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Planar Quadratic Vector Fields with Finite Saddle Connection on a Straight Line (Convex Case)

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We classify the phase portraits of planar quadratic vector fields with a invariant straight line passing through two finite saddles, when the singularities of the field form a convex quadrilateral.

Key Words: Quadratic Vector Field, Saddle Connection

1. INTRODUCTION

A variety of applications in many areas of science require the study of polynomial systems of ordinary differential equations in the plane. These systems have attracted the attention of many pure and applied mathematicians. Since H. Poincaré (1880), the study of the topological behavior of the configurations of polynomial systems has been developed from several points of view. The topological behavior of the configuration can change when critical points and closed orbits change in number or type, or when two saddle points are connected by an orbit. The present paper sheds some light on the later kind of bifurcation for quadratic systems.

Orbits connecting critical points of dynamical systems are important for a number of reasons. For instance, they specify the physically admissible discontinuities in solutions of nonlinear systems of conservation laws. In the

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theory of nonlinear conservation laws, weak solutions are required for an existence theory, but they are not uniquely determined by the Cauchy initial data. Supplementary conditions, known as entropy conditions, are then imposed on the solutions to obtain uniqueness (see Lax [11]). Gel'fand [6] showed that the existence of some discontinuous solutions, called shock solutions, are related to connections between two critical points of an associated dynamical system. These critical points are states representing the long distance behavior of Cauchy initial data for conservation laws. Therefore, we are interested in determining which two saddle points in the plane are connected by an orbit. Consequently, a global analysis of the configuration of dynamical systems is required; see for instance Gomes [7], Isaacson, Marchesin and Plöhr [9, 10] and Shaeffer and Shearer [13].

Chicone [4] proved that for quadratic gradient systems all saddle connections are straight lines. In this paper we classify all possible configurations for a planar quadratic dynamical system containing saddle connection supported on straight line segments, when the singularities of the quadratic field form a convex quadrilateral (Theorem 1). Systems with saddle connections are important as they are in the boundary between stable configurations. These stable configurations have been object of intense study (see [12]), and a complete classification of all structurally stable quadratic vector vector fields, modulo limit cycles, was obtained by Artés, Kooij and Llibre [3]

The technique used in this work combines several different pieces of information concerning global and local behavior of the dynamical system. In our classification we use strongly Coppel's result which says that: If a quadratic system has straight line path then it either has a centre or at most one closed path (stable or unstable) [5].

This paper is organized as following: in Section 2 we obtain normal form for planar quadratic vector fields which have more than two singularities and an invariant straight line passing through two finite saddles. In sections 3 and 4 we restrict ourselves to the case of convex distribution of four finite singularities and use Poincaré's compactification [1, 8] to give a complete description of the singularities of these fields at infinity, and describe all possible configurations and local behavior of the finite singularities. In the final section we analyze the changes of these fields according to a parameter d of the field in normal form and conclude with the following Theorem:

THEOREM 1. *Let χ be a quadratic field on plane with four finite singularities forming a convex quadrilateral and such that the straight line passing through the two finite saddles is invariant. Then the phase portrait of χ is given by one of the the following pictures:*

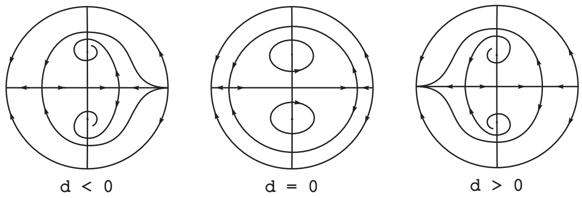

FIG. 1. Phase Portrait of χ_d when $m = 0$

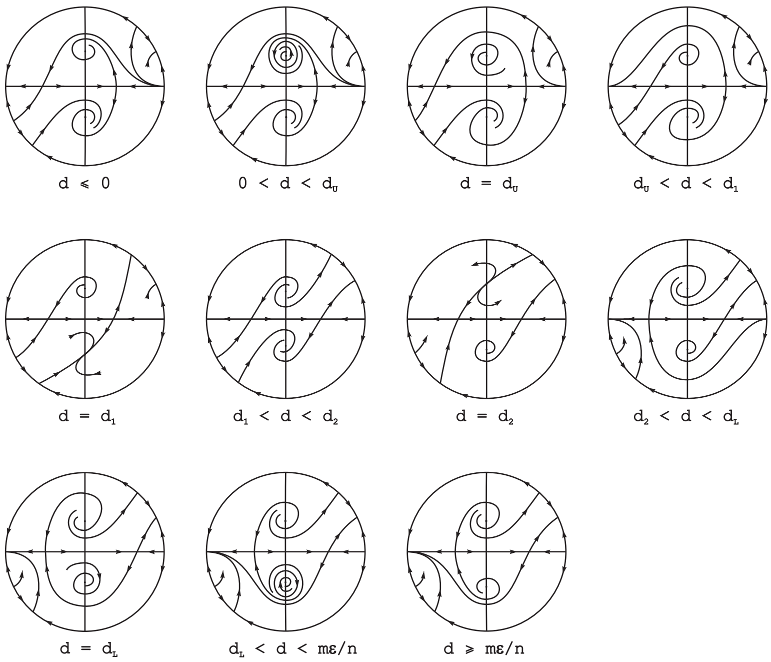

FIG. 2. Phase Portrait of χ_d when $m^2 - 4n(l-1) > 0$

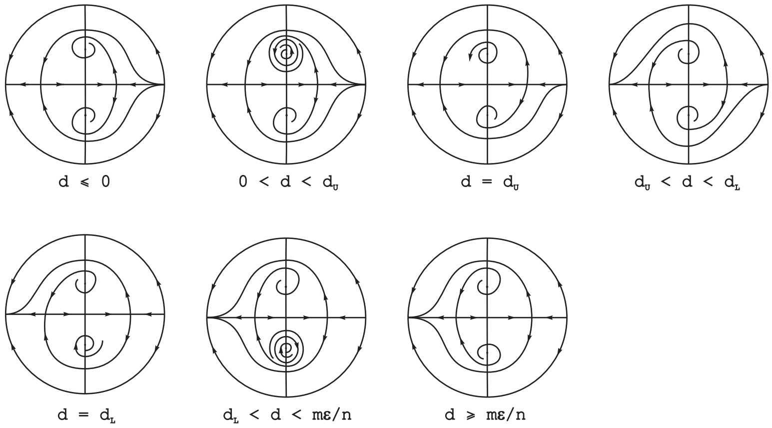

FIG. 3. Phase Portrait of χ_d when $m^2 - 4n(l-1) < 0$ and $m \neq 0$

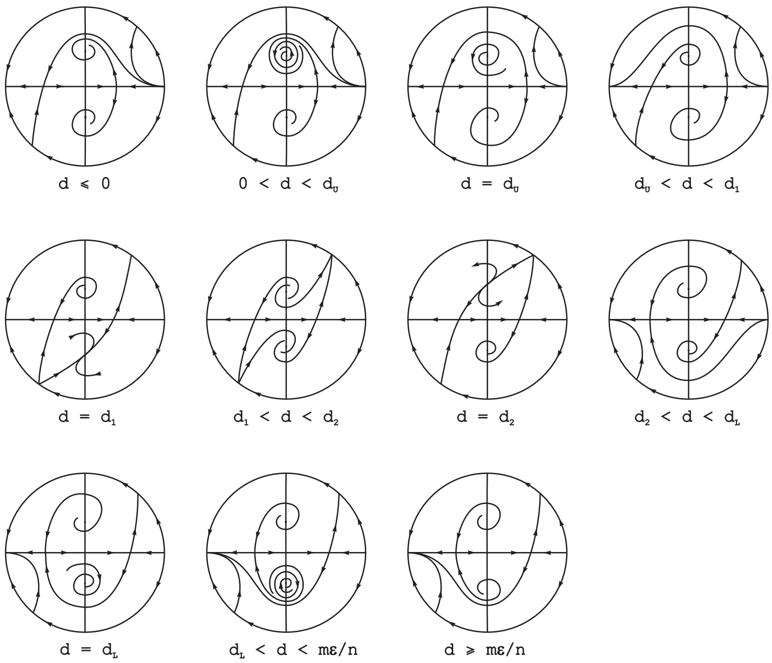

FIG. 4. Phase Portrait of χ_d when $m^2 - 4n(l-1) = 0$

2. NORMAL FORM

In this section we present normal form for planar quadratic vector fields which have more than two singularities and an invariant straight line passing through two finite saddles.

We consider the quadratic field:

$$\chi : \begin{cases} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{cases}$$

where $P(x,y) = \sum_{i+j=0}^{2} a_{ij}x^i y^j$ and $Q(x,y) = \sum_{i+j=0}^{2} b_{ij}x^i y^j$, with a finite number of singularities and a saddle connection supported on a straight line.

LEMMA 2. *If the saddle connection of χ is horizontal then $b_{11} \neq 0$.*

Proof. Let $y = c$ be the invariant line that contains the saddles (x_1, c) and (x_2, c).

We have $\frac{\partial Q}{\partial x}(x, c) = 0$ because the straight line $y = c$ is invariant. Since x_1 and x_2 are the roots of the quadratic equation $P(x, c) = 0$, it follows that $\frac{\partial P}{\partial x}(x_1, c)$ and $\frac{\partial P}{\partial x}(x_2, c)$ have opposite signs.

If $b_{11} = 0$ then $\frac{\partial Q}{\partial y}(x, c) = b_{01} + 2b_{02}c$, a constant, and then the determinants $|d\chi(x_1, c)|$ and $|d\chi(x_2, c)|$ would be positive, contradicting the fact that the singularities (x_1, c) and (x_2, c) are both saddles. ∎

LEMMA 3. *If χ has more than two singularities then it can be written in the form:*

$$\chi : \begin{cases} \dot{x} &= dx + \varepsilon y + lx^2 + mxy + ny^2 \\ \dot{y} &= x(1+y) \end{cases}$$

where $l < 0$, $\varepsilon < n$ and $\varepsilon \leq 0$.

Moreover, we will have 3 singularities if $n\varepsilon = 0$ and 4 singularities if $n\varepsilon \neq 0$. In the last case the singular quadrilateral will be convex if $n < 0$ and not convex if $n > 0$.

Proof. Since χ has at least three singularities, one of them is not a saddle. Supposing that this one is the origin and that the straight line supporting the saddle connection is $y = -1$ we get

$$\chi : \begin{cases} \dot{x} &= \sum_{i+j=1}^{2} a_{ij}x^i y^j \\ \dot{y} &= (\alpha x + \beta y)(1+y) \end{cases}$$

We have $\alpha \neq 0$ by Lemma 2, and using the change of coordinates

$$\begin{cases} X &= \alpha x + \beta y \\ Y &= y \end{cases}$$

we can write the field χ as

$$\chi : \begin{cases} \dot{x} &= dx + \varepsilon y + lx^2 + mxy + ny^2 \\ \dot{y} &= x(1+y) \end{cases}$$

Since the invariant line $y = -1$ contains two singularities (both saddles), the equation $lx^2 + (d-m)x + (n-\varepsilon) = 0$ has two distinct real roots

$$x_1 = \frac{m - d - \sqrt{\Delta}}{2l} \quad \text{and} \quad x_2 = \frac{m - d + \sqrt{\Delta}}{2l}, \tag{1}$$

where $\Delta = (m-d)^2 - 4l(n-\varepsilon)$.

The linear part of χ at the saddle point $(x_1, -1)$ is

$$d\chi(x_1, -1) = \begin{bmatrix} -\sqrt{\Delta} & \varepsilon + mx_1 - 2n \\ 0 & x_1 \end{bmatrix}$$

and therefore $x_1 > 0$. With the same arguments we prove that $x_2 < 0$, and we get $l(n-\varepsilon) < 0$. Since $x_2 - x_1 = \frac{\sqrt{\Delta}}{l} < 0$ we conclude that $l < 0$ and $n > \varepsilon$.

Finally we have $\varepsilon \leq 0$ because $|d\chi(0,0)| = -\varepsilon$ and $(0,0)$ is not a saddle.

The last statements are easy to prove, since

$$O = (0,0),\ S_1 = (x_1, -1),\ S_2 = (x_2, -1) \text{ and } F = (0, -\varepsilon/n)$$

are the singularities in the case $n\varepsilon \neq 0$, and

$$O = (0,0),\ S_1 = (x_1, -1) \text{ and } S_2 = (x_2, -1)$$

are the singularities in the case $n\varepsilon = 0$. ▮

We observe that the functions $x_1 = x_1(d)$ and $x_2 = x_2(d)$ given by (1) are strictly increasing, as can be easily verified.

3. SINGULARITIES AT INFINITY

To see the asymptotic behavior of the unbounded orbits of the field

$$\begin{cases} \dot{x} &= \sum_{i+j=0}^{2} a_{ij} x^i y^j \\ \dot{y} &= \sum_{i+j=0}^{2} b_{ij} x^i y^j \end{cases} \tag{2}$$

we will use the Poincaré compactification described in [1, pg 216].

Consider the new coordinates

$$u = y/x \ \text{ and } \ w = 1/x \ \text{ if } \ x \neq 0 \tag{3}$$

$$v = x/y \ \text{ and } \ w = 1/y \ \text{ if } \ y \neq 0 \tag{4}$$

Using the coordinates (3) in (2) and making $\tau = w\,t$ we get

$$\begin{cases} \frac{du}{d\tau} &= \phi(u) + wF(u,w) \\ \frac{dw}{d\tau} &= -wH(u,w) \end{cases} \tag{5}$$

where

$$\begin{aligned} \phi(u) &= -a_{02}u^3 + (b_{02} - a_{11})u^2 + (b_{11} - a_{20})u + b_{20} \ , \\ F(u,w) &= -a_{01}u^2 + (b_{01} - a_{10})u + b_{10} + b_{00}w - a_{00}uw \ \text{ and} \\ H(u,w) &= a_{02}u^2 + a_{11}u + a_{20} + a_{10}w + a_{01}uw + a_{00}w^2 \ . \end{aligned}$$

Similarly, for the coordinates (4) we get

$$\begin{cases} \frac{dv}{d\tau} &= \psi(v) + wG(v,w) \\ \frac{dw}{d\tau} &= -wK(v,w) \end{cases} \tag{6}$$

where

$$\begin{aligned} \psi(v) &= -b_{20}v^3 + (a_{20} - b_{11})v^2 + (a_{11} - b_{02})v + a_{02} \ , \\ G(v,w) &= -b_{10}v^2 + (a_{10} - b_{01})v + a_{01} + a_{00}w - b_{00}vw \ \text{ and} \\ K(v,w) &= b_{20}v^2 + b_{11}v + b_{02} + b_{01}w + b_{10}vw + b_{00}w^2 \ . \end{aligned}$$

The points $(u,0)$ and $(v,0)$ represent the points at the infinity for the field (2). We observe that the points $(u,0)$ and $(u^{-1},0)$ with $u \neq 0$ represent the same point at infinity.

The points $(u_*,0)$ (respectively $(v_*,0)$) for which $\phi(u_*) = 0$ (respectively $\psi(v_*) = 0$) are critical points of the system (5) (respectively (6)), which means that u_* (respectively v_*) is an asymptotic direction for the field (2). We will call u_* (respectively v_*) critical point at infinity and $\phi(u) = 0$ (respectively $\psi(v) = 0$) the equation of the critical points at infinity of the field (2).

The Jacobian matrix of the system (5) at a point $(u,0)$ is given by

$$J(u) = \begin{bmatrix} \phi'(u) & F(u,0) \\ 0 & -H(u,0) \end{bmatrix} \tag{7}$$

and the Jacobian matrix of the system (6) at the point $(0,0)$ is given by

$$L_0 = \begin{bmatrix} a_{11} - b_{02} & a_{01} \\ 0 & -b_{02} \end{bmatrix} \tag{8}$$

We study now the possibilities for the singularities at the infinity for χ given by Lemma 3 in the case of convex distribution of four finite singularities. We have

$$\phi(u) = -u(nu^2 + mu + l - 1)$$

and

$$\psi(v) = (l-1)v^2 + mv + n$$

and therefore $u = 0$ is always a singularity at the infinity. Moreover, since

$$J(0) = \begin{bmatrix} 1-l & * \\ 0 & -l \end{bmatrix}$$

and $l < 0$, this singularity will be always a hyperbolic source.

Furthermore, if $u \neq 0$ is a root of $\phi(u) = 0$ then

$$J(u) = \begin{bmatrix} \phi'(u) & * \\ 0 & -1 \end{bmatrix},$$

and we conclude that u will be a non-hyperbolic singularity at infinity only if $\phi'(u) = 0$.

We observe also that $v = 0$ will be a singularity at the infinity for χ if and only if $n = 0$.

The following Lemma describes all the possibilities for the singularities at the infinity for χ:

Lemma 4. *Consider the field χ given by Lemma 3, with four singularities and convex distribution. We have the following possibilities for the singularities at infinity distinct of $u = 0$:*

If $m^2 - 4n(l-1) > 0$ we have two singularities $u_1 < u_2 < 0$, with u_1 a saddle and u_2 an attractor when $m < 0$, and $0 < u_1 < u_2$, with u_1 an attractor and u_2 a saddle when $m > 0$.

If $m^2 - 4n(l-1) = 0$ we have one singularity u_1, a saddle-node, with $u_1 < 0$ when $m < 0$ and $0 < u_1$ when $m > 0$.

If $m^2 - 4n(l-1) < 0$ we do not have another singularity besides $u = 0$.

Proof. The cases with hyperbolic singularities at infinity are easy to verify; it is enough to observe the signs of the roots of the equation $\phi(u) = 0$ and the sign of ϕ' at these points. We then have to analyze only the case when $m^2 - 4n(l-1) = 0$.

We follow the notation of Andronov et al [1].

With the change of coordinates $u = x - \alpha y - \frac{m}{2n}$, $w = y$ and $\tau = -t$, where $\alpha = \frac{\varepsilon m^2}{4n^2} - \frac{dm}{2n} - 1$, the system (5) becomes

$$\begin{cases} \dot{x} &= \tilde{P}(x,y) = -\frac{m}{2}x^2 + (m\alpha + d - \frac{\varepsilon m}{2} + n\alpha)xy \\ & \quad + \frac{m\alpha}{2}(\varepsilon - \alpha)y^2 + nx^3 + \varepsilon(1 - n\alpha)x^2 y - n\alpha^3 y^3 \\ \\ \dot{y} &= y + \tilde{Q} = y + (d - \frac{\varepsilon m}{2n})y^2 + nx^2 y \\ & \quad + (\varepsilon - 2n\alpha)xy^2 - \alpha(\varepsilon - n\alpha)y^3 \end{cases}$$

Let $y = \varphi(x)$ be a solution of $y + \tilde{Q}(x,y) = 0$. Then $\varphi(0) = \varphi'(0) = 0$ and so

$$\psi(x) = \tilde{P}(x, \varphi(x)) = -\frac{m}{2}x^2 + \cdots.$$

By Theorem 65 [1, pg 340] it follows that u_1 is a saddle-node. ▮

4. THE FINITE SINGULARITIES

We now describe the singularities of χ when it has four of them with convex distribution. By Lemma 3 we have

$$\chi : \begin{cases} \dot{x} &= dx + \varepsilon y + lx^2 + mxy + ny^2 \\ \dot{y} &= x(1+y) \end{cases}$$

where $l < 0$ and $\varepsilon < n < 0$.

Since $S_1 = (x_1, -1)$ and $S_2 = (x_2, -1)$ are saddle points by hypothesis, we must analyze the singularities $O = (0,0)$ and $F = (0, -\varepsilon/n)$ (in case $n\varepsilon \neq 0$).

The linear part of χ at these points is given by

$$d\chi(0,0) = \begin{bmatrix} d & \varepsilon \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad d\chi(0, -\varepsilon/n) = \begin{bmatrix} d - m\varepsilon/n & -\varepsilon \\ 1 - \varepsilon/n & 0 \end{bmatrix} \tag{9}$$

LEMMA 5. *If $d = 0$ then χ doesn't have a limit cycle and O is*
a) a center if $m(n - l\varepsilon) = 0$;
b) a weak attractor if $m(n - l\varepsilon) < 0$;
b) a weak source if $m(n - l\varepsilon) > 0$.

Proof. With the change of variables $x = |b|\sqrt{-\varepsilon}X$, $y = bY$ and $t = \frac{b}{|b|\sqrt{-\varepsilon}}\tau$, the field χ becomes

$$\begin{cases} \dot{X} &= -Y + blX^2 + \frac{|b|m}{\sqrt{-\varepsilon}}XY - \frac{bn}{\varepsilon}Y^2 \\ \dot{Y} &= X(1+bY) \end{cases}$$

By Theorem 16.1 [12, page 359] it follows that χ doesn't have limit cycles and the origin is a center if $m(n - l\varepsilon) = 0$. Computing the focal value [2, pg 252], we get

$$\alpha_3 = \frac{\pi}{4\varepsilon^2\sqrt{-\varepsilon}} m(n - l\varepsilon) = 0,$$

and this finishes the Lemma. ▌

Lemma 6. *If χ is such that $\varepsilon < n < 0$ we have the following possibilities for the origin O and the singularity F:*

$$O : \begin{cases} d = 0 \begin{cases} m = 0 \implies \textit{center} \\ m < 0 \implies \textit{weak source} \\ m > 0 \implies \textit{weak attractor} \end{cases} \\ \\ d > 0 \implies \textit{hyperbolic source} \\ d < 0 \implies \textit{hyperbolic attractor} \end{cases}$$

$$F : \begin{cases} d = m\varepsilon/n \begin{cases} m = 0 \implies \textit{center} \\ m < 0 \implies \textit{weak source} \\ m > 0 \implies \textit{weak attractor} \end{cases} \\ \\ d > m\varepsilon/n \implies \textit{hyperbolic source} \\ d < m\varepsilon/n \implies \textit{hyperbolic attractor} \end{cases}$$

Proof. The case O follows from Lemma 5 because $n - l\varepsilon < 0$. The case F is proved the same way, by way of a change of coordinates which exchanges the singularities O and F. ▌

5. PHASE PORTRAIT IN CONVEX QUADRILATERAL CASE

We describe in this section the possible configurations for the phase space of χ when the finite singularities form a convex quadrilateral. By Lemma 3, the field is written as

$$\chi : \begin{cases} \dot{x} = dx + \varepsilon y + lx^2 + mxy + ny^2 \\ \dot{y} = x(1 + y) \end{cases} \tag{10}$$

with $l < 0$, and $\varepsilon < n < 0$, and the singularities will be the saddle points $S_1 = (x_1, -1)$ and $S_2 = (x_2, -1)$, $x_2 < 0 < x_1$, and the non-saddle points $O = (0, 0)$ and $F = (0, -\varepsilon/n)$.

With the change of coordinates

$$\begin{cases} x = \bar{x} \\ y = (1 - \frac{\varepsilon}{n})\bar{y} - \frac{\varepsilon}{n} \end{cases}$$

we get

$$\bar{\chi} : \begin{cases} \dot{\bar{x}} = \bar{d}\bar{x} + \bar{\varepsilon}\bar{y} + \bar{l}\bar{x}^2 + \bar{m}\bar{x}\bar{y} + \bar{n}\bar{y}^2 \\ \dot{\bar{y}} = \bar{x}(1 + \bar{y}) \end{cases} \tag{11}$$

where $\bar{d} = d - \frac{m\varepsilon}{n}$, $\bar{l} = l < 0$, $\bar{\varepsilon} = \varepsilon\frac{\varepsilon - n}{n} < \bar{n} = n(\frac{n-\varepsilon}{n})^2 < 0$ and $\bar{m} = m(1 - \varepsilon/n)$,and we can assume that $m \geq 0$. From now on we will also assume that $m > 0$; we will perform later the analysis of the case $m = 0$.

We denote by $U_1(d)$ (respectively $U_2(d)$) the unstable (respectively stable) separatrix of the saddle S_1 (respectively S_2) located in the semi-plane $y > -1$ and by $L_1(d)$ (respectively $L_2(d)$) the unstable (respectively stable) separatrix of the saddle S_1 (respectively S_2), in the semi-plane $y < -1$.

The singularities at infinity in the coordinates (u, w) will be, according Lemma 4, the points $A = (u = 0, w = 0)$, $B = (u = u_1, w = 0)$ and $C = (u = u_2, w = 0)$ when $m^2 - 4n(l-1) > 0$, A and B when $m^2 - 4n(l-1) = 0$ and only A when $m^2 - 4n(l-1) < 0$. We denote by $\bar{A}$, $\bar{B}$ and $\bar{C}$ the symmetric singularities at infinity, and by Z_1 and Z_2 the separatrices of the saddles C and $\bar{C}$ at infinity (or the saddle-nodes B and $\bar{B}$ when $m^2 - 4n(l-1) < 0$). We observe that χ always presents the orientations given by the figure 5 bellow, for any value of d (the invariant line $y = -1$ is drawn as $y = 0$ to improve the image):

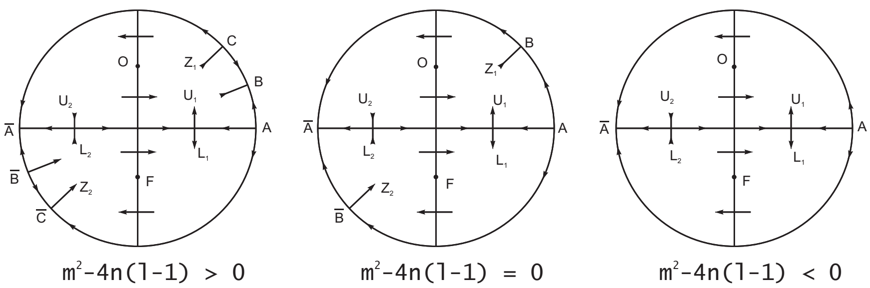

FIG. 5. Orientation of the orbits of χ in the convex case

We will use from now on the notation χ_d for the field χ, using d as a parameter, and fixing the other coefficients ε, l, m and n.

To describe χ we must know the ω-limit and the α-limit of the separatrices of the saddles S_1 and S_2, the α-limit of Z_1 and the ω-limit of Z_2.

The following Lemma gives the values of d for which we have a saddle connection between a finite saddle and a saddle at infinity.

LEMMA 7. *Consider the field χ_d given by* (10)*, and let $u = r$ be the singularity at infinity which is a saddle* (in case $m^2 - 4n(l-1) > 0$) *or saddle-node* (in case $m^2 - 4n(l-1) = 0$)*, as in Lemma 4. Then there exist two values d_1 and d_2 for d, with the following properties:*
(a) *The line $y = rx$ is invariant by the flow of χ_{d_2}. More precisely, for any $p = (x, rx)$, the function $h_2(d) = \langle \chi_d^\perp(p), (1, r)\rangle$ satisfies $h_2'(d) = rx$ and $h_2(d_2) = 0$.*
(b) *The line $y = rx - \varepsilon/n$ is invariant by the flow of χ_{d_1}. More precisely, for any $p = (x, rx - \varepsilon/n)$, the function $h_1(d) = \langle \chi_d^\perp(p), (1, r)\rangle$ satisfies $h_1'(d) = rx$ and $h_1(d_1) = 0$.*
(c) $0 < d_1 < d_2 < \frac{m\varepsilon}{n}$.

Proof. Consider any line $y = rx + s$, and let p be one point on this line. Then

$$\chi_d(p) = (dx + \varepsilon(rx + s) + lx^2 + mx(rx + s) + n(rx + s)^2, x + x(rx + s)).$$

Since r is a singularity at infinity, we have $nr^2 + mr + l = 1$ and so

$$\chi_d(p) = (dx + \varepsilon rx + \varepsilon s + x^2 + msx + 2nrsx + ns^2, x + rx^2 + sx),$$

$$\langle \chi_d^\perp(p), (1, r)\rangle = (dr + \varepsilon r^2 + mrs + 2nr^2 s - s - 1)x + rs(ns + \varepsilon).$$

If $s = 0$, letting $d = d_2 = \frac{1}{r} - \varepsilon r$ we get $\langle \chi_d^\perp(p), (1, r)\rangle = 0\ \forall x$, which proves statement (a).

In the same way, if $s = -\varepsilon/n$, and making $d = d_1 = (1 - \frac{l\varepsilon}{n})\frac{1}{r}$ we prove statement (b).

The inequality $d_1 > 0$ is clear because $\frac{l\varepsilon}{n} < 0$.

To prove that $d_2 > d_1$ we observe first that $r \geq \frac{-m}{2n} > 0$ and that $m^2 - 4n(l-1) \geq 0$. Then

$$\frac{(d_1 - d_2)r}{\varepsilon} = r^2 - \frac{l}{n} \geq \frac{m^2}{4n^2} - \frac{l}{n} > \frac{m^2}{4n^2} - \frac{4ln}{4n^2} + \frac{4n}{4n^2} = \frac{m^2 - 4n(l-1)}{4n^2} \geq 0,$$

concluding the proof of the inequality.

Finally, we prove that $d_2 < \frac{m\varepsilon}{n}$. We have

$$d_2 - \frac{m\varepsilon}{n} = \frac{1}{r} - \varepsilon r - \frac{m\varepsilon}{n} = \frac{\varepsilon}{nr}(\frac{n}{\varepsilon} - nr^2 - mr)$$

But $\frac{n}{\varepsilon} < 1$ because $\varepsilon < n < 0$, and $nr^2 + mr = 1 - l$, and hence

$$d_2 - \frac{m\varepsilon}{n} = \frac{\varepsilon}{nr}(\frac{n}{\varepsilon} + l - 1) < \frac{l\,\varepsilon}{nr} < 0.$$

∎

Lemma 8. *Let χ_d be the vector field given by* (10)*, and let* $(0, R(d))$ $[(0, L(d))]$ *denote the first* [*last*] *point of intersection of* $U_1(d)$ $[U_2(d)]$ *with the axis* Oy.
(a) The function $R(d)$ *is continuous, positive, strictly increasing, and defined on an interval* $[0, b_R)$*, with* $lim_{d \to b_R} R(d) = \infty$.
(b) The function $L(d)$ *is continuous, positive, strictly decreasing, and defined on an interval* $[0, b_L)$*, with* $lim_{d \to b_L} L(d) = 0$.

Proof. The function $R(d)$ is continuous and defined on a convex set by the continuous variation of the unstable manifold of S_1 with the parameter d. Let $a < b$ be points for which the function $R(d)$ is defined. For each point $p = (x, y) \in \mathbf{R}^2$ we have $\chi_b(p) = \chi_a(p) + ((b - a)x, 0)$ and therefore, for every point $p = (x, y) \in \mathbf{R}^2$ with $y > -1$, we have $\langle \chi_b, (\chi_a)^{\perp} \rangle < 0$. Since $x_1(b) > x_1(a)$ we must have $R(b) > R(a)$, For otherwise $\langle \chi_b, (\chi_a)^{\perp} \rangle \geq 0$ at some point $p \in U_1(a) \cup U_1(b)$.

For the proof of the rest of statement (a), we analyze first the case $m^2 - 4n(l - 1) \geq 0$.

Observe that by Lemma 7 the separatrix $U_1(d)$ cannot be at the right side of the line $y = rx - \varepsilon/n$ if $d < d_1$. Also, we cannot have $U_1(d) \subset \{x > 0\}$ for $d \geq 0$ because, by Lemma 6, the singularity O is a source if $d > 0$ and a weak attractor if $d = 0$. Then $U_1(d)$ must cross the Oy-axis at one point above the singularity O for all $d \in [O, d_1)$. The last statement is proved by observing that if $d = d_1$ we have $\omega(U_1(d)) = r$ and so, for d near to d_1 and less than d_1, we have $R(d)$ arbitrarily large.

Suppose now that $m^2 - 4n(l-1) < 0$. In this case we only have $u = 0$ as an asymptotic direction for χ, and so. With same argumentation we can prove that U_1 crosses the Oy axis at one point above the singularity O for all $d \in [O, \infty)$. It is easy to see that in this case $\lim_{d \to \infty} R(d) = \infty$.

We define then b_R as d_1 in the case $m^2 - 4n(l - 1) \geq 0$ and ∞ in the case $m^2 - 4n(l - 1) < 0$.

By analogous arguments, the function $L(d)$ is continuous, strictly decreasing and defined on a convex set.

The orientation of χ at the Oy-axis shows that $L(d) > 0$ for the values of d where the function is defined.

Since we do not have singularities in the quadrant $\{(x, y) \in \mathbf{R}^2; y > -1, x < 0\}$ nor negative asymptotic directions , $L(d)$ is not defined only for the values of the parameter d for which $U_2(d)$ is completely contained in this quadrant. Since the origin O in a weak attractor for $d = 0$, $U_2(0)$ cannot be contained in this quadrant and therefore L is defined at $d = 0$.

We then can assume that L is defined on an interval $[0, b_L)$ and not defined at the point $d = b_L$. In this case $\alpha(U_2(b_L)) = 0$ and $U_2(b)$ will be completely contained in this quadrant. It is now easy to conclude that if $d < b_L$ is near to b_L, then $L(d)$ will be near to 0. ∎

The same arguments prove the following Lemma.

LEMMA 9. *Consider the vector field* χ_d *given by* (10), *and let* $(0, \bar{R}(d))$ $[(0, \bar{L}(d))]$ *be the first* [*last*] *point of intersection of* $L_1(d)$ $[L_2(d)]$ *with the* Oy *axis.*
(a) The function $\bar{R}(d)$ *is continuous,* $\bar{R}(d) < -1$*, strictly decreasing, and defined on an interval* $(a_R, \frac{m\varepsilon}{n}]$*, with* $lim_{d \to a_R} \bar{R}(d) = -1$.
(b) The function $\bar{L}(d)$ *is continuous,* $\bar{L}(d) < -1$*, strictly increasing, and defined on an interval* $(a_L, \frac{m\varepsilon}{n}]$*, with* $lim_{d \to a_L} \bar{L}(d) = -\infty$.

For any d where the functions $L(d)$ and $R(d)$ are defined, let $G(d)$ be the region of the plane limited by the piece of the separatrix U_1 between S_1 and $(0, R(d))$, by the piece of the separatrix U_2 between S_2 and $(0, L(d))$, by the segment of straight line between S_1 and S_2 and by the segment of straight line $I(d)$, between $(0, L(d))$ and $(0, R(d))$.

Notice that if p is an interior point of the segment $I(d)$ then the ω-limit (respectively α-limit] of p will be located inside the region G if and only if $R(d) < L(d)$ (respectively $L(d) < R(d)$). We will say in this case that G is an attracting [repelling] region.

Analogously, with the functions $\bar{L}(d)$ and $\bar{R}(d)$ we define a region $\bar{G}(d)$ contained in the lower half-plane $y \leq -1$, which will be attracting if $\bar{L}(d) < \bar{R}(d)$ and repelling if $\bar{R}(d) < \bar{L}(d)$.

We will now see for which values of d the field χ_d presents a saddle connection not contained in a straight line.

LEMMA 10. *Let* f *and* $\bar{f}$ *be the functions defined by:*

$$\begin{aligned} f(d) &= R(d) - L(d) \\ \bar{f}(d) &= \bar{R}(d) - \bar{L}(d) \end{aligned}$$

where $R, L, \bar{R}$ *and* $\bar{L}$ *are defined by Lemmas 8 and 9. Then,*
(a) f *and* $\bar{f}$ *are continuous,* f *is increasing and* $\bar{f}$ *is decreasing.*
(b) There exist a unique value d_U *for which* $f(d_U) = 0$.
(c) There exist a unique value d_L *for which* $\bar{f}(d_L) = 0$.
(d) $0 < d_U < m < d_L$.

Proof. The statement (a) is clear by the definition of the functions $L(d)$ and $R(d)$. To prove (b) it is enough to show that $f(0) < 0$ and $f(d) > 0$ for some point $d > 0$.

If $f(0) > 0$ it follows that $G(0)$ is a repelling region. So, if p is an interior point of this region, distinct from the origin, we have $\alpha(p) \subset G(0)$. Since the origin a weak attractor, $\alpha(p)$ cannot be the origin. We conclude that there exist a repelling closed orbit around the origin. But the existence of such an orbit is impossible because a quadratic field with an invariant line and a weak focus does not admit limit cycles [12].

The eigenvalues of the saddle S_1 are $-\sqrt{\Delta}$ and x_1 and the eigenvalues of the saddle S_2 are $\sqrt{\Delta}$ and x_2. For $d = 0$ we have $|x_2| > |x_1|$; this shows that if $f(0) = 0$, then the region $G(0)$ would be bounded by an attractor graphic. The same argumentation as before also proves that $f(0) = 0$ is not possible.

So we must have $f(0) < 0$.

Using the fact that $\lim_{d \to b_R} R(d) = \infty$ and $\lim_{d \to b_L} L(d) = 0$ it is easy to show that $f(d) > 0$ for some $d < \min\{b_R, b_L\}$. Since f is strictly increasing, there exists a unique value d_U for which $f(d_U) = 0$.

The statement (c) is proved the same way, by using the region $\bar{G}$ and the fact that the singularity F is weakly repelling if $d = 0$.

To prove (d) we consider $\chi_*(x, y) = (\varepsilon x + lx^2 + ny^2, x + xy)$ the field χ given by (10) with $d = m = 0$. This field has the singularities O and F as centers, $U_1 = U_2$ and $L_1 = L_2$ [see Section 5]. We then have $f(0) = \bar{f}(0) = 0$ if $m = d = 0$.

We observe now that $\langle \chi_m(x, y), \chi_*^{\perp}(x, y) \rangle = -mx^2(1 + y)^2$ is always negative if $m > 0$ and $x(1 + y) \neq 0$. It is easy now to conclude, as in Lemma 7, that $f(m) > 0$ if $f(m)$ is defined, and so, $m > d_U$ in any case. In the same way we prove that $m < d_L$. ▮

By Lemma 10 we see that χ_{d_U} presents a connection $\mathcal{L}_U$, between the saddles S_1 and S_2, contained in the semi-plane $y > -1$, χ_{d_L} presents a connection $\mathcal{L}_L$, between the saddles S_1 and S_2, contained in the semi-plane $y < -1$ and, if d is not equal to d_U or d_L, that χ_d does not presents saddle connection between the saddles S_1 and S_2 besides the one contained in the line $y = -1$.

In the next lemma we will use strongly the fact that a quadratic field with an invariant line has either a center or at most one closed orbit, hyperbolic in the last case [5].

LEMMA 11. *Let χ_d be the vector field given by* (10) *and let d_U and d_L be the values defined in the Lemma 10. Then*
(a) χ_d will have one closed orbit (hyperbolic attracting) $\mathcal{O}_U$ around the origin if and only if d belongs to the interval $(0, d_U)$.
(b) χ_d will have one closed orbit (hyperbolic repelling) $\mathcal{O}_L$ around the singularity F if and only if d belongs to the interval $(d_L, \frac{m\varepsilon}{n})$.

Proof. If $d \in (0, d_U)$, then $f(d) < 0$ and therefore $\omega(p) \subset G(d)$ for all point $p \in G(d)$. But $\omega(p) \neq O$ because the origin is repelling if $d > 0$, and so we can conclude that there exist a closed orbit $\mathcal{O}_U$ contained in $G(d)$. By [5], this orbit is unique and hyperbolic.

If we suppose that χ_d has one closed orbit $\mathcal{O}$ around the origin for some $d > d_U$, then this orbit will be attracting because the origin will be repelling and $\mathcal{O}$ will be the unique closed orbit of the vector field. Let p be one point in the attraction basin of $\mathcal{O}$, located outside the region limited by $\mathcal{O}$. We must then have $\omega(p) = \mathcal{O}$ and $\alpha(p)$ must be the saddle S_1 or the singularity at infinity A. In any case this implies that $\omega(U_1) = \mathcal{O}$ and we have $f(d)$ defined and satisfying $f(d) < 0$, contradicting the hypothesis $d > d_U$.

Suppose now that χ_d has one closed orbit $\mathcal{O}$ around the origin for some $d < 0$. This orbit will be a repelling one, because the origin will be attracting and $\mathcal{O}$ will be the unique closed orbit of the vector field. Then $L(d)$ is defined since we cannot have $\alpha(U_2) = O$. We also have $R(d)$ defined because we cannot have $\omega(U_1) = O$, and obviously $d < b_R$, and so $f(d)$ will be defined. We didn't have $f(d) \geq 0$ because $d < 0$ and f is increasing, with $f(0) < 0$ by Lemma 10. Therefore $f(d) < 0$ and the region $G(d)$ will be attracting, and consequently $\mathcal{O}$ is attracting, a contradiction.

The vector field χ_d cannot exhibit a closed orbit if $d = 0$ or $d = d_U$, because this orbit would be hyperbolic and so persistent by small variations of d, implying the existence of a closed orbit for χ_d when $d < 0$ or when $d > d_U$, a contradiction.

The proof of statement (b) is analogous. ▮

Based on the lemmas we will describe now the behavior of the six saddle separatrices, U_1, U_2, L_1, L_2, Z_1 and Z_2, pointed out in figure 5 of page 197.

In any case, independently on the sign of $m^2 - 4n(l-1)$, the α-limit of U_2 is S_1 if $d = d_U$ by Lemma 10, and the same lemma applies to show that if we decrease d we will get $\alpha(U_2) = A$ and if we increase d we will have $\alpha(U_2) = O$. With similar arguments we also show that $\omega(L_1)$ is S_2, F or $\bar{A}$ if d is respectively equal, lower or greater than d_L.

From Lemma 7, when $m^2 - 4n(l-1) \geq 0$, it follows that the α-limit of Z_1 is S_1 when $d = d_1$ and the ω-limit of Z_2 is S_2 when $d = d_2$. It is then easy to conclude that $\alpha(Z_1) = A$ if $d > d_1$, $\alpha(Z_1) = O$ if $d < d_1$, $\omega(Z_2) = \bar{A}$ if $d > d_2$ and $\omega(Z_2) = F$ if $d < d_2$.

The ω-limit of separatrix U_1 and the α-limit of separatrix L_2 will depend on the sign of $m^2 - 4n(l-1)$.

First we note that, by Lemma 6 and Lemmas 7 to 11, we have the following cases:

1. If $m^2 - 4n(l-1) \geq 0$ (χ_d with two or three pairs of singularities at infinity), then χ_d will change behavior at d when $d \in \{0, d_U, d_1, d_2, d_L, \frac{m\varepsilon}{n}\}$.

2. If $m^2 - 4n(l-1) < 0$ (χ_d with one pair of singularities at infinity), then χ_d will change behavior at d when $d \in \{0, d_U, d_L, \frac{m\varepsilon}{n}\}$.

By observing the behavior of χ_d when the parameter d is close to a point in the set $\{0, d_U, d_1, d_2, d_L, \frac{m\varepsilon}{n}\}$, we conclude that when $m^2 - 4n(l-1) > 0$ we have the following behavior of the two saddle separatrices $\omega(U_1)$ and $\alpha(L_2)$:

$$\omega(U_1)\begin{cases} O & \text{if } d \le 0 \\ \mathcal{O}_U & \text{if } 0 < d < d_U \\ S_2 & \text{if } d = d_U \\ \bar{A} & \text{if } d_U < d < d_1 \\ C & \text{if } d = d_1 \\ B & \text{if } d > d_1 \end{cases}$$

$$\alpha(L_2)\begin{cases} \bar{B} & \text{if } d < d_2 \\ \bar{C} & \text{if } d = d_2 \\ A & \text{if } d_2 < d < d_L \\ S_1 & \text{if } d = d_L \\ \mathcal{O}_L & \text{if } d_L < d < \frac{m\varepsilon}{n} \\ F & \text{if } d = \frac{m\varepsilon}{n} \end{cases}$$

When $m^2 - 4n(l-1)=0$, the singularities at infinity $B[\bar{B}]$ and $C[\bar{C}]$ collapse $B[\bar{B}]$ (see figure 5 at page 197) and the only changes on the above table is $\omega(U_1) = B$ when $d \ge d_1$ and $\alpha(L_2) = \bar{B}$ when $d \le d_2$.

If $m^2 - 4n(l-1) < 0$ and $m \ne 0$ we do not have the singularities B and $\bar{B}$ (so we do not have the separatrices Z_1 and Z_2) and $\omega(U_1)$ and $\alpha(L_2)$ are

$$\omega(U_1)\begin{cases} O & \text{if } d \le 0 \\ \mathcal{O}_U & \text{if } 0 < d < d_U \\ S_2 & \text{if } d = d_U \\ \bar{A} & \text{if } d > d_U \end{cases}$$

$$\alpha(L_2)\begin{cases} A & \text{if } d < d_L \\ S_1 & \text{if } d = d_L \\ \mathcal{O}_L & \text{if } d_L < d < \frac{m\varepsilon}{n} \\ F & \text{if } d = \frac{m\varepsilon}{n} \end{cases}$$

Finally, when $m = 0$ we also have only one pair of singularities at infinity, and by Lemma 6 the singularities O and F will be centers when $d = 0$, hyperbolic sources when $d > 0$ and hyperbolic attractors if $d < 0$. We observe that in case $m = 0$ the phase portrait of χ_d when $d < 0$ and $d > 0$ is the same as the phase portrait of χ_d when $d \le 0$ and $d \ge \frac{m\varepsilon}{n}$, respectively, in case of $m \neq 0$ and $m^2 - 4n(l-1) < 0$. This finishes the proof of Theorem 1.

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