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A Note on the Completeness of Homogeneous Quadratic Vector Fields on the Plane

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In this paper we give a full description of the complete homogeneous quadratic vector fields defined on the plane.

Key Words: Quadratic vector fields, completeness.

1. INTRODUCTION

The theory of polynomial vector fields defined on the plane has a long history that can be traced, at least, to Hilbert's problems (see, for instance, [4] where more than 800 papers are listed addressing problems within this theory). Homogeneous quadratic vector fields arise naturally in the context of Lie groups. In fact, the equation for the geodesics of a left invariant pseudo-Riemannian metric defined on a Lie group can be transferred to the dual of Lie algebra of the group, giving rise to a homogeneous quadratic vector field. The completeness of this vector field implies the completeness of the geodesic flow [1]. The purpose of this paper is to give a full description of the complete homogeneous quadratic vector fields defined on the plane.

2. PRELIMINARIES

DEFINITION 1. A *homogeneous quadratic vector field* on the Euclidean space $\mathbf{R}^n$ is a vector field $F : \mathbf{R}^n \to \mathbf{R}^n$, $F = (F_1, \ldots, F_n)$, where F_i is a homogeneous polynomial of degree 2, $i = 1, \ldots, n$.

Consider the differential equation

$$\dot{X} = F(X). \tag{1}$$

Remark 2. Let F be a homogeneous quadratic vector field and $t \mapsto \gamma(t)$ be a solution of Eq. (1). The homogeneity of F implies that $t \mapsto \lambda\gamma(\lambda t)$ is also a solution.

DEFINITION 3. The vector field F is called *complete* when the solutions of Eq. (1) are defined for all $t \in \mathbf{R}$.

The following definition provide the simplest case for completeness for homogeneous quadratic vector fields:

DEFINITION 4. An *affine-quadratic* vector field is a homogeneous quadratic vector field such that Eq. (1) is equivalent by a linear change of coordinates to the equation

$$\dot{U} = A(V)U + B(V), \ \ \dot{V} = 0,$$

where A is linear and the coordinate functions of B are homogeneous polynomials of degree 2.

Remark 5. Every affine-quadratic vector field is complete.

DEFINITION 6. A non trivial solution of the equation $F(X) = 0$ is called a *zero* of F. A non trivial solution of the equation $F(X) = X$ is called an **idempotent** of F.

Remark 7. A simple calculation shows that any idempotent of F gives an incomplete solution for Eq. (1). In fact, the solution of Eq. (1) with initial condition an idempotent X_0 is given by $t \mapsto \alpha(t)X_0$ where α satisfies $\dot{\alpha} = \alpha^2$.

LEMMA 8. *[3] Let F be a homogenous quadratic vector field. Then, F has a zero or an idempotent.*

3. THE MAIN RESULT

Let F be a homogeneous quadratic vector field on the plane.

THEOREM 9. *The following assertions are equivalent:*

(i)F is complete;
(ii)F is quadratic-affine or the equation (1) is equivalent, by a linear change of coordinates, to a system

$$\dot{x} = y(ax + by), \;\; \dot{y} = y(cx + dy),$$

where $(a + d)^2 - 4(ad - bc) < 0$ *(condition* C).

The following two lemmas and Remark 5, prove Theorem 9.

Lemma 10. *Any complete homogeneous quadratic field on the plane is either affine-quadratic or satisfies condition* C.

Proof. Let F be a complete homogeneous quadratic field. Remark 7 implies that F has no idempotents. Hence, by Lemma 8, F has a zero. Let $X_0 \neq 0$ such that $F(X_0) = 0$. By a linear change of coordinates we can assume that $X_0 = (1, 0)$. Then, Eq. (1) is given by

$$\dot{x} = y(ax + by), \;\; \dot{y} = y(cx + dy),$$

which is complete since the original equation is. The completeness implies that if $c = 0$ then $d = 0$, and, in this case, F is affine-quadratic. Suppose $c \neq 0$. Then, the change of coordinates

$$u = cx - ay, \;\; v = y,$$

transforms the last equation into

$$\dot{u} = -\Delta v^2, \;\; \dot{v} = v(u + \mathrm{T}v), \tag{2}$$

where $\Delta = ad - bc$ and $\mathrm{T} = a + d$. Since Eq.(2) is complete, it has no idempotents. Hence, the only solution of the system

$$u = -\Delta v^2, \;\; v = v(u + \mathrm{T}v),$$

is the trivial one, implying that the equation

$$\Delta x^2 - \mathrm{T}x + 1 = 0$$

has no real solutions. Then, either

$$\mathrm{T}^2 - 4\Delta < 0,$$

i.e. F satisfies condition C, or

$$\mathrm{T} = \Delta = 0,$$

in which case F is affine-quadratic. ∎

Lemma 11. *Any equation satisfying condition* C *is complete.*

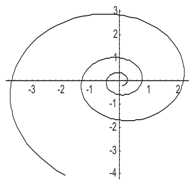
Proof. The condition implies that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

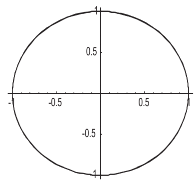
has two complex conjugate eigenvalues. Hence, the solutions of the system obtained rescaling the time

$$\dot{x} = ax + by, \;\; \dot{y} = cx + dy,$$

go round the origin:



$T := a + d \neq 0$



$T := a + d = 0$

Since the solutions of the original equation cannot intersect the x axis (because every point in the x axis is an equilibrium point), and are traced on the solutions of the rescaled equation, the solutions lie on bounded sets, hence are defined for all t. ▮

Note that the integral curves of complete homogeneous quadratic vector fields of type C are bounded.

4. REMARKS ON FIRST INTEGRALS

It is known that any homogeneous vector field on the plane has a first integral ([2]). If in addition the vector field is complete, it is possible to be more precise. The first integral for an affine-quadratic vector field is a linear map. For homogeneous quadratic vector fields of type C, there are two cases. If $T = 0$, the rescaled system has a center and a first integral which is a quadratic form, which is also a first integral for the original vector field. If $T \neq 0$, the first integral is defined on $\mathbf{R}^2$ except for one ray (weak first integral in the sense of [2]). Assuming that the line of zeros is $y = 0$, a first integral is given by the intersection of the integral curve passing through a point with the positive x axis. The first integral is analytic in $\mathbf{R}^2 \setminus \{(x, 0); x \leq 0\}$ and, by Remark 2, homogeneous of degree 1.

5. IDEMPOTENTS AND ZEROS

The condition of not having idempotents is necessary for the completeness of a homogeneous quadratic vector field. It is, however, not sufficient. Consider for instance, the homogeneous quadratic vector field:

$$F(x, y) = (y(x + 2y), y^2),$$

which is not complete (since the solutions contain the solutions of the incomplete equation $\dot{y} = y^2$) and which has no idempotents.

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