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An Eigenvalue Condition for the Injectivity and Asymptotic Stability at Infinity *

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Let $X : U \to \mathbb{R}^2$ be a differentiable vector field defined on the complement of a compact set. We study the intrinsic relation between the asymptotic behavior of the real eigenvalues of the differential DX_z and the global injectivity of the local diffeomorphism given by X. This set U induces a neighborhood of ∞ in the Riemann Sphere $\mathbb{R}^2 \cup \{\infty\}$. In this work we prove the existence of a sufficient condition which implies that the vector field $X : (U, \infty) \to (\mathbb{R}^2, 0)$, —which is differentiable in $U \setminus \{\infty\}$ but not necessarily continuous at ∞,— has ∞ as an attracting or a repelling singularity. This improves the main result of Gutiérrez–Sarmiento: Asterisque, **287** (2003) 89–102.

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1. INTRODUCTION

Given U an open subset of the real plane and a differentiable vector field (or map) $X : U \to \mathbb{R}^2$, we shall denote by $\mathrm{Spec}(X)$ the set of all eigenvalues of the differential DX_z, when z varies in whole the domain U. This domain may be the complement of a disk $\overline{D}_\sigma$ where this compact set is given by $\{z \in \mathbb{R}^2 : ||z|| \leq \sigma\}$, for some $\sigma > 0$. This notation was introduced to support [19], by Gutiérrez and Sarmiento, and afterwards used in [13]. These papers are related with the study of the injectivity of a local diffeomorphism on the plane but in the last article the map is supposed just to be differentiable. Specifically, in this paper was proved the following result, which led to a positive solution of the global asymptotic stability conjecture in the two–dimensional differentiable case (see also [7]).

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Theorem 1. *Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable map (not necessarily of class C^1). If for some $\epsilon > 0$, $Spec(Y) \cap [0, \epsilon) = \emptyset$, then Y is injective.*

Theorem 1 is optimal, because Pinchuck [30] proved that there are non–injective polynomial maps $Y : \mathbb{R}^2 \to \mathbb{R}^2$ such that $0 \notin \text{Spec}(Y)$. A weaker C^1-version of Theorem 1 had already been proved in [9]. Nevertheless, Smith–Xavier ([35], Theorem 4) proved that there exists integers $n > 2$ and non–injective polynomial maps $P : \mathbb{R}^n \to \mathbb{R}^n$ with $\text{Spec}(P) \cap [0, +\infty) = \emptyset$. But, under additional assumptions, there is an extension of Theorem 1 for maps from $\mathbb{R}^n$ to itself ([14], Theorem 1).

Theorem 1 is closer related to the problem of characterizing the injectivity of differentiable maps. This characterization in terms of spectral conditions for maps on $\mathbb{R}^n$ has been studied, for instance, in [6], [35], [28] and [14]. These works are related to the Keller Jacobian Conjecture, that is: "Any polynomial map from $\mathbb{R}^n$ to itself, whose jacobian determinant is constant and equal one, is injective" ([2, 36]).

Theorem 1 is the deepest statement added to a long sequence of results on both the asymptotic stability of C^1-planar vector fields and the injectivity of C^1-maps. This was initiated in 1962 by C. Olech in [26] (see also [24] and [25]) who proved that the two dimensional case of the global asymptotic stability conjecture [23] can be reduced to the statement: "If Y is of class C^1 and $\text{Spec}(Y) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$, then Y is injective". In 1988, Olech–Meister [27] gave a positive answer for the polinomial case. In 1995 Gutiérrez [20] and Feßler [15] obtained this fact from the result that the C^1-map Y is injective if Y is a local diffeomorphis whose differentials DY_z do not have positive real eigenvalues for all z with $|z|$ large enough. It has already been proved that the global asymptotic stability fails in $\mathbb{R}^3$, even for polynomial vector fields [10].

The asymptotic stability at infinity in global C^1-vector fields of the plane was studied for Gutiérrez–Teixeira in [21] (see also [24]). In this paper the authors consider a C^1-vector field $Y : \mathbb{R}^2 \to \mathbb{R}^2$ such that (i) $\det(DY_z) > 0$ and (ii) $\text{Trace}(DY_z) < 0$ in an neighborhood of infinity. They show that "if such Y has a singularity then, the infinity is either a repeller or an attractor." Moreover, they present the Index $\mathcal{I}(Y) = \int \text{Trace } (DY)$ and show that if Y has a singularity and $\mathcal{I}(Y) < 0$ (resp. $\mathcal{I}(Y) \geq 0$), then Y is topologically equivalent to $z \mapsto -z$ that is, "the infinity is a repeller" (resp. to $z \mapsto z$ that is, "the infinity is an attractor"). Recently, Alarcón-Guíñez-Gutiérrez in [1] has been studied a one-parameter family X_μ of C^1-vector fields; they show the bifurcation given by the change in the sign of this Index. In [17] the definition of such Index for differentiable vector fields has been extended to vector fields not necessarily globally defined. Moreover, by using that Gutiérrez–Teixeira's paper [21] the authors of [19] prove the next.

THEOREM 2 (Gutiérrez-Sarmiento). *Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a C^1-map, where $\sigma > 0$ and $\overline{D}_\sigma = \{z \in \mathbb{R}^2 : ||z|| \leq \sigma\}$. The following is satisfied:*

(i) if for some $\epsilon > 0$, Spec(X) is disjoint of $(-\epsilon, \infty)$, then there exists $s \geq \sigma$ such that $X|_{\mathbb{R}^2 \setminus \overline{D}_s}$ is injective;

(ii) if for some $\epsilon > 0$, Spec(X) is disjoint of $(-\epsilon, 0] \cup \{z \in \mathbb{C} : \Re(z) \geq 0\}$, then there exist $p_0 \in \mathbb{R}^2$ such that the point ∞ of the Riemann Sphere $\mathbb{R}^2 \cup \{\infty\}$ is either an attractor or a repellor of $z' = X(z) + p_0$.

The asymptotic stability at infinity study the system induced by a vector field $X : U \to \mathbb{R}^2$, defined on the complement of a compact set, so the set $V = U \cup \{\infty\}$ is a neighborhood of ∞ in the Riemann Sphere $\mathbb{R}^2 \cup \{\infty\}$. We study when the condition Spec$(X) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$ implies that the vector field $X : (V, \infty) \to (\mathbb{R}^2, 0)$ (which is differentiable in $V \setminus \{\infty\}$ but not necessarily continuous at ∞) has ∞ as an attracting or a repelling singularity. Moreover, the methods used in this work are related to those used in the study of planar vector fields, see for instance Chicone [8], Dumortier–Maesschalck [11], Roussarie [33] and Dumortier et. al [12].

In the present article, we extend the theorems of Gutiérrez–Van Chau given in [16]. Furthermore, here various results contained in the articles [13], [14], [17] and [18] are extended to the case of maps (or induced vector fields) satisfying the, so–called, $B-$condition. The structure of the proof in our main results is similar to that [19]. Nevertheless, most of the arguments had to be reconstructed. The basic difficulty was that, in this new case, the eigenvalues of DX_z can be approach to zero.

Throughout this paper, we shall denote by $(\mathbb{R}^2 \setminus \overline{D}_\sigma) \cup \{\infty\}$ the subspace of the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$ with the induced topology. Moreover, given a topological circle $C \subset \mathbb{R}^2$, the compact disc (resp. open disc) bounded by C, will be denoted by $\overline{D}(C)$ (resp. $D(C)$).

2. STATEMENT OF THE RESULTS

We will consider a differentiable map (or vector field) $Y : \mathbb{R}^2 \to \mathbb{R}^2$ whose jacobian determinant at any point is different from zero.

Let us say a little more about Theorem 1. The proof of this result can be divided into two steps. The first step is the following result "If for some $\epsilon > 0$, Spec$(Y) \cap (-\epsilon, \epsilon) = \emptyset$, then Y is injective." The second it to obtain Theorem 1 by regarding a map Y, which satisfies the eigenvalue condition of Theorem 1, as the limit of a sequence of injective maps each of which is of the form $Y_t(z) = Y(z) - tz$, where $t \in \mathbb{R} \setminus \{0\}$ is small, and thus Y_t satisfies the eigenvalue condition Spec$(Y_t) \cap (-\epsilon_t, \epsilon_t) = \emptyset$, where every $\epsilon_t > 0$.

We will use the following condition on the real eigenvalues of DY_z.

Definition 3. ($B-$condition) We say that Y satisfies the $B-$condition if there does not exist a sequence $\mathbb{R}^2 \ni (x_k, y_k) \to \infty$ such that $Y((x_k, y_k)) \to p \in \mathbb{R}^2$ and $DY_{(x_k,y_k)}$ has a real eigenvalue λ_k satisfing $|x_k|\lambda_k \to 0$.

Observe that, if for some $\epsilon > 0$ we have that $\mathrm{Spec}(Y)$ is disjoint of $(-\epsilon, \epsilon)$, the map Y satisfies the $B-$condition.

2.1. Injectivity of global maps.

By Local Inverse Function Theorem [5, 4], the map Y is locally injective at any point of $\mathbb{R}^2$, but this local condition is not sufficient to guarantee the global injectivity. Even in the polynomial case, as shown [30]. Consequently, the goal is to give sufficient conditions on such a map Y to insure that it is globally injective.

Theorem A. *Let $Y = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a local homeomorphism such that for some $s > 0$, $Y|_{\mathbb{R}^2 \setminus D_s}$ is differentiable. If Y satisfies the $B-$condition, then it is globally injective and $Y(\mathbb{R}^2)$ is a convex set.*

The map Y of Theorem A is not necessarily a homeomorphism of $\mathbb{R}^2$; it is a differentiable embedding, the image of which may be properly contained in $\mathbb{R}^2$.

Theorem A improves the main result of Gutiérrez–van Chau [16], where present the so–called (*) condition: "There does not exist a sequence $\mathbb{R}^2 \ni z_n \to \infty$ such that $Y(z_n) \to p \in \mathbb{R}^2$ and DY_{z_n} has a real eigenvalue $\lambda_n \to 0$"

Remark 4. Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be as in Theorem A. If the graph Y is an algebraic set by using the principle "Injectivity $\Rightarrow$ Bijectivity" (which asserts that every continuous injective mapping $\mathbb{R}^n \to \mathbb{R}^n$ whose graph is algebraic must be surjective, [31] and [32]) we obtain that such map will be bijective.

Let us proceed to give an idea of the proof of Theorem A, it shall be present in Section 3. First it will be used that the assumptions imply that the Local Inverse Function Theorem is true [5, 4] (see also the references of [13]). As a consequence, the level curves $\{f = \text{constant}\}$ (resp. $\{g = \text{constant}\}$) make up a $\mathcal{C}^0$-foliation $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) on the plane, without singularities, such that every leaf L of $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) is a differentiable curve and $g|_L$ (resp. $f|_L$) is strictly monotone; in particular $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are (topologically) transversal to each other.

To prove Theorem A, it will be seen that the foliation $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) is topologically equivalent to the foliation, on the (x, y)-plane, induced by the form dx —this foliation is made up by all the vertical straight lines—; that is $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) has no "Half-Reeb component" (or inseparable leaves). The injectivity of Y will follow from the fact that $\mathcal{F}(f)$ and $\mathcal{F}(g)$

are topologically transversal everywhere. The last result will obtain by the study of the geometrical behavior of Y.

2.2. Injectivity at infinity

Theorem 2 is also devoted to study the injectivity of C^1-maps $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ whose Spec(X) is disjoint of $[0, +\infty)$.

THEOREM B. *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable map which satisfies the $B-$condition. If Spec$(X) \cap [0, +\infty) = \emptyset$, then there exists $s \geq \sigma$ such that $X|_{\mathbb{R}^2 \setminus \overline{D}_s}$ can be extended to an injective local homeomorphism $\widetilde{X} = (\widetilde{f}, \widetilde{g}) : \mathbb{R}^2 \to \mathbb{R}^2$.*

Theorem B is valid for maps X such that Spec$(X) \cap (-\infty, 0] = \emptyset$ and it satisfies the $B-$condition. In fact, if in Theorem B we change the pair $\{X; [0, \infty)\}$ by $\{-X; (-\infty, 0]\}$, we may see that its conclusions remain valid. Also, if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an arbitrary invertible linear map, Theorem B applies to the map $T \circ X \circ T^{-1}$.

Let us say a little more about the proof of Theorem B. In Section 4 we prove

PROPOSITION 21 *Let $X = (f, g)$ be as in Theorem B. There exists a topological circle C such that $\mathcal{F}(f)$, restricted to $\mathbb{R}^2 \setminus D(C)$, is topologically equivalent to the foliation, on $\mathbb{R}^2 \setminus D_1$, induced by dx.*

Observe that the foliation, on $\mathbb{R}^2 \setminus D_1$, induced by dx has exactly two tangencies with $\partial \overline{D}_1$ (at $(-1, 0)$ and $(1, 0)$) which are "external".

In order to obtain Proposition 21 in Subsection 4.2, we see that given a topological circle $C_1 \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ surrounding the origin, and having "contact" with $\mathcal{F}(f)$, the number of "external" tangencies of $\mathcal{F}(f)$ with C_1 is equal to 2 plus the number of "internal" tangencies of $\mathcal{F}(f)$ with C_1. We show, in Subsection 4.2, that the circle C_1 can be deformed to a new topological circle C_2 so that the referred "external" and "internal" tangencies cancel in pairs yielding exactly 2 tangencies which are "external". Moreover, by using the results of [18] it will be seen that, under conditions of Proposition 21, the circle C can be deformed so that, for the resulting new circle, still denoted by C:

(i) $\mathcal{F}(f)|_{\mathbb{R}^2 \setminus D(C)}$, is topologically equivalent to the foliation, on $\mathbb{R}^2 \setminus D_1$, induced by dx;

(ii) X takes C homeomorphically to a circle; and

(iii) $X|_{\mathbb{R}^2 \setminus D(C)}$ can be extended to a local homeomorphism $\widetilde{X} : \mathbb{R}^2 \to \mathbb{R}^2$.

Under these conditions, we conclude the proof of Theorem B from Theorem A.

2.3. Differentiable vector fields

Let us consider the system

$$z' = X(z) \tag{1}$$

where X is a differentiable vector field defined over an open subset U of $\mathbb{R}^2$. Since, each point on this open set can be an initial condition, such point jointly to (1) give an autonomous differential equation, which may have many solutions defined on their maximal interval of existence. Nevertheless, for every of those trajectories — through the same point, kept fixed — all their local funnel sections are compact connected sets (see [34]); moreover, each trajectory has its two limit sets, α and ω respectively, which are well defined in the sense that only depend of such solution. Notice that we called trajectory to the curve determined by any solution defined on its maximal interval of existence. If γ_q denotes a trajectory through a point $q \in U$, then γ_q^+ (resp. γ_q^-) will denote the positive (resp. negative) semi-trajectory of X, contained in γ_q and starting at q. In this way $\gamma_q = \gamma_q^- \cup \gamma_q^+$ and $\gamma_q^- \cap \gamma_q^+ = \{q\}$.

A C^0-vector field $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2 \setminus \{0\}$ (without singularities) can be extended to a map

$$\widehat{X} : ((\mathbb{R}^2 \setminus \overline{D}_\sigma \cup \infty), \infty) \longrightarrow (\mathbb{R}^2, 0)$$

(which takes ∞ to 0). In this manner, all questions concerning the local theory of isolated singularities of planar vector fields can be formulated and studied in the case of the vector field $\widehat{X}$. For instance, if γ_p^+ (resp. γ_p^-) is an unbounded semi-trajectory of $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ passing through $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$ such that, its $\omega-$limit (resp. $\alpha-$limit) set is empty, we will also say that γ_p^+ *goes to infinity* (resp. γ_p^- *comes from infinity*), it will be denoted by $\omega(\gamma_p^+) = \infty$ (resp. $\alpha(\gamma_p^-) = \infty$). In this context, we may also talk about the phase portrait of X in a neighborhood of ∞. Like in [17], we will need the following definition.

Definition 5. (attractor and repellor) We will say that the infinity ∞, is an attractor (resp. a repellor) for the differentiable vector field $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ if

1.- There is a sequence of transversal circles to X tending to infinity, that is for every $r \geq \sigma$ there exists a circle C with $D_r \subset D(C)$ and transversal to X.

2.- For some $s \geq \sigma$, all trajectories γ_p through a point $p \in \mathbb{R}^2 \setminus \overline{D}_s$, satisfy $\omega(\gamma_p) = \infty$ that is, γ_p^+ goes to infinity (resp. $\alpha(\gamma_p) = \infty$ that is, γ_p^- comes from infinity).

Definition 6. (global attractor and global repellor) We will say that the infinity ∞, is a global attractor (resp. global repellor) for the differentiable vector field $X : \mathbb{R}^2 \setminus \overline{D}_s \to \mathbb{R}^2$ if ∞ is an attractor (resp. a repellor) for X, and this vector field has only non-periodic regular trajectories in $\mathbb{R}^2 \setminus \overline{D}_s$.

Theorem C. *Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ a differentiable vector field which satisfies the $B-$condition. If $Spec(X) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$. Then,*

(i) for all $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, there is a unique positive semi-trajectory of X starting at p; and the infinity ∞ of the Riemann Sphere $\mathbb{R}^2 \cup \{\infty\}$ is either a repeller or an attractor of the vector field X. Moreover,

(ii) there are $s > 0$ and $v \in \mathbb{R}^2$, such that the infinity is either a global attractor or a global repellor of the vector field $X + v : \mathbb{R}^2 \setminus \overline{D}_s \to \mathbb{R}^2$.

This theorem joint Theorem B improve Theorem 2. The differentiable version of the Gutiérrez–Sarmiento's result was proved in [17] which is also motivated by the results of [21] and [1] related to C^1 vector fields. The proof of Theorem C will be completed in Section 5 by using Theorem B and some results of [17].

3. INJECTIVITY AND MAPS WITH CONVEX IMAGE

This section is devoted to prove Theorem A.

Let $a > 0$ and let $\beta, \gamma : (-a, a) \to \mathbb{R}^2$ be injective C^0-curves such that $\beta(0) = \gamma(0)$. We will say that β is *transversal* (resp *tangent*) to γ at $\beta(0) = \gamma(0)$, if there exist a local C^0-coordinates in a neighborhood of $\beta(0) \in \mathbb{R}^2$ such that in these coordinates $\beta(t) = (t, t)$ and $\gamma(t) = (t, 0)$ (resp. $\beta(t) = (t, \phi(t))$ where $\phi(t) \geq 0$). In particular, when $\sigma([-a, a)) = C$ is a topological circle, we will say that the tangency in $p = \sigma(0) = \gamma(0)$ is *external* (resp. *internal*) if we have that $\gamma(t) \in \mathbb{R}^2 \setminus D(C)$ (resp. $\gamma(t) \in \overline{D}(C)$) for all $t \neq 0$ small enough.

Following [13] we orient $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) in agreement that if L_p is an oriented leaf (or trajectory) of $\mathcal{F}(f)$ (resp. $\mathcal{F}(g)$) thought the point p, then the restriction $g|_{L_p}$ (resp. $f|_{L_p}$) is an increasing function in conformity with the orientation of L_p. Notice that, this function $g|_{L_p}$ (resp. $f|_{L_p}$) is strictly monotone.

Remark 7. Let $a > 0$ and let $\alpha : (-a, a) \to L_p$ be a local differential curve with $\alpha(0) = p$ (L_p is a leaf of $\mathcal{F}(f)$). It is not difficult to obtain that $\frac{d}{dt} g(\alpha(t)) > 0$. Moreover, by using the local inverse of X at $X(p)$ it is easy to see that, for each $t \in (-a, a)$ there exists $\eta_t > 0$ such that $\alpha'(t) = \eta_t X_f(\alpha(t))$ where $X_f(\alpha(t)) := (-f_y(\alpha(t)), f_x(\alpha(t)))$.

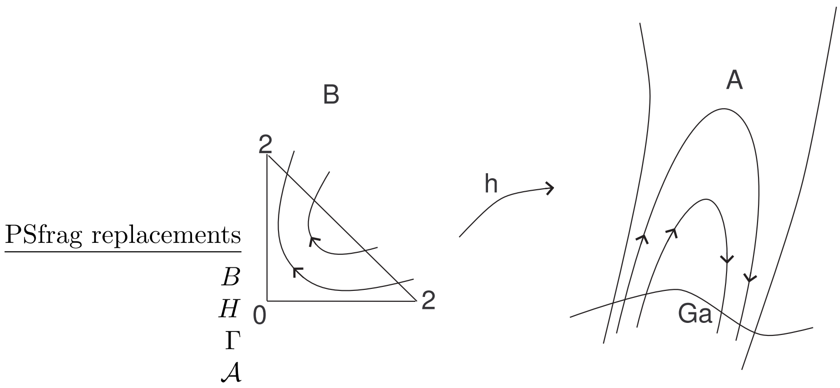

FIG. 1. A half-Reeb component, `hrc`.

Let $h_0(x, y) = xy$ and consider the set

$$B = \{(x, y) \in [0, 2] \times [0, 2] : 0 < x + y \leq 2\}.$$

Definition 8. (Half-Reeb component) Let $X = (f, g) : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be such that $0 \notin \operatorname{Spec}(X)$, so there exist $\mathcal{F}(f)$ and $\mathcal{F}(g)$. Given $h \in \{f, g\}$, we will say that $\mathcal{A} \subset U$ is a half-Reeb component for $\mathcal{F}(h)$ (or simply a `hrc` for $\mathcal{F}(h)$) if there is a homeomorphism $H : B \to \mathcal{A}$ which is a topological equivalence between $\mathcal{F}(h)|_{\mathcal{A}}$ and $\mathcal{F}(h_0)|_B$ such that:

1.- The segment $\{(x, y) \in B : x + y = 2\}$ is sent by H onto a transversal section for the foliation $\mathcal{F}(h)$ in the complement of the point $H(1, 1)$; this section is called the compact edge of $\mathcal{A}$.

2.- Both segments $\{(x, y) \in B : x = 0\}$ and $\{(x, y) \in B : y = 0\}$ are sent by H onto full half-trajectories of $\mathcal{F}(h)$. These two semi-trajectories of $\mathcal{F}(h)$ are called the non–compact edges of $\mathcal{A}$.

Observe that $\mathcal{A}$ may not be a closed subset of $\mathbb{R}^2$. Moreover, the homeomorphism of its definition, does not need to be extended to infinity.

Remark 9. (Geometrical properties) The geometry of any half-Reeb component $\mathcal{A}$ is simple, however it is very useful to examine the behavior of $X|_{\mathcal{A}}$ around infinity. More precisely:

(a) Both non-compact edges of $\mathcal{A}$ are subsets of one level of f, say $\{f = c\}$. The map $X = (f, g)$ sends diffeomorphically these two edges onto a pair of disjoint half-open intervals of the vertical line $u = c$ say $I_1 = \{c\} \times [\alpha, \beta)$ and $I_2 = \{c\} \times (\gamma, \delta]$ with $\beta < \gamma$.

(b) The map X sends diffeomorphically the compact edge of $\mathcal{A}$ to a compact path Σ lying on one-side of the vertical line $u = c$ and intersecting it only at the points (c, α) and (c, δ).

(c) The image $X(\mathcal{A})$ is a simply connected domain bounded by $\Sigma \cup I_1 \cup I_2 \cup I_3$, where $I_3 = \{c\} \times [\beta, \gamma]$.

(d) The vertical foliation in $\mathbb{R}^2$, induces in $\text{int}(X(\mathcal{A}))$ (the interior of $X(\mathcal{A})$) a trivial fibration by open–interval–fibers; furthermore, $X|_{\text{int}(\mathcal{A})}$: $\text{int}(\mathcal{A}) \to \text{int}(X(\mathcal{A}))$ is a homeomorphism giving a topological equivalence between this fibration and the foliation $\mathcal{F}(f)$ restricted to $\text{int}(\mathcal{A})$.

(e) The essencial point is that while every leaf of f, restricted to $\text{int}(\mathcal{A})$, is connected; so the intersection of the level $\{f = c\}$ with the closure $\bar{\mathcal{A}}$ of $\mathcal{A}$ must have at lest two components contained in the boundary $\partial\mathcal{A}$ of $\mathcal{A}$.

For each $\theta \in \mathbb{R}$ we will denote by R_θ the usual linear rotation given by $(x, y) \mapsto (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$ and $X_\theta := (f_\theta, g_\theta) = R_{-\theta} \circ X \circ R_\theta$. In other words, $X_\theta := (f_\theta, g_\theta)$ is the representation of the map X in the linear coordinates of $\mathbb{R}^2$ associated with the rotation R_θ.

It statement can be deduced from [13] and the corollary from [16].

Proposition 10. *If some level curve $\{f = c\}$ is disconnected, then $\mathcal{F}(f)$ has a Half-Reeb component.*

Corollary 11. *Let $Y = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a map as in Definition 8. If $\mathcal{F}(f)$ and $\mathcal{F}(g)$ have no* `hrc`*'s, then $Y(\mathbb{R}^2)$ is a convex set.*

Proof. Let $p, q \in Y(\mathbb{R}^2)$ and let $[p, q] = \{(1-t)p + tq : 0 \le t \le 1\}$. Take $\theta \in \mathbb{R}$ so that $R_\theta([p, q])$ is contained in the vertical line $x = c$. By Remark 9, Proposition 10 implies that the level curve $\{f_\theta = c\}$ is a connected subset of the straight line $x = c$ connecting $R_\theta(p)$ with $R_\theta(q)$; that is $R_\theta([p, q]) \subset Y_\theta(\mathbb{R}^2)$ which implies that $[p, q] \subset Y(\mathbb{R}^2)$ and conclude this proof. ∎

3.1. Global injectivity result

By using that "If $\theta \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$ then, $\mathcal{F}(f_\theta)$ and $\mathcal{F}(g_\theta)$ are transversal to both $R_\theta(\mathcal{F}(f))$ and $R_\theta(\mathcal{F}(g))$", in [13] we prove the following proposition.

Proposition 12. *Let $X = (f, g) : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a map as in Definition 8 such that $\mathcal{F}(f)$ has an unbounded* `hrc`*, $\mathcal{A}$. Let $(f_\theta, g_\theta) = R_\theta \circ X \circ R_{-\theta}$, $\theta \in \mathbb{R}$. If $\Pi(\mathcal{A})$ is bounded, where $\Pi : \mathbb{R}^2 \to \mathbb{R}$ is given by $\Pi(x, y) = x$ then, there is an $\epsilon > 0$ such that, for all $\theta \in (-\epsilon, 0) \cup (0, \epsilon)$, $\mathcal{F}(f_\theta)$ has a* `hrc` *$\mathcal{A}_\theta$ for which $\Pi(\mathcal{A}_\theta)$ is an interval of infinite length.*

An analogous statement to the proposition below was proved in [18], by using the condition $\text{Spec}(X) \cap (-\epsilon, \epsilon) = \emptyset$. The proof of proposition below can be done in a similar way to [13].

Proposition 13. *Let $X = (f, g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a map as in Definition 8. If X satisfies the $B-$condition. Then,*

(a) any half-Reeb component of either $\mathcal{F}(f)$ or $\mathcal{F}(g)$ is a bounded subset of $\mathbb{R}^2$;

(b) when X extends to a local homeomorphism $\tilde{X} = (\tilde{f}, \tilde{g}) : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{F}(\tilde{f})$ and $\mathcal{F}(\tilde{g})$ have no **hrc***'s.*

Proof. Consider only the case of $\mathcal{F}(f)$. Suppose by contradiction that $\mathcal{F}(f)$ has an unbounded half-Reeb component. By Proposition 12, we may assume that $\mathcal{F}(f)$ has a half-Reeb component $\mathcal{A}$ such that $\Pi(\mathcal{A})$ is an unbounded interval. To simplify matters, let us suppose that $[b, +\infty) \subset \Pi(\mathcal{A})$. Then, when $a > b$ is large enough, for any $x \geq a$, the vertical line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x \subset \mathcal{A}$ of $\mathcal{F}(f)|_{\mathcal{A}}$ such that $\Pi(\alpha_x) \cap [x, +\infty) = \{x\}$. In other words, x is the maximum for the restriction $\Pi|_{\alpha_x}$. The leaf α_x is a continuous curve, it follows that; if $x \geq a$, $\alpha_x \cap \Pi^{-1}(x)$ is a compact subset of $\mathcal{A}$. So we can define the functions $H : (a, +\infty) \to \mathbb{R}$ by

$$H(x) = \sup\{y : (x, y) \in \alpha_x \cap \Pi^{-1}(x)\},$$

and $\varphi : (a, +\infty) \to \mathcal{A}$ $\varphi(x) = f(x, H(x))$.

As proved in [13], φ is bounded and a strictly monotone function such that, for some full measure subset $M \subset (a, +\infty)$ such function φ is differentiable on M and for all $x \in M$

$$DF_{(x,H(x))} = \begin{pmatrix} \varphi'(x) & 0 \\ g_x(x, H(x)) & g_y(x, H(x)) \end{pmatrix}.$$

In other words, if $x \in M$, then $\varphi'(x) = f_x(x, H(x)) \in \text{Spec}(X)$.

To proceed we shall only consider the case in which $\varphi'(x) \geq 0$.

If $\liminf_{x\to\infty} x\varphi'(x) = 0$, there exists a sequence $(x_k, H(x_k)) \to \infty$ such that $DY_{(x_k,H(x_k))}$ has a real eigenvalue $\lambda_k = \varphi'(x_k)$ for which $\lim x_k\lambda_k = 0$ and $Y(x_k, H(x_k))$ tends to a finite value in the closure $\overline{Y(\mathcal{A})}$ (which is compact). This contradics the $B-$condition.

If $\liminf_{x\to\infty} x\varphi'(x) \neq 0$, then $\liminf_{x\to\infty} x\varphi'(x) > 0$, this implies that there are constants $a_0 \geq a$ and $\ell > 0$ such that $\ell \leq x\varphi'(x)$ if $x \geq a_0$. As $f|_{\mathcal{A}}$ is bounded, φ is bounded too. Hence, there is a constant $K > 0$ such that for all $x > a$, $0 \leq \varphi(x) - \varphi(a) \leq K$. Take $c_0 > a_0$ so that $K < \int_{a_0}^{c_0} \frac{\ell}{x} dx$. Then

$$K < \int_{a_0}^{c_0} \frac{\ell}{x} dx \leq \int_{a_0}^{c_0} \varphi'(x) dx \leq \varphi(c_0) - \varphi(a_0) < K.$$

This contradiction proves the proposition. ∎

Proof of Theorem A. By Proposition 13, Corollary 11 implies that $Y(\mathbb{R}^2)$ is a convex set. In order to prove, that Y is injective we assume, by contradiction, the existence of two (different) points $p, q \in \mathbb{R}^2$ such that $Y(p) = Y(q) = (c, d)$. As the restriction as $g|_L$ to any leaf of $\mathcal{F}(f)$ is strictly monotone (see Remark 7) p and q belong to different connected componentes of $\{f = c\}$. Therefore, Proposition 10 implies that $\mathcal{F}(f)$ has a half-Reeb component. This contradiction with Proposition 13 concludes the proof.

4. EXTENDING MAPS TO A TOPOLOGICAL EMBEDDING

This section is devoted to prove Theorem B. Notice that we not only prove the injectivity at infinity of X.

4.1. A local flow associated to $\mathcal{F}(f)$

Let $X : \mathbb{R}^2 \backslash \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable map such that for all $p \in \mathbb{R}^2 \backslash \overline{D}_\sigma$, DX_p is non-singular (see Remark 7). Let L_p be the connected component of the level curve $\{f = f(p)\}$ passing through p. Since $g|_{L_p}$ is strictly monotone, given $q \in L_p$ and $t = g(q) - g(p)$ we define $\varphi(t, p)$ as the unique point which is the intersection of L_p with the level curve $\{g = g(q)\}$. For each $p \in \mathbb{R}^2$, let $a_m(p) = \inf\{g(q) : q \in L_p\}$ and $a_M(p) = \sup\{g(q) : q \in L_p\}$. If $p \in \mathbb{R}^2$ and $t \in (a_m(p) - g(p), a_M(p) - g(p))$ then $\varphi(t, p)$ is well defined and determines a continuous local flow around any point of $\mathbb{R}^2$. This map φ will be called the *local flow associated to* $\mathcal{F}(f)$.

For the proof of the following result we refer the reader to Proposition 3.1 of [18].

Proposition 14. *Let* $X = (f, g) \colon \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ *be a differentiable map with* Spec$(X) \cap [0, +\infty) = \emptyset$. *If* $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ *is a topological circle surrounding the origin, there exists* $\varepsilon_0 > 0$ *such that:*

(i) the local flow φ *associated to* $\mathcal{F}(f)$ *is defined in* $(-\varepsilon_0, \varepsilon_0) \times C$.

(ii) Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. *If* $u \in (-\varepsilon_0, 0) \cup (0, \varepsilon_0)$ *and* $Z_u = (A_u, B_u) : C \to \mathbb{S}^1$ *is defined as*

$$Z_u(p) = \frac{\varphi(u, p) - p}{\| \varphi(u, p) - p \|}.$$

Then $A_u(p_0) = 0$*, for some* $p_0 \in C$*, implies that* $B_u(p_0) < 0$*. In particular, the degree of* Z_u *is zero.*

4.2. Avoiding internal tangencies

Let $C \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a topological circle surrounding the origin. We say that the vector field $X : U \to \mathbb{R}^2$ (resp. $\mathcal{F}(f)$) has *contact* (resp. tangency with; resp. transversal to; etc) with C at $p \in C$ if every small local integral curve of X (resp. X_f, defined in Remark 7) at p has such property.

Definition 15. (General position) We say that a topological circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ is in general position with $\mathcal{F}(f)$ (resp. with X) if there exists a set $T \subset C$, at most finite such that:

1.- $\mathcal{F}(f)$ (resp. X) is transversal to $C \setminus T$,

2.- $\mathcal{F}(f)$ (resp X) has a tangency with C at every point of T, and

3.- a leaf of $\mathcal{F}(f)$ (resp. any integral curve of X) can meet tangentially C at most at one point.

Denote by $\mathcal{GP}(f) = \mathcal{GP}(f, \sigma)$ the set of all topological circles $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ in general position with $\mathcal{F}(f)$ and surrounding the origin.

Definition 16. Let $C \in \mathcal{GP}(f)$. The Index of $\mathcal{F}(f)$ along C is the integer number

$$I_{\mathcal{F}(f)}(C) := \frac{2 - n^e(f, C) + n^i(f, C)}{2}$$

where $n^e(f, C)$ (resp. $n^i(f, C)$) is the number of tangencies of $\mathcal{F}(f)$ with C, which are external (resp. internal).

It is well known that if C is in general position with Z_u,

$$\deg(Z_u) = \frac{2 - n^e(Z_u, C) + n^i(Z_u, C)}{2}$$

where $n^i(Z_u, C)$ (resp. $n^e(Z_u, C)$) is the number of internal tangency (resp. external tangency) of Z_u with C (see [22, Theorems 9.1 and 9.2, p. 166-174]).

By using a standard homotopy argument we may conclude that

Lemma 17. *If $Z_u : C \to \mathbb{S}^1$ is as in Proposition 14,*

$$deg(Z_u) = I_{\mathcal{F}(f)}(C).$$

As a consequence

Corollary 18. *Let $C \in \mathcal{GP}(f)$ be such that $n^i(f, C) = 0$. If $n^e(f, C)$ is greater than two, the degree of Z_u is different from zero.*

Next proposition will shows us that we can always select $C \in \mathcal{GP}(f)$ such that, $\mathcal{F}(f)$ has no internal tangencies with C and exactly two external ones. We shall need two lemmas the firs of which is proved in [19, Lemma 2].

Lemma 19. *Let $C \in \mathcal{GP}(f)$. Suppose that a leaf γ of $\mathcal{F}(f)$ meets C transversally somewhere and with an external tangency at a point $p \in C$. Then, γ contains a closed subinterval $[p,r]_f$ which meets C exactly at $\{p,r\}$ (doing it transversally at r) and the following is satisfied:*

(i) If $[p,r]$ denotes the closed subinterval of C such that $\Gamma = [p,r] \cup [p,r]_f$ bounds a compact disc $\overline{D}(\Gamma)$ contained in $\mathbb{R}^2 \setminus D(C)$, then points of $\gamma \setminus [p,r]_f$ nearby p do not belong to $\overline{D}(\Gamma)$.

(ii) Let $(\tilde{p},\tilde{r})$ and $[\tilde{p},\tilde{r}]$ be subintervals of C satisfying $[p,r] \subset (\tilde{p},\tilde{r}) \subset [\tilde{p},\tilde{r}]$. If $\tilde{p}$ and $\tilde{r}$ are close enough to p and r, respectively, then we may deform C into $C_1 \in \mathcal{GP}(f)$ in such a way that the deformation fixes $C \setminus (\tilde{p},\tilde{r})$ and takes $[\tilde{p},\tilde{r}] \subset C$ to a closed subinterval $[\tilde{p},\tilde{r}]_1 \subset C_1$ which is close to $[p,r]_f$. Furthermore, the number of generic tangencies of $\mathcal{F}(f)$ with C_1 is smaller than that of $\mathcal{F}(f)$ with C.

By using Proposition 13, it statement can be deduced from [18].

Lemma 20. *Let $X = (f,g)\colon \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be as in Theorem B. If $C \in \mathcal{GP}(f)$ minimizes the number of tangencies with $\mathcal{F}(f)$, then every tangency is external.*

Proposition 21. *Let $X = (f,g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be as in Theorem B. There exists a topological circle $C \in \mathcal{GP}(f)$ and there are two points $a, b \in C$, with $f(a) < f(b)$, such that $\mathcal{F}(f)$ is tangent to C exactly at a and b; moreover, these tangencies are external.*

Proof. Take $C \in \mathcal{GP}(f)$ as in Lemma 20, so $n^i(f,C) = 0$. If $a, b \in C$ are such that $f(C) = [f(a), f(b)]$, the circle C has two external tangencies: one at a and the other at b.

In fact, suppose by contradiction that a and b are not the only tangencies; so $n^e(f,C)$ is greater than two. This implies, by Corollary 18, that the degree of Z_u is different from zero, contradicting Proposition 14. ▌

We shall say that a collar neighborhood U of a topological circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ is *interior* (resp. *exterior*), if U is contained in $\overline{D}(C)$ (resp. $\mathbb{R}^2 \setminus D(C)$). By using Proposition 21 and Proposition 13, it result follows from [18].

Proposition 22. *Let $X = (f,g) : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable map as in Theorem B. There exists a topological circle $C \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ surrounding the origin such that: (i) $X(C)$ is a topological circle; (ii) for some exterior collar neighborhood U of C, its image $X(U)$ is an exterior collar neighborhood of $X(C)$ and (iii) $X|_U : U \to X(U)$ is a homeomorphism.*

Proof of Theorem B. Let C and U be as in Proposition 22. By Schoenflies Theorem (Theorem III.6.B, [3]), the map $X|_C : C \to X(C)$, can be extended to a homeomorphism $Y_1 : \overline{D}(C) \to \overline{D}(X(C))$. In this way, we extend $X : \mathbb{R}^2 \setminus D(C) \to \mathbb{R}^2$ to $\widetilde{X} : \mathbb{R}^2 \to \mathbb{R}^2$ by defining $\widetilde{X}|_{\overline{D}(C)} = Y_1$. As $\widetilde{X}|_U : U \to X(U)$ is a homeomorphism and U and $X(U)$ are exterior collar neighborhoods of C and $X(C)$, respectively, $\widetilde{X}$ is a local homeomorphism everywhere. By Theorem A $\widetilde{X}$ is globally injective and conclude the proof.

5. ASYMPTOTIC STABILITY AT INFINITY

The proof of Theorem C will be finished at the end of this section.

Let $X^* = (-g, f) : \mathbb{R}^2 \to \mathbb{R}^2$. Certainly X^* is orthogonal to X. In the following, the same notation as that for intervals of $\mathbb{R}$ will be used for oriented arcs of trajectory $[p,q], [p,q), \cdots$ (resp. $[p,q]^*, [p,q)^*, \cdots$), connecting the points p and q, of X (resp. of X^*). The orientation of these arcs is that induced by X (resp. by X^*). For any arc of trajectory $[p,q]^*$ we have the function $L([p,q]^*)$, given by

$$L([p,q]^*) = |\int_{[p,q]^*} ||X|| ds|$$

where ds denotes the arc length element.

Lemma 23. *Let $R(p_1,q_1;p_2,q_2)$ be a compact rectangle such that its boundary is make up of (oriented) arcs of trajectory: $[p_1,q_1]$, $[p_2,q_2]$ of X and $[p_1,p_2]^*$, $[q_1,q_2]^*$ of X^*. Then*

$$L([q_1,q_2]^*) - L([p_1,p_2]^*) = \int_{R(p_1,q_1;p_2,q_2)} Trace(DX) dx \wedge dy \qquad (2)$$

Proof. Since $\text{Trace}(DX) : \mathbb{R}^2 \to \mathbb{R}$ is bounded from above everywhere, it is Lebesgue integrable in $R(p_1,q_1;p_2,q_2)$, so (2) follows from the Green's formula, as presented in [29, Corollary 5.7]. ▌

Corollary 24. *For all $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, there is a unique positive semi-trajectory of X starting at p.*

Proof. Suppose, by contradiction, that there are in $\mathbb{R}^2 \setminus \overline{D}_s$ two positive half-trajectories σ_p^+ and γ_p^+ starting at p. Since $X(p) \neq 0$, there are $q_1 \in \sigma_p^+$, $q_2 \in \gamma_p^+$ and one arc of trajectory $[q_1,q_2]^*$ of X^* such that the triangle A (i.e. a degenerate rectangle) limited by $[p,q_1]$, $[q_1,q_2]^*$ and $[p,q_2]$ is as in Lemma 23. But since $\text{Trace}(DY) < 0$ everywhere in A, then we will obtain

$$L([q_1,q_2]^*) = \int_A \text{Trace}(DX) < 0.$$

This contradiction proves the lemma. ▮

COROLLARY 25. *Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be a global differentiable vector field such that $Spec(Y|_{\mathbb{R}^2 \setminus \overline{D}_s}) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$, for some $s > 0$. If $Y : \mathbb{R}^2 \setminus \overline{D}_s \to \mathbb{R}^2$ has only non-periodic and regular trajectories, then for every compact set $K \subset \mathbb{R}^2 \setminus \overline{D}_s$, there is no positive (resp. negative) semi-trajectory of Y contained in K.*

Proof. In the case of a positive semi–trajectory the proof follows from Corollary 24 and the Poincaré–Bendixson Theorem. In the case of a negative semi–trajectory, we will give an explicit proof by using that Trace(DX) < 0. Let us assume, by contradiction, that γ^- is a negative semi–trajectory of X contained in a compact set $K \subset U$. Let $p \in \alpha(\gamma^-)$ and let Σ be a compact orthogonal section to X passing through p. We know that no negative semi–trajectory can intersect itself, otherwise it would contain a periodic trajectory. So γ^- intersects Σ monotonically and infinitely many times. Let $(p_n)_1^\infty$ denote the corresponding sequence of intersection points, where $p_n \to p$ as $n \to \infty$. Then, from equation (2):

$$L([p_{j-1}, p_j]^*) - L([p_j, p_{j+1}]^*) < 0, \quad \forall j \in \mathbb{N}^*,$$

where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Hence, $\forall n \in \mathbb{N}^*$,

$$L([p_0, p_1]^*) - L([p_n, p_{n+1}]^*) = \sum_{j=1}^{n} L([p_{j-1}, p_j]^*) - L([p_j, p_{j+1}]^*) < 0.$$

That is,

$$0 < L([p_0, p_1]^*) < L([p_n, p_{n+1}]^*).$$

But this is an absurd since $L([p_n, p_{n+1}]^*) \to 0$ as $n \to \infty$. So $\alpha(\gamma^-) = \emptyset$. As K is a compact and $\gamma^- \subset K$, $\alpha(\gamma^-)$ cannot be empty. This contradiction finishes the proof. ▮

Remark 26. An immediate consequence of Theorem B is that if X is as in Theorem C, then outside a larger disk $\overline{D}_s$ with $s \geq \sigma$, the vector field X has no singularity. In addition, as Trace(DX) is negative by Green's formula (i) X does not have any periodic trajectory γ with $D(\gamma)$ contained in $\mathbb{R}^2 \setminus \overline{D}_\sigma$ and, (ii) X admits at most one periodic trajectory, say γ, such that $\overline{D}(\gamma) \supset \overline{D}_\sigma$. Therefore, there exit $s > \sigma$ such that, for any circle $C \subset \mathbb{R}^2 \setminus \overline{D}_s$ with $D(C) \supset \overline{D}_s$, the vector field X has only non-periodic and regular trajectories in $\mathbb{R}^2 \setminus D(C)$.

The following theorem will be need. For the proof we refer the reader to [17].

Theorem 27. *Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable global vector field. If there exist $s > 0$ such that $Spec(Y|_{\mathbb{R}^2 \setminus \overline{D}_s}) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$ and $Y|_{\mathbb{R}^2 \setminus \overline{D}_s}$ is injective, then for each $r \geq s$ there exist C a circle with $\overline{D}_r \subset D(C)$ and transversal to Y.*

Proof of Theorem C. Let $s \geq \sigma$ and $Y_1 : \mathbb{R}^2 \to \mathbb{R}^2$ the (injective) vector field given by Theorem B. By Remark 26, Theorem 27 implies that the first item of Definition 5 is true. In order to prove (i) of Theorem C, we use Corollary 24 and consider C_r $C_{\tilde{r}}$ two circles obtained from Theorem 27 for some $r > \tilde{r} \geq \sigma$. As $K = \overline{D}(C_r) \setminus D(C_{\tilde{r}})$ is a compact subset of $\mathbb{R}^2 \setminus D_\sigma$ Corollary 25 shown the second item of Definition 5. Thus (i) is true, because $X = Y_1$ on $\mathbb{R}^2 \setminus D_s$.

Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field given by $Y(z) = \tilde{X}(z) + v$ where $\tilde{X}$ is given by Theorem B and $v = -\tilde{X}(0)$. As Y satisfies Theorem 27 and Corollary 25, the infinity ∞ is either an attractor or a repellor for the vector field $Y : \mathbb{R}^2 \setminus D_\sigma \to \mathbb{R}^2$. This concludes the proof because Y is injective with $Y(0) = 0$ and it is equal to $X + v$ in $\mathbb{R}^2 \setminus D_s$.

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