QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **6**, 273–287 (2007)
ARTICLE NO. 102

A Note on Higher Order Melnikov Functions

Ahmed Jebrane

Institut de Mathématiques de Bourgogne, UMR 5584 CNRS, Université de Bourgogne, 9 Av. Alain Savary, BP 47870, 21078 Dijon Cedex, France.
E-mail: jebrane@u-bourgogne.fr

and

Henryk Żołądek*

Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland.
E-mail: zoladek@mimuw.edu.pl

We present several classes of planar polynomial Hamilton systems and their polynomial perturbations leading to vanishing of the first Melnikov integral. We discuss the form of higher order Melnikov integrals. In particular, we present new examples where the second order Melnikov integral is not an Abelian integral.

Key Words: Higher order Melnikov integral, Abelian integral.

1. INTRODUCTION

It is well known that the problem of limit cycles for a polynomial perturbation $X_H + \varepsilon V$, of a polynomial Hamiltonian vector field X_H leads to the so called Poincaré–Pontryagin integral (see [1])

$$I_\omega(h) = \int_{\gamma_h} \omega, \tag{1}$$

where γ_h is a continuous family of ovals of the curves $H(x, y) = h$ and the polynomial-form ω is associated to the vector field V via the Pfaff equation

$$dH - \varepsilon\omega = 0 \tag{2}$$

* Supported by Polish MEN Grant No 1 P03A 015 29

for phase curves. But there are situations (which we present in detail in further sections) where the integral (1) vanishes identically, but the Poincaré return map is not identity. In such cases one obtains an expansion of the increment ΔH (along a trajectory near γ_h) in powers of ε of the form

$$\Delta H = \varepsilon^k M_k(h) + O(\varepsilon^{k+1})$$

where $M_k \not\equiv 0$ is called the *higher-order Melnikov function*, or the *principal Poincaré–Pontryagin integral* (like in [14] or [5]).

It is easy to express M_k via iterated integrals. Consider M_2 for example. Then equation (2) has the approximate first integral $H_\varepsilon = H - \varepsilon H_1$, where $H_1(P) = \int_{P_0}^{P} \omega$ and the integration is along a path in the curve γ_h joining a fixed point $P_0 = P_0(h)$ (the intersection point of γ_h with a fixed section S) with P. Then the increment of H_ε equals

$$\Delta H_\varepsilon \approx \varepsilon \int_{H_\varepsilon = h} \omega = \varepsilon^2 \int_{\gamma_h} H_1 \frac{d\omega}{dH} + O(\varepsilon^3) = \varepsilon^2 \int_{\gamma_h} \frac{d\omega}{dH} \int_{P_0}^{P} \omega + O(\varepsilon^3). \quad (3)$$

(Above $\frac{d\omega}{dH}$ is the Gelfand–Leray residue form, see [2].)

The interest in the higher order Melnikov functions arose when in 1996 J.-P. Françoise [4] gave an extremely simple algorithm for successive calculation of the higher Melnikov functions in form of Abelian integrals. The *Françoise's algorithm* works under so-called *$*$-property*:

$$\text{if} \quad I_\omega(h) \equiv 0 \quad \text{then} \quad \omega = f dH + dH_1 \qquad (4)$$

for some polynomials f and H_1. Then the function H_1 in (3) becomes a polynomial, $\frac{d\omega}{dH} = df$, and we get

$$M_2 = \int_{\gamma_h} H_1 df = -\int_{\gamma_h} f dH_1 = -\int_{\gamma_h} f\omega. \qquad (5)$$

If $M_2 \equiv 0$ then we can again use the $*$-property and repeat the above calculations. In essence the Françoise's algorithm allows to calculate successive approximations to the supposed first integral and integrating factor.

Probably the simplest example of a Hamiltonian function without the $*$-property is the following elliptic Hamiltonian

$$H(x, y) = y^2 \pm (x^2 - 1)^2 = f_1 f_2, \qquad (6)$$

where $f_{1,2} = y \pm \sqrt{\mp 1}(x^2 - 1)$. Then the 1-form $\eta = xydx$ satisfies $\int_{\gamma_h} \eta \equiv 0$, but $\int_{\delta_\pm(h)} \eta \not\equiv 0$, where γ_h is a cycle in the complex curve defined by $H = h$

vanishing at $x = y = 0$ and $\delta_\pm(h)$ are cycles vanishing at $x = \pm 1, y = 0$ (see [2]). A. Jebrane, P. Mardešić and M. Pelletier [10, 11] extended the Françoise's algorithm to polynomial perturbations of the equation $dH = 0$ for the Hamiltonian (6). (In fact they considered $H(x, y) = y^2 + (x^2 - 1)^2$, but there is no essential difference between the two cases.) They showed the property (4) but with f and H_1 depending on x, y, H^{-1} and on the auxiliary angle function

$$\phi = \log(f_1/f_2). \tag{7}$$

The latter function satisfies $d\phi = 8\eta - 2d[y(x^2-1)]$. Moreover, it turned out any higher order Melnikov function is an Abelian integral (see also [5, 6]).

Another case is the so-called *Hamiltonian triangle*

$$H = f_1 f_2 f_3 \tag{8}$$

where f_i are affine functions. It was was considered by M. Uribe [13]. He showed that the higher order Melnikov functions are combinations of tree basic integrals I_0, I_1, and I^*, where I_0, I_1 are Abelian integrals and $I^* = \int \log f_1 d(\log f_2/f_3)$. The coefficients of these combinations depend on h and $1/h$.

The case of quadratic perturbations of the situation (8) was firstly studied by the second author in [15]. There the integral I^* appeared for the first time. I. Iliev in [7] showed that the integral I^* has a singularity at $h = 0$ of the form

$$c_0 + c_1 h \log^2 h + \dots . \tag{9}$$

(see also [8]). Since any Abelian integral of the type (1) with a rational 1-form ω has singularities of the form

$$\alpha_0(h) + \alpha_1(h) \log(h - h_c) + \dots \tag{10}$$

(where the functions $\alpha_{0,1}(h)$ are meromorphic near a singular point h_c), it follows that the function $I^*(h)$ is not an Abelian integral. (The expansion (10) is a well known consequence of the Picard–Lefeschetz formula, see [2].)

In the subsequent paper [14] Uribe generalized the result from [13] to the case

$$H = f_1 f_2 \dots f_d \tag{11}$$

where f_i are affine linear functions in general position. He proved existence of basic Abelian integrals I_i and non-Abelian integrals

$$I^*_{i,j} = \int \log(f_i) d \log(f_j), \quad 2 \le i < j \le d, \tag{12}$$

such that M_k belongs to the $\mathbb{R}[h, 1/h]-$module generated by I_i and $I_{i,j}^*$.

On the other hand, L. Gavrilov proved in [5] that the function $M_k(h)$ satisfies a linear differential equation with rational coefficients (see also [3]). In particular, he presented explicitly equations satisfied by the tree basic integrals for the Hamiltonian (8).

In this paper we look at the higher order Melnikov functions from rather geometrical point of view. Namely, we try to describe the (linear) space

$$\mathcal{X} = \{\omega : I_\omega(h) \equiv 0\}.$$

When H has $*$-property, this space is generated by two subspaces

$$\mathcal{X}_1 = \{gdH : g \in \mathbb{R}[x,y]\}, \quad \mathcal{X}_2 = \{df : f \in \mathbb{R}[x,y]\}. \tag{13}$$

Each of them corresponds to forms ω such that the Pfaff equation (2) has center.

We give examples when $\mathcal{X}$ is an algebraic sum of several spaces $\mathcal{X}_i$ such that forms $\omega \in \mathcal{X}_i$ lead to equation (2) with center. (For instance, this holds in the case (13).) In such situation sometimes M_2 is an Abelian integral and sometimes it is not.

Another source of vanishing of the linear Poincaré–Pontryagin integrals is a symmetry of the Hamiltonian and of the form ω. This symmetry may act on dH and on ω in the same or in different ways. Some symmetries of order 2, namely the reflections, are generalized to a so-called rational reversibility property (introduced in [16]). In the symmetric case the corresponding equation (2) may have not center. Depending on a kind of the symmetry the function M_2 is or is not an Abelian integral.

2. THE MAP *INT* AND THE STAR PROPERTY

Let us fix the Hamilton function $H(x,y)$ and a family of its ovals $\{\gamma_h : h_1 \leq h \leq h_2\}$, where $h_{1,2}$ are suitable critical values of H. The linear Melnikov integral defines a linear map

$$Int_\gamma : \Lambda^1\mathbb{R}^2 \otimes \mathbb{R}[x,y] \longrightarrow C^\infty((h_1,h_2)) \tag{14}$$

where $\Lambda^1\mathbb{R}^2 \otimes \mathbb{R}[x,y]$ denotes the space of real polynomial 1-forms. Our problem is to describe the subspace

$$\mathcal{X} = \ker(Int). \tag{15}$$

The space $\mathcal{X}$ contains the subspaces

$$\mathcal{X}_1 = \mathbb{R}[x, y] \cdot dH, \quad \mathcal{X}_2 = d(\mathbb{R}[x, y]) \tag{16}$$

with the intersection $\mathcal{X}_1 \cap \mathcal{X}_2 = \mathbb{R}[H] \cdot dH$. The $*$-property states that $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$.

We note the following

LEMMA 1. *(i) If $\omega = gdH \in \mathcal{X}_1$ then the equation $dH - \varepsilon\omega = 0$ has H as first integral with the integral factor $R = 1 + \varepsilon g$.*

(ii) If $\omega = dH_1 \in \mathcal{X}_2$, then the equation $dH - \varepsilon\omega = 0$ has the first integral $H - \varepsilon H_1$ with the integral factor $R = 1$ (it is a Hamiltonian equation).

Yu. Il'yashenko [9] proved that the $*$-property is typical.

THEOREM 2 (Il'yashenko). *If the Hamiltonian H of degree d is generic, i.e. has Morse critical points with different critical values and d points at infinity, then $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$.*

EXAMPLE 3. Any quadratic polynomial with center satisfying the assumption of the Il'yashenko theorem is equivalent to $x^2 + y^2$. Here the $*$-property can be checked directly.

PROBLEM. Take a cubic Hamiltonian $H = x^2 + y^2 + Ax^3 + Bxy^2 + Cy^3$ which satisfies the assumptions of the Il'yashenko theorem and a 1-form

$$(ax+by)dH + d(\alpha x^3 + \beta x^2 y + \gamma xy^2 + \delta y^3 + \varepsilon x^4 + \nu x^3 y + \theta x^2 y^2 + \zeta xy^3 + \eta y^4)$$

where $a, b, \alpha, \dots, \eta$ are small. Apply the Françoise's algorithm and detect the center conditions near $\mathcal{X}_H$ in the class of cubic vector fields. These are algebraic conditions for $a, b, \dots, \eta_0$.

In the case $H = x^2 + y^2$ and quadratic ω the Françoise's algorithm leads to the well known Dulac's center conditions (see [4], [15]).

3. DARBOUX FIRST INTEGRALS

One possibility of not satisfying the assumptions of the Il'yashenko theorem is that some critical level is reducible:

$$H - h_c = f_1 \dots f_d, \quad d \geq 2, \tag{17}$$

where $f_1, \dots, f_d$ are irreducible polynomials. Assume also that the factors f_i are pairwise different (below we remove this restriction). Then the *Darboux function*

$$f_1^{1+\delta_1} \dots f_d^{1+\delta_d} \tag{18}$$

is a perturbations of the function (17) and constitutes a first integral of the Pfaff equation

$$dH + (H - h_c) \sum \delta_j d(\log f_j) = 0.$$

Therefore, for any polynomial P the integral of the form $P(H)(H - h_c)d(\log f_j)$ along a cycle γ_h is zero.

So we can consider a general perturbation

$$dH + \sum_{j=1}^{d} \delta_j P_j(H)(H - h_c)d(\log f_j) + \varepsilon_1 g dH + \varepsilon_2 dH_1 = 0 \tag{19}$$

of the equation $dH = 0$. H. Movasatti [12] proved that, in the case of affine linear functions f_j in generic position, any perturbation of dH from $\mathcal{X} = \ker(int)$ is of the form (19) (see also [14]).

Theorem 4. *Take H like in (17) and the perturbation (19).*

If $d = 2$ then M_2 is an Abelian integral.

If $d \geq 3$ then M_2 can be not an Abelian integral. It can contain some integral I_{ij}^* of the form (12) with a nonzero coefficient, which depends rationally on h.

Remark 5. Recall that Uribe [13, 14] proved the representation of M_k as linear combination, with coefficients being Laurent polynomials of $h - h_c$, of some basic Abelian integrals and the integrals (12), provided f_j's are affine-linear functions in generic position. He gave an example when the coefficient before I_{ij}^* does not vanish. But in the case of Hamiltonian triangle (i.e. $d = 3$) and its quadratic perturbations the coefficient before $I^* = I_{12}^* - I_{13}^*$ in M_2 vanishes. In M_3 this coefficient is nonzero. The reason is that the perturbation leading to nonzero I^*-part in M_2 must be of sufficiently high degree (see the proof of Theorem2).

Remark 6. The Hamiltonian (6) considered by Jebrane, Mardešić and Pelletier is a particular case of (17) with $d = 2$. Recall that in [11] it was proved that M_k is an Abelian integral for any k. The proof is essentially algebraic.

Maybe in general case with $H = f_1 f_2$ and such that

$$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 + \mathbb{R}[H] \cdot Hd(\log f_1) + \mathbb{R}[H] \cdot Hd(\log f_2) \ ,$$

M_k is an Abelian integral for any k.

Before proving Theorem 4 we interpret the perturbations appearing in (19).

Lemma 7. *The Pfaff equation $dH + \delta_1 P_1(H)(H - h_c)d(\log f_1) = 0$ has center.*

Proof. We rewrite this equation on the form

$$\frac{dH}{P_j(H)(H-h_c)} + \delta_1 d(\log f_1) = 0.$$

Here the variables H and f_1 are separated and integration gives a first integral of the type $W + \delta_1 \log f_1$ (with a rational function W). ∎

Proof of Theorem 4. The form $\sum \delta_j (H-h_c) d(\log f_j)$ for $\delta_1 = \cdots = \delta_d$ is proportional to $(H-h_c)d(\log H) = dH$. Therefore in the representation (19) we can choose $\delta_d = 0$ (like in [14]).

If $d = 2$, then we get

$$dH + \delta_1 P_1(H)(H-h_c)d(\log f_1) + \varepsilon_1 g dH + \varepsilon_2 dH_1.$$

By Lemma 7 it is enough to show that the terms in M_2 before $\delta_1\varepsilon_1$ and $\delta_2\varepsilon_2$ are Abelian integrals. They are equal to

$$-P_1(h)\int_{H=h} g(H-h_c)d(\log f_1), \qquad \int_{H=h} H_1 \frac{d[P_1(H)(H-h_c)d(\log f_1)]}{dH}$$

respectively.

Let $d \geq 3$ and assume that $h_c = 0$. Take the perturbation

$$dH + \delta_1 H d(\log f_1) + \delta_2 H^2 d(\log f_2) = 0$$

and calculate the term before $\delta_1\delta_2$ in M_2 (note the different powers of H).

Since

$$dH + \delta_1 H d(\log f_1) = f_1^{-\delta_1} d(f_1^{1+\delta_1} f_2 \ldots f_d) = f_1^{-\delta_1} d(H_{\delta_1}),$$

$H_{\delta_1} \approx H \cdot (1 + \delta_1 \log f_1)$, we get the Pfaff equation

$$dH_{\delta_1} + f_1^{\delta_1} \cdot \delta_2 H^2 d(\log f_2) = 0.$$

From this we find that the increment of the function H_{δ_1} equals

$$-\delta_2 \Phi(h, \delta_1) = -\delta_2 \int_{H_{\delta_1}=h} f_1^{2+\delta_1} f_2^2 f_3^2 \ldots f_d^2 d(\log f_2)$$

Here $\Phi(h, 0) \equiv 0$. The linear in δ_1 part of Φ equals

$$\delta_1 \int_{H=h} (f_1 f_2 \ldots f_d)^2 \log f_1 d(\log f_2) - \delta_1 \int_{H=h} \frac{d[(f_1 f_2 \ldots f_d)^2 d(\log f_2)]}{dH} H \log f_1$$

$$= -\delta_1 h^2 \int_{H=h} \log f_1 d(\log f_2).$$

(Here we have used the formula $d[f_1^2 \dots f_d^2 d(\log f_2)] = 2HdH \wedge d(\log f_2)$. Note that for $d = 2$ we have $\log f_2 = -\log f_1$ along γ_h and this proof does not work.)

In [14] it is proved that in the case of affine-linear f_j's the integrals I_{ij}^* are non-Abelian. From this it is easy to conclude that they are non-Abelian in our case. ▮

Consider now the non-reduced case of the factorization (17):

$$H - h_c = f_1^{1+r_1} \dots f_d^{1+r_d} \tag{20}$$

where some $r_i > 1$, say $r_1 > 1$.

Now some other terms may appear in (19). Namely the factor $f_1^{1+\delta_1}$ in the Darboux integral (18) can replaced by $f_1^{1+r_1+\delta_1} \exp(\nu_1 Q(1/f_1)$ where Q is a polynomial of degree $\leq r_1$ and δ_1, ν_1 are small. The general term which can be added to (19) is

$$\varepsilon_0 (H - h_c) d\left(\frac{g}{f_1^{r_1} \dots f_d^{r_d}}\right) = \varepsilon_0 \omega_0$$

where g is a general polynomial and ε_0 is a small parameter.

LEMMA 8. *The Pfaff equation $dH + \varepsilon_0 \omega_0 = 0$ has center.*

Proof. This Pfaff equation is equivalent to the equation

$$\frac{d(H - h_c)}{H - h_c} + \varepsilon_0 d(g f_1^{-r_1} \dots f_d^{-r_d}) = 0.$$

It has the first integral

$$f_1^{\alpha_1} \dots f_d^{\alpha_d} \exp\left[\varepsilon_0 g f_1^{-r_1} \dots f_d^{-r_d}\right]. \tag{21}$$

▮

Functions of the form (21) are called the *generalized Darboux functions.* They (and their logarithms) are limits of the Darboux functions (18).

We finish this subsection by remarking that, when the polynomial has several 'atypical' values h_{c_i} such that the polynomials $H - h_{c_i}$ are reducible, then the expansion of the type (19) should contain suitable summands associated with each critical value.

4. ROTATIONAL SYMMETRY

There are two series of finite group of symmetries of the plane: the cyclic groups C_m, of rotations by angles $\frac{2\pi j}{m}$, and the dihedral groups D_m, generated by two reflections with respect to lines with angle $\frac{2\pi}{m}$ between them.

Vanishing of some linear Melnikov integrals may be caused by such a symmetry.

Consider firstly the rotation group C_m. Denote by $\rho : z \to e^{2\pi i/m} z$ its generator where $z = x+iy$ is the complex coordinate, after an identification of $\mathbb{R}^2$ with $\mathbb{C}$. Therefore we assume that

$$\rho^* H = H, \quad \rho(\gamma_h) = \gamma_h, \tag{22}$$

where $\rho^*(H) = \rho \circ H$. Note that we get $\rho^* dH = dH$.

The real 1-forms $\omega = Adx + Bdy$ can be written in the form

$$\omega = C(z, \bar{z})dz + \overline{C(z, \bar{z})}d\bar{z} = 2\Re(Cdz).$$

We say that a complex form η is a *semi-invariant* (with respect to the group C_m) if

$$\rho^* \eta = \chi_\eta \cdot \eta, \tag{23}$$

where

$$\chi_\eta = \chi_\eta(\rho) : C_m \to S^1 \subset \mathbb{C} \text{ is a } \textit{character} \text{ (of } C_m).$$

Of course, $\chi_\eta(\rho) = e^{2\pi i l/m}$ for some integer l and $\chi_\eta(\rho^k) = \chi_\eta^k$.

PROPOSITION 9. *If ω is a real part of a complex form η which is a semi-invariant with nontrivial character $\chi \neq 1$, then the linear Melnikov integral of ω along γ_h vanishes.*

Proof. We divide the oval γ_h which surrounds the origin into m pieces $\gamma_h^0, \dots \gamma_h^{m-1}$ such that $\rho(\gamma_h^j) = \gamma_h^{j+1}$ with preserved orientation. Then

$$I_\eta(h) = \sum_{j=0}^{m-1} \int_{\rho^j(\gamma_h^0)} \eta = \sum_{j=0}^{m-1} \int_{\gamma_h^0} (\rho^*)^j \eta = \left(\sum_{j=0}^{m-1} \chi_\eta^j \right) \int_{\gamma_h^0} \eta = 0.$$

∎

EXAMPLE 10. Let $H(x,y) = x^2+y^2+K(x,y)$, where K is a homogeneous quartic polynomial, and let γ_h for $h > 0$ be the ovals ovals of $H = h$ around $x = y = 0$ for small h. Of course, H and γ_h are invariant with respect to the group C_2 generated by the central symmetry, $\rho(x,y) = (-x,-y)$. Any homogeneous quadratic form ω is a semi-invariant of the group C_2 with $\chi_\omega = -1$. Therefore $\omega \in \ker(Int_\gamma)$.

Let us consider the second order Melnikov function associated with perturbations of dH by semi-invariant forms.

Let H be C_m-invariant and such that the cycle γ_h at a critical value h_0 degenerates to a polycycle consisting of non-degenerate saddle points $P_0, \ldots, P_{m-1}$ and m connections $\gamma_{h_c}^0, \ldots, \gamma_{h_c}^{m-1}$ between the consecutive saddles points. Let $\delta_0, \ldots, \delta_{m-1}$ be the cycles vanishing at $P_0, \ldots, P_{m-1}$ respectively and such that $\rho(\delta_j) = \delta_{j+1}$ (see Figure 1).

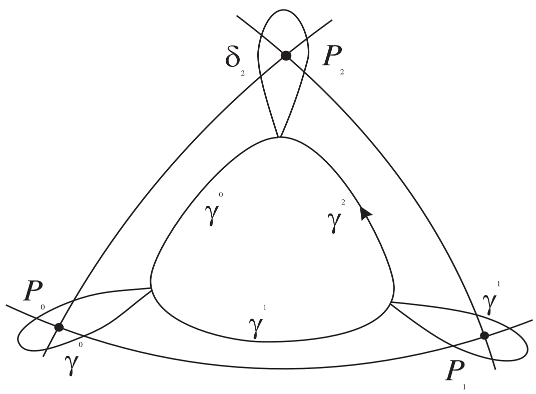

FIG. 1.

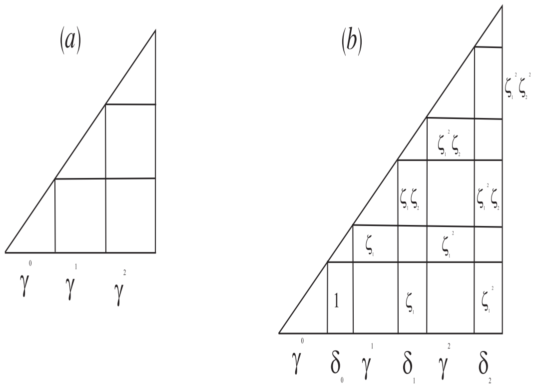

FIG. 2.

Let η_1 and η_2 be two complex 1-forms quasi-invariant with respect to C_m, with the characters $\chi_{\eta_1} = \zeta_1 \neq 1$ and $\chi_{\eta_2} = \zeta_2 \neq 1$ respectively. Assume also that

$$\begin{aligned} J_1' &= \int_{\gamma^0} \frac{d\eta_1}{dH} \not\equiv 0, \quad J_2 = \int_{\gamma^0} \eta_2 \not\equiv 0, \\ K_1' &= \int_{\delta_0} \frac{d\eta_1}{dH} \not\equiv 0, \quad K_2 = \int_{\delta_0} \eta_2 \not\equiv 0, \end{aligned} \tag{24}$$

(in fact, it is enough that $K_1' \not\equiv 0$, $K_2 \not\equiv 0$) and

$$\begin{aligned} \mu \;:=\; & (1+\zeta_1)\{[1+\zeta_1\zeta_2+\cdots+(\zeta_1\zeta_2)^{m-2}]+\zeta_1[1+\cdots+(\zeta_1\zeta_2)^{m-3}]+\ldots \\ & +\zeta_1^{m-2} \neq 0. \end{aligned} \tag{25}$$

Theorem 11. *Under the above assumptions the term before $\varepsilon_1\varepsilon_2$ in M_2 for the complex Pfaff perturbation $dH + \varepsilon_1\eta_1 + \varepsilon_2\eta_2 = 0$ is not an Abelian integral.*

Proof. Using the iterated integrals (see (3)) we represent our function in the form

$$M_2 = \int_{\gamma_h} \frac{d\eta_1}{dH}(P) \int_{P_0(h)}^{P} \eta_2 = \int\int_{\Delta} \frac{d\eta_1}{dH}(P)\eta_2(P).$$

Here the initial point $P_0 = P_0(h)$ is the intersection of γ_h with the ray $\mathbb{R}_+ P_{m-1}$ and Δ is a suitable triangular domain like in Figure 2(a). Choosing suitable orientation of δ_0 we can assume that the Picard–Lefschetz transformation corresponding to moving h around the critical value h_0 can be specified as follows (see [2]):

$$\gamma^j \to \gamma^j + \delta_j, \quad \delta_j \to \delta_j, \quad j = 0, 1, \ldots, m-1.$$

The resulting change in the domain Δ is presented at Figure 2(b). The variation of M_2 is the union of integrals over the squares $\delta_i \times \delta_j$, the triangles $\frac{1}{2}\delta_i \times \delta_j$ and the rectangles.

The integrals over the squares and the triangles are analytic. We denote their sum by $\Phi_1(h)$.

The integrals over the vertical rectangles equal

$$\int_{\delta_i} \frac{d\eta_1}{dH} \cdot \int_{\gamma^j} \eta_2 = \zeta_1^i \zeta_2^j \cdot K_1' J_2,$$

whereas the integrals over the horizontal rectangles are equal

$$\int_{\gamma^i} \frac{d\eta_1}{dH} \cdot \int_{\delta_j} \eta_2 = \zeta_1^i \zeta_2^j \cdot J_1' K_2.$$

Denote by $\Phi_2(h)$ the sum of integrals over the rectangles. Therefore

$$\mathrm{Var}(M_2) = \Phi_1 + \Phi_2. \tag{26}$$

Since $\mathrm{Var}(K_1' J_2) = \mathrm{Var}(J_1' K_2) = K_1' K_2$, the variation of Φ_2 equals $K_1' K_2$ times the sum of the coefficients $\zeta_1^i \zeta_2^j$ associated with the rectangles. It is easy to see that the latter sum equals the number μ defined in (25) (see Figure 2(b)). Thus

$$\mathrm{Var}(\Phi_1) = 0, \quad \mathrm{Var}(\Phi_2) = \mu \cdot K_1' K_2, \tag{27}$$

where $K_1' K_2$ is analytic. We obtain that under assumptions (23) and (24) the function Φ_2 is not analytic, it has a singularity of the form

$$\Phi_2(h) = \psi_2(h) + \frac{\mu}{2\pi i} K_1' K_2 \cdot \log(h - h_0). \tag{28}$$

From (26), (27) and (28) it follows that M_2 has singularity of the type $\log^2(h - h_0)$, which excludes the possibility of being an Abelian integral (compare(10)). ▮

It remains to construct an example satisfying the assumptions of Theorem 11. Unfortunately, in Example 10 above one must have $\zeta_1 = -1$ which implies $\mu = 0$; we do not know whether the integral M_2 is Abelian in this situation.

Example 12. Consider vector fields with the rational symmetry of order 3. Here $\mu = (1 + \zeta_1)(1 + \zeta_1 + \zeta_1 \zeta_2)$ is nonzero for $\zeta_2 = \zeta_1^2 = \zeta^{-1} = e^{-2\pi i/3}$. Take the following C_3-symmetric Hamiltonian function

$$H(x, y) = 3z\bar{z} - z^3 - \bar{z}^3 + az^4\bar{z} + \bar{a}z\bar{z}^4.$$

For small $a \in \mathbb{C} \setminus \{0\}$ it is close to the Hamiltonian triangle. The corresponding vector field has the saddle vertices P_0, P_1, P_2, which are close to $z = 1$, $z = \zeta = \zeta_1 = \frac{1 \pm \sqrt{3}}{2}$ and $z = \zeta^2$ respectively. The 1-forms

$$\eta_1 = \bar{z}^3 dz, \quad \eta_2 = \bar{\omega_1}$$

are semi-invariants with the characters $\zeta_1 = \zeta$ and $\zeta_2 = \zeta^2$; so $\mu \neq 0$. It is not difficult to check that the conditions (23) hold true.

This and Theorem 11 imply that M_2 is not an Abelian integral for the perturbation

$$dH - \varepsilon(\eta_1 + \bar{\eta}_1) = 0.$$

Remark 13. In contrast of the cases (of vanishing of I_ω) from Section 3, the (real) equation

$$dH - \varepsilon\omega = 0,$$

where ω is a quasi-invariants of C_m, generally does not have center.

5. REFLECTION SYMMETRY AND REVERSIBILITY

Any dihedral group D_m is generated by reflections, so it contains the subgroup $D_1 \approx \mathbb{Z}/2$. We can assume that the generator of D_1 is the following

$$\sigma : (x, y) \to (-x, y). \tag{29}$$

Any D_1-invariant Hamilton function is of the form

$$H = \tilde{H}(x^2, y) \tag{30}$$

and the form $dH = 2\tilde{H}'_{x^2}dx + \tilde{H}'_y dy$ satisfies $\sigma^*(dH) = dH$, i.e. it is *time–reversible.* We assume that $\sigma(\gamma_h) = \gamma_h$.

We say that a 1-form ω is *time–reversible* if $\sigma^*\omega = \omega$.

Note the following easy

LEMMA 14. *If ω is time–reversible then the Pfaff equation $dH - \varepsilon\omega = 0$ has center. If, additionally, H satisfies* $\ker_\gamma(Int) = \{\omega_0 + gdH + dH_1 : \sigma^*\omega_0 = \omega_0\}$ *then M_2 is an Abelian integral.*

Remark 15. The hyperelliptic Hamilton functions

$$H(x, y) = y^2 + S(x) \tag{31}$$

are D_1-invariant with the reflection $\tau : (x, y) \to (x, -y)$.

They satisfy the $*$-property. Indeed, any time-reversible 1-form equals $\omega = A(x, y^2)dx + yB(x, y^2)dy$. When we replace y^2 by $H - S(x)$ and ydy by $\frac{1}{2}(dH - S'(x)dx)$, we shall see that $\omega = C(x, H)dx \,(\mod (dH) = d(D(x, h)) \,(\mod dH)$.

The Jebrane–Mardešić–Pelletier's hamiltonian (6) is invariant with respect to the reflections σ and τ. Therefore the Hamiltonian (6) is degenerate in three ways. Here the $\sigma-$invariant 1-form $xydx$ does not belong to $\mathcal{X}_1 + \mathcal{X}_2 = \mathbb{R}[x, y] \cdot dH + d(\mathbb{R}[x, y])$.

It would be interesting to know whether M_3 is Abelian for generic Hamiltonian of the form $H = y^2 + S(x^2)$; probably not.

A natural generalization of the condition (30) is following. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial mapping which has a *fold-type singularity*

$$\Phi(x, y) = (X, Y) = (x^2 + \dots, y + \dots)$$

near $(0, 0)$. Denote by $\Gamma = \{\det d\Phi = 0\}$ the *fold curve* of Φ. We can put $H(0, 0) = 0$.

Assume that

$$H = \tilde{H} \circ \Phi, \tag{32}$$

where $\widetilde{H}(X, Y)$ is a polynomial such that $\tilde{H}(0, 0) = 0$ and the curve $\{\widetilde{H} = 0\}$ is tangent to $\Phi(\Gamma)$ at $(0, 0)$ from outside of the domain $\Phi(U)$, where U is a neighborhood of $(0, 0)$. In this case we say that H is *rationally reversible by means of* Φ.

Following [16] we say that a 1-form ω is *rationally reversible by means of* Φ if

$$\omega = \Phi^* \tilde{\omega}, \tag{33}$$

where $\tilde{\omega}(X, Y)$ is a polynomial 1-form in the target space.

Proposition 16. *The statements of Lemma 14 hold in the case when H and ω are both rationally revertible by means of Φ.*

Remark 17. Above we presented all known to us mechanisms leading to the vanishing of I_ω. Natural question is whether they generate $\ker(Int_\gamma)$ for any Hamiltonian function H and any family $\{\gamma_h\}$ of its ovals. We think that this is the case.

ACKNOWLEDGMENT

The results of this paper were obtained during visiting of the second author the Institute Mathématique de Bourgogne in Dijon. He wants to express his gratitude to the institute and to Ahmed Jebrane, Pavao Mardešić and Michelle Pelletier for hospitality and stimulating discussions.

REFERENCES

1. V. I. Arnold, *Geometrical methods in the theory of ordinary differential equations*, Springer–Verlag, New York, 1988.
2. V. I. Arnold, A. N. Varchenko and S. M. Gusein–Zade, *Singularities of Differential Maps*, v. 2, Monographs in Mathematics, Birkhäuser, Boston, 1988.
3. M. Bobieński, *Contour integrals as Ad-invariant functions on the fundamental group*, preprint, University of Warsaw, 2005.

4. J.-P. Françoise, *Successive derivatives of a first return map, application to the study of quadratic vector fields*. Ergodic Theory Dynam. Syst. **16** (1996), 87–96.

5. L. Gavrilov, *Higher order Poincaré–Pontryagin functions and iterated path integral*, Annales de la Faculté des Sciences de Toulouse **14** (2005), 663–682.

6. L. Gavrilov, I. D. Iliev, *The displacement map associated to polynomial unfoldings of planar Hamiltonian vector fields*, American Journal of Mathematics **12** (2005), 1153–1190.

7. I. D. Iliev, *The cyclicity of the period annulus of the quadratic Hamiltonian triangle*, J. Differential Equations **128** (1996), 309-326.

8. I. D. Iliev, *Higher-order Melnikov functions for degenerate cubic hamiltonians*, Adv. Differential Equations **1** (1996), 689–708.

9. Yu. S. Il'yashenko, *The origin of limit cycles under perturbation of the equation $dw/dz = -R_z/R_w$, where $R(z,w)$ is a polynomial*, Math. USSR Sbornik **78** (1969) 360–373 [Russian: Mat. Sbornik (N. S.) **83(125)** (1970), 452–455].

10. A. Jebrane, P. Mardešić and M. Peletier, *A generalization of Françoise's algorithm for calculating higher order Melnikov functions*, Bull. Sci. Math. **126** (2002), 705–732.

11. A. Jebrane, P. Mardešić and M. Peletier, *A note on a generalization of Françoise's algorithm for calculating higher order Melnikov functions*, Bull. Sci. Math. **128** (2004), 749–760.

12. H. Movasati, *Center conditions: rigidity of logarithmic differential equations*, J. Differential Equations **197** (2004), 197–217.

13. M. Uribe, *Principal Poincaré–Pontryagin function of polynomial perturbations of the Hamiltonian triangle*, J. Dyn. Control Syst. **12** (2006), 109–134.

14. M. Uribe, *Principal Poincaré–Pontryagin function associated to polynomial perturbations of a product of (d + 1) straight lines*, preprint, Université de Bourgogne, 2005.

15. H. Żołądek, *Quadratic systems with center and their perturbations*, J. Differential Equations **109** (1994), 223–273.

16. H. Żołądek, *The classification of reversible cubic systems with center*, Topol. Methods Nonlinear Anal., **4** (1994), 79–136.