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Limit Cycles Bifurcated from a Reversible Quadratic Center

Jinming Li

Department of mathematics, Institute of Science,
China Agricultural University,
Beijing100080, China.
E-mail: lijm327@sohu.com

In this paper, we consider the quadratic perturbations of the one parameter family of reversible quadratic system that write in the complex form as

$$\dot{z} = -iz(1 + a\bar{z})$$

being $a \neq 0$ a complex number. We prove that the exact upper bound of the number of limit cycles produced by the period annulus system is two.

Key Words: Quadratic system, reversible center, Abelian integral.

1. INTRODUCTION AND MAIN RESULTS

The problem of finding the maximal number of limit cycles for polynomial system is the second part of the Hilbert's 16th Problem, and relatively little progress is made to the final solution of this problem. A weak form of this problem, proposed by Arnold [1, 2], can be expressed as follows:

Let H, f and g be real polynomials in x, y of degree $n+1$, at most m and m respectively, $\triangle$ be a set of h such that the real algebraic curve $H(x, y) = h$ has a compact component $\delta(h)$, and $\omega = gdx - fdy$ be a polynomial 1-form. Denote by $I(h)$ the Abelian integral of ω over $\delta(h)$, then which is the least upper bound $Z(n, m)$ of the number of zeros of $I(h)$ ($h \in \triangle$) for fixed n, m and arbitrary H, f and g?

The number $Z(n, m)$ is closely related to the number of limit cycles of the perturbed Hamiltonian system $X_H + \varepsilon(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y})$, where $X_H = H_y\frac{\partial}{\partial x} - H_x\frac{\partial}{\partial y}$ and $0 < \varepsilon \ll 1$. It is known that $Z(n, m)$ is finite ([13, 17]), but there is no concrete estimation of $Z(n, m)$. Up to now the number $Z(n, m)$ was given only for $n = m = 2$, see [7, 9, 18, 15, 6] and the references therein.

To study the Hilbert's 16th problem, it is also necessary to consider the perturbations of integrable but non-Hamiltonian systems (the extended version of the weakened Hilbert's 16th problem). Unfortunately, few works go in this direction, due to the complex structure of the first integral of the unperturbed systems: the first integral is not algebraic or algebraic with a large genus. Hence, in general it does not admit a Picard-Fuchs equation or the order of the Picard-Fuchs equation is very high or even infinite.

In the quadratic case, the integrable systems admitting period annulus must have at least one center. Besides the Hamiltonian class (Q_3^H), the integrable systems with centers can be classified as the reversible (Q_3^R), the Lotka-Volterra (Q_3^{LV}) and the codimension 4 (Q_4) classes, see [16, 19, 11]. Most mathematicians working in this field believe that the perturbations of the reversible systems may give the richest dynamical behavior, see [3] for instance.

It is well known that a quadratic reversible system can be written in the form

$$\begin{aligned} \dot{x} &= -y + ax^2 + by^2, \\ \dot{y} &= x(1 + cy), \end{aligned} \tag{1}$$

where a, b and c are real constants. There are some papers dealing with the case $c \neq 0$ (see [12, 5] for instance), and in these cases the Picard-Fuchs equations are of high order or infinite order. As far as we know, there is no work concerning to the case $c = 0$, and in this case it does not admit the Picard-Fuchs equation, since the first integral contains the exponential function.

In this paper we consider the case $c = 0$ and $a = b \neq 0$. Note that by a scaling it can be changed to the case $a = b = 1$. After perturbations in quadratic systems we will prove that the least upper bound of limit cycles, bifurcating from the period annulus, is two.

This system can be seen from another point of view. In [8] the authors study the period function of the systems that in the complex form write as

$$\dot{z} = -if(z)g(\bar{z}).$$

If one studies the number of limit cycles under perturbations of this kind of systems, it is natural to start from the system

$$\dot{z} = -iz(1 + A\bar{z}), \tag{2}$$

where A is a nonzero complex number. Changing to the real form, and making a scaling, from (2) we obtain system (1) with $c = 0$ and $a = b = 1$.

Theorem 1. *Under quadratic perturbations, the exact upper bound of the number of limit cycles produced by the period annulus of system (2) is 2.*

Theorem 1 follows basically from a careful estimate of the number of zeros of the associate Abelian integrals. Since there is no Picard-Fuchs equation associated to the quadratic perturbations of (2) , we use different technique to give an estimation of zeros of the Abelian integral.

2. PROOF OF THE MAIN RESULTS

After a complex scaling $z \to z/A$, in real coordinates, system (2) looks as:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -(x + x^2 + y^2). \end{aligned} \tag{3}$$

It has the first integral

$$H(x, y) = (x^2 + y^2)e^{2x},$$

with integrating factor $M(x) = 2e^{2x}$. We observe that system (3) has a homoclinic loop connecting saddle point $(-1, 0)$, which is the outer boundary of the period annulus. Hence, in the period annulus, the Hamiltonian function takes values between 0 and $h^* = e^{-2}$, where $H = h^*$ corresponds to the saddle and its separatrices connecting it. So in the period annulus we have $H(x, y) = h \in (0, h^*)$.

To prove theorem 1, we will use the following results given in [11].

LEMMA 2. *(1) (See page 108 of [11]) A quadratic system possessing a reversible center at the origin can be reduced to the complex form as*

$$\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2,$$

where a, b are arbitrary real constants. Moreover, the origin is a generic reversible center if and only if $a \neq -1$, 4 or $a = 4$, $b \neq \pm 2$.

(2) (See Theorem 1(ii) on page 114 of [11]) The upper bound of limit cycles produced by the period annulus of a generic reversible center under quadratic perturbations is realizable by the following essential perturbation

$$\dot{z} = (\lambda_1\varepsilon - i)z + (a + i\lambda_3\varepsilon)z^2 + 2|z|^2 + (b + i\lambda_5\varepsilon)\bar{z}^2,$$

where λ_k are arbitrary real constants independent on ε.

(3) (See Theorem 2(ii)(3) on page 116 and Appendix (ii) (3) on page 157 of [11]) The exact upper bound for the number of limit cycles produced by the period annulus of a generic reversible center with $a = b$ under quadratic perturbations is equal to the maximum number of zeros in (h_c, h_s), counting multiplicity, of the following related Melnikov integral

$$M_1(h) = \int\int_{F(x,y)<h} M(x)(\mu_1 + \mu_2 x + \mu_3 x^2)dxdy,$$

where $F(x,y)$ is the first integral:

$$F(x,y)=e^{-4(a+1)x}\left(\frac{y^2}{2}-\frac{a-1}{2(a+1)}x^2-\frac{a}{2(a+1)^2}-\frac{a}{8(a+1)^3}\right),$$

and $M(x)=e^{-4(a+1)x}$ is the integrating factor. h_c and h_s are the critical values of $F(x,y)$ corresponding to the center and the outer boundary of the period annulus, respectively. In the Melnikov integral, the μ_j are independent constants which are linear combination of the former λ_i.

From (2) we have that after a scaling $z\to-\frac{2}{A}z$, the origin of the system (3) is a generic center in the sense of lemma 2 (1). For system (3), the integrating factor and the first integral are given in the first paragraph of this section. Hence by lemma 2 (3), we need only consider the maximal number of zeros of the following integral

$$I(h)=\int\int_{H<h}e^{2x}(\alpha+\beta x+\gamma x^2)dxdy,$$

where α,β,γ are real constants dependent only on λ_k (see lemma 2 (2)) but not on ε. Note that $h\in(0,h^*)=(0,e^{-2})$.

Let $(x_{1h},0)$ and $(x_{2h},0)$ be the intersection points of the closed curve $\delta(h)$ with the x-axis, satisfying $x_{1h}<0<x_{2h}$, $h\in(0,e^{-2})$. Noticing that $\delta(h)$ is clockwise oriented. We have, by Green's formula, that

$$\begin{aligned} I(h) &= \oint_{H=h}e^{2x}(\alpha+\beta x+\gamma x^2)ydx \\ &= 2\int_{x_{1h}}^{x_{2h}}(\alpha+\beta x+\gamma x^2)e^x\sqrt{h-x^2e^{2x}}dx. \end{aligned} \tag{4}$$

Write $z=f(x)=xe^x$. Since $f'(x)=(x+1)e^x$, it defines a homeomorphism for $x\in(-1,x^*)$, where x^* satisfies $x^*e^{x^*}=e^{-1}$ and hence $0<x^*<0.28$. By the Implicit Function Theorem, an inverse function $x=x(z)=f^{-1}(z)$ is defined in the domain $|z|<e^{-1}$. We shall use the notation $x(z)$ for simplicity. Under this transformation, the generalized Abelian integral looks as follows:

$$I(h)=2\int_{-\sqrt{h}}^{\sqrt{h}}(\alpha+\beta\,x(z)+\gamma\,x^2(z))\sqrt{h-z^2}/(1+x(z))dz.$$

Let $z=\sqrt{h}\sin\theta$, then

$$I(h)=2h\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(\alpha+\beta\,x(\sqrt{h}\sin\theta)+\gamma\,x^2(\sqrt{h}\sin\theta))\cos^2\theta}{1+x(\sqrt{h}\sin\theta)}d\theta.$$

By using the above expressions, it is not difficult to prove:

LEMMA 3. *It follows that* $I(h) = 2hJ(h)$, *where* $J(h) = \bar{\alpha} + \bar{\beta}J_0(h) + \bar{\gamma}J_2(h)$ *with*

$$\begin{aligned} J_0(h) &= \int_0^{\frac{\pi}{2}} [\frac{1}{1+x(\sqrt{h}\sin\theta)} + \frac{1}{1+x(-\sqrt{h}\sin\theta)}]\cos^2\theta d\theta, \\ J_2(h) &= \int_0^{\frac{\pi}{2}} [x(\sqrt{h}\sin\theta) + x(-\sqrt{h}\sin\theta)]\cos^2\theta d\theta, \end{aligned} \tag{5}$$

and $\bar{\alpha} = \pi(\beta - \gamma)$, $\bar{\beta} = \alpha - \beta + \gamma$, $\bar{\gamma} = \gamma$. *The fact that* $\bar{\alpha}, \bar{\beta}$ *and* $\bar{\gamma}$ *are independent constants follows from the independence of* α, β, γ *(cf lemma 2 (3)).*

To prove theorem 1, we need the following proposition.

PROPOSITION 4. *If* $\bar{\alpha}^2 + \bar{\beta}^2 + \bar{\gamma}^2 \neq 0$, *then* $J(h)$ *has at most two zeros in the interval* $(0, h^*)$.

Proof. Firstly we note that If $\bar{\beta} = \bar{\gamma} = 0$, then $J(h) = \bar{\alpha} \neq 0$. So we assume without loss of generality that $\bar{\beta}^2 + \bar{\gamma}^2 \neq 0$. Secondly, we consider the number of zeros of $J'(h)$. For this purpose, we prove that for $h \in (0, h^*)$ it holds that $J_0'(h) > 0$. This fact allows to introduce a new function $P(h) := J_2'(h)/J_0'(h)$ so that $J'(h) = J_0'(h)(\bar{\beta} + \bar{\gamma}P(h))$. Hence to prove proposition 4, it suffices to prove the monotonicity of $P(h)$. Let us start by showing that $J_0'(h) > 0$.

By definition of $x(z)$ we have that the following inequalities hold for $0 < z < e^{-1}$:

$$\begin{aligned} &0 < \pm x(\pm z) < 1, \\ &x'(z) = (x(z)+1)^{-1}\exp(-x(z)), \\ &x(z) + x(-z) < 0. \end{aligned} \tag{6}$$

Let

$$z = \sqrt{h}\sin\theta, \quad \xi_k(x) = \frac{e^{-x}}{(1+x)^k}, \quad k = 1, 2, \cdots,$$

then we have that

$$\begin{aligned} J_0'(h) &= -\frac{1}{2\sqrt{h}}\int_0^{\frac{\pi}{2}} (\xi_3(x(z)) - \xi_3(x(-z)))\cos^2\theta\,\sin\theta d\theta, \\ J_2'(h) &= \frac{1}{2\sqrt{h}}\int_0^{\frac{\pi}{2}} (\xi_1(x(z)) - \xi_1(x(-z)))\cos^2\theta\,\sin\theta d\theta. \end{aligned} \tag{7}$$

From $\xi_k'(x) = -(x+1+k)[(x+1)^{k+1}e^x]^{-1}$ we know that $\xi_k(x)$ are decreasing for $x > -1$ and $k \geq 0$. On the other hand, for $0 < h < h^* = e^{-2}$, we have that $|z| < e^{-1}$. Therefore, for $\theta \in (0, \pi/2)$, we have that $-1 < x(-z) < 0 < x(z) < 1$. Hence from (5)–(7), we have that $J_0(h) > 0, J_0'(h) > 0, h \in (0, h^*)$, as we want to see.

Suppose that $P(h)$ is defined as above, then

$$P'(h) = (J_0')^{-2}(J_2''J_0' - J_2'J_0'')$$

which shows that $P'(h)$ has the same sign that $J_2''(h)J_0'(h) - J_2'(h)J_0''(h)$.

Let

$$k_0(h) = -2\sqrt{h}J_0'(h), \quad k_2(h) = 2\sqrt{h}J_2'(h).$$

From (4), we observe that

$$I'(h) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\alpha + \beta\, x(z) + \gamma\, x^2(z)}{1 + x(z)} d\theta.$$

Hence we have

$$\begin{aligned} I''(h) \ &= 4J'(h) + 2hJ''(h) \\ &= \frac{1}{2\sqrt{h}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\beta - \alpha + 2\gamma\, x(z) + \gamma\, x^2(z)}{(1 + x(z))^3 \exp(x(z))} \sin\theta\, d\theta. \end{aligned}$$

Choosing $\alpha = 1,\ \beta = \gamma = 0$ in the last equality, we have

$$\frac{d}{dh}\left(h^2 J_0'(h)\right) = -\frac{\sqrt{h}}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\exp(-x(z))}{(1 + x(z))^3} \sin\theta\, d\theta.$$

Similarly we have

$$\frac{d}{dh}\left(h^2 J_2'(h)\right) = \frac{\sqrt{h}}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\exp(-x(z))}{1 + x(z)} \sin\theta\, d\theta.$$

Therefore, by definition of $k_0(h)$ and $k_2(h)$, we obtain

$$\begin{aligned} k_0'(h) \ &= \ -\frac{3}{2h}k_0(h) + \frac{1}{2h}\int_0^{\frac{\pi}{2}} \left(\xi_3(x(z)) - \xi_3(x(-z))\right)\sin\theta d\theta; \\ k_2'(h) \ &= \ -\frac{3}{2h}k_2(h) + \frac{1}{2h}\int_0^{\frac{\pi}{2}} \left(\xi_1(x(z)) - \xi_1(x(-z))\right)\sin\theta d\theta. \end{aligned}$$

Let us make a comment here. Although we can compute the derivatives by using their definitions, we cannot be more effective to get the above formulae.

Hence we have

$$
\begin{aligned}
& J_2''(h)J_0'(h) - J_2'(h)J_0''(h) \\
&= \frac{1}{4h}\left(k_0'(h)k_2(h) - k_0(h)k_2'(h)\right) \\
&= \frac{1}{16h^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} G(\theta,\phi)\sin\theta\sin\phi(\cos^2\phi - \cos^2\theta)d\theta d\phi,
\end{aligned}
\tag{8}
$$

where

$$G(\theta,\phi) = \varphi(\theta)\varphi(\phi)(\psi(\theta) - \psi(\phi)) \tag{9}$$

is defined by letting $g(\theta) = f^{-1}(\sqrt{h}\sin\theta)$ and thus

$$
\begin{aligned}
\varphi(\theta) &= \frac{e^{-x}}{1+x} - \frac{e^{-\tilde{x}}}{1+\tilde{x}}, \\
\psi(\theta) &= \left(\frac{e^{-x}}{(1+x)^3} - \frac{e^{-\tilde{x}}}{(1+\tilde{x})^3}\right)\left(\frac{e^{-x}}{1+x} - \frac{e^{-\tilde{x}}}{1+\tilde{x}}\right)^{-1},
\end{aligned}
$$

with

$$x = g(\theta), \quad \tilde{x} = g(-\theta), \quad y = g(\phi), \quad \tilde{y} = g(-\phi), \tag{10}$$

Remember that $f(x) = xe^x$. Note that in (8), we express the product of integrals as double integral in which the variable θ becomes ϕ alternatively.

By definitions of $f(x)$ and $g(\theta)$, it can be seen that $xe^x = -\tilde{x}e^{\tilde{x}} = \sqrt{h}\sin\theta$ under the above notations. Therefore using (10) we have

$$\psi(\theta) = \frac{(1+x)^3\tilde{x} + x(1+\tilde{x})^3}{(1+x)^2(1+\tilde{x})^2((1+x)\tilde{x} + x(1+\tilde{x}))}.$$

From $\xi e^\xi = -\eta e^\eta$ we obtain a function $\eta = f^{-1}(-f(\xi)) \overset{\text{def}}{=} p(\xi)$, $\xi \in [0, x^*)$ satisfying

$$p'(\xi) = \frac{d\eta}{d\xi} = -\frac{(\xi+1)e^\xi}{(\eta+1)e^\eta} = \frac{\eta(\xi+1)}{\xi(\eta+1)},$$

with $-1 < \eta = p(\xi) < 0 < \xi < x^*$. Denoting

$$\kappa(\xi) = \frac{(1+\xi)^3 p(\xi) + \xi(1+p(\xi))^3}{(1+\xi)^2(1+p(\xi))^2((1+\xi)p(\xi) + \xi(1+p(\xi)))},$$

then we have that $\psi(\theta) = \kappa(g(\theta))$. Since $g(\theta)$ is monotonely increasing, to consider the monotonicity of $\psi(\theta)$, we need only consider the monotonicity of $\kappa(\xi)$. Now we have that

$$\kappa'(\xi) = \kappa_1(\xi)\left[\xi(1+\xi)^3(1+\eta)^4(\xi+\eta+2\,\xi\,\eta)^2\right]^{-1}.$$

Let $u = \xi - \eta = \xi - p(\xi)$, then

$$\frac{du}{d\xi} = 1 - p'(\xi) = 1 - \frac{\eta(\xi+1)}{\xi(\eta+1)} > 0,$$

since $-1 < \eta = p(\xi) < 0 < \xi < x^*$. Hence u is an increasing function of ξ and $0 < u < 1 + x^* < 1.28$. Using the relationship between ξ and $\eta = p(\xi)$ we have that $\xi = u/(e^u+1), \eta(\xi) = -ue^u/(e^u+1)$. Under this transformation, $\kappa_1(\xi)$ becomes:

$$\kappa_1(\xi) = \frac{u^3}{(1+e^u)^9}M(u),$$

where

$$M(u) = \sum_{i=0}^{6} p_i(e^u)u^i,$$

and

$$\begin{aligned}
p_0(v) &= 2(v-1)(v^2+1)(v+1)^6,\\
p_1(v) &= 2v(1+v)^5(5-14v+5v^2),\\
p_2(v) &= 4\,(v-1)\,v\,(1+v)^4\left(2+19\,v+2\,v^2\right),\\
p_3(v) &= -v\,(1+v)^3\left(5+16\,v-138\,v^2+16\,v^3+5\,v^4\right),\\
p_4(v) &= (v-1)\,v\,(1+v)^2\left(1+6\,v-50\,v^2+6\,v^3+v^4\right),\\
p_5(v) &= -v^2\,(1+v)\left(1+v^2\right)\left(1-22\,v+v^2\right),\\
p_6(v) &= 4v^3(1-v^3).
\end{aligned}$$

So in $0 < u < 1 + x^*$, $M(u)$ has the same sign that $\psi'(x)$ in $0 < x < x^*$. Now we have

$$M(u) = \frac{8192}{3}u^3 + O(u^4), \quad 0 < u \ll 1, \tag{11}$$

and therefore $M(u) > 0$ for u positive and small enough. Furthermore we have that

$$M'(u) = e^u \sum_{i=0}^{6} l_i(e^u)u^i,$$

where

$$\begin{aligned}
l_0(v) &= 18\,v\,(-1+v)\,(1+v)^6,\\
l_1(v) &= 6\,(1+v)^5\,(-1-21\,v+16\,v^2),\\
l_2(v) &= (1+v)^4\,(-23-201\,v+411\,v^2+49\,v^3),\\
l_3(v) &= -9\,(1+v)^3\,(1+7\,v-69\,v^2+9\,v^3+4\,v^4),\\
l_4(v) &= (1+v)^2\,(-1-17\,v+275\,v^2-229\,v^3+28\,v^4+8\,v^5),\\
l_5(v) &= -v\,(1+v)\,(2-89\,v+9\,v^2-109\,v^3+7\,v^4),\\
l_6(v) &= -12\,v^2\,(2\,v^3-1).
\end{aligned}$$

For $u>0$, we apply the inequality $e^u-1>u+\dfrac{u^2}{2}+\dfrac{u^3}{6}+\dfrac{u^4}{24}+\dfrac{u^5}{120}+\dfrac{u^6}{720}$ in the factor e^u-1 of $l_0(e^u)$ obtaining that

$$M'(u)>\frac{1}{40}u\,e^u\sum_{i=0}^{5}l_i^1(e^u)u^i,$$

where $l_i^1(v)$ are polynomials of v and

$$l_0^1(e^u)=240(e^u-1)(1+e^u)^5(1+19e^u).$$

Similarly, applying the inequality $e^u-1>u+\dfrac{u^2}{2}+\dfrac{u^3}{6}+\dfrac{u^4}{24}+\dfrac{u^5}{120}(u>0)$ to the factor e^u-1 of $l_0^1(e^u)$, we can prove that

$$M'(u)>\frac{1}{40}u^2e^uM_1(u),$$

where $M_1(u)>0$, as can be shown easily by using the inequality $0<u<1+x^*<1.28$. We omit these computations here.

Finally, since $M(u)>0$ for $0<u\ll 1$ and $M'(u)>0$ we have that $M(u)>0$ for all $u\in(0,1.28)$ and hence that $\kappa_1(\xi)>0$. For $\xi>0$ we have $\kappa'(\xi)>0$. As a function of θ, the derivative of $\psi(\theta)$ with respect to θ is given by the chain formula as

$$\frac{d\psi(\theta)}{d\theta}=\left.\frac{d\kappa(\xi)}{d\xi}\right|_{\xi=g(\theta)}\cdot\frac{dg(\theta)}{d\theta},$$

which combining the fact that $\dfrac{dg(\theta)}{d\theta}=\sqrt{h}\cos\theta/f'(x(\sqrt{h}\sin\theta))>0$ gives $\dfrac{d\psi(\theta)}{d\theta}>0$. Noticing that $-1<\tilde{x}<0<x<x^*$, by (9), we have that $G(\theta,\phi)$ has the same sign as $x-y=g(\sqrt{h}\sin\theta)-g(\sqrt{h}\sin\phi)$ by the

monotonicity of $g(z)$ and therefore has the same sign as $\theta-\phi$. On the other hand, $\cos^2\phi-\cos^2\theta$ also has the same sign as $\theta-\phi$ since $\theta,\phi\in[0,\frac{\pi}{2}]$. The above discussions show that the integrand in the double integral in equality (8) is nonnegative and so we have proved the convexity of $P(w)$. Hence the proof of lemma 4 is completed. ▮

Remark 5. In [14], the authors give a general method of proving the monotonicity of the ratio of two integrals. In proving proposition 4, though the idea is the same as that in [14], but the theorems given in [14] are not applicable here.

Proof of Theorem 1 From lemma 2(1) we know that system (3) is generic. Therefore, lemma 2 and proposition 4 together show that an upper bound of limit cycles produced by the period annulus is two. From the proof of proposition 4, we have that $J_0'(h)>0$. Hence we can define $u=J_0(h)$ and its inverse $h=h(u)$. Write $\varphi(u)=J_2(h(u))$, then $\varphi'(u)=P(h(u))$ and $\varphi''(u)=P'(h(u))\frac{dh}{du}=P'(h(u))/J_0'(h(u))>0$ by proposition 4, which means that the curve $v=\varphi(u)$ is strictly convex. Hence a zero of $J(h)$ corresponds uniquely to an intersection point of the straight line $\bar\alpha+\bar\beta u+\bar\gamma v=0$ with the curve $v=\varphi(u)$ in the (u,v)-plane. By the strict convexity of the curve $v=\varphi(u)$, we can choose appropriate $\bar\alpha,\bar\beta,\bar\gamma$ (since they are independent constants), such that the number of intersection points of the above curve and the corresponding straight line is two, which gives two zeros of $J(h)$ in $h\in(0,h^*)$ and hence of $I(h)$. Hence the theorem follows. ▮

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