation explains the anomalous precession of Mercury's perihelion. This more accurate post-Newtonian approximation is a particular case of the PPN formalism which also has associated retarded solutions, see [19].

## 3. Weber-type potentials

Before the beginnings of the General Relativity of Eintein, several tries to explain the anomalous precession of Mercury's perihelion had place. At the end of the 19th century, theoretical physicists were investigating modifications of the Coulomb inverse–square law. For instance, Gauss and Weber introduced a velocity–dependent potential to represent the electromagnetic field, consistent with the finite propagation speed of changes in the field. The application of this velocity–dependent potential to the gravitation was immediately. Several physicists proposed different gravitational potentials based on finite propagation speed in order to account for Mercury's orbital precession, see for instance [20, 21] for a review of these proposals.

In 1870 F.G. Holzmuller [6] proposed a law of gravitation of the same form that the electrodynamic Weber's law, given by

$$F = \frac{Gm_1m_2}{r^2}\left(1 - \frac{\dot{r}^2}{h^2} + \frac{2r\ddot{r}}{h^2}\right).$$

where $h$ is the finite propagation speed. Later, F. Tisserand [25] had used this law to study the anomalous precession of Mercury's perihelion and he explained only 14.1 arc seconds per century. In 1898, P. Gerber proposed a velocity–dependent potential that predicts exactly the observed value for the anomalous precession of Mercury's perihelion, see [7, 8].

The form of the proposed force laws are based, in general, in to do a parallelism between the electromagnetism and the gravitation and to propose what is known as gravitational field with a gravitoelectric component and with a gravitomagnetic component, see [3, 16] and references therein. These lines of research were abandoned when it was definitively implanted Einstein's Relativity Theory. Recently, in [12, 13] it have been shown that all these laws are, in fact, developments until a certain order of a retarded potential. The basic equation of motion proposed by the Weber models is

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM}{r^3}\left[1 + \frac{\xi}{c^2}(r\,\ddot{r} - \alpha\,\dot{r}^2)\right]\mathbf{r}\,, \tag{9}$$

where $\mathbf{r}$ is the relative radius vector of the particle with respect to the central mass, see [24]. The last term on the right-hand side of (9) is called the gravitational Weber force per unit of mass. We remark that the Weber's law of electrodynamic action corresponds to $\alpha = 1/2$ and $\xi = 1$, see [2]. With $\alpha = 1/2$, (9) is the expression adopted by Assis, see [1], who needed to fix $\xi = 6$ to obtain the right advance of the perihelion of the planets. Moreover, in [12] it was shown that (9) with $\alpha = 1/2$ and $\xi = 6$ is an approximation of the Gerber's potential proposed in 1898 which is, in fact, an approximation of the simple retarded potential

$$\begin{aligned} V &= -\frac{M}{r(t-\tau-\frac{r(t-\tau)}{c})} \approx -\frac{M}{r(t-\tau)}\frac{r(t)}{r(t-\tau)} \\ &\approx -\frac{M}{r}\frac{1}{\left(1-\frac{\dot{r}}{c}\right)^2} \approx -\frac{M}{r}\left[1+\frac{2\dot{r}}{c}+\frac{3\dot{r}^2}{c^2}\right], \end{aligned} \tag{10}$$

where the delay $\tau$ is equal to $r(t)/c$. Taking into account that the term $\dot{r}/c$ of (10) cancels in the computation of the associated force, we have that

$$V \approx -\frac{M}{r}\left[1+\frac{3\dot{r}^2}{c^2}\right]. \tag{11}$$

A law of motion of type (9) was first proposed by Tisserand [25] with $\alpha = 1/2$ and $\xi = 2$. Moreover, in [13], it was shown that this Weber type force (9) with $\alpha = 1/2$ and $\xi = 2$ gives the correct value of the gravitational deflection of fast particles of General Relativity.

The expansion in powers of $\dot{r}/c$ of the gravitational force law associated to the velocity-dependent potential function (10) is given by

$$f = -\frac{M}{r^2}\left(1 - \frac{3\dot{r}^2}{c^2} + \frac{6r\ddot{r}}{c^2} + \dots\right). \tag{12}$$

The radial and tangential components of the point's acceleration in polar coordinates are:

$$a_r = \ddot{r} - r\omega^2, \qquad a_t = r\dot{\omega} + 2\dot{r}\omega.$$

The conservation of the angular momentum gives $a_t = 0$. Equating the radial acceleration with the radial specific force, we get

$$\ddot{r} - r\omega^2 = -\frac{M}{r^2}(1 - 3\dot{r}^2 + 6r\ddot{r}). \tag{13}$$

If we define, as before, $u(\theta) = 1/r(t)$, then equation (13) becomes

$$(1 + 6Mu)\frac{d^2u}{d\theta^2} + u = \frac{M}{h^2}\left(1 - 3h^2\left(\frac{du}{d\theta}\right)^2\right), \tag{14}$$

The quantities inside the parentheses are both nearly equal to unity, because the terms added to or subtracted from 1 are many orders of magnitude less than 1 (for astronomical orbits). Moreover, as before, the quantity $3h^2(du/d\theta)^2$ will be negligible for planetary motions, see the computations in [18].

## 4. How is it possible that two so different theories explain the same phenomenon?

In General Relativity the gravitational field is described by 10 functions, the components of the symmetrical tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. On the other hand, the Weber-type potentials derived from retarded potentials are scalar potentials. The answer to this question is straightforward because in terms of the variable $u = 1/r$ the equations of motion (5) and (14) are only quite different in the negligible terms, taking into account that $r \approx h^2/m$ for small perturbations of closed orbits.

## 5. Is it possible a coherent inertia theory?

In [14], the inertial force proposed by Sciama, in a simple case, is derived from the Assis' inertia theory based in the introduction of a Weber type force. The origin of the inertial force is totally justified taking into account that the Weber force is, in fact, an approximation of a simple retarded potential, see [12, 13]. The way how inertial forces are also derived from some solutions of the general relativistic equations is presented in [26], see also [14]. The consequences of the present work show that the theory of inertia of Assis is not included in the framework of General Relativity. The perturbation of the Newton's inverse square law that gives the Schwarzschild solution for the case of a punctual or spherical and homogeneous mass has a similar form as the development with respect to the delay of the retarded scalar potential proposed in [12, 13]. However, it is not exactly the same as we will see in the following section. Anyway, the limit of the weak gravitational field of the General Relativity gives an effective scalar potential which is the approximation to the Einstein's field equations for this case.

## 6. The effective scalar potential

We have seen (see section 1) that in the particular case of the Mercury's perihelion problem, the General Relativity reduces to a Newtonian-like potential. The solution associated to the Schwarzchild space-metric gives the following radial equation component

$$\ddot{r} - r\omega^2 = -\frac{M}{r^2} - \frac{3Mh^2}{c^2 r^4},$$

see equation (4). Hence, we obtain an effective radial force perturbation of the Newton's inverse-square law of the form

$$F_r = -\frac{M}{r^2} - \frac{3Mh^2}{c^2 r^4}. \tag{15}$$

This radial force (15) has the associated potential energy given by

$$U = -\frac{M}{r} - \frac{Mh^2}{c^2 r^3},$$

where $F_r = -dU/dr$. If we consider that $h = r^2\omega$ we have that

$$U = -\frac{M}{r}\left(1 + \frac{r^2\omega^2}{c^2}\right). \tag{16}$$

We now consider the unperturbed Keplerian ellipse

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{p}{1+e\cos\theta}, \tag{17}$$

where $a$ is the semi-major axis, $e$ is the eccentricity of the ellipse, and $p$ is the semi-lactus rectum of the Keplerian ellipse. Notice that $e \in [0, 1)$ and when $e = 0$ we have a circle. We are going to evaluate the value of the tangential velocity $v_t = r\omega$ in the unperturbed Keplerian ellipse which will be an approximation of the tangential velocity of the perturbed orbit. From equation (17) we have

$$\dot{r} = \frac{p\ e\ \dot{\theta}\sin\theta}{(1+e\cos\theta)^2} = \frac{r^2\, e\ \omega\sin\theta}{p} = \frac{h\ e\sin\theta}{p}\,. \tag{18}$$

Therefore, we obtain that

$$r^2\omega^2 = \frac{h^2}{r^2} = \frac{h^2}{p^2}(1+e\cos\theta)^2 = \frac{(1+e\cos\theta)^2\dot{r}^2}{e^2\sin^2\theta}\,. \tag{19}$$

Hence, the potential energy (16) takes the form

$$U \approx -\frac{M}{r}\left[1+\left(\frac{1+e\cos\theta}{e\sin\theta}\right)^2\frac{\dot{r}^2}{c^2}\right], \tag{20}$$

which has a similar form to (11). On the other hand, the potential energy (16) is the first approximation of the potential energy

$$U = -\frac{M}{r\left(1-\frac{r^2\omega^2}{c^2}\right)} = -\frac{M}{r\left(1-\frac{v^2\sin^2\theta}{c^2}\right)}\,, \tag{21}$$

where $r\omega = r\dot{\theta} = v_t = v\sin\theta$. This potential function (21) is also similar to the Gravitational Lienard-Wiechert potential given in [3] page 5, for a flat Minkowski space-time.

Therefore, in the limit case of weak gravitational field the General Relativity gives an effective scalar potential of the form (16). This effective potential (16) explains the anomalous precession of Mercury's perihelion. The correct value of the gravitational deflection of fast particles of General Relativity is also explained from equation (4) taking $\varepsilon = 0$ to consider a null orbit, see [23].

## 7. The retarded potentials from General Relativity

The Parametrized Post-Newtonian approximation (PPN formalism) to Einstein's equation gives a PPN space-metric which has special retarded solutions where the delay associated to the speed of the gravitational interaction appears. The more accurate post-Newtonian approximation to General Relativity (which is a particular case of the PPN formalism) is obtained in [19] page 1089. Setting $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We have $h_{00} = 2U - 2U^2 + 4\Psi + O(\epsilon^6)$, $h_{0j} = -7V_j/2 - W_j/2 + O(\epsilon^5)$, and $h_{ij} = 2U\delta_{ij} + O(\epsilon^4)$ where

$$U(\mathbf{x},t) = \int \frac{\rho_0(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\ d^3x',$$

$$\Psi(\mathbf{x},t) = \int \frac{\rho_0(\mathbf{x}',t)\psi(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\, d^3x',$$

$$V_j(\mathbf{x},t) = \int \frac{\rho_0(\mathbf{x}',t)v_j(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\, d^3x',$$

$$W_j(\mathbf{x},t) = \int \frac{\rho_0(\mathbf{x}',t)[(\mathbf{x}-\mathbf{x}')\cdot\nu(\mathbf{x}',t)](x_j-x'_j)}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x',$$

with $\psi = \nu^2 + U + \Pi/2 + 3p/(2\rho_0)$ and where $\rho_0$ is the baryon "mass" density, $\Pi$ is the specific internal energy density and $p$ is the pressure. All these potentials appear because we approximate the retarded integrals by expansions in the time derivatives; thus we have for instance

$$\int \frac{\rho_0(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' = \int \left[\frac{\rho_0(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} - \frac{\partial\rho_0(\mathbf{x}',t)}{\partial t} + \cdots\right] d^3x'.$$

Hence, all the potentials $U$, $\Psi$, $V_j$ and $W_j$ come from the Einstein's field equation where the delay appears in a natural way. Everything points out that, for amenable situations, General Relativity would reduce to retarded Newtonian-like potentials if we do not do expansions with respect to the delay.

## 8. Concluding remarks

The present paper is the last in a series of works where it has been shown that the retarded potentials can also explain some primordial phenomena of the General Relativity and the Quantum mechanics and the connections between them, see [9, 10, 11, 12, 13]. This paper show that, in fact, these exact retarded potentials do not appear in the classical General Relativity Theory. The possibility that these retarded potentials would be an alternative theory to the General Relativity is an absurdity due to the quantity of experimental tests in accordance with the General Relativity. Anyway, in the classical General Relativity, the delay due to the finite propagation velocity appears in a natural way and this fact suggests the possibility to find the connection between the General Relativity and the Quantum mechanics and also to demonstrate that, in fact, the gravitational quantization is another consequence of General Relativity. This conviction is based in what happens in the alternative simple theory of retarded potentials of Weber-type. In this theory, the approximation to the retarded potential is given by the velocity dependent potential (11). Moreover, this potential function (11) is of Weber-type which has associated a Weber-type force of the form (9) with $\alpha = 1/2$ and $\xi = 6$. On the other hand, as we have seen, this potential function (11) is the approximation of the the simplest retarded potential (10). And this retarded potential (10) has the associated force of the form

$$f = -\frac{M}{[r(t-\tau)]^2}(1 + \cdots), \tag{22}$$

The form of this associated force justifies the introduction in [10, 11] of the retarded inverse square force to explain the the quantized Bohr atomic model and the gravitational quantification with an explanation of the modified Titius–Bode law, respectively. These two works give models too simple to show that it is possible to obtain the quantization from a retarded force of the form (22). For the electromagnetic classical field, in the ideal case of a point–charge particle, the resulting retarded potentials are the Liénard–Wiechert potentials. From the Lorentz force associated to these Liénard–Wiechert potentials we must find the origin of the quantum mechanics in the general case. However, the correct quantization of the gravitational field must be obtained using the Einstein's field equation and the delay, which appears in a natural way in the Einstein's field equation. However, it is difficult to work with the Einstein's field equations to deduce this consequence and it is still an open problem.

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