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Box Dimension of Spiral Trajectories of some Vector Fields in $\mathbb{R}^3$

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We study the behaviour of Minkowski content of bounded sets under bi-Lipschitzian mappings. Applications include Minkowski contents and box dimensions of spirals in $\mathbb{R}^3$, dynamical systems, and singular integrals.

Key Words: bi-Lipschitz mapping, Minkowski content, box dimension, spiral, vector field

1. INTRODUCTION AND NOTATION

We are interested in finding the box dimension of spiral trajectories appearing as solutions of some dynamical systems in $\mathbb{R}^3$. This value can be viewed as a measure of "dimensional concentration" of the spiral near its limit set. We deal with a class of systems such that the corresponding linear part has a pure imaginary pair and a simple zero eigenvalues, see Theorems 18 and 20. It is interesting that it is possible to construct dynamical systems in $\mathbb{R}^3$ such that the box dimension of their spiral trajectories is sensitive on the coefficients of the system, see Proposition 21.

Important role in our study of Minkowski content and box dimension of a spiral trajectory in $\mathbb{R}^3$ is played by the natural two-dimensional surface containing the spiral near its limit set. According to whether the surface is of Lipschitz or of Hölder type near the limit set, we distinguish the following two cases: Lipschitzian spirals (see Theorem 7) and Hölderian spirals (see Theorem 9). We also study spirals of focus and limit cycle type. We thus obtain four types of spirals in $\mathbb{R}^3$: Lipschitz-focus spirals, Lipschitz-cycle spirals, Hölder-focus spirals, and Hölder-cycle spirals. See Figures 1 and 2 on p. 257. Basic tool in the study of spirals in the space is provided by a general result concerning behaviour of Minkowski contents of a bounded set under bi-Lipschitz mappings. It turns out that nondegeneracy or degeneracy of a set defined below is not affected by bi-Lipschitz mappings, see Theorem 1.

It is interesting that the orthogonal projection of the Hölder-focus spiral Γ defined in Theorem 9(a) onto (y,z)-plane is the curve $\Gamma_{y,z}$ defined by $y = z^{1/\beta}\sin(z^{-1/\alpha\beta})$. Using the formula of Tricot [15, p. 122] we obtain that box dimensions of Γ and its projection $\Gamma_{y,z}$ coincide. In the case of Lipschitz-focus spiral in Theorem 7(a) this is not the case, see Corollary 11. From Corollary 11 we also see that only one of coordinate projections P_{xy} and P_{yz} preserves box dimension of the spiral in $\mathbb{R}^3$. Generalizations of graphs of functions like $\Gamma_{y,z}$ appear in Pašić [13] and Pašić, Županović [12] in the study of box dimensions of graphs of weak solutions of the one-dimensional p-Laplace equation.

This work is related to the study of spiral trajectories of planar vector fields, see [19]. Let us mention that in Dupain, Mendès France, and Tricot [3] the Steinhaus dimension of planar spirals is studied. The question of box dimension of planar spirals is considered in Tricot [15, p. 122]. General properties of Minkowski contents of fractal strings and related problems are treated in He, Lapidus [8], and in Lapidus, van Frankenhuysen [10]. The role of fractal dimensions in dynamics is described in a survey article by Županović and Žubrinić [21]. Some of the results of this paper have been announced without proofs in [20].

The paper is organized as follows:

(1) Introduction and notation
(2) Bi-Lipschitz mappings and Minkowski content
(3) Lipschitzian and Hölderian spirals of focus and limit cycle types in $\mathbb{R}^3$
(4) Box dimension of spiral trajectories of some vector fields in $\mathbb{R}^3$
(5) Singular integrals generated by spirals in $\mathbb{R}^3$.

Now we introduce some notation. Let A be a bounded set in $\mathbb{R}^N$, and let $d(x,A)$ be Euclidean distance from x to A. Then the *Minkowski sausage* of radius ε around A is defined as ε-neighbourhood of A, that is, as the set $A_\varepsilon := \{y \in \mathbb{R}^N : d(y,A) < \varepsilon\}$. By *lower s-dimensional Minkowski content* of A, $s \geq 0$, we mean the following quantity: $\mathcal{M}_*^s(A) := \liminf_{\varepsilon\to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}}$, where $|\cdot|$ denotes N-dimensional Lebesgue measure. Analogously for the *upper s-dimensional Minkowski content* of A. The corresponding lower and upper box dimensions are denoted by $\underline{\dim}_B A$ and $\overline{\dim}_B A$. See Falconer [5] or Mattila [11].

If A is such that $\underline{\dim}_B A = \overline{\dim}_B A$, the common value is denoted by $d := \dim_B A$. Furthermore, if both the upper and lower d-dimensional Minkowski contents of A are different from 0 and ∞, we say that the set A has *nondegenerate* Minkowski contents (or shorter, that A is nondegenerate). If in addition to this we have $\mathcal{M}_*^d(A) = \mathcal{M}^{*d}(A) =: \mathcal{M}^d(A) \in (0, \infty)$, we say that A is *Minkowski measurable.* Nondegeneracy assumption on A is important in the study of singular integrals of the form $\int_{A_\varepsilon} d(x, A)^{-\gamma} dx$, see [17] and [18].

We say that a function $F : \Omega \to \Omega'$, where $\Omega, \Omega' \subseteq \mathbb{R}^N$ are open sets, is bi-Lipschitzian if there exist two positive constants $\underline{C}$ and $\overline{C}$ such that $\underline{C}|x - y| \leq |F(x) - F(y)| \leq \overline{C}|x - y|$ for all x, $y \in \Omega$. The constants $\underline{C}$ and $\overline{C}$ will be called lower and upper Lipschitz constants of F respectively. The Jacobian of a Lipschitz mapping F is $J_F(x) := \det F'(x)$, and it is easy to see to be defined for a.e. $x \in \Omega$ and measurable. Its L^∞-norm will be denoted by $\|J_F\|_\infty$.

For two functions $f, g : I \to (0, \infty)$, where I is a subset of $\mathbb{R}$, we say to be comparable, and write $f(r) \simeq g(r)$, if there exist two positive constants $\underline{c}$ and $\overline{c}$ such that $\underline{c}g(r) \leq f(r) \leq \overline{c}g(r)$ for all $r \in I$, that is, $f(r)/g(r) \in [\underline{c}, \overline{c}]$. A constant C in proofs may change its value from line to line.

2. BI-LIPSCHITZ MAPPINGS AND MINKOWSKI CONTENT

The following result refines the known fact that box dimension of a set is preserved under bi-Lipschitzian mappings, see Falconer [5, p. 44]. In particular, it shows that nondegeneracy and degeneracy of fractals are preserved under bi-Lipschitz mappings.

Theorem 1. (Minkowski content under bi-Lipschitz mappings) *Let Ω and Ω' be open sets in $\mathbb{R}^N$, and let $F : \Omega \to \Omega'$ be a bi-Lipschitz mapping with lower and upper Lipschitz constants equal to $\underline{C}$ and $\overline{C}$ respectively. Let A be a bounded set such that $\overline{A} \subseteq \Omega$. Then for all $s \geq 0$ we have*

$$\frac{1}{\overline{C}^{N-s}\|J_{F^{-1}}\|_\infty} \mathcal{M}^{*s}(A) \leq \mathcal{M}^{*s}(F(A)) \leq \frac{\|J_F\|_\infty}{\underline{C}^{N-s}} \mathcal{M}^{*s}(A), \tag{1}$$

and analogously for the lower s-dimensional Minkowski content, that is, for $\mathcal{M}^{*s}$ replaced with $\mathcal{M}_*^s$. In particular, we have $\overline{\dim}_B F(A) = \overline{\dim}_B A$ and $\underline{\dim}_B F(A) = \underline{\dim}_B A$. Furthermore, for all bi-Lipschitz mappings F we have

$$\|J_F\|_\infty \leq N!\overline{C}^N, \quad \|J_{F^{-1}}\|_\infty \leq N!\underline{C}^{-N}. \tag{2}$$

Remark 2. Estimate (2) is rough. In some concrete examples the value of $\|J_F\|_\infty$ can be better estimated, or even precisely computed. For example, if $F : \mathbb{R}^N \to \mathbb{R}^N$ is a similarity of the form $F(x) = c \cdot Sx + b$, where $c > 0$, $b \in \mathbb{R}^N$, and S is orthogonal matrix, then using Theorem 1 we conclude that for any bounded set A and all $s \geq 0$,

$$\mathcal{M}^{*s}(F(A)) = c^s \mathcal{M}^{*s}(A), \tag{3}$$

and analogously for $\mathcal{M}_*^s$. This follows easily from $\underline{C} = \overline{C} = c$ and $J_F(x) \equiv c^N$. In particular, if A is Minkowski measurable, so is $F(A)$ in this case.

The following result, that we shall need in the proof of Theorem 1, is a simple consequence of the change of variables formula. It is very probably known in this generality, but we were not able to find it in the literature.

LEMMA 3. *Let $F : \Omega \to \Omega'$, where $\Omega, \Omega' \subseteq \mathbb{R}^N$ are open sets, be an injective Lipschitz mapping. Then for any measurable set E in Ω such that $|E| < \infty$ we have*

$$|F(E)| = \int_E |J_F(x)|\, dx. \tag{4}$$

Proof. We use the change of variables formula, see e.g. Evans and Gariepy [4, p. 108, the case of $m = n = N$, modified to arbitrary open domain Ω]: if $F : \Omega \to \mathbb{R}^N$ is a Lipschitz function then for any $u \in L^1(\Omega)$ we have

$$\int_\Omega u(x)\, |J_F(x)|\, dx = \int_{F(\Omega)} \left[\sum_{x \in F^{-1}(y)} u(x) \right] dy, \tag{5}$$

and the set $F^{-1}(y)$ is at most countable for a.e. $y \in F(\Omega)$ (so that the sum is meaningful for a.e. y). Let us define a function u as the characteristic function of E. Since it is Lebesgue integrable we can apply the change of variables formula. Note that the sum on the right-hand side is zero if $y \notin F(E)$, and 1 if $y \in F(E)$, since F is injective on Ω. Therefore,

$$\int_E |J_F(x)|\, dx = \int_{F(E)} 1\, dy = |F(E)|.$$

∎

PROOF OF THEOREM 1. It is easy to see that for any $\varepsilon > 0$

$$F(A)_{\underline{C}\varepsilon} \subseteq F(A_\varepsilon) \subseteq F(A)_{\overline{C}\varepsilon}.$$

Indeed, $F(A_\varepsilon) = F(\cup_{x\in A} B_\varepsilon(x)) = \cup_{x\in A} F(B_\varepsilon(x)) \subseteq \cup_{x\in A} B_{\overline{C}\varepsilon}(F(x))$. The left-hand inclusion is obtained similarly. Changing ε to $\varepsilon/\overline{C}$ in the right-hand inclusion, and to $\varepsilon/\underline{C}$ in the left-hand inclusion, and taking Lebesgue measures, we obtain

$$|F(A_{\varepsilon/\overline{C}})| \le |F(A)_\varepsilon| \le |F(A_{\varepsilon/\underline{C}})|.$$

Now we apply Lemma 3 with $E = A_{\varepsilon/\underline{C}}$:

$$|F(A_{\varepsilon/\underline{C}})| = \int_{A_{\varepsilon/\underline{C}}} |J_F(x)|\,dx \le \|J_F\|_\infty |A_{\varepsilon/\underline{C}}|.$$

Dividing this inequality by $(\varepsilon/\underline{C})^{N-s}$ and taking $\limsup$ as $\varepsilon \to 0$, we obtain

$$\mathcal{M}^{*s}(F(A)) \le \frac{\|J_F\|_\infty}{\underline{C}^{N-s}}\,\mathcal{M}^{*s}(A), \tag{6}$$

which is the right-hand side inequality in (1).

To prove the left-hand side inequality in (1), note that since F is bi-Lipschitz, than F^{-1} is bi-Lipschitz too with lower and upper Lipschitz constants equal to $\underline{D} := \overline{C}^{-1}$ and $\overline{D} := \underline{C}^{-1}$ respectively. Applying the same proof as above to the function F^{-1} instead of F and to the set $B = F(A)$ instead of A, we get

$$\mathcal{M}^{*s}(F^{-1}(B)) \le \frac{\|J_{F^{-1}}\|_\infty}{\underline{D}^{N-s}}\,\mathcal{M}^{*s}(B),$$

which immediately yields the left-hand side inequality in (1).

To prove the last claim in the theorem, first note that by the Lagrange expansion of the determinant $J_F(x)$ we obtain

$$|J_F(x)| \le \sum_{(j_1,\dots,j_N)} \prod_{j=1}^{N} \left|\frac{\partial F_j}{\partial f_{i_j}}\right|,$$

where the sum runs over all $N!$ permutations of $(1,2,\dots,N)$, and $F = (F_1,\dots,F_N)$. Since $|F_j(x) - F_j(y)| \le |F(x) - F(y)| \le \overline{C}|x-y|$, we have $\left|\frac{\partial F_j}{\partial x_k}\right| \le \overline{C}$ for a.e. $x \in \Omega$. Hence, $|J_F(x)| \le N!\overline{C}^N$ a.e. in Ω. ∎

Lemma 4. *Let $g : \mathbb{R}^{N-1} \to \mathbb{R}$ be any Lipschitz function, $N \ge 2$, and define $F : \mathbb{R}^N \to \mathbb{R}^N$ by $F(x,z) := (x, z + g(x))$, where $x \in \mathbb{R}^{N-1}$ and $z \in \mathbb{R}$. Then F is bi-Lipschitzian and measure preserving, that is, for any mesurable set E in $\mathbb{R}^N$ of bounded measure we have*

$$|F(E)| = |E|. \tag{7}$$

Furthermore, for any bounded set A in $\mathbb{R}^N$ we have that $\overline{\dim}_B F(A) = \overline{\dim}_B A$ and $\underline{\dim}_B F(A) = \underline{\dim}_B A$. The set A is nondegenerate if and only if $F(A)$ is nondegenerate.

Proof. It is easy to see that F is Lipschitzian, as well as F^{-1}, since $F^{-1}(x,z) = (x, z - g(x))$. Next, it is easy to check that the Jacobian J_F is equal to the determinant of a lower triangular matrix with 1's on the diagonal, hence, $J_F \equiv 1$. The claim follows from Lemma 3 and Theorem 1. ▌

Remark 5. The conclusions of the above lemma hold also for more general bi-Lipschitz functions $F : \mathbb{R}^N \to \mathbb{R}^N$ of "cascade type":

$$F(x) = (x_1, x_2 + g_1(x_1), x_3 + g_2(x_1, x_2), \dots, x_N + g_{N-1}(x_1, \dots, x_{N-1})).$$

where $g_i : \mathbb{R}^i \to \mathbb{R}$ are Lipschitz functions. For $N = 3$, by defining

$$F(x,y,z) := (x, y + g_1(x), z + g_2(x,y)),$$

where $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz functions, it is easy to check that F is bi-Lipschitzian, with $F^{-1}(x,y,z) = (x, y - g_1(x), z - g_2(x, y - g_1(x)))$.

Let A be a bounded subset of $\mathbb{R}^N$. Its upper Minkowski content is defined with respect to the Lebesgue measure in $\mathbb{R}^N$, that is, it depends on N as well. Since also $A \subseteq \mathbb{R}^{N+1}$, it is of interest to know how the upper Minkowski content of A, defined with respect to $\mathbb{R}^{N+1}$, denoted by $\mathcal{M}^{*s}(A, \mathbb{R}^{N+1})$, is related to the standard value $\mathcal{M}^{*s}(A, \mathbb{R}^N) := \mathcal{M}^{*s}(A)$. It follows from the following result that nondegeneracy and degeneracy of A are not affected by the dimension of the ambient Euclidean space.

Proposition 6. *Let A be a bounded set in $\mathbb{R}^N$. Then for any $s \geq 0$,*

$$\frac{2}{\sqrt{N+1}^{N+1-s}} \mathcal{M}^{*s}(A, \mathbb{R}^N) \leq \mathcal{M}^{*s}(A, \mathbb{R}^{N+1}) \leq 2\,\mathcal{M}^{*s}(A, \mathbb{R}^N),$$

and the same for $\mathcal{M}_*^s$. In particular, if a planar spiral Γ_0 is nondegenerate in $\mathbb{R}^2$, then it is also nondegenerate as a subset of $\mathbb{R}^3$.

Proof. Denote by $A_{\varepsilon,N}$ the Minkowski sausage of radius ε around A in $\mathbb{R}^N$. Then clearly $A_{\varepsilon,N+1} \subseteq A_{\varepsilon,N} \times (-\varepsilon, \varepsilon)$. For $T \in A$ we consider the $N+1$-dimensional ball $B_\varepsilon(T) \subseteq A_{\varepsilon,N+1}$. The open cube contained in $B_\varepsilon(T)$ centered at T has maximal possible side $2\varepsilon/\sqrt{N+1}$, hence

$$A_{\frac{\varepsilon}{\sqrt{N+1}},N} \times \left(-\frac{\varepsilon}{\sqrt{N+1}}, \frac{\varepsilon}{\sqrt{N+1}}\right) \subseteq A_{\varepsilon,N+1} \subseteq A_{\varepsilon,N} \times (-\varepsilon, \varepsilon),$$

so that

$$|A_{\frac{\varepsilon}{\sqrt{N+1}},N}| \cdot \frac{2\varepsilon}{\sqrt{N+1}} \leq |A_{\varepsilon,N+1}| \leq |A_{\varepsilon,N}| \cdot 2\varepsilon.$$

The claim follows by dividing by $\varepsilon^{(N+1)-s}$ and taking $\limsup$ as $\varepsilon \to 0$. ∎

3. LIPSCHITZIAN AND HÖLDERIAN SPIRALS OF FOCUS AND LIMIT CYCLE TYPES IN $\mathbb{R}^3$

The notions of spatial spirals of focus and limit cycle type in three dimensional space should be clear from the context. We do not intend to define the notion of spiral in $\mathbb{R}^3$ precisely. We assume that every spiral in the space is contained in a two dimensional surface M, and obtained as the graph of a suitable function $h : (\varphi_1, \infty) \to \mathbb{R}^3$, where $h(\varphi) \in M$ for each φ. We define ω-set of a spiral Γ in the usual way, as the set $\omega(\Gamma)$ of accumulation points of all sequences $h(\varphi_n) \in \mathbb{R}^3$ obtained by letting $\varphi_n \to \infty$. If $\omega(\Gamma)$ is a single point, we say that the spiral Γ is of *focus type*, see Figure 1. If $\omega(\Gamma)$ is bi-Lipschitz homeomorphic to a circle in $\mathbb{R}^3$, we say that the spiral is of the *limit cycle type* (the corresponding surface is illustrated on Figure 2).

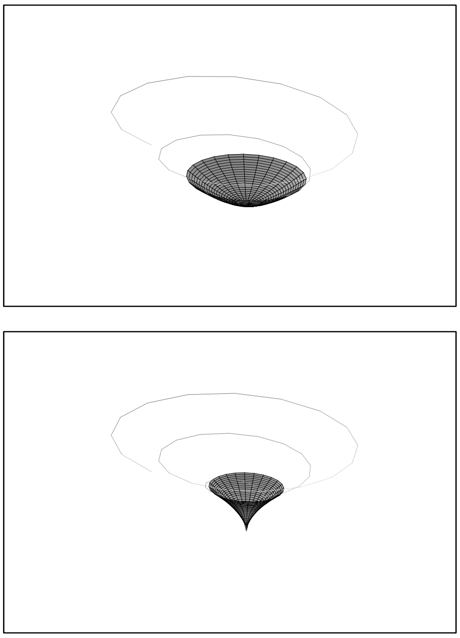

FIG. 1. Focus spirals of Lipschitz and Hölder types

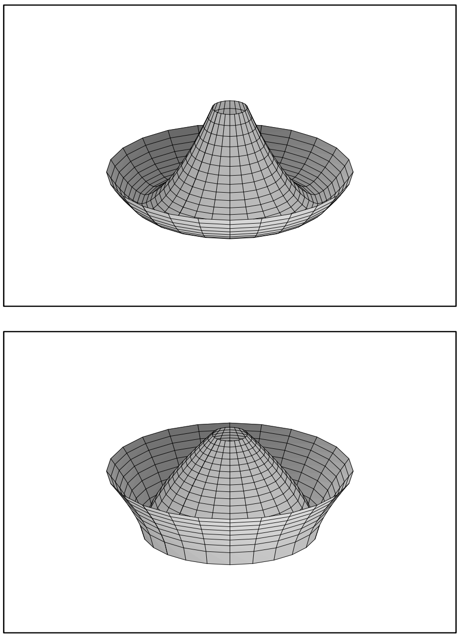

FIG. 2. Surfaces containing cycle spirals of Lipschitz and Hölder types

Depending on whether the surface M containing Γ is Lipschitzian (that is, locally equal to a graph of a Lipschitz function) or Hölderian near $\omega(\Gamma)$, we say that the spiral is *Lipschitzian* or *Hölderian*. We can thus speak about four types of spirals: *Lipschitz-focus spirals*, *Lipschitz-cycle spirals*, *Hölder-focus spirals*, and *Hölder-cycle spirals*. They will be treated in Theorems 7 and 9, where the standard choice is $f(\varphi) = \varphi^{-\alpha}$ and $g(r) = r^{\beta}$.

Now we assume that h is such that for each $\varphi > \varphi_1$ the set $\{h(\varphi + 2\pi t) \in \Gamma : t \in [0,1]\}$ corresponds to one "cycle" of the spiral Γ. If the rate of convergence of a spiral Γ to its ω-set is of power type, that is, $d(h(\varphi), \omega(\Gamma)) \simeq \varphi^{-\alpha}$, with $\alpha > 0$, then we say that the spiral Γ is of *power type* (or power spiral). Analogously we define *exponential* and *logarithmic spirals* if $d(h(\varphi), \omega(\Gamma)) \simeq e^{-\gamma\varphi}$ and $\simeq (\log\varphi)^{-\gamma}$ for some $\gamma > 0$ respectively.

THEOREM 7. (Lipschitzian spirals) *Assume that* $f : [\varphi_1, \infty) \to (0, \infty)$, $\varphi_1 > 0$, *is a decreasing function of class* C^2, *such that*

$$f(\varphi) \simeq \varphi^{-\alpha}, \quad |f'(\varphi)| \simeq \varphi^{-\alpha-1}, \quad |f''(\varphi)| \le M\varphi^{-\alpha}, \tag{8}$$

for some positive constant M.

(a) Let Γ *be a Lipschitz-focus spiral in* $\mathbb{R}^3$ *defined in cylindrical coordinates by*

$$r = f(\varphi), \quad \varphi \geq \varphi_1, \quad z = g(r\cos\varphi, r\sin\varphi), \tag{9}$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ *is any given Lipschitz function defined in Cartesian coordinates. If* $\alpha \in (0,1)$ *then*

$$\dim_B \Gamma = \frac{2}{1+\alpha}, \tag{10}$$

and Γ *is nondegenerate.*

(b) Let Γ *be a Lipschitz-cycle spiral on the cylinder* $r = 1$ *in* $\mathbb{R}^3$*, defined by*

$$r = 1, \quad \varphi \geq \varphi_1, \quad z = f(\varphi). \tag{11}$$

Then for any $\alpha > 0$ *we have that*

$$\dim_B \Gamma = \frac{2+\alpha}{1+\alpha}, \tag{12}$$

and Γ *is nondegenerate.*

Proof. (a) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (x, y, z + g(x, y))$. Denoting by Γ_0 the spiral in $\mathbb{R}^2$ defined in polar coordinates by $r = f(\varphi)$, it is clear that $\Gamma = F(\Gamma_0)$. Since $\dim_B \Gamma_0 = \frac{2}{1+\alpha}$ and Γ_0 is nondegenerate set as a subset of $\mathbb{R}^2$ (see [19, Theorem 5]), hence also in $\mathbb{R}^3$ (see Proposition 6), the claim follows from Theorem 1.

(b) Let Ω be $\frac{1}{2}$-neighbourhood of the circle defined by $r = 1$, $z = 0$ in $\mathbb{R}^3$. This set is a three-dimensional solid torus. Let us define a mapping $F : \Omega \to \Omega$ which represents a rotation of $x \in \Omega$ around central circle of Ω for the angle $\pi/2$ in positive direction with respect to standard orientation of the circle. The rotation of $x \in \Omega$ is performed inside two dimensional plane spanned by x and the z-axis (vertical axis). It is clear that this mapping is bi-Lipschitzian. Furthermore, if Γ_0 is the spiral in (x, y)-plane defined by $r = 1 - f(\varphi)$, where $\varphi \geq \varphi_1$, and φ_1 is large enough, so that $f(\varphi) < 1/2$, then $\Gamma_0 \subseteq \Omega$ and $\Gamma = f(\Gamma_0)$. The claim follows from the fact that $\dim_B \Gamma_0 = \frac{2+\alpha}{1+\alpha}$, see [19, Theorem 5], using Theorem 1. ▮

Remark 8. Note that the dimension $2/(1+\alpha)$ for spirals in $\mathbb{R}^3$ represents an extension of the corresponding result due to Tricot for planar spirals $r = \varphi^{-\alpha}$, $\varphi \geq \varphi_1 > 0$, when $\alpha \in (0,1)$, see [15, p. 121].

From Theorem 7 we know that for functions $z = g(x, y)$ of the form $z = r^\beta = (x^2+y^2)^{\beta/2}$ with $\beta \geq 1$ (Lipschitzian case) we have that for any spiral

Γ_0 in (x,y)-plane having box dimension equal to d_0, the corresponding spiral $\Gamma := F(\Gamma_0)$, where $F(x,y,z) := (x,y,z+g(x,y))$, has the same box dimension. This is not the case for $\beta\in(0,1)$, that is, when g is Hölderian function in two variables, as the following result shows.

THEOREM 9. (Hölderian spirals) *Assume that $\beta\in(0,1)$. Let $f:[\varphi_1,\infty)\to(0,\infty)$, $\varphi_1>0$, be a decreasing function of class C^2 satisfying (8). Assume that $g:(0,f(\varphi_1))\to(0,\infty)$ is a function of class C^2 such that*

$$g(r)\simeq r^{\beta},\quad g'(r)\simeq r^{\beta-1},\quad |g''(r)|\simeq r^{\beta-2}. \tag{13}$$

(a) Let Γ be a Hölder-focus spiral defined by $r=f(\varphi)$, $\varphi\in[\varphi_1,\infty)$, $z=g(r)$, and $\alpha\in(0,1)$ in (8). Then

$$\dim_B\Gamma=\frac{2-\alpha(1-\beta)}{1+\alpha\beta}, \tag{14}$$

and Γ is nondegenerate.

(b) Let Γ be a Hölder-cycle spiral defined by $r=1-f(\varphi)$, $\varphi\in[\varphi_1,\infty)$, $z=g(|1-r|)$, $\alpha>0$ in (8). Then

$$\dim_B\Gamma=\frac{2+\alpha\beta}{1+\alpha\beta}, \tag{15}$$

and Γ is nondegenerate.

Proof. (a) Let us consider the height function of the spiral Γ, defined by $z(\varphi):=g(f(\varphi))$. We say that the Minkowski sausage Γ_ε has the property of vertical separation at φ if $z(\varphi)-z(\varphi+2\pi)=g(f(\varphi))-g(f(\varphi+2\pi))\ge 2\varepsilon$. By the Lagrange mean value theorem the condition of vertical separation is fulfilled when $c(\varphi+2\pi)^{-\alpha\beta-1}\ge 2\varepsilon$, that is, for $\varphi\le\underline{\varphi}(\varepsilon):=C\varepsilon^{-1/(\alpha\beta+1)}$. The condition of horizontal separation of Γ_ε is then also satisfied for such φ since the distance of the corresponding point on Γ from z-axis is $f(\underline{\varphi}(\varepsilon))\ge C\varepsilon^{\alpha/(\alpha\beta+1)}>\varepsilon$ (note that $\alpha<\alpha\beta+1$). Now we estimate the volume of Γ_ε from below, denoting $r(\varphi):=f(\varphi)$, and using Weyl's tube formula, see Gray [6, p. 6]:

$$\begin{aligned}
|\Gamma_\varepsilon| &\ge \varepsilon^2\pi\int_{\varphi_1}^{\underline{\varphi}(\varepsilon)}\sqrt{r(\varphi)^2+r'(\varphi)^2+z'(\varphi)^2}\,d\varphi\\
&\ge \varepsilon^2\pi\int_{\varphi_1}^{\underline{\varphi}(\varepsilon)}r(\varphi)\,d\varphi\ge C\varepsilon^2\int_{\varphi_1}^{\underline{\varphi}(\varepsilon)}\varphi^{-\alpha}\,d\varphi\\
&\ge C\varepsilon^2\cdot(\varepsilon^{-1/(\alpha\beta+1)})^{1-\alpha},
\end{aligned}\tag{16}$$

where we have used that $\alpha\in(0,1)$. For the applicability of Weyl's formula we have to check that the radius of curvature $R(\Gamma,\varphi)$ of spiral Γ at the point corresponding to φ is larger than ε for all $\varphi \leq \underline{\varphi}(\varepsilon)$. To this end, denoting $\vec{r}(\varphi) = x(\varphi)\vec{i} + y(\varphi)\vec{j} + z(\varphi)\vec{k}$, and $x(\varphi) = f(\varphi)\cos\varphi$, $y(\varphi) = f(\varphi)\sin\varphi$, we use the following well known formula from differential geometry:

$$R(\Gamma, \varphi) = \frac{|\vec{r}(\varphi)'|^3}{|\vec{r}(\varphi)' \times \vec{r}(\varphi)''|}.$$

From $|\vec{r}(\varphi)'|^3 = (f(\varphi)^2 + f'(\varphi)^2 + z'(\varphi)^2)^{3/2} \geq f(\varphi)^3 \geq C \cdot \varphi^{-3\alpha}$ and $|\vec{r}(\varphi)' \times \vec{r}(\varphi)''| \leq |\vec{r}(\varphi)'| \cdot |\vec{r}(\varphi)''| \leq C \cdot \varphi^{-2\alpha}$ we conclude that

$$R(\Gamma, \varphi) \geq C \cdot \varphi^{-\alpha} \geq C \cdot \underline{\varphi}(\varepsilon)^{-\alpha} = C \cdot \varepsilon^{\alpha/(\alpha\beta+1)} > \varepsilon, \quad \forall \varphi \leq \underline{\varphi}(\varepsilon),$$

where C is a positive constant independent of ε. Using (16) we get

$$\mathcal{M}_*^s(\Gamma) := \liminf_{\varepsilon \to 0} \frac{|\Gamma_\varepsilon|}{\varepsilon^{3-s}} \geq C \cdot \varepsilon^{s - \frac{2-\alpha(1-\beta)}{\alpha\beta+1}}.$$

Since $\mathcal{M}_*^s(\Gamma) = \infty$ for $s < \frac{2-\alpha(1-\beta)}{\alpha\beta+1}$, we have

$$\underline{\dim}_B \Gamma \geq \frac{2 - \alpha(1-\beta)}{\alpha\beta + 1}. \tag{17}$$

To obtain the upper bound of $|\Gamma_\varepsilon|$ we first consider φ's for which we have vertical overlapping: $z(\varphi) - z(\varphi + 2\pi) \leq 2\varepsilon$. Using the Lagrange theorem we obtain that this holds for $\varphi \geq \overline{\varphi}(\varepsilon) := C\varepsilon^{-1/(\alpha\beta+1)}$. Since for all $\varphi \geq \varphi_1$ we have $r(\varphi) = f(\varphi) \leq C\varphi^{-\alpha}$, $|r'(\varphi)| \leq C\varphi^{-\alpha-1} \leq C\varphi^{-\alpha}$, and $|z'(\varphi)| = Cf(\varphi)^{\beta-1}|f'(\varphi)| \leq C\varphi^{-\alpha\beta-1} \leq C\varphi^{-\alpha}$, we get

$$\begin{aligned} |\Gamma(\varphi_1, \overline{\varphi}(\varepsilon))_\varepsilon| &\leq \pi\varepsilon^2 \int_{\varphi_1}^{\overline{\varphi}(\varepsilon)} \sqrt{r(\varphi)^2 + r'(\varphi)^2 + z'(\varphi)^2}\, d\varphi + C\varepsilon^3 \\ &\leq C\varepsilon^2 \int_{\varphi_1}^{\overline{\varphi}(\varepsilon)} \varphi^{-\alpha}\, d\varphi + C\varepsilon^3 \leq C(\varepsilon^{2-\frac{1-\alpha}{\alpha\beta+1}} + \varepsilon^3), \end{aligned}$$

where $\Gamma(\varphi_1, \overline{\varphi}_1(\varepsilon))$ denotes the part of Γ corresponding to $\varphi \in (\varphi_1, \overline{\varphi}_1(\varepsilon))$.

Let us denote by $S(z_1, z_2)$, $z_1 < z_2$, the part of the surface $z = g(r)$ for which $z \in (z_1, z_2)$. Its radial ε-neighbourhood (radial with respect to z-axis) will be denoted by $S(z_1, z_2)_{\varepsilon,rad}$:

$$\begin{aligned} S(z_1, z_2)_{\varepsilon,rad} := \{&(r, \varphi, z) \in \mathbb{R}^3 : z_1 < z < z_2,\ \varphi \geq \varphi_1, \\ &r \in (\max\{g^{-1}(z) - \varepsilon, 0\}, g^{-1}(z) + \varepsilon)\}. \end{aligned} \tag{18}$$

Due to $\beta \in (0, 1)$ we have that there exists $\gamma > 0$ such that $g'(r) \geq \gamma$ for all $r \in (0, f(\varphi_1))$. Therefore, for a fixed $\varepsilon_0 > 0$ there exists $a > 1$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\overline{\varphi}_1$, $\overline{\varphi}_2 \in (0, z(\varphi_1))$ we have

$$|\Gamma(\overline{\varphi}_1, \overline{\varphi}_2)_\varepsilon| \leq |S(z(\overline{\varphi}_2) - \varepsilon, z(\overline{\varphi}_1) + \varepsilon)_{a\varepsilon,rad}|. \tag{19}$$

The set $\Gamma(\overline{\varphi}(\varepsilon), \infty)_\varepsilon$ can be included in the set $S(\overline{\varphi}(\varepsilon)-\varepsilon, \infty)_{a\varepsilon,rad} \cup B_{a\varepsilon}(0)$, having the shape of a funnel (note that $g'(+0) = \infty$ due to $\beta < 1$, so that the part of $\Gamma(\overline{\varphi}(\varepsilon), \infty)_\varepsilon$ corresponding to $z < 0$ is contained in $B_{a\varepsilon}(0)$). Using (19) we obtain that its part where $f(\varphi) \geq a\varepsilon$, with $a > 1$ fixed, that is, for $\varphi \leq \overline{\varphi}_1(\varepsilon) := C\varepsilon^{-1/\alpha}$, has the volume (here $\overline{\varphi}(\varepsilon) < \overline{\varphi}_1(\varepsilon)$ for ε small enough, and the spiral Γ is on the surface $r = g^{-1}(z)$):

$$\begin{aligned}
|\Gamma(\overline{\varphi}(\varepsilon), \overline{\varphi}_1(\varepsilon))_\varepsilon| &\leq |S(z(\overline{\varphi}_1(\varepsilon)) - \varepsilon, z(\overline{\varphi}(\varepsilon)) + \varepsilon)_{a\varepsilon,rad}| \\
&= \pi \int_{z(\overline{\varphi}_1(\varepsilon))-\varepsilon}^{z(\overline{\varphi}(\varepsilon))+\varepsilon} [(r(z) + a\varepsilon)^2 - (r(z) - a\varepsilon)^2]\, dz \\
&\leq C\varepsilon \int_{C\varepsilon^\beta - \varepsilon}^{C\varepsilon^{\alpha\beta/(\alpha\beta+1)}+\varepsilon} g^{-1}(z)\, dz \\
&\leq C\varepsilon \int_{C\varepsilon^\beta/2}^{2C\varepsilon^{\alpha\beta/(\alpha\beta+1)}} z^{1/\beta}\, dz \leq C\varepsilon^{1+\frac{\alpha(1+\beta)}{\alpha\beta+1}},
\end{aligned}$$

where we have again used that $\beta \in (0, 1)$.

The volume of the trunk of the funnel can be estimated by

$$\begin{aligned}
|\Gamma(\overline{\varphi}_1(\varepsilon), \infty)_\varepsilon| &\leq \pi \int_0^{z(\overline{\varphi}_1(\varepsilon))+\varepsilon} (r(z) + a\varepsilon)^2 dz + |B_{a\varepsilon}(0)| \\
&\leq C \int_0^{C\varepsilon^\beta} (z^{1/\beta} + a\varepsilon)^2 dz + C\varepsilon^3 = C(\varepsilon^{2+\beta} + \varepsilon^3).
\end{aligned}$$

Dividing $|\Gamma_\varepsilon| \leq |\Gamma(\varphi_1, \overline{\varphi}(\varepsilon))_\varepsilon| + |\Gamma(\overline{\varphi}(\varepsilon), \overline{\varphi}_1(\varepsilon))_\varepsilon| + |\Gamma(\overline{\varphi}_1(\varepsilon), \infty)_\varepsilon|$ by ε^{3-s} and taking lim sup as $\varepsilon \to 0$, we obtain $\mathcal{M}^{*s}(\Gamma) = 0$ for $s > \frac{2-\alpha(1-\beta)}{\alpha\beta+1}$, hence $\overline{\dim}_B \Gamma \leq \frac{2-\alpha(1-\beta)}{\alpha\beta+1}$. This together with (17) proves the claim.

(b) We extend the definition of g to $r = 0$, defining $g(0) = 0$. It is easy to see that if we "project" our spatial spiral Γ onto the cylinder $r = 1$ in radial direction (that is, the line joining a point on Γ and the corresponding projection on the cylinder is perpendicular with and intersecting z-axis), then we obtain the spatial spiral Γ_1 defined by $r = 1$, $\varphi \geq \varphi_1$, $z = g(f(\varphi))$. Therefore using Theorem 7(b) we obtain that $\dim_B \Gamma_1 = \frac{2+\alpha\beta}{1+\alpha\beta}$. Indeed, conditions (8) are satisfied with $g \circ f$ instead of f, and with $\alpha\beta$ instead of α. For example, let us check the condition corresponding to the last one in (8):

$$|(g \circ f)''(\varphi)| \leq |g''(f(\varphi))|\, f'(\varphi)^2 + g'(f(\varphi))\, |f''(\varphi)|$$

$$\begin{aligned}&\le C(f(\varphi)^{\beta-2}\varphi^{2\alpha-2}+f(\varphi)^{\beta-1}\varphi^{-\alpha-1})\\&\le C(\varphi^{\alpha\beta-2}+\varphi^{\alpha\beta-1})\le C\varphi^{-\alpha\beta}.\end{aligned}$$

Now consider the mapping of the part of the cylinder $r=1$, $z\in(0,1)$ onto the part of the surface $|z|=g(|1-r|)$, $r\in(1/2,1)$, using radial projection F from the cylinder to the surface, defined in cylindrical coordinates (r,φ,z):

$$F(1,\varphi,z):=(1-g^{-1}(|z|),\varphi,z).$$

We extend F to the mapping of a neighbourhood of the cylinder onto a neighbourhood of the surface. Defining

$$\Omega:=\{(r,\varphi,z)\in\mathbb{R}^3: r\in(\frac{1}{2},\frac{3}{2}),\ \varphi\in\mathbb{R},\ z\in(-1,1)\}$$

we let

$$F:\Omega\to F(\Omega),\quad F(r,\varphi,z):=(r-g^{-1}(|z|),\varphi,z).$$

Since also $F^{-1}(r,\varphi,z):=(r+g^{-1}(|z|),\varphi,z)$, then due to $\beta\in(0,1)$ we have that F is bi-Lipschitzian, see Lemma 4. Indeed, the function $r=g^{-1}(z)$, $z\in(0,z_1)$ is Lipschitzian, since

$$[g^{-1}(z)]'=g'(r)^{-1}\le Cr^{1-\beta}\le Cz^{\frac{1}{\beta}-1},$$

so that $\mathrm{Lip}(g^{-1})\le C\sup_{z\in(0,z_1)}z^{\frac{1}{\beta}-1}<\infty$ due to $\beta\in(0,1)$. Hence, using Theorem 1 we conclude that $\dim_B\Gamma=\dim_B\Gamma_1$, and Γ is nondegenerate. ∎

Remark 10. Comparing explicit expressions of box dimensions, it is easy to see that under conditions of Theorem 9(a) we have $\dim_B\Gamma>\dim_B\Gamma_0$. Since Γ_0 is equal to the orthogonal projection of Γ onto (x,y)-plane, namely, $\Gamma_0=P(\Gamma)$, this inequality is in accordance with the fact that $\dim_B P(\Gamma)\le\dim_B\Gamma$ for Lipschitz mapping P, see Falconer [5, p. 44].

The following corollary shows an interesting phenomenon: when projecting a spiral Γ appearing in Theorems 7 and 9 onto horizontal and vertical planes, then one of these projections has box dimension equal to $\dim_B\Gamma$, while the other projection has box dimension less than $\dim_B\Gamma$.

COROLLARY 11. (Projections of spirals) *Let Γ be a spiral in $\mathbb{R}^3$. Let $\Gamma_{y,z}$ be its orthogonal projection onto (y,z)-plane (or onto any vertical plane), and let Γ_0 be the orthogonal projection of Γ onto (x,y)-plane.*

(a) For Γ in Theorem 7(a) we have $\dim_B\Gamma=\dim_B\Gamma_0=\frac{2}{1+\alpha}$ and $\dim_B\Gamma>\dim_B\Gamma_{y,z}$.

(b) For Γ in Theorem 9(a) we have $\dim_B\Gamma=\dim_B\Gamma_{y,z}=2-\frac{\alpha(1+\beta)}{1+\alpha\beta}$ and $\dim_B\Gamma>\dim_B\Gamma_0$.

(c) For Γ *in Theorem 7(b) we have* $\dim_B \Gamma = \dim_B \Gamma_{y,z} = \frac{2+\alpha}{1+\alpha}$ *and* $\dim_B \Gamma > \dim_B \Gamma_0$.

(d) For Γ *in Theorem 9(b) we have* $\dim_B \Gamma = \dim_B \Gamma_{y,z} = \frac{2+\alpha\beta}{1+\alpha\beta}$ *and* $\dim_B \Gamma > \dim_B \Gamma_0$.

Proof. For the sake of simplicity we provide the proof for $f(\varphi) := \varphi^{-\alpha}$ and $g(r) = r^\beta$. We consider the case (b) only. Other cases are treated similarly. The orthogonal projection of Hölder-focus spiral Γ defined in Theorem 9(a) onto (y, z)-plane is the curve $\Gamma_{y,z}$ defined by $y = z^{1/\beta} \sin(z^{-1/\alpha\beta})$. According to the formula of Tricot [15, p. 122] box dimension of the graph of the function $y = z^{\alpha_1} \sin(z^{-\beta_1})$ is equal to

$$\dim_B(\text{Graph } y(z)) = 2 - \frac{1+\alpha_1}{1+\beta_1},$$

provided $0 < \alpha_1 < \beta_1$. Using this result we obtain that

$$\dim_B \Gamma_{y,z} = 2 - \frac{\alpha(1+\beta)}{1+\alpha\beta},$$

which is equal to $\dim_B \Gamma$, see Theorem 9(a). ∎

Remark 12. Note that the limiting case of $\beta = 1$ in Theorems 9(a) and (b) corresponds to the Lipschitzian case in Theorems 7(a) and (b), so that $\dim_B \Gamma = \frac{2}{1+\alpha}$ and $\frac{2+\alpha}{1+\alpha}$ respectively. It is interesting that for $\beta \to 0$ we have that $\dim_B \Gamma \to 2 - \alpha$ in Theorem 9(a), and $\dim_B \Gamma \to 2$ in Theorem 9(b).

Remark 13. It is easy to see that box dimensions of Hölderian spirals computed in Theorem 9, see (14) and (15), viewed as functions of $\beta \in (0, 1)$, are decreasing. In other words, the more irregular the surface $z = g(r) \simeq r^\beta$ near the origin (i.e. the sharper the spike at $r = 0$), the larger the dimension of the spiral Γ.

Proposition 14. *Let* Γ *be a spiral in* $\mathbb{R}^3$ *defined by* $r = \varphi^{-\alpha}$, $\varphi \geq \varphi_1$, $z = r^\beta$, *with* $\alpha > 1$ *and* $\beta > 0$. *Then* Γ *is rectifiable. In particular,* $\dim_B \Gamma = 1$. *The same conclusion holds in the case of exponential spiral, that is,* $r = e^{c_0\varphi}$ *with* $c_0 < 0$, $\varphi \geq \varphi_1$, $z = r^\beta$. *The result also holds for more general spirals of power or exponential type defined via* $r = f(\varphi) \simeq e^{c_0\varphi}$, *where* $c_0 < 0$, $\varphi \in [\varphi_1, \infty)$, *and* $z = g(r) \simeq r^\beta$.

Proof. It is clear that the planar spiral Γ_0 defined by $r = \varphi^{-\alpha}$, $\varphi \geq \varphi_1$, is rectifiable, hence for $\beta \geq 1$ (Lipschitzian case) the spiral Γ is also rectifiable, see Theorem 1. It remains to show by direct computation that also in Hölderian case, that is, for $\beta \in (0, 1)$, the spiral Γ remains rectifiable.

Parametrizing Γ by $\varphi \geq \varphi_1$ we have that its length is

$$\begin{aligned} l(\Gamma) &= \int_{\varphi_1}^{\infty} \sqrt{r(\varphi)^2 + r'(\varphi)^2 + z'(\varphi)^2}\, d\varphi \\ &= \int_{\varphi_1}^{\infty} \varphi^{-\alpha-1} \sqrt{\varphi^2 + \alpha^2 + \beta^2 \varphi^{2\alpha(1-\beta)}}\, d\varphi. \end{aligned}$$

We assume without loss of generality that $\varphi_1 \geq 1$. In the case when $\alpha(1-\beta) \geq 1$ we have $\varphi^2 \leq \varphi^{2\alpha(1-\beta)}$, so that $l(\Gamma) \leq C \int_{\varphi_1}^{\infty} \varphi^{-\alpha-1} \varphi^{\alpha(1-\beta)} d\varphi < \infty$. In the case of $\alpha(1-\beta) < 1$ we have $\varphi^2 \geq \varphi^{2\alpha(1-\beta)}$, so that $l(\Gamma) \leq C \int_{\varphi_1}^{\infty} \varphi^{-\alpha} d\varphi < \infty$, since $\alpha > 1$. ▮

Remark 15. Using Theorem 1 we can obtain a large class of new spatial spirals. It suffices to take any bi-Lipschitz mapping F defined on a neighbourhood of spiral Γ appearing in Theorem 7 or 9, and to consider the spiral $F(\Gamma)$.

Remark 16. Versions of Theorems 7 and 9 can be stated for discontinous functions f as well, say locally constant. Hence, Γ may consist of a countable family of circles around z-axis, lying on the radial surface $z = g(r)$.

Remark 17. For logarithmic spirals Γ defined by $r = f(\varphi) \simeq (\log \varphi)^{-\gamma}$, where $\gamma > 0$, $\varphi \geq \varphi_1$, and $z = g(r) \simeq r^{\beta}$, $\beta > 0$, we have $\dim_B \Gamma = 2$. This follows from Theorem 1 using [19, Theorem 11].

4. BOX DIMENSION OF SPIRAL TRAJECTORIES OF SOME VECTOR FIELDS IN $\mathbb{R}^3$

This section is devoted to some dynamical systems with spiral solutions. In particular, we consider a system such that its linear part in Cartesian coordinates has a conjugate pair $\pm\omega i$ of pure imaginary eigenvalues with $\omega > 0$, and the third eigenvalue is equal to zero. The corresponding normal form in cylindrical coordinates is:

$$\begin{aligned} \dot{r} &= a_1 r z + a_2 r^3 + a_3 r z^2 + O(|r,z|)^4 \\ \dot{\varphi} &= \omega + O(|r,z|)^2 \\ \dot{z} &= b_1 r^2 + b_2 z^2 + b_3 r^2 z + b_4 z^3 + O(|r,z|)^4, \end{aligned} \tag{20}$$

where a_i and $b_i \in \mathbb{R}$ are coefficients of the system. Such systems are treated in Guckenheimer-Holmes [7, Section 7.4]. In the sequel the value of ω will be normalized to $\omega = 1$.

We generalize our results concerning spiral trajectories of planar vector fields from [19] to systems in $\mathbb{R}^3$ using the normal form for such systems, see [7]. We shall describe a class of dynamical systems in $\mathbb{R}^3$ for which the novelty consists in nontrivial dependence of box dimension of spiral trajectories on the coefficients of the systems, see Proposition 21. We were not able to detect a planar system having this property.

We consider the following system, which is a special case of (19) when $c_0 = 0$:

$$\begin{aligned} \dot{r} &= c_0 r + c_1 r^3 + \ldots + c_m r^{2m+1} \\ \dot{\varphi} &= 1 \\ \dot{z} &= d_2 z^2 + \ldots + d_n z^n. \end{aligned} \tag{21}$$

where $m, n \in \mathbb{N}$ and $c_i, d_i \in \mathbb{R}$. The first two equations represent a standard planar Hopf-Takens bifurcation model obtained from polynomial system in Cartesian coordinates, see Takens [14]. See also Caubergh, Dumortier [1], Caubergh, Françoise [2] for some generalizations of Takens' results. Our aim is to generalize the results stated in [19, Theorems 9 and 10] concerning box dimensions of spiral trajectories of planar vector fields to the case of vector fields in $\mathbb{R}^3$.

THEOREM 18. (Spiral solutions of focus type) *Let Γ be a part of a trajectory of (20) near the origin.*

1.Assume that $c_0 \neq 0$. Then the spiral Γ is of exponential type, hence $\dim_B \Gamma = 1$.

2.Let k and p be fixed positive integers, $1 \leq k \leq m$, $2 \leq p \leq n$, such that $c_0 = \ldots = c_{k-1} = 0$, $d_2 = \ldots = d_{p-1} = 0$, and $c_k d_p > 0$.

(i)If $2k+1 \geq p$ then Γ is a power, Lipschitz-focus and nondegenerate spiral with

$$\dim_B \Gamma = 2 - \frac{2}{2k+1}.$$

(ii)If $2k+1 < p$ then Γ is a power, Hölder-focus and nondegenerate spiral with

$$\dim_B \Gamma = 2 - \frac{2k+p-1}{2kp}.$$

Proof. We use Theorems 7(a) and 9(a). Let us assume that initial point of the spiral Γ is (φ_1, r_1, z_1). We first consider the case of (1). Assume that $c_k < 0$ and $d_p < 0$, in which case the spiral tends to the origin and the spiral Γ_0, obtained as vertical projection of Γ onto the (x, y)-plane, has positive orientation when $\varphi \to \infty$. The proof is based on the idea explained in [19, the proof of Theorem 9]. The solution of the planar system defined by the first two equations in (20) is

$$\varphi = \int_{r_1}^{r} \frac{dr}{r^{2k+1}(c_k + c_{k+1}r^2 + \ldots + c_m r^{2(m-k)})} + \varphi_1, \quad r \in (0, r_1],$$

that is, $\varphi = \Phi(r)$, $\Phi \in C^2(0, r_1)$. In the same way as in [19, Theorem 9] we conclude that

$$\Phi(r) \simeq r^{-2k}, \quad |\Phi'(r)| \simeq r^{-2k-1}, \quad |\Phi''(r)| \simeq r^{-2k-2} \tag{22}$$

for all $r \in (0, r_1]$. The spiral Γ_0 in (x, y)-plane defined by $r = f(\varphi)$, where $f := \Phi^{-1}$, satisfies conditions of Theorem 7(a), see the proof of [19, Theorem 9].

Now we analogously consider the system defined by the last two equations in (20). Let $\varphi = \Psi(z)$ be the solution, where

$$\Psi(z) := \int_{z_1}^{z} \frac{dz}{z^p(d_p + d_{p+1}z + \ldots + d_n z^{n-p})} + \varphi_1, \quad z \in (0, z_1].$$

We have that

$$\Psi(z) \simeq z^{-p+1}, \quad |\Psi'(z)| \simeq z^{-p}, \quad |\Psi''(z)| \simeq z^{-p-1}, \tag{23}$$

for all $z \in (0, z_1]$. Let us introduce the surface $z = g(r)$ containing the spiral Γ, defined by $\Phi(r) = \Psi(z)$ near the origin, that is,

$$z = g(r), \quad g := \Psi^{-1} \circ \Phi.$$

We consider the corresponding curve $z = g(r)$ in a fixed (r, z)-plane defined in parametric form by $\Phi(r) = \varphi$, $\Psi(z) = \varphi$, that is, $r = \Phi^{-1}(\varphi)$, $z = \Psi^{-1}(\varphi)$, by viewing φ as a parameter. Using (22) and (23) we check that g satisfies the conditions of Theorem 7(a). First, denoting $\beta := 2k/(p-1)$ we have

$$g(r) = \Psi^{-1}(\Phi(r)) \simeq \Phi(r)^{-\frac{1}{p-1}} \simeq r^{\beta}.$$

Next,

$$g'(r) = \frac{\Phi'(r)}{\Psi'(z)} \simeq \frac{r^{-2k-1}}{z^{-p}} \simeq r^{-2k-1+\beta p} = r^{\beta-1}.$$

Furthermore,

$$|g''(r)| \;\le\; \frac{|\Phi''(r)\Psi'(z)| + |\Phi'(r)\Psi''(z)z'(r)|}{\Psi'(z)^2}$$

$$\begin{aligned}
&\le C\frac{r^{-2k-2}z^{-p}+r^{-2k-1}z^{-p-1}r^{\beta-1}}{z^{-2p}}\\
&\le C(r^{-2k-2}z^{p}+r^{\beta-2k-2}z^{p-1})\le C(r^{-2k-2+\beta p}+r^{\beta-2k-2+(p-1)\beta})\\
&\le Cr^{\beta-2}.
\end{aligned}$$

Claims (2a) and (2b) in the theorem now follow from Theorems 7(a) and 9(a) respectively.

The case of $c_k > 0$ and $d_p > 0$, where Γ_0 has negative orientation when $\varphi \to -\infty$, is treated similarly. If $c_k d_p < 0$ then Γ does not tend to the origin.

If we have exponential spiral in case (1) of the theorem, it is treated similarly as above. The only difference in the proof is the moment when we need to check that the rectifiable spiral Γ_0 in (x, y)-plane remains rectifiable even though the surface can be of Hölder type, see Proposition 14. ▮

Remark 19. Theorem 18 can be proved also for more general systems than the one treated in (20). Here we concentrate only on polynomial vector fields in Cartesian coordinates in $\mathbb{R}^3$, so that we can have only odd exponents on r-s on the right-hand side of the first equation in (20).

THEOREM 20. (Spiral solutions of limit cycle type) *Let the system (20) have limit cycle $r = a$ of multiplicity j, $1 \le j \le m$. By Γ_1 and Γ_2 we denote the parts of two trajectories of (20) near the limit cycle from outside and inside respectively. Then the trajectories Γ_1 and Γ_2*

1.are exponential spirals of limit cycle type if $j = 1$, and in this case $\dim_B \Gamma_i = 1$, $i = 1, 2$*;*

2.are power spirals of limit cycle type if $j > 1$. Precisely,

(i)for $j \ge p$ the spirals Γ_i are power Lipschitz nondegenerate, and

$$\dim_B = 2 - \frac{1}{j}, \quad i = 1, 2;$$

(ii)for $j < p$ the spirals Γ_i are power Hölder nondegenerate, and

$$\dim_B \Gamma_i = 2 - \frac{1}{p}.$$

Proof. The proof is similar to that of Theorem 18, we only use the results related to limit cycle spirals of Lipschitz and Hölder types stated in Theorems 7(b) and 9(b) respectively. ▮

EXAMPLE. Note that box dimensions computed in Theorems 18 and 20 depend only on the exponents appearing on right-hand sides of (20), and not on the coefficients c_i or d_i. On the other hand, there is a special case of system (19), namely

$$\dot{r} = a_1 r z, \quad \dot{\varphi} = 1, \quad \dot{z} = b_2 z^2, \tag{24}$$

for which the box dimension of spiral trajectories depends on the coefficients a_1 and b_2. The solution of system (24) is

$$r = C_1(-b_2 t + C_3)^{-a_1/b_2}, \quad \varphi = t + C_2, \quad z = \frac{1}{-b_2 t + C_3}.$$

Note that the corresponding spiral Γ near the origin is on the surface $z = C \cdot r^{b_2/a_1}$. Let Γ_0 be the orthogonal projection of Γ onto (x, y)-plane.

(i) If $a_1/b_2 \in (0, 1]$ then Γ_0 is nonrectifiable power spiral (note that $\alpha = a_1/b_2 \leq 1$), and since the corresponding surface is Lipschitzian (note that $\beta = b_2/a_1 \geq 1$) we have that $\dim_B \Gamma = \frac{2}{1+a_1/b_2}$, see Theorem 7(a).

(ii) If $a_1/b_2 > 1$ then we have that Γ_0 is rectifiable power spiral (here $\alpha = a_1/b_2 > 1$) and the corresponding surface is Hölderian (note that $\beta = b_2/a_1 \in (0, 1)$). Using Proposition 14 we obtain that Γ is also rectifiable, and in particular, $\dim_B \Gamma = 1$.

(iii) If $a_1/b_2 < 0$ then the origin is not an accumulation point for Γ.

We can provide a slight generalization of the previous example. We indicate a class of dynamical systems in $\mathbb{R}^3$ with the following surprising property: its spiral solutions have box dimensions that depend also on the coefficients of the system.

Proposition 21. (Sensitivity of box dimension of spirals on the coefficients of a system) *Let a system*

$$\dot{r} = a_1 r^i z^j, \quad \dot{\varphi} = 1, \quad \dot{z} = b_2 r^{i-1} z^{j+1}, \tag{25}$$

be given, where $i, j \in \mathbb{N}$. Let Γ be the part of any of its trajectories viewed near the origin.

(a) If $b_2/a_1 \geq 1$ then Γ is Lipschitz-focus spiral (nondegenerate for $i + j(b_2/a_1) > 2$ and degenerate for $i + j(b_2/a_1) = 2$), and

$$\dim_B \Gamma = 2 - \frac{2a_1}{ia_1 + jb_2}.$$

(b) If $b_2/a_1 \in (0, 1)$ and $i + j(b_2/a_1) \geq 2$ then Γ is Hölder-focus spiral (nondegenerate for $i + j(b_2/a_1) > 2$ and degenerate for $i + j(b_2/a_1) = 2$), and

$$\dim_B \Gamma = 2 - \frac{a_1 + b_2}{(i-1)a_1 + (j+1)b_2}.$$

Proof. Solutions of (25) are spirals to which Theorem 7(a) and Theorem 9(a) apply with $\alpha := (i + j(b_2/a_1) - 1)^{-1}$, and $\beta := b_2/a_1$. ∎

5. SINGULAR INTEGRALS GENERATED BY SPIRALS IN $\mathbb{R}^3$

It has been shown in [17, Theorem 2] that for any bounded set A in $\mathbb{R}^N$ which is nondegenerate (that is, $\underline{\dim}_B A = \overline{\dim}_B A := d$ and d-dimensional lower and upper Minkowski contents of A are different from 0 and ∞), then for any $\varepsilon > 0$

$$\int_{A_\varepsilon} d(x, A)^{-\gamma} dx < \infty \quad \Longleftrightarrow \quad \gamma < N - \dim_B A. \tag{26}$$

Furthermore, for such γ we have the following asymptotic behaviour of singular integral taken over the Minkowski sausage of radius ε around the set A:

$$\int_{A_\varepsilon} d(x, A)^{-\gamma} dx \simeq \varepsilon^{N - \dim_B A - \gamma}, \quad \text{as } \varepsilon \to 0.$$

For generalizations of this result involving generalized Minkowski contents see [18]. The condition of nondegeneracy is indeed essential for (26) to hold, see [18, Theorem 4.2]. In Sections 3 and 4 we have obtained many spirals $A = \Gamma$ in $\mathbb{R}^3$ having the desired nondegeneracy property. The following result concerning Lebesgue integrability of functions with singular sets equal to spirals follows immediately from (26), using Theorems 7 and 9.

Theorem 22. *Let Γ be a spiral in $\mathbb{R}^3$ defined in Theorems 7 or 9, and let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set in $\mathbb{R}^3$ containing $\overline{\Gamma}$. Then*

$$\int_\Omega d(x, \Gamma)^{-\gamma} dx < \infty$$

if and only if

(a) $\gamma < 1 + \frac{2\alpha}{1+\alpha}$, provided Γ is a Lipschitz-focus spiral from Theorem 7(a);

(b) $\gamma < 1 + \frac{\alpha}{1+\alpha}$, provided Γ is a Lipschitz-cycle spiral from Theorem 7(b);

(c) $\gamma < 1 + \frac{\alpha(1+\beta)}{1+\alpha\beta}$, provided Γ is a Hölder-focus spiral from Theorem 9(a);

(d) $\gamma < 1 + \frac{\alpha\beta}{1+\alpha\beta}$, provided Γ is a Hölder-cycle spiral from Theorem 9(b).

Furthermore, for such γ we have the following asymptotic behaviour:

$$\int_{\Gamma_\varepsilon} d(x,\Gamma)^{-\gamma}dx \simeq \varepsilon^{3-\dim_B\Gamma-\gamma}, \quad \text{as } \varepsilon \to 0.$$

Remark 23. Using functions of the form $f(x) = d(x,\Gamma)^{-\gamma}$, $x \in \Omega \subseteq \mathbb{R}^3$, and $f(x) = 0$ otherwise, with $\gamma < \frac{1}{p}(3 - \dim_B \Gamma)$, we can easily obtain new nontrivial examples of Sobolev functions defined by $G_a * f \in L^{a,p}(\mathbb{R}^3)$, $1 < p < \infty$, where $L^{a,p}(\mathbb{R}^3) := \{G_a * f : f \in L^p(\mathbb{R}^3)\}$ is the Bessel potential space defined by the kernel G_a, $a > 0$. For $a = 1$ and $p = 2$ we obtain Sobolev functions in the standard Sobolev space $H^1(\mathbb{R}^3)$, and for γ such that $1 < \gamma < \frac{1}{2}(3 - \dim_B \Gamma)$ we have that $G_1 * f \in H^1(\mathbb{R}^3)$ is a Sobolev function which is singular on the spiral Γ, more precisely, $(G_1 * f)(x) \geq C \cdot d(x,\Gamma)^{1-\gamma}$, see [16, inequality (12)]. Maximally singular Sobolev functions, that is, such that the Hausdorff dimension of their singular sets is maximal possible, are treated in [9].

Remark 24. Instead of continuous spirals appearing in Theorems 7 and 9 we can consider "discrete spirals" Γ obtained as countable unions of circles concentric with respect to z-axis:

$$\begin{aligned} \Gamma &:= \cup_{k=1}^{\infty}\Gamma_k, \\ \Gamma_k &:= \{(r,\varphi,z) \in \mathbb{R}^3 : r = k^{-\alpha},\ \varphi \in [0,2\pi),\ z = r^\beta\}. \end{aligned} \tag{27}$$

The same conclusions as in Theorems 7(a) and 9(a) hold for these discrete spirals of focus type. We may also consider discrete spirals of limit cycle type defined by changing $r = k^{-\alpha}$ to $r = 1 - k^{-\alpha}$ in (26), and we obtain the same conclusions as in Theorems 7(b) and 9(b). Such discrete spirals can be easily obtained in the form $r = f(\varphi)$, $\varphi \geq 2\pi$, $r = r^\beta$, where f is a piecewise constant function, equal to $k^{-\alpha}$ on interval $[2\pi k, 2\pi(k+1))$ for $k \in \mathbb{N}$. It is clear that Theorem 22 holds for such discrete spirals too.

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