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On Vector Fields with Prescribed Limit Cycles

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Let $H : \mathbb{R}^2 \to \mathbb{R}^1$ be a real analytic function. Let c be a real number. Denote level $\{(x, y) \in \mathbb{R}^2 \mid H(x, y) = c\}$ of function H as L^c. Let C be a connected component of level L^c, which is diffeomorphic to the standard circle S^1. What is an ordinary differential equation, for which component C is a limit cycle? In this paper we present such differential equations, found in literature, as well as offer one more such equation.

Key Words: Vector fields, limit and compound limit cycles, perturbation of multiple limit cycles.

1. INTRODUCTION

We start by representing a little history of equations with prescribed limit cycles.

1) One can immediately guess the answer to the question stated in the Abstract. Let $H(x, y)$ be the function described in the Abstract, which defines a compact connected component C of level $L^c = \{(x, y) \in \mathbb{R}^2 \mid H(x, y) = c\}$. Turning to the complex notation, let $z = x + iy$ be a chart in the plane, and $\nabla H = H_x + iH_y$. Let $\varphi : \mathbb{R}^1 \to \mathbb{R}^1$ be an analytic function with $\varphi(c) = 0$. It is easy to see that every component of level L^c is an integral curve of equation

$$\dot{z} = i \exp[i\varphi(H)]\nabla H, \tag{1}$$

and if $C \subset L^c$ is a component homeomorphic to the standard circle S^1, then one can see that C is a limit cycle of equation (1).

2) In paper [2] Bautin offered equation

$$\frac{dy}{dx} = -\frac{(y + \lambda)H_x + H}{(y + \lambda)H_y}, \tag{2}$$

where $H(x, y)$ is a polynomial of degree d, whose zero level L^0 has connected component C diffeomorphic to the circle S^1, and line $y+\lambda = 0$ is picked so that it does not intersect component C. He proved that C is a limit cycle of equation (2). The polynomial vector field (PVF) of Bautin equation (2) has degree d. As a corollary, Bautin proved that if $H(x, y) = 0$ is an M-curve of degree d, then the PVF of (2) has $\frac{1}{2}\left[d^2 - 3d + 3 + (-1)^d\right]$ limit cycles.

3) Bautin equation (2) was the subject of generalization in series of papers of Dolov and coauthors [4], [5], [6] (see also [3]). In paper [4] authors investigated limit cycles of equation

$$\frac{dy}{dx} = -\frac{NH_x + bH}{NH_y - aH}, \tag{3}$$

where $H(x, y)$ is an analytic function in a domain G, $N(x, y)$ is a linear function, chosen so that $C \subset L^0$ diffeomorphic to the circle S^1, and a, b are real constants. They proved *if line* $N(x, y) = 0$ *does not intersect the ovals of curve* $H(x, y) = 0$ *and* $aN_x + bN_y \neq 0$*, then domain* G *does not contain limit cycles of equation (3) others than ovals of curve* $H(x, y) = 0$. (Christopher [3] proved the same theorem for the case when $H(x, y)$ is a polynomial, but his proof is different.)

In paper [5] authors investigated equation

$$\frac{dy}{dx} = -\frac{FH_x - b\Omega(H)}{FH_y + a\Omega(H)}, \tag{4}$$

where a, b, F, H are functions of x and y, and Ω is a function of one variable. For some types of (4), they (i) proved that if c is a root of function $\Omega(H)$, then every circle of level L^c is a limit cycle of equation (4), and (ii) found out the conditions when equation (4) does not have other limit cycles.

4) In paper [10] Winkel offered equation

$$\frac{dy}{dx} = -\frac{H_x - HH_y}{H_y + HH_x}, \tag{5}$$

where $H(x, y)$ is a polynomial of degree d (or an analytic function[1] in $\mathbb{R}^2$). He proved that every component of zero level L^0, which is diffeomorphic

[1]Winkel in [10] considers function $H(x, y)$ to be a C^∞-function. We require function $H(x, y)$ to be analytic by the following reason. Consider C^∞-function

$$H(x, y) = \begin{cases} \exp\left[-\frac{1}{(x^2+y^2-1)^2}\right] & \text{if } x^2 + y^2 \neq 1 \\ 0 & \text{if } x^2 + y^2 = 1 \end{cases}.$$

From topological point of view, circle $x^2+y^2 = 1$ is a limit cycle of the Winkel equation. On the other hand, this circle represents the set of zeros of respective vector field. In this paper we consider vector fields with no more than countable sets of zeros.

to the circle S^1, is an attractive limit cycle of equation (5). The PVF of Winkel equation (5) has degree $2d-1$.

Note that the Winkel equation follows from equation (1). Namely, if we pick up $\varphi(x)=x$, and for a thin enough tubular neighborhood of a limit cycle $C\subset L^0$ of equation (1), we replace in (1) the exponent $\exp(iH)$ by $1+iH$ and write it in the real form, then we obtain the Winkel equation (having the same limit cycles in zero level L^0). On the other hand, the Winkel equation is a particular case of Dolov-Mulko equation (4), where $F=1$, $\Omega(H)=H$, $a=H_x$, and $b=H_y$, but is not a generalization of the Bautin equation.

5) In this paper we offer one more equation,

$$\frac{dy}{dx}=-\frac{H_x+\beta(y-b)\xi(H)}{H_y+\alpha(x-a)\eta(H)},\tag{6}$$

where α,β,a,b are real numbers, and $\xi(x)$ and $\eta(x)$ are analytic functions. In Section 5 we prove, if c is a common real root of functions $\xi(x)$ and $\eta(x)$, i.e. $\xi(c)=\eta(c)=0$, and $\alpha\eta(H)-\beta\xi(H)$ is not zero function, then every compact component of level L^c is a limit set for some trajectories of equation (6). If $H(x,y)$ is a polynomial of degree d and both functions $\xi(x)$ and $\eta(x)$ are polynomials of degree n, then equation (6) describes a PVF of degree $nd+1$.

Equation (6) was inferred in the following manner. A prototype of (6) is well known equation

$$\frac{dy}{dx}=-\frac{\omega x+\mu y(x^2+y^2-1)^k}{\omega y-\mu x(x^2+y^2-1)^k},\tag{7}$$

where $\omega\neq 0$, $\mu\neq 0$, and $k\in\mathbb{N}$. One can find equation (7), for example, in [8]. Equation (7) has a limit cycle $x^2+y^2=1$ of multiplicity k. This limit cycle is attractive (resp., repelling) when k is odd and $\mu<0$ (resp., $\mu>0$), and is attractive-repelling limit cycle when k is even. If in equation (7) we set $\omega=2$ and denote $H=x^2+y^2-1$, then we can write this equation in the form

$$\frac{dy}{dx}=-\frac{H_x+\mu yH^k}{H_y-\mu xH^k}.\tag{8}$$

Now we consider equation (8) independently from (7). Namely, if C is a component of the set $\{(x,y)\in\mathbb{R}^2\mid H(x,y)=0\}$ diffeomorphic to S^1, where now H is a given real analytic function in $\mathbb{R}^2$, then component C is a limit cycle of multiplicity k of equation (8), which satisfies the same conditions to be attractive or repelling limit cycle like limit cycle $x^2+y^2=1$ of equation (7). Now one can infer that equation (6) is just a generalization of equation (8).

Related topics were considered in papers [7] and [9], where authors use another tools.

2. PRELIMINARIES

For integral curves of a vector field, the following terminology is used. A zero of a field is the point where the field vanishes. An integral curve, homeomorphic to the real line, is called a trajectory. An integral curve, homeomorphic to the standard circle S^1, is called an orbit. An orbit is called a limit cycle if there exists its tubular neighborhood, which does not contain another orbits.

Let $F : \mathbb{R}^2 \to \mathbb{R}^1$ and $G : \mathbb{R}^2 \to \mathbb{R}^1$ be real analytic functions, which satisfy two properties:

(1) each of the functions F and G has no multiple analytic factors,

(2) functions F and G have no common analytic factors.

Let $Z = \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = G(x, y) = 0\}$ be the set of common zeros of functions F and G. Consider the mapping

$$H : \mathbb{R}^2 \setminus Z \to \mathbb{R}P^1 \tag{9}$$

defined by formula $H = (F : G)$ and mapping $K : \mathbb{R}^2 \setminus Z \to \mathbb{R}P^1$ defined by formula $K = (G : F)$. For every point $c = (c_1 : c_2) \in \mathbb{R}P^1$, consider the set $L^c = \{(x, y) \in \mathbb{R}^2 \mid c_2F(x, y) - c_1G(x, y) = 0\}$. One can see that $Z \subset L^c$ for every $c \in \mathbb{R}P^1$.

In the set $\mathbb{R}^2 \setminus L^{(1:0)}$ (resp., $\mathbb{R}^2 \setminus L^{(0:1)}$), ratio $(F : G)$ (resp., $(G : F)$) is identified by fraction $\frac{F}{G}$ (resp., $\frac{G}{F}$), and formula $H = \frac{F}{G}$ (resp., $K = \frac{G}{F}$) is used for mapping H (resp., K). If $(c_1 : c_2) \neq (1 : 0)$, then $L^{(c_1:c_2)}$ and $L^{\frac{c_1}{c_2}}$ are used as the same notation of level L^c.

Let $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$ be real analytic vector fields defined in domains $D_1, D_2 \subset \mathbb{R}^2$, respectively. Fields $\mathbf{u}$ and $\mathbf{v}$ are called *equivalent in* $D_1 \cap D_2$ if they have the same set of integral curves in $D_1 \cap D_2$. Vector field $\mathbf{u}$ is called an*integral continuation* of $\mathbf{v}$ into the sets $D_2 \setminus (D_1 \cap D_2)$. Fields $\mathbf{u}$ and $\mathbf{v}$ are equivalent in $D_1 \cap D_2$ iff there exists a real analytic function $A : D_1 \cap D_2 \to \mathbb{R}^1$ such that identity $\mathbf{v} = A\mathbf{u}$ holds for all points in $D_1 \cap D_2$.

Vector fields $\mathcal{H} = (H_y, -H_x)$ and $\mathcal{K} = (K_y, -K_x)$ are defined in domains $\mathbb{R}^2 \setminus L^{(1:0)}$ and $\mathbb{R}^2 \setminus L^{(0:1)}$, respectively. Integral curves of fields $\mathcal{H}$ and $\mathcal{K}$ satisfy exact differential equations

$$H_x dx + H_y dy = 0 \tag{10}$$

and

$$K_x dx + K_y dy = 0. \tag{11}$$

Analytic vector field $\mathcal{L} = (F_yG - FG_y,\ -(F_xG - FG_x))$ is defined in domain $\mathbb{R}^2$. Since $G^2 d\left(\frac{F}{G}\right) = -F^2 d\left(\frac{G}{F}\right) = (F_xG - FG_x)dx + (F_yG - FG_y)dy$, then vector fields $\mathcal{L}$ and $\mathcal{H}$ are equivalent in $\mathbb{R}^2 \smallsetminus L^{(1:0)}$, and vector fields $\mathcal{L}$ and $\mathcal{K}$ are equivalent in $\mathbb{R}^2 \smallsetminus L^{(0:1)}$. Vector field $\mathcal{L}$ is an integral continuation of fields $\mathcal{H}$ and $\mathcal{K}$ into $\mathbb{R}^2$. Integral curves of field $\mathcal{L}$ satisfy differential equation

$$(F_xG - FG_x)dx + (F_yG - FG_y)dy = 0. \tag{12}$$

A general solution of equation (12) is $c_2F(x,y) - c_1G(x,y) = 0$, where ratio $(c_1 : c_2) \in \mathbb{R}P^1$ is considered as an arbitrary constant. For every $c = (c_1 : c_2) \in \mathbb{R}P^1$, level L^c consists of entire integral curves of field $\mathcal{L}$. For every pair $(\mu_1, \mu_2) \neq (0,0)$, expression $\frac{\mu_1}{F^2} + \frac{\mu_2}{G^2}$ is an integrating factor for equation (12).

LEMMA 1. *If for some value $(c_1 : c_2) \in \mathbb{R}P^1$, function $c_2F - c_1G$ can be represented in the form $c_2F - c_1G = f^r g$, where $f, g : \mathbb{R}^2 \to \mathbb{R}^1$ are analytic functions, and r is a positive integer, then functions $F_xG - FG_x$ and $F_yG - FG_y$ have common factor f of multiplicity $r - 1$.*

Proof. If identity $c_2F - c_1G = f^r g$ holds, then $c_1 \neq 0$ and $c_2 \neq 0$, because functions F and G have no multiple factors. The passage to the differentials of both sides of identity $\frac{F}{G} - \frac{c_1}{c_2} = \frac{1}{c_2}\frac{f^r g}{G}$ gives the desired result. ▮

Now on, mapping $H = (F : G)$ is considered to satisfy the property: *components $F_xG - FG_x$ and $F_yG - FG_y$ of field $\mathcal{L}$ have no common non-constant analytic factors.* This property implies that the measure of the set of zeros of analytic vector field $\mathcal{L}$ (and thus of $\mathcal{H}$ and $\mathcal{K}$) is equal to zero.

3. COMPOUND ORBITS

Let $\mathbf{v}$ be an analytic vector field defined in a domain $D \subset \mathbb{R}^2$. Let C be a connected union of a finite number of integral curves of field $\mathbf{v}$, which can be zeros and heteroclinic or homoclinic trajectories. The set C is called a *unicursal cycle* if there exists continuous mapping $g : S^1 \to \mathbb{R}^2$ satisfying the following three properties:

(1) $g(S^1) = C$,

(2) if $M_0 \in C$ is a zero of field $\mathbf{v}$, then its preimage $g^{-1}(M_0)$ is a finite set, and

(3) if $\gamma \subset C$ is an integral curve other than a zero of field $\mathbf{v}$, then restriction $g|_{g^{-1}(\gamma)} : g^{-1}(\gamma) \to \gamma$ is a homeomorphism.

A neighborhood $U(C)$ of unicursal cycle C is called *regular* if

(1) C is a strong deformation retract of closure $\mathrm{Cl}U(C)$,

(2) closure $\mathrm{Cl}U(C)$ has no zeros of field $\mathbf{v}$ other than the points which already belong to C, and

(3) closure $\mathrm{Cl}U(C)$ has no points of limit cycles of field $\mathbf{v}$.

Every connected component of the set $U(C) \smallsetminus C$ is homeomorphic to the standard open annulus $(0,1) \times S^1$. If $U_1(C)$ is a connected component of $U(C) \smallsetminus C$, then its boundary $\partial U_1(C)$ consists of two connected components, one of which is contained in C, and another one is disjoint with C and homeomorphic to the standard circle S^1.

Unicursal cycle C is called a *compound orbit* of field $\mathbf{v}$ if there exist regular neighborhood $U(C)$ and connected component $U_1(C)$ of $U(C) \smallsetminus C$ such that the following property ($\mathbf{p}$) holds:

($\mathbf{p}$) *there exists an open segment* $\{C_t\}_{t \in (0,1)}$ *of orbits of field* $\mathbf{v}$ *such that* $\bigcup\limits_{t \in (0,1)} C_t \subset U_1(C)$ *and* $\lim\limits_{t \to 0} C_t = C$.

If C is a compound orbit, then for every regular neighborhood $U(C)$, there exist no more than two connected components of $U(C) \smallsetminus C$, for which property ($\mathbf{p}$) holds. If there exists one (two) connected component of $U(C) \smallsetminus C$, which satisfies property ($\mathbf{p}$), then compound orbit C is called *unilateral* (*bilateral*). Every orbit of an analytic field is bilateral.

Examples. 1) Let $H = \big((x^2 + y^2 - 1) : (x^2 + y^2 - 4)\big)$. If $c \in [-\infty, \frac{1}{4}) \cup (1, +\infty]$, then every level L^c consists of one orbit. If $c = \frac{1}{4}$, level L^c consists of one singular point $(0,0)$, and mapping H has local maximum at this point. If $c \in (\frac{1}{4}, 1)$, then level L^c is empty.

2) Consider Cassini mapping $H = (((x^2 + y^2)^2 - r^2(x^2 - y^2)) : 1)$, $r \neq 0$. If $c \in (\infty, -\frac{r^4}{4})$, then level L^c is empty. If $c = -\frac{r^4}{4}$, then level L^c consists of two points $(-\frac{r}{\sqrt{2}}, 0)$ and $(\frac{r}{\sqrt{2}}, 0)$, which are local minima of H. If $c \in (-\frac{r^4}{4}, 0)$, then level L^c consists of two orbits. These orbits are the Cassini ovals: one is situated inside the left loop $L = \{(r \cos\varphi \sqrt{\cos 2\varphi}, r \sin\varphi \sqrt{\cos 2\varphi}) \in \mathbb{R}^2 \mid \varphi \in [\frac{3\pi}{4}, \frac{5\pi}{4}]\}$, and another is situated inside the right loop $R = \{(r \cos\varphi \sqrt{\cos 2\varphi}, r \sin\varphi \sqrt{\cos 2\varphi}) \in \mathbb{R}^2 \mid \varphi \in [-\frac{\pi}{4}, \frac{\pi}{4}]\}$. If $c = 0$, then level L^c is the Bernoulli lemniscate; it has three unilateral compound orbits R, L, and $R \cup L$. If $c \in (0, \infty)$, then level L^c has one orbit, – the Cassini oval, – which embraces the Bernoulli lemniscate.

3) The zero level of mapping $H = ((y^2 + x^3(x - r)) : 1)$, where $r > 0$, represents an example of bilateral compound orbit.

4) The zero level of mapping $H = ((y^2 + x^3(x - r)) : (x^2 + xy + y^2))$, where $r > 0$, represents an example of unilateral compound orbit.

4. LIMIT CYCLES AND COMPOUND LIMIT CYCLES

Let L be a unicursal cycle of analytic vector field $\mathbf{v}$ defined in domain $D \subset \mathbb{R}^2$.

Unicursal cycle L is called a *compound limit cycle* of field $\mathbf{v}$ if there exist regular neighborhood $U(L)$ and connected component $U_1(L)$ of $U(L) \setminus L$ such that the following property ($\mathbf{q}$) holds:

($\mathbf{q}$) *for every integral curve* Γ *of field* $\mathbf{v}$ *such that* $\Gamma \cap U_1(L) \neq \varnothing$, *one of connected components of boundary* $\partial\Gamma = \mathrm{Cl}\Gamma \setminus \Gamma$ *coincides with unicursal cycle* L.

For every regular neighborhood $U(L)$, there exist no more than two connected components of $U(L) \setminus L$, for which property ($\mathbf{q}$) holds. If there exists one (two) connected component of $U(L) \setminus L$, which satisfies property ($\mathbf{q}$), then the compound limit cycle is called *unilateral* (*bilateral*). Every limit cycle is bilateral.

If L is a limit cycle of field $\mathbf{v}$, then the set $\mathbb{R}^2 \setminus L$ consists of two connected components: one of them, say D_1, is homeomorphic to the standard open disk, and another one, say D_2, is homeomorphic to the standard open annulus. If $U(L)$ is a regular neighborhood of L, then the compliment $U(L) \setminus L$ consists of two connected components, which are homeomorphic to the standard annulus. The connected component of $U(L) \setminus L$, which belongs to D_1 (resp., to D_2), is called the *inner* (resp., *outer*) component and is denoted as $U_{\text{in}}(L)$ (resp., as $U_{\text{out}}(L)$).

Every vector field defines an orientation of its trajectories and orbits, and in particular, of limit cycles. If the orientation of a limit cycle L, defined by field $\mathbf{v}$, induces the standard (resp., opposite) orientation of the plane defined by ordered pair of basis vectors $(\mathbf{e}_1, \mathbf{e}_2)$, then L is called the *right* (resp., *left*) limit cycle; and one can assign to limit cycle L the index $i(L) = 1$ (resp., $i(L) = -1$).

The boundary of each annulus $U_{\text{in}}(L)$ and $U_{\text{out}}(L)$ consists of two circles, and $\partial U_{\text{in}}(L) \cap \partial U_{\text{out}}(L) = L$. Denote circles $\partial U_{\text{in}}(L) \setminus L$ and $\partial U_{\text{out}}(L) \setminus L$ as γ_{in} and γ_{out}, respectively. Choose the orientation of γ_{in} and γ_{out} such that both of them and L induce the same orientation of the plane $\mathbb{R}^2$. The signs of the outward flux of field $\mathbf{v} = (v_1, v_2)$ along curves γ_{in} and γ_{out}, namely,

$$i_{\text{in}} = sign \oint\limits_{\gamma_{\text{in}}} (-v_2 dx + v_1 dy) \quad \text{and} \quad i_{\text{out}} = sign \oint\limits_{\gamma_{\text{out}}} (-v_2 dx + v_1 dy),$$

are called the *inner* and *outer* indices of L, respectively. Triple $(i, i_{\text{in}}, i_{\text{out}})$ is called the *signature* of pair $(U(L), L)$ (or of limit cycle L).

The signature of a limit cycle realizes an element of group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{+1, -1\}$ is considered as the group with multiplication. There are eight possible values of the signatures of limit cycles; and one can check that each of the values is realizable by some limit cycle.

Traditionally, limit cycles

(1) with signatures $(1, 1, -1)$ and $(-1, -1, 1)$ are called the right and left *attractive* limit cycles, respectively,

(2) with signatures $(1, -1, 1)$ and $(-1, 1, -1)$ are called the right and left *repelling* limit cycles, respectively, and

(3) with signatures $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, 1)$, $(-1, -1, -1)$ are called *attractive-repelling* limit cycles.

One can apply these definitions to bilateral compound limit cycles. In the same sense, one can apply the notions *attractive* and *repelling* to unilateral compound limit cycles.

5. AN EQUATION WITH PRESCRIBED LIMIT CYCLES

Return to mapping (9), which in the set $\mathbb{R}^2 \setminus L^{(1:0)}$, is defined by formula $H(x, y) = \frac{F(x,y)}{G(x,y)}$. Let $\xi : \mathbb{R}^1 \to \mathbb{R}^1$ and $\eta : \mathbb{R}^1 \to \mathbb{R}^1$ be real analytic functions, which have common real root $c \in \mathbb{R}^1$ of multiplicities $m \geq 1$ and $n \geq 1$, respectively. It means that there exist analytic functions $\xi_1(t)$ and $\eta_1(t)$ with $\xi_1(c) \neq 0$ and $\eta_1(c) \neq 0$ such that identities

$$\xi(t) = (t - c)^m \xi_1(t) \quad \text{and} \quad \eta(t) = (t - c)^n \eta_1(t) \tag{13}$$

hold. Consider differential equation

$$[H_x + \beta(y - b)\xi(H)]\, dx + [H_y + \alpha(x - a)\eta(H)]dy = 0, \tag{14}$$

where $a, b, \alpha, \beta \in \mathbb{R}^1$. After substitution of functions (13) into equation (14), it can be written in the following form

$$[H_x + \beta(y - b)(H - c)^m \xi_1(H)]\, dx + [H_y + \alpha(x - a)(H - c)^n \eta_1(H)]dy = 0. \tag{15}$$

Substitution of $H = c$ into equation (15) shows that every integral curve of field $\mathcal{H}$, which belongs to level L^c, satisfies this equation. Denote vector field

$$\Big(H_y + \alpha(x - a)(H - c)^n \eta_1(H),\ -[H_x + \beta(y - b)(H - c)^m \xi_1(H)]\Big), \tag{16}$$

which corresponds to equation (15), as $\mathcal{H}^* = \mathcal{H}^*(\alpha, \beta, x, y)$. One can consider $\mathcal{H}^*(\alpha, \beta, x, y)$ as a two-parametric family of vector fields, which contains field $\mathcal{H}(x, y) = \mathcal{H}^*(0, 0, x, y)$.

The number $\min\{m, n\}$ is called the *multiplicity* of every integral curve of equation (15), which belongs to level L^c. In particular, if $L \subset (L^c \setminus Z)$

is a unicursal cycle of equation (15), then number $\min\{m,n\}$ is called the multiplicity of cycle L.

Consider the case, when identity $\alpha\eta(t) \equiv \beta\xi(t)$ holds for all $t \in \mathbb{R}^1$. Equation (14) becomes

$$[H_x + \beta(y-b)\xi(H)]\,dx + [H_y + \beta(x-a)\xi(H)]dy = 0. \tag{17}$$

Equation (17) has an integrating factor $\frac{1}{\xi(H)}$, which turns it into the exact form

$$d\left[\int \frac{dH}{\xi(H)} + \beta(x-a)(y-b)\right] = 0. \tag{18}$$

The integral curve of equation (18), which passes through point $(x_0, y_0) \in \mathbb{R}^2 \setminus L^{(1:0)}$, satisfies equation

$$\int\limits_{(x_0,y_0)}^{(x,y)} \frac{dH}{\xi(H)} + \beta\left[(x-a)(y-b) - (x_0-a)(y_0-b)\right] = 0$$

if $\xi(H(x_0,y_0)) \neq 0$ and equation $H(x,y) = H(x_0,y_0)$ if $\xi(H(x_0,y_0)) = 0$.

One can define the integral continuation of vector field $\mathcal{H}^*$ into the set $L^{(1:0)} \setminus Z$. Consider real analytic functions (13), which satisfy the additional property: there exists a neighborhood $U \in \mathbb{R}^1$ of zero such that functions $\sigma(t) = \xi\left(\frac{1}{t}\right)$ and $\varsigma(t) = \eta\left(\frac{1}{t}\right)$ are either real analytic in U or possibly have the unique pole of order one at zero. Thus one can represent these functions as Laurent series

$$\sigma(t) = a_{-1}t^{-1} + a_0 + a_1 t + a_2 t^2 + ... + a_p t^p + ... \tag{19}$$

and

$$\varsigma(t) = b_{-1}t^{-1} + b_0 + b_1 t + b_2 t^2 + ... + b_p t^p + ... \tag{20}$$

Consider vector field

$$\mathcal{L}^*(\alpha,\beta,x,y) = G^2\mathcal{H}^*(\alpha,\beta,x,y) = (A,B), \tag{21}$$

whose components are $A = F_yG - FG_y + \alpha(x-a)G^2\varsigma\left(\frac{G}{F}\right)$ and $B = -\left[F_xG - FG_x + \beta(y-b)G^2\sigma\left(\frac{G}{F}\right).\right]$ Field $\mathcal{L}^*$ is defined in the set $\mathbb{R}^2 \setminus L^{(0:1)}$ and is an integral continuation of field $\mathcal{H}^*$ into the set $L^{(1:0)} \setminus Z$. Equation (14) for vector field $\mathcal{L}^*$ can be written in the form

$$-Bdx + Ady = 0. \tag{22}$$

Every integral curve of equation (12), which belongs to the set $L^{(1:0)} \setminus Z$ and thus satisfies the equation $G(x,y)=0$, is an integral curve of equation (22). Thus every orbit or compound orbit $C \subset L^{(1:0)} \setminus Z$ satisfies equation (22).

Using notation of function $K=\frac{G}{F}$, one can write equation (22) in the form

$$\Big[K_x+\beta(y-b)K^2\sigma(K)\Big]\,dx+\Big[K_y+\alpha(x-a)K^2\varsigma(K)\Big]\,dy=0. \quad (23)$$

If $\alpha a_i-\beta b_i=0$ for $i=-1,0,1,2,...,p-1$, and $\alpha a_p-\beta b_p\neq 0$, then the number $p+2$ is the multiplicity of every integral curve of equation (22), which belongs to $L^{(1:0)} \setminus Z$.

Consider the case, when identity $\alpha\varsigma(t)\equiv\beta\sigma(t)$ holds. For every pair $(\mu_1,\mu_2)\in\mathbb{R}^2\setminus\{(0,0)\}$, equation (22) has an integrating factor $\frac{\mu_1}{F^2}+\frac{\mu_2}{G^2}$. If one picks $(\mu_1,\mu_2)=(0,1)$, then equation (22) turns into equation (17). If one picks $(\mu_1,\mu_2)=(1,0)$, then equation (22) turns into the exact form

$$d\left[\int\frac{d\left(\frac{G}{F}\right)}{\left(\frac{G}{F}\right)^2\sigma\left(\frac{G}{F}\right)}+\beta(x-a)(y-b)\right]=0.$$

Equation of the integral curve of (22) passing through point $(x_0,y_0)\in\mathbb{R}^2\setminus L^{(0:1)}$ is

$$\int\limits_{(x_0,y_0)}^{(x,y)}\frac{d\left(\frac{G}{F}\right)}{\left(\frac{G}{F}\right)^2\sigma\left(\frac{G}{F}\right)}+\beta\left[(x-a)(y-b)-(x_0-a)(y_0-b)\right]=0$$

if $\left(\frac{G(x_0,y_0)}{F(x_0,y_0)}\right)^2\sigma\left(\frac{G(x_0,y_0)}{F(x_0,y_0)}\right)\neq 0$ and $G(x_0,y_0)F(x,y)-F(x_0,y_0)G(x,y)=0$ if $\left(\frac{G(x_0,y_0)}{F(x_0,y_0)}\right)^2\sigma\left(\frac{G(x_0,y_0)}{F(x_0,y_0)}\right)=0$.

Let $C\subset L^c\setminus Z$ be a compound orbit of vector field $\mathcal{L}$, and $M_0\in C$ be a singular point of level L^c. If $c\neq(1:0)$, then M_0 is the zero of both fields $\mathcal{H}$ and $\mathcal{H}^*$. If $c=(1:0)$, then M_0 is the zero of both fields $\mathcal{L}$ and $\mathcal{L}^*$. Consider the following condition on the behavior of vector fields $\mathcal{H}^*$ and $\mathcal{L}^*$ at point $M_0\in C\subset L^c\setminus Z$.

Condition 2. *If $c\neq(1:0)$ (resp., $c=(1:0)$) then vector fields $\mathcal{H}$ and $\mathcal{H}^*$ (resp., $\mathcal{L}$ and $\mathcal{L}^*$) are topologically equivalent at M_0. This means that for every neighborhood $D_R=\{(x,y)\in\mathbb{R}^2 \mid (x-x_0)^2+(y-y_0)^2<R^2\}$ of M_0 such that $\mathrm{Cl}D_R\cap Y=M_0$, there exist an open disk $D_r=\{(\alpha,\beta)\in\mathbb{R}^2 \mid \alpha^2+\beta^2<r^2\}$ and a family of homeomorphisms $h^{(\alpha,\beta)}:D_R\to D_R$ such that for every $(\alpha,\beta)\in D_r$, the homeomorphism $h^{(\alpha,\beta)}$ transfers integral curves of vector field $\mathcal{H}|_{D_R}(x,y)$ (resp., $\mathcal{L}|_{D_R}(x,y)$) onto integral curves*

of vector field $\mathcal{H}^*|_{D_R}(\alpha,\beta,x,y)$ *(resp.,* $\mathcal{L}^*(\alpha,\beta,x,y)$*) and such that the restriction* $h^{(\alpha,\beta)}|_{L^c\cap D_R}$ *is the identity mapping.*

Conjecture 3. Vector fields $\mathcal{H}$ and $\mathcal{H}^*$ (resp., $\mathcal{L}$ and $\mathcal{L}^*$) are topologically equivalent at every common zero.

Lemma 4. *If point* $M_0 \in L^c$ *is a generic saddle point of vector field* $\mathcal{H}$ *(or* $\mathcal{L}$*), then vector fields* $\mathcal{H}$ *and* $\mathcal{H}^*$ *(resp.,* $\mathcal{L}$ *and* $\mathcal{L}^*$*) are topologically equivalent at point* M_0*.*

Proof. 1) Let $c \neq (1:0)$ and point $M_0 \in L^c$ have coordinates (x_0, y_0). Point M_0 satisfies the equation of level L^c, namely, $H(x_0,y_0) - c = 0$. Linearizations of equations (10) and (15) in a neighborhood of point M_0 coincide and can be written in the form

$$\left[H^0_{xx}(x-x_0) + H^0_{xy}(y-y_0)\right]dx + [H^0_{xy}(x-x_0)+H^0_{yy}(y-y_0)]dy = 0, \quad (24)$$

where $H^0_{xx} = H_{xx}(x_0,y_0)$, $H^0_{xy} = H_{xy}(x_0,y_0)$, and $H^0_{yy} = H_{yy}(x_0,y_0)$, and the characteristic equation of (24) is

$$\lambda^2 + H^0_{xx}H^0_{yy} - \left(H^0_{xy}\right)^2 = 0. \quad (25)$$

If point M_0 is a generic saddle point of vector field $\mathcal{H}$, then $H^0_{xx}H^0_{yy} - \left(H^0_{xy}\right)^2 < 0$. The latter implies that characteristic equation (25) has two real roots of opposite signs. Thus for all $\alpha, \beta \in \mathbb{R}^1$, the point M_0 is a generic saddle point of both vector fields $\mathcal{H}(x,y)$ and $\mathcal{H}^*(\alpha,\beta,x,y)$.

2) To prove this lemma in the case $M_0 \in L^{(1:0)}$, one can use equations (11) and (23) and similarly repeat the proof of item 1). ▮

Let C be an orbit or uni- or bilateral compound orbit of vector field $\mathcal{H}$, which belongs to level L^c and thus satisfies equation $H - c = 0$. According to the Bertini-Sard theorem (see, for example, [1]) there exist a neighborhood $(c'', c') \subset \mathbb{R}^1$ of c such that segments (c'', c) and (c, c') have no critical values of mapping H. Numbers c'' and c' can be either critical values of mapping H or $c'' = -\infty$ or $c' = \infty$. If C is an orbit or bilateral compound orbit, then there exist two families of orbits of vector field $\mathcal{H}$, namely, $\{\gamma^t\}_{t\in(c'',c)}$ in $\bigcup_{t\in(c'',c)} L^t$ and $\{\gamma^t\}_{t\in(c,c')}$ in $\bigcup_{t\in(c,c')} L^t$ such that

$$\lim_{t\to c,\ t\in(c'',c)} \gamma^t = C \quad \text{and} \quad \lim_{t\to c,\ t\in(c,c')} \gamma^t = C. \quad (26)$$

If C is a unilateral compound orbit, then there exists one family of orbits of vector field $\mathcal{H}$ either $\{\gamma^t\}_{t\in(c'',c)}$ in $\bigcup_{t\in(c'',c)} L^t$ or $\{\gamma^t\}_{t\in(c,c')}$ in $\bigcup_{t\in(c,c')} L^t$ such that the respective limit of (26) holds.

Using previous notations, consider real analytic function $E : \mathbb{R}^1 \to \mathbb{R}^1$, defined by formula $E(\varepsilon) = \alpha\varepsilon^n \eta_1(c+\varepsilon) - \beta\varepsilon^m \xi_1(c+\varepsilon)$. This means, in particular, that the values of parameters $m, n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}^1$ and analytic functions $\xi_1(t)$ and $\eta_1(t)$ are such that function $E(\varepsilon)$ is a nonzero function. Let $Rr(E)$ be the set of real roots of function $E(\varepsilon)$. Let us pick the connected component of the set $\mathbb{R}^1 \setminus (Rr(E) \setminus \{0\})$, which contains the zero root. This component is an open segment, say $(\varepsilon_1, \varepsilon_2)$, where $\varepsilon_1 < 0$ and $\varepsilon_2 > 0$ can be either roots of function $E(\varepsilon)$ or $\varepsilon_1 = -\infty$ or $\varepsilon_2 = \infty$. Denote the segments $(c+\varepsilon_1, c+\varepsilon_2) \cap (c'', c)$ and $(c+\varepsilon_1, c+\varepsilon_2) \cap (c, c')$ as (c^{**}, c) and (c, c^*) respectively. Function $E(\varepsilon)$ does not have roots in segments (c^{**}, c) and (c, c^*).

Let $C \subset L^c$ be a unilateral compound orbit of the field $\mathcal{H}$, and let, for definiteness, $\{\gamma^t\}_{t\in(c,c^*)} \subset \{\gamma^t\}_{t\in(c,c')}$ be the subfamily of orbits, for which $\lim\limits_{t\to c,\ t\in(c,c^*)} \gamma^t = C$ holds.

Theorem 5. *If (1) $C \subset L^c \setminus Z$ is a unilateral compound orbit of field $\mathcal{H}$, and (2) the zeros of $\mathcal{H}$ belonging to C satisfy Condition 2, then the set C is a unilateral compound limit cycle of field $\mathcal{H}^*$.*

Proof. To prove this theorem, it is sufficient to show that for every orbit $\gamma^{c+\varepsilon} \in \{\gamma^t\}_{t\in(c,c^*)}$, outward flux $Fl(\varepsilon) = \oint\limits_{\gamma^{c+\varepsilon}} (\mathcal{H}^* \cdot \mathbf{n})\, ds$ has the same sign. Orbit $\gamma^{c+\varepsilon}$ satisfies equation $H = c+\varepsilon$. One can make the following calculation via Green's formula:

$$Fl(\varepsilon) = \oint\limits_{\gamma^{c+\varepsilon}} (\mathcal{H}^* \cdot \mathbf{n})\, ds =$$

$$= \oint\limits_{\gamma^{c+\varepsilon}} [H_y + \alpha(x-a)(H-c)^n \eta_1(H)]dy + [H_x + \beta(y-b)(H-c)^m \xi_1(H)]\, dx =$$

$$= \oint\limits_{\gamma^{c+\varepsilon}} [H_y + \alpha(x-a)\varepsilon^n \eta_1(c+\varepsilon)]dy + [H_x + \beta(y-b)\varepsilon^m \xi_1(c+\varepsilon)]\, dx =$$

$$= \iint\limits_{R^\varepsilon} [\alpha\varepsilon^n \eta_1(c+\varepsilon) - \beta\varepsilon^m \xi_1(c+\varepsilon)] dxdy = S(\varepsilon)[\alpha\varepsilon^n \eta_1(c+\varepsilon) - \beta\varepsilon^m \xi_1(c+\varepsilon)],$$

where R^ε is the inner component of $\mathbb{R}^2 \setminus \gamma^{c+\varepsilon}$, and $S(\varepsilon) = \iint\limits_{R^\varepsilon} dxdy$ is the area of disk R^ε oriented by its boundary $\partial R^\varepsilon = \gamma^{c+\varepsilon}$. According to the choice of segment (c, c^*), flux $Fl(\varepsilon)$ has the same sign on every orbit of $\{\gamma^t\}_{t\in(c,c^*)}$. ▮

It is clear that product $S(\varepsilon)i(\gamma^{c+\varepsilon})$ is always positive.

THEOREM 6. *If $C \subset L^{(1:0)} \setminus Z$ is a unilateral compound orbit of field $\mathcal{L}$, zeros of $\mathcal{L}$ belonging to C satisfy Condition 2, and function $E_1(\varepsilon) = \varepsilon^2 [\alpha\varsigma(t) - \beta\sigma(t)]$ is analytic, then set C is a unilateral compound limit cycle of field $\mathcal{L}^*$.*

Proof. The proof of this theorem repeats the proof of Theorem 5. It is just enough to apply the proof of Theorem 5 to the vector field of equation (23). ▮

Under Condition 2, Lemmas 7 – 12 are simple consequences of Theorem 5.

LEMMA 7. *Let $C \subset L^c \setminus Z$ be a unilateral compound orbit of field $\mathcal{H}$.*
(1) If $\gamma^{c+\varepsilon}$ is situated in the inner (resp., outer) component of $\mathbb{R}^2 \setminus C$ and $i(\gamma^{c+\varepsilon})Fl(\varepsilon) > 0$ (resp., $i(\gamma^{c+\varepsilon})Fl(\varepsilon) < 0$), then C is an attractive unilateral limit cycle of field $\mathcal{H}^*$.
(2) If $\gamma^{c+\varepsilon}$ is situated in the inner (resp., outer) component of $\mathbb{R}^2 \setminus C$ and $i(\gamma^{c+\varepsilon})Fl(\varepsilon) < 0$ (resp., $i(\gamma^{c+\varepsilon})Fl(\varepsilon) > 0$), then C is a repelling unilateral limit cycle of field $\mathcal{H}^*$.

LEMMA 8. *If $C \subset L^c$ is a bilateral compound orbit (resp., an orbit) of field $\mathcal{H}$, and function $E(\varepsilon)$ is a nonzero function, then set C is a bilateral compound limit cycle (resp., limit cycle) of field $\mathcal{H}^*$.*

LEMMA 9. *If $C \subset L^c$ is an orbit of field $\mathcal{H}$, then for every pair (α, β) such that function $E(\varepsilon)$ is a nonzero function, set C is a limit cycle of field $\mathcal{H}^*(\alpha, \beta, x, y)$.*

LEMMA 10. *Let $C \subset L^c$ be a bilateral compound orbit [resp., an orbit] of field $\mathcal{H}$.*
(1) If $\gamma^{c+\varepsilon}$ is situated in the inner (resp., outer) component of $\mathbb{R}^2 \setminus C$ and inequalities $Fl(\varepsilon)Fl(-\varepsilon) < 0$ and $i(\gamma^{c+\varepsilon})Fl(\varepsilon) > 0$ (resp., $i(\gamma^{c+\varepsilon})Fl(\varepsilon) < 0$) hold, then C is an attractive compound limit cycle [resp., attractive limit cycle] of field $\mathcal{H}^*$.
(2) If $\gamma^{c+\varepsilon}$ is situated in the inner (resp., outer) component of $\mathbb{R}^2 \setminus C$ and inequalities $Fl(\varepsilon)Fl(-\varepsilon) < 0$ and $i(\gamma^{c+\varepsilon})Fl(\varepsilon) < 0$ (resp., $i(\gamma^{c+\varepsilon})Fl(\varepsilon) > 0$) hold, then C is a repelling compound limit cycle [resp., repelling limit cycle] of field $\mathcal{H}^*$.
(3) If $Fl(\varepsilon)Fl(-\varepsilon) > 0$, then C is an attractive-repelling compound limit cycle [resp., attractive-repelling limit cycle] of field $\mathcal{H}^*$.

Let $Rr(\xi)$ and $Rr(\eta)$ be the sets of real roots of functions $\xi(t)$ and $\eta(t)$ respectively, and $Rr(\xi) \cap Rr(\eta) \neq \varnothing$. If $c \in Rr(\xi) \cap Rr(\eta)$ is a root of function $\xi(t)$ of multiplicity $m \geq 2$ and of function $\eta(t)$ of multiplicity $n \geq 2$. Functions $\xi(t)$ and $\eta(t)$ can be represented in form (13); and every limit cycle $C \subset L^c$ of field (16) has multiplicity $\min\{m,n\} \geq 2$. Let, for definiteness, $\min\{m,n\} = m$. Let segment (c_1, c_2) be the connected component of $\mathbb{R}^1 \setminus [(Rr(\xi) \cup Rr(\eta)) \setminus \{c\}]$, which contains root c. Let $\widetilde{\xi}(t)$ and $\widetilde{\eta}(t)$ be some perturbations of functions $\xi(t)$ and $\eta(t)$ respectively, and

$$\Big(H_y + \alpha(x-a)\widetilde{\eta}(H),\ -[H_x + \beta(y-b)\widetilde{\xi}(H)]\Big) \tag{27}$$

be the corresponding perturbation of vector field (16).

Lemma 11. *If $\widetilde{\xi}(t) = (t-c')^{m_1}(t-c'')^{m_2}\xi_1(t)$ and $\widetilde{\eta}(t) = (t-c')^{m_1}(t-c'')^{m_2}(t-c)^{n-m}\eta_1(t)$, where $c', c'' \in (c_1, c_2)$ are two distinct numbers, and positive integers m_1 and m_2 are such that $m_1 + m_2 = m$, then*

(1) if $\widetilde{c} \in Rr(\xi) \cap Rr(\eta)$ and $\widetilde{c} \neq c$, then every limit or compound limit cycle of field (16), which satisfies equation $H - \widetilde{c} = 0$, is preserved under perturbation (27),

(2) every limit cycle of field (16), which satisfies equation $H - c = 0$, splits under perturbation (27) into two limit cycles of multiplicities m_1 and m_2 with equations $H - c' = 0$ and $H - c'' = 0$, respectively.

Lemma 12. *If $\widetilde{\xi}(t) = (t-c)^{m-2}(t^2 + 2ct + c^2 + \delta^2)\xi_1(t)$ and $\widetilde{\eta}(t) = (t^2 + 2ct + c^2 + \delta^2)(t-c)^{n-2}\eta_1(t)$, where $\delta \neq 0$ is a real number, then*

1) if $\widetilde{c} \in Rr(\xi) \cap Rr(\eta)$ and $\widetilde{c} \neq c$, then every limit or compound limit cycle of field (16), which satisfies equation $H - \widetilde{c} = 0$, is preserved under perturbation (27),

2) every limit cycle of field (16) (of multiplicity m), which satisfies equation $H - c = 0$, is a limit cycle of multiplicity $m-2$ of the perturbation (27) with the same equation $H - c = 0$.

Using Lemmas 11 and 12 one can split a limit cycle of multiplicity $m \geq 2$ of vector field (16) into k simple limit cycles, where $\frac{1-(-1)^m}{2} \leq k \leq m$ and $k \equiv m (\bmod 2)$.

Example. Consider mapping

$$H = (F : G) = \big((x^2 + y^2 - 1) : (x^2 + y^2 - 4)\big).$$

If $\xi(t) = t$ and $\eta(t) = t$, then one can check that vector field

$$\Big(H_y + \alpha(x-a)H,\ -[H_x + \beta(y-b)H]\Big)$$

has one limit cycle of multiplicity 1, which satisfies equation $F = x^2 + y^2 - 1 = 0$. Functions $\sigma(t) = \xi\left(\frac{1}{t}\right) = \frac{1}{t}$ and $\varsigma(t) = \eta\left(\frac{1}{t}\right) = \frac{1}{t}$ represent the Laurent series with $p = -1$. Integral continuation of vector field (21) can be written in the form

$$\Big(F_yG - FG_y + \alpha(x-a)FG,\ -[F_xG - FG_x + \beta(y-b)FG]\Big).$$

According to Theorem 6, this field has the limit cycle of multiplicity 1 with equation $G = x^2 + y^2 - 4 = 0$. Thus, equation (14) has two limit cycles.

Example. Consider Cassini mapping $H = (((x^2+y^2)^2 - r^2(x^2-y^2)) : 1)$, where $r \neq 0$. If $c_1 \in (-\infty, -\frac{r^4}{4})$, $c_2 \in (-\frac{r^4}{4}, 0)$, $c_3, c_4 \in (0, \infty)$, $\xi(t) = t^2(t-c_1)^2(t-c_2)^3(t-c_3)^{11}(t+c_4^2)$ and $\eta(t) = t^5(t-c_1)^3(t-c_2)^7(t-c_3)^6(t+c_4^2)$, then vector field $\Big(H_y + \alpha(x-a)\eta(H), \; -[H_x + \beta(y-b)\xi(H)]\Big)$ has three unilateral compound limit cycles R, L, and $R \cup L$ of multiplicity 2, which satisfy equation $H = 0$; two limit cycles of multiplicity 3, which satisfy equation $H - c_2 = 0$; and the limit cycle of multiplicity 6, which satisfies equation $H - c_3 = 0$.

Example. Consider mapping $H = ((\sin x \sin y) : 1)$. If $\xi(t) = t^2(t - \frac{1}{2})^3$ and $\eta(t) = t^5(t - \frac{1}{2})^4$, then one can check vector field

$$(H_y + \alpha(x-a)\eta(H), -[H_x + \beta(y-b)\xi(H)])$$

has infinite number of unilateral compound limit cycles of multiplicity 2, which satisfy equation $H = 0$, and infinite number of limit cycles of multiplicity 3, which satisfy equation $H - \frac{1}{2} = 0$.

One can check that if in Lemmas 7 – 12, one replaces L^c to $L^{(1:0)}$, $\mathcal{H}$ to $\mathcal{L}$, and $\mathcal{H}^*$ to $\mathcal{L}^*$, then one can obtain similar lemmas about $C \subset L^{(1:0)} \setminus Z$.

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