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Structurally Stable Discontinuous Vector Fields in the Plane

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Let M be a plane domain, partitioned into sub domains N and S, with common border D. In N and S are defined vector fields X and Y, respectively, leading to a discontinuous vector field $Z = (X, Y)$. This work pursues the stability and transition analysis of solutions between N and S, started by Filippov and Kozlova and reformulated by Sotomayor and Teixeira in terms of the regularization method. This method consists in defining a one parameter family of continuous vector fields Z_ϵ, by averaging X and Y. This family approaches Z when the parameter goes to zero. The results of Sotomayor and Teixeira providing conditions for the regularized vector fields to be structurally stable are extended and shown to be generic.

Key Words: structural stability, discontinuous vector fields, regularization.

1. INTRODUCTION

Among the most accomplished stability theories for dynamical systems is that of Andronov, Pontryagin [2] and Peixoto [7, 6] for C^1 vector fields in the plane and on surfaces. Elements of this theory provide characterization and genericity results for structurally stable vector fields. Extensions to the class of discontinuous, piecewise smooth, vector fields have been provided by Filippov [3] and Kozlova [4]. The need for such an extended theory goes back to Andronov et al. [1].

In [3], Filippov defined the rules (revisited below) for the transition of the orbits crossing the line D of discontinuity which separates two regions

N and S on which the field is smooth, given by X and Y respectively. He also established when the orbit slides along D. This leads to an orbit structure that is not always a flow on the surface obtained glueing N and S along D. The work of Kozlova [4, 3] pursues the setting established by Filippov.

In [10], Sotomayor and Teixeira developed the *regularization method*, taking as domain the sphere S^2. This method consists in defining a one parameter family of continuous vector fields that approaches the discontinuous one, when the parameter goes to zero. To this end, a transition function φ is used to average X and Y in order to get the family of continuous vector fields, called *regularized vector fields*. Sotomayor and Teixeira provided conditions on $Z = (X, Y)$, which guarantee that the regularized vector fields are structurally stable.

In this paper the method outlined above is applied to the case of a planar region M, with border ∂M. The conditions given in [10] are adjusted to this case and detailed proofs of some results merely outlined there are provided here. Furthermore the genericity not discussed in [10] is established explicitly here.

Let M be a compact and connected region in $\mathbb{R}^2$ and $f : M \to \mathbb{R}$ be a C^∞ function having 0 as a regular value. We assume for simplicity that $D = f^{-1}(0)$ has a single connected component so that $M \setminus D$ has two connected components, denoted by $N = f^{-1}(0, \infty)$ and $S = f^{-1}(-\infty, 0)$.

Designate by χ^r the space of C^r vector fields on M, $r \geq 1$.

Let $\Omega^r = \Omega^r(M, f)$ be the space of vector fields Z on M defined by:

$$Z(q) = \begin{cases} X(q) & \text{if} \quad f(q) \geq 0. \\ Y(q) & \text{if} \quad f(q) \leq 0. \end{cases}$$

where $X, Y \in \chi^r$. We write $Z = (X, Y)$, which we will accept to be bi-valued in the points of D. Given any $Z = (X, Y) \in \Omega^r$, following Filippov's terminology (in [3]), we distinguish in D:

- Sewing Arc (SW): characterized by $(Xf)(Yf) > 0$.
- Escaping Arc (ES): given by the inequalities $Xf > 0$ and $Yf < 0$.
- Sliding Arc (SL): given by the inequalities$Xf < 0$ and $Yf > 0$.

As usual, here and in what follows, Xf will denote the derivative of the function f in the direction of the vector X.

On the arcs ES and SL we define the *Filippov vector field* F_Z associated to $Z = (X, Y)$, as follows: if $p \in SL$ or ES, then $F_Z(p)$ denotes the vector tangent to D in the cone spanned by $X(p)$ and $Y(p)$.

By a *transition function* we mean a C^∞ function $\varphi : \mathbb{R} \to \mathbb{R}$ such that: $\varphi(t) = 0$ if $t \leq -1$, $\varphi(t) = 1$ if $t \geq 1$ and $\varphi'(t) > 0$ if $t \in (-1, 1)$.

DEFINITION 1. A *φ_ϵ-regularization of* $Z = (X, Y) \in \Omega^r$ is the one parameter family of vector fields $Z_\epsilon \in \chi^r$ given by

$$Z_\epsilon(q) = (1 - \varphi_\epsilon(f(q))Y(q) + \varphi_\epsilon(f(q))X(q).$$

where $\varphi_\epsilon(t) = \varphi(\frac{t}{\epsilon})$.

In this paper we give sufficient conditions on $Z = (X, Y)$ which determine the global phase portrait of its regularization and guarantee the structural stability of Z_ϵ, for any transition function and small ϵ. This will be achieved using the following characterization of the class Σ^r of the structurally stable vector fields on smooth submanifolds of M, due to Andronov-Pontryagin and Peixoto [6].

DEFINITION 2. We call $\Sigma^r(M)$ the class of all vector fields $X \in \chi^r$ which satisfy the following conditions:

1. $\Sigma^r(1)$: all singular points are hyperbolic and contained in the interior of M;
2. $\Sigma^r(2)$: all periodic orbits of X are hyperbolic and contained in the interior of M;
3. $\Sigma^r(3)$: any tangency between a trajectory of X and ∂M is quadratic;
4. $\Sigma^r(4)$: X does not have saddle or tangency connections.

Recall that a *tangency connection* is a saddle separatrix with a parabolic contact or an orbit with two parabolic contacts with ∂M.

In Section 4 of this paper will be defined a set $G^r = G^r(M)$ of discontinuous vector fields that satisfy the above mentioned sufficient conditions for having a structurally stable regularization. In Section 5, the genericity of G^r will be established. A preliminary analysis of relevant local aspects of discontinuous vector fields is developed in Sections 2 and 3. There is studied the effect of regularization on singular points, closed orbits and tangencies between the orbits and ∂M.

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2. REGULARIZATION OF SINGULAR POINTS AND CLOSED POLY-TRAJECTORIES

In this section we define the regular and singular points of Z, then we study the effects of the regularization on vector fields around these points. After this, we define the closed *poly-trajectories* and study their regularization. The main goal in this section is to determine the conditions for the regularization to posses only regular points, hyperbolic singularities and hyperbolic closed orbits.

2.1. Regular and Singular Points

DEFINITION 3. A point $p \in D$ is called a *D-regular point of* Z if one of the following conditions hold:

1. $Xf(p).Yf(p) > 0$. This means that $p \in SW$;
2. $Xf(p).Yf(p) < 0$ but $det[X,Y](p) \neq 0$. This means that p belongs either to SL or ES and it is not a critical point of F_Z.

DEFINITION 4. A point $p \in D$ is called a *critical point of* F_Z if $Xf(p).Yf(p) < 0$ and $det[X,Y](p) = 0$. If we have $d(det[X,Y]|_S)(p) \neq 0$, then p is called a *hyperbolic critical point of* F_Z.

Let $p \in D$ be a hyperbolic critical point of F_Z. The point p is called a *saddle* if $p \in SL$ and $d(det[X,Y]|_D)(p) > 0$ or $p \in ES$ and $d(det[X,Y]|_D)(p) < 0$. The point p is called a *node* if $p \in SL$ and $d(det[X,Y]|_D)(p) < 0$ or if $p \in ES$ and $d(det[X,Y]|_D)(p) > 0$.

DEFINITION 5. A point $p \in D$ is an *elementary D-singular point of* $Z = (X,Y)$ if one of the following conditions is satisfied:

1. The point p is a *fold point of* $Z = (X,Y)$. This means that: either p is a fold point of X: $Yf(p) \neq 0, Xf(p) = 0$ and $X^2f(p) \neq 0$; or p is a fold point of Y: $Xf(p) \neq 0, Yf(p) = 0$ and $Y^2f(p) \neq 0$;
2. The point p is a hyperbolic critical point of F_Z.

PROPOSITION 6. *Let $p \in M$ be a D-regular point of $Z = (X,Y) \in \Omega^r$. Then, given a transition function φ, there exists a neighborhood V of $p \in M$ and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, Z_ϵ has no critical points in V.*

Proof. By definition, p could be a sewing, escaping or a sliding point that is not a critical of F_Z. In a local chart (x,y) around p we write $p = (0,0)$, $D = \{y = 0\}$, $f(x,y) = y$, $X(x,y) = (a(x,y), b(x,y))$ and $Y(x,y) = (c(x,y), d(x,y))$.

Case (i): p is a sewing point. In this case we have $Xf(p)Yf(p) > 0$. We can consider that $Xf(p) > 0$ and $Yf(p) > 0$, the other case is treated similarly. So $Xf(p) = b(0) > 0$, $Yf(p) = d(0) > 0$ and $Z_\epsilon(.) = ((1 - \varphi_\epsilon(y))c(.) + \varphi_\epsilon(y))a(.), (1 - \varphi_\epsilon(y))d(.) + \varphi_\epsilon(y))b(.))$.

Define $V = \{(x,y) \in M; b(x,y) > 0 \text{ and } d(x,y) > 0\}$. Of course, V is a neighborhood of p that has not critical points of Z_ϵ, for any small $\epsilon > 0$.

Case (ii): p is an escaping or sliding point that is not a critical point of F_Z. So we have $Xf(p)Yf(p) = b(0)d(0) < 0$ with $det[X,Y](p) = a(0)d(0) - b(0)c(0) \neq 0$. We can assume that p is a sliding point, this means that

$b(0) > 0$ and $d(0) < 0$. Let V be the set $\{(x,y) \in M; a(x,y)d(x,y) - b(x,y)c(x,y) \neq 0\}$. It is clear that V is a neighborhood of p that has no critical points of Z_ϵ, for any $\epsilon > 0$. In fact, for a point (x,y) to be a critical point of Z_ϵ, the following vector field must vanish

$$Z_\epsilon(.) = ((1-\varphi_\epsilon(y))c(.) + \varphi_\epsilon(y))a(.), (1-\varphi_\epsilon(y))d(.) + \varphi_\epsilon(y))b(.)).$$

This implies that $\varphi_\epsilon = \frac{c}{c-a} = \frac{d}{d-b} \Rightarrow d(c-a) = c(d-b) \Rightarrow ad - bc = det[X,Y] = 0$. Therefore, Z_ϵ will not have singularities in V since the last equality would not be satisfied for points of V, by definition. ▮

Lemma 7. *Let $Z = (X,Y) \in \Omega^r$ be a discontinuous vector field such that $X = (a,b)$, $Y = (c,d)$ and $a(0)c(0) < 0$. Given a number $K > 0$, there is $\epsilon_0 > 0$ and an interval $[h_0,h_1] \subset (0,1)$ so that if $\varphi_\epsilon(y) \in [h_0,h_1]$, then $\varphi'_\epsilon(y) > K$, for $\epsilon \in (0,\epsilon_0]$.*

Proof. As $a(0)c(0) < 0$, then there is $\epsilon_0 > 0$ such that $a(0,y)$ and $c(0,y)$ do not vanish and have different signs for $|y| \leq \epsilon_0$. So, defining $h(y) = \frac{c(0,y)}{c(0,y)-a(0,y)}$, we have $h(0,y) \in (0,1)$. As h is continuous in the interval $[-\epsilon_0,\epsilon_0]$, the constants $h_0 = min\{h(y); y \in [-\epsilon_0,\epsilon_0]\}$ and $h_1 = max\{h(y); y \in [-\epsilon_0,\epsilon_0]$ satisfy $[h_0,h_1] \subset (0,1)\}$.

Let ϵ' and ϵ'' be such that $0 < \epsilon' < \epsilon'' < \epsilon_0$ and let y' and y'' be such that $0 < \varphi_{\epsilon'}(y') = \varphi_{\epsilon''}(y'') < 1$. We assert that $\varphi'_{\epsilon'}(y') > \varphi'_{\epsilon''}(y'')$. Indeed, $\varphi_{\epsilon'}(y') = \varphi(\frac{y'}{\epsilon'}) = \varphi(\frac{y''}{\epsilon''}) = \varphi_{\epsilon''}(y'')$. As φ is injective, we have $\frac{y'}{\epsilon'} = \frac{y''}{\epsilon''}$. So, $\varphi'(\frac{y'}{\epsilon'}) = \varphi'(\frac{y''}{\epsilon''})$ and, as $\epsilon' < \epsilon''$, $\varphi'_{\epsilon'}(y') = \frac{1}{\epsilon'}\varphi'(\frac{y'}{\epsilon'}) > \frac{1}{\epsilon''}\varphi'(\frac{y''}{\epsilon''}) = \varphi'_{\epsilon''}(y'')$. Diminishing ϵ, φ' increases. It is easy to see that given $\hat{\epsilon} > 0$ we can take y' such that $\varphi'_{\hat{\epsilon}}(y') = \min\{\varphi'_{\hat{\epsilon}}(y); \varphi_{\hat{\epsilon}}(y) \in [h_0,h_1]\}$ and define $\epsilon_0 = \hat{\epsilon}\frac{\varphi'_{\hat{\epsilon}}(y')}{K}$, and the lemma follows. ▮

Proposition 8. *Given $Z = (X,Y) \in \Omega^r$, let p be a hyperbolic critical point of F_Z. Then, given a transition function φ, there is a neighborhood V of p in M and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, Z_ϵ has near p a unique critical point which is a hyperbolic saddle or a hyperbolic node.*

Proof. By assumption, we have $Xf(p).Yf(p) < 0$, $det[X,Y](p) = 0$ and $d(det[X,Y]|_S)(p) \neq 0$. We will consider the case that $Xf(p) < 0$ and $Yf(p) > 0$, which means that $p \in SL$. The other case is similar.

Consider coordinates around p so that $f(x,y) = y$, $X(.) = (a(.),b(.))$ and $Y(.) = (c(.),d(.))$.

With this notation, we have $Xf(p) = b(0) < 0$, $Yf(p) = d(0) > 0$, $det[X,Y](p) = a(0)d(0) - b(0)c(0) = 0$ and

$$d(det[X,Y]|_S)(p) = \frac{\partial}{\partial x}(det[X,Y](p)) =$$

$$a_x(0)d(0) + a(0)d_x(0) - b_x(0)c(0) - b(0)c_x(0) \neq 0.$$

The regularized vector field takes the form

$$Z_\epsilon = ((1-\varphi_\epsilon)c + \varphi_\epsilon a, (1-\varphi_\epsilon)d + \varphi_\epsilon b).$$

The critical points of Z_ϵ are the solutions of

$$\begin{cases} (1-\varphi_\epsilon)c + \varphi_\epsilon a = 0 \\ (1-\varphi_\epsilon)d + \varphi_\epsilon b = 0 \end{cases}.$$

So, we have $\varphi_\epsilon = \frac{c}{c-a} = \frac{d}{d-b}$. We claim that there is a curve $(\alpha(y), y)$ such that it is true that $\frac{c}{c-a} = \frac{d}{d-b}$, and, for ϵ enough small, this curve crosses the graph of φ_ϵ only once. Indeed, from the last equality we have that any singularity of Z_ϵ will satisfy $det[X,Y] = 0$. Since $d(det[X,Y])(p) \neq 0$, by the Implicit Function Theorem, there is an open neighborhood $I \times J$ of $p = (0,0)$ and a differentiable function $\alpha : J \to I$ such that $det[X,Y](\alpha(y), y) = 0$. At the points of the curve $(\alpha(y), y)$ it is true that $\frac{c}{c-a} = \frac{d}{d-b}$. We define $g(y) = \frac{d(\alpha(y),y)}{d(\alpha(y),y)-b(\alpha(y),y)}$. Let k be the number $\frac{d(0)}{d(0)-b(0)}$ and define $m = min\{k, 1-k\}$. As $k \in (0,1)$, we can diminish ϵ_0 so that $\frac{d(x,y)}{d(x,y)-b(x,y)} \in (k - \frac{m}{2}, k + \frac{m}{2})$, if $(x,y) \in V$.

We assert that, decreasing ϵ_0 if necessary, the graphs of g and φ_ϵ cross each other only once, at a point p_ϵ near p. As $\varphi_{\epsilon_0}(-\epsilon_0) = 0$, $\varphi_{\epsilon_0}(\epsilon_0) = 1$ and φ_ϵ and g are continuous, there are at least one point of crossing between φ_ϵ and g. As g does not depend of ϵ and φ_{ϵ_0} is strictly increasing in the interval $[-\epsilon_0, \epsilon_0]$, we can diminish ϵ_0 if necessary to have only one point of crossing between φ and g, and this is true also for $\epsilon \in (0, \epsilon_0]$. Let y_ϵ be the crossing point of φ_ϵ and g, So, the point $p_\epsilon = (\alpha(y_\epsilon), y_\epsilon)$ is the unique singularity of Z_ϵ in V.

It can be also shown that p_ϵ is a hyperbolic saddle of Z_ϵ if p is a saddle for F_Z or a hyperbolic node of Z_ϵ if p is a node for F_Z. Indeed, for a singularity with eigenvalues given by $\lambda_1 = \frac{1}{2}(-b_\lambda + \sqrt{\Delta_\lambda})$ and $\lambda_2 = \frac{1}{2}(-b_\lambda - \sqrt{\Delta_\lambda})$ to be a hyperbolic saddle, we must have $\Delta_\lambda > b_\lambda^2$, and to be a hyperbolic node, we must have $0 \leq \Delta_\lambda < b_\lambda^2$. So, we have to evaluate $\Delta_\lambda - b_\lambda^2 = b_\lambda^2 - 4a_\lambda c_\lambda - b_\lambda^2 = -4a_\lambda c_\lambda$. Calculating the eigenvalues of the matrix DZ_ϵ in p_ϵ we find $a_\lambda = 1$ and $c_\lambda = -\varphi_\epsilon'(p_\epsilon)d(det[X,Y](p_\epsilon)) + L(p_\epsilon)$, where $L(p_\epsilon)$ is bounded. So, it remains to prove that $\varphi_\epsilon'(p_\epsilon)$ goes to infinity when ϵ goes to zero and we will have that $-c_\lambda$ has the same sign as $d(det[X,Y](p_\epsilon))$, and therefore p_ϵ is a hyperbolic saddle or a hyperbolic node, as p. Indeed, we prove this using Lemma 7 and the facts that $\varphi_\epsilon(p_\epsilon) \in (k - \frac{m}{2}, k + \frac{m}{2})$ and that k does not depend of ϵ. ∎

PROPOSITION 9. *Let p be a fold point of $Z = (X, Y) \in \Omega^r$. Then, given a transition function φ, there is a neighborhood V of p and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0, Z_\epsilon$ has no critical points in V.*

Proof. Assume for instance that $p \in D$, $Xf(p) = 0$, $X^2 f(p) > 0$ and $Yf(p) > 0$. The other cases are treated similarly. In a local chart (x, y) around p we write $p = (0,0)$, $D = \{y = 0\}$, $f(x, y) = y$, $X(x, y) = (a(x, y), b(x, y))$ and $Y(x, y) = (c(x, y), d(x, y))$. So, we have $Yf(p) = d(0) > 0$, $Xf(p) = b(0) = 0$ and $X^2 f(p) = a(0) b_x(0) > 0$, then $a(0) \neq 0$. The regularized vector field Z_ϵ takes the form

$$Z_\epsilon = ((1 - \varphi_\epsilon)c + \varphi_\epsilon a, (1 - \varphi_\epsilon)d + \varphi_\epsilon b).$$

For (x, y) to be a singularity of Z_ϵ, we must have $det[X, Y](x, y) = 0$. At the point p, we have $det[X, Y](p) = a(0)d(0) - b(0)c(0) = a(0)d(0) \neq 0$. Let $\epsilon_0 > 0$ be such that $det[X, Y](x, y) \neq 0$ if $|x| \leq \epsilon_0$ and $|y| \leq \epsilon_0$. Let V be the set $\{(x, y) \in M; |x| \leq \epsilon_0 \text{ and } |y| \leq \epsilon_0\}$. Of course, V is a neighborhood of p and if $0 < \epsilon \leq \epsilon_0$, Z_ϵ does not have singularities in V. ▮

2.2. Closed Poly-Trajectories

DEFINITION 10. A continuous curve γ consisting of regular trajectory arcs of X and/or of Y and/or of F_Z is called a *poly-trajectory* if:

1. γ has arcs of at least two fields among X, Y and F_Z, or consists of a single arc of F_Z;
2. the transition between arcs of X and Y is made across the sewing arc;
3. the transition between arcs of X or Y and F_Z is made at fold points or regular points of the sliding or the escaping arcs, preserving the sense of the arcs.

DEFINITION 11. Let γ be a closed poly-trajectory of $Z = (X, Y)$.

1. γ is called a *closed poly-trajectory of type 1* if γ meets D only in sewing points;
2. γ is called a *closed poly-trajectory of type 2* if $\gamma = D$;
3. γ is called a *closed poly-trajectory of type 3* if γ has at least one fold point of Z.

DEFINITION 12. Let γ be a closed poly-trajectory of $Z = (X, Y) \in \Omega^r$. It is called *elementary* if one of the cases below holds:

1. γ is of type 1 and has a first return map η with $\eta' \neq 1$;
2. γ is of type 2;

3. γ is of type 3 and all arcs of F_Z are sliding or all are escaping.

PROPOSITION 13. *Let γ be an elementary closed poly-trajectory of $Z = (X, Y) \in \Omega^r$. Then, given a transition function φ, there is a neighborhood V of γ and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, Z_ϵ has only one periodic orbit in V, and this orbit is hyperbolic.*

Proof. Case where γ is of type 1: We have to consider an annulus V around γ such that the poly-trajectories of Z which have points of V are all entering or exiting V. Then, it will shown that this annulus persists to a similar annulus for Z_ϵ, if ϵ is enough small. The Poincaré-Bendixson Theorem will imply the existence of at least one periodic orbit of Z_ϵ. As γ is an elementary closed poly-trajectory, we will have that the periodic orbit of Z_ϵ in the annulus is unique and hyperbolic.

We can choose coordinates (x, y) such that $X(x, y) = (a(x, y), b(x, y))$, $Y(x, y) = (c(x, y), d(x, y))$ and $f(x, y) = y$. The regularized vector field Z_ϵ takes the form

$$Z_\epsilon = ((1 - \varphi_\epsilon)c + \varphi_\epsilon a, (1 - \varphi_\epsilon)d + \varphi_\epsilon b).$$

For simplicity, consider that γ is made of only two components, γ_0 and γ_1. Let $D \cap \gamma$ be the points p_0 and p_1, L_0 and L_1 be orthogonal sections to γ_0 at p_0 and p_1, and L_2 and L_3 be orthogonal sections to γ_1 at p_1 and p_0 (see Figure 1). Denote by $\pi_0 : L_0 \to L_1$ and by $\pi_1 : L_2 \to L_3$ the transitions maps defined by the flow of X and Y, respectively. Let $\eta = \pi_1 \circ \pi_0$ be the first return map of Z. Let θ_0, θ_1, θ_2 and θ_3 the angles between L_0, L_1, L_2 and L_3 with D, respectively (see Figure 2).

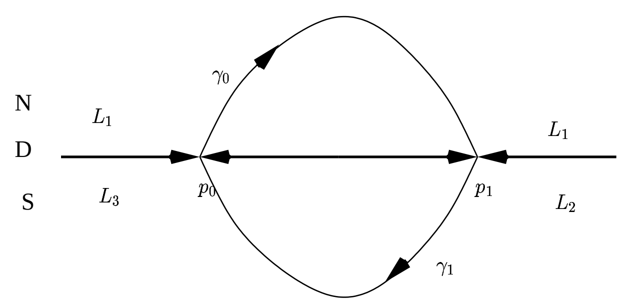

FIG. 1. Poly-trajectory of type 1

The derivative of η in the point p_0 is given by (see [2, 9])

$$\frac{\cos\theta_2 |Y(p_1)| exp[\int_{t_1}^{t_2} divY(\gamma_1(t))\, dt] \cos\theta_0 |X(p_0)| exp[\int_{t_0}^{t_1} divX(\gamma_0(t))\, dt]}{\cos\theta_3 |Y(p_0)| \cos\theta_1 |X(p_1)|}$$

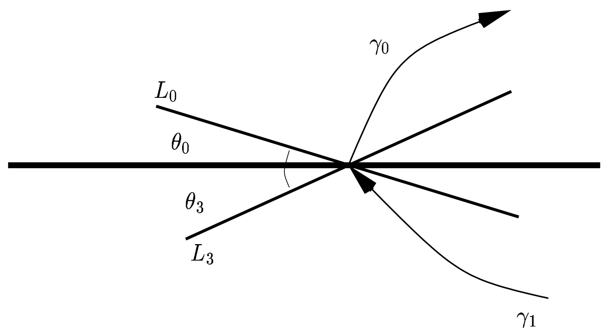

FIG. 2. Poly-trajectory of type 2

We can suppose that $\eta'(p_0) < 1$. Fix an orientation in L_0 and the origin at the point p_0. Let s_0 and s_1 be points of L_0 such that $s_0 < 0 < s_1$. As we have $\eta'(p_0) < 1$, we can assert that $s_0 < \eta(s_0) < 0$ and $0 < \eta(s_1) < s_1$. Let V be an annulus formed by the poly-trajectories of Z by s_0 and s_1 between L_0 and $\eta(L_0)$ and by the arcs $\widehat{s_0\eta(s_0)}$ and $\widehat{\eta(s_1)s_1}$ in L_0. By construction, the poly-trajectories of Z that cross the arcs $\widehat{s_0\eta(s_0)}$ and $\widehat{\eta(s_1)s_1}$ enter in V. Let $\epsilon_0 > 0$ be small enough that $\eta_\epsilon(s_0) > s_0$ and $\eta_\epsilon(s_1) < s_1$, where $0 < \epsilon \leq \epsilon_0$ and $\eta_\epsilon : L_0 \to L_0$ is the first return map of Z_ϵ. Let V_ϵ be an annulus consisting of the trajectories of Z_ϵ by s_0 and s_1 between L_0 and $\eta_\epsilon(L_0)$ and by the arcs $\widehat{s_0\eta_\epsilon(s_0)}$ and $\widehat{\eta_\epsilon(s_1)s_1}$ in L_0. By construction, the orbits of Z_ϵ enter in V_ϵ and there are no singularities of Z_ϵ in V_ϵ. By the Poincaré-Bendixson Theorem, there exists at least one periodic orbit γ_ϵ of Z_ϵ in V_ϵ. We assert that γ_ϵ is hyperbolic. Indeed, we have to show that $\eta'_\epsilon(p_\epsilon)$, the first return map of γ_ϵ, converge to $\eta'(p)$ when ϵ goes to zero. We have $\eta'_\epsilon(p_\epsilon) = exp[\int_{t_{0\epsilon}}^{t_{1\epsilon}} div Z_\epsilon(\gamma_\epsilon(t))\, dt] = exp[\int_{t_{0\epsilon}}^{t_{1\epsilon}} [(1-\varphi_\epsilon) div Y + \varphi_\epsilon div X + \frac{1}{\epsilon}\varphi'(\frac{y}{\epsilon})(b-d)](\gamma_\epsilon(t))\, dt]$. Only in the time interval in which γ_ϵ is inside the regularization strip we have the complete expression given here. Decomposing $\eta'_\epsilon(p_\epsilon)$ in the sum of the derivatives in the arcs defined by the regularization strip, as seen in Figure 3, we have that in the parts exterior to the regularization strip, $\eta'_\epsilon(p_\epsilon)$ converges to $\eta'(p)$.

So, it remains to prove that when ϵ goes to zero the expression

$$A = exp[\int_{t_{0\epsilon}}^{t_{\epsilon'}} (\frac{1}{\epsilon})\varphi'(\frac{y}{\epsilon})(b-d)](\gamma_\epsilon(t))\, dt] exp[\int_{t_{\epsilon''}}^{t_{\epsilon'''}} (\frac{1}{\epsilon})\varphi'(\frac{y}{\epsilon})(b-d)(\gamma_\epsilon(t))\, dt]$$

$$exp[\int_{t_{\epsilon''''}}^{t_{1\epsilon}} (\frac{1}{\epsilon})\varphi'(\frac{y}{\epsilon})(b-d)(\gamma_\epsilon(t))\, dt]$$

goes to

$$\frac{\cos\theta_2|Y(p_1)|\cos\theta_0|X(p_0)|}{\cos\theta_3|Y(p_0)|\cos\theta_1|X(p_1)|},$$

where $[t_{0\epsilon}, t_{\epsilon'}] \cup [t_{\epsilon''}, t_{\epsilon'''}] \cup [t_{\epsilon''''}, t_{1\epsilon}]$ is the time that γ_ϵ spends inside the regularization strip.

Indeed, we have

$$\frac{\cos\theta_2|Y(p_1)|\cos\theta_0|X(p_0)|}{\cos\theta_3|Y(p_0)|\cos\theta_1|X(p_1)|} = \frac{-d(p_1)|Y(p_0)||Y(p_1)|b(p_0)|X(p_1)||X(p_0)|}{|Y(p_1)|d(p_0)|Y(p_0)||X(p_0)|(-b(p_1))|X(p_1)|}$$

$$= \frac{d(p_1)b(p_0)}{d(p_0)b(p_1)}.$$

From the expression of Z_ϵ we have $\frac{dy}{dt} = (1-\varphi_\epsilon)d + \varphi_\epsilon b$, that is: $\frac{dt}{dy} = \frac{1}{(1-\varphi_\epsilon)d+\varphi_\epsilon b}$. Supposing that b and d are constants in the interval of integration, we have

$$A = exp[\int_{-\epsilon}^{\epsilon} \frac{(\frac{1}{\epsilon})\varphi'(\frac{y}{\epsilon})(b-d)}{(1-\varphi_\epsilon)d+\varphi_\epsilon b}\, dy + \int_{\epsilon}^{-\epsilon} \frac{(\frac{1}{\epsilon})\varphi'(\frac{y}{\epsilon})(b-d)}{(1-\varphi_\epsilon)d+\varphi_\epsilon b}\, dy] =$$

$$exp[ln((1-\varphi_\epsilon(\epsilon))d(p_0)+\varphi_\epsilon(\epsilon)b(p_0)) - ln((1-\varphi_\epsilon(-\epsilon))d(p_0)+\varphi_\epsilon(-\epsilon)b(p_0))$$

$$+ln((1-\varphi_\epsilon(-\epsilon))d(p_1)+\varphi_\epsilon(-\epsilon)b(p_1)) - ln((1-\varphi_\epsilon(\epsilon))d(p_1)+\varphi_\epsilon(\epsilon)b(p_1))] =$$

$$= exp[ln(b(p_0)) - ln(d(p_0)) + ln(d(p_1)) - ln(b(p_1))] = \frac{d(p_1)b(p_0)}{d(p_0)b(p_1)}.$$

And the result follows.

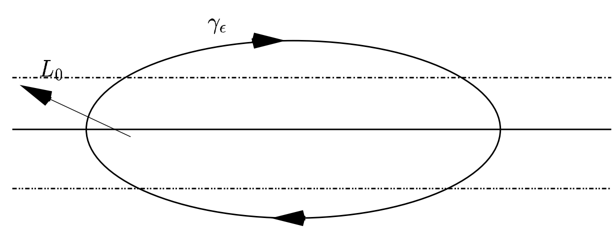

FIG. 3. Regularized Poly-trajectory of type 1

Case where γ is of type 2: We can suppose that $D = SL$, which means that all points of D are sliding points. We will use polar coordinates (θ, ρ) around D in M such that $D = \{\rho = 0, 0 \leq 2\pi\}$. In this case, $f(\theta, \rho) = \rho$. Let $X(\theta, \rho) = (a(\theta, \rho), b(\theta, \rho))$ and $Y(\theta, \rho) = (c(\theta, \rho), d(\theta, \rho))$ be the components of the discontinuous vector field. We can claim that there exists $\epsilon_0 > 0$ such that, if $0 < \epsilon < \epsilon_0$, then the field Z_ϵ has a hyperbolic periodic orbit γ_ϵ in $V = \{(\theta, \rho); |\rho| \leq \epsilon_0\}$. Indeed, as all the points of D are sliding points, we have $Xf(\theta, 0) = b(\theta, 0) < 0$ and $Yf(\theta, 0) = d(\theta, 0) > 0$. By continuity, we can choose $\epsilon_0 > 0$ such that $b(\theta, \rho) < 0$ and $d(\theta, \rho) > 0$ if $|\rho| \leq \epsilon_0$. So, if $0 < \epsilon < \epsilon_0$, the orbits of Z_ϵ are entering in V, and by the Poincaré-Bendixson Theorem, there exists at least one periodic orbit in V (it is easy to show that there are not singularities in V). We assert that if γ_ϵ is a periodic orbit of Z_ϵ in V, then γ_ϵ is attracting and hyperbolic, so it is unique. This prove is made showing that if $p \in \gamma_\epsilon$, L is a transversal section to γ_ϵ in p and $\eta : L \to L$ is the first return map, then $0 < \eta'(p) < 1$. Indeed, we have

$$\eta'(p) = exp[\int_0^{2\pi} divZ_\epsilon(\gamma_\epsilon(\theta))\, d\theta],$$

where

$$divZ_\epsilon = -\varphi'_\epsilon(d-b) + \varphi_\epsilon(a_\theta + b_\rho) + (1-\varphi_\epsilon)(c_\theta + d_\rho) = -\varphi'_\epsilon(d-b) + L,$$

and L is a bounded function in V. It is obvious that $(d - b) > 0$ in γ_ϵ, so it remains to show that $\varphi'_\epsilon(\rho_\epsilon(\theta))$ goes to infinity when ϵ goes to zero, for any θ. To do this, we first considerer a simplified discontinuous field such $X(\theta, \rho) = (1, b_0)$ and $Y(\theta, \rho) = (0, 1)$, where $b_0 < 0$ is constant. In this case, the regularized vector field is $Z(\theta, \rho) = (\varphi_\epsilon(\rho), 1 - \varphi_\epsilon(\rho) + \varphi_\epsilon(\rho)b_0)$. Changing coordinates $\widehat{\rho} = \frac{\rho}{\epsilon}$ and $\widehat{\theta} = \frac{\theta}{\epsilon}$, we have $\frac{d\widehat{\rho}}{d\widehat{\theta}} = \frac{1-\varphi(\widehat{\rho})+\varphi(\widehat{\rho})b_0)}{\varphi(\widehat{\rho})}$. Making zero this last expression, we find a periodic solution of this system, since it does not depend of θ. This periodic solution is implicitly given by $\varphi(\widehat{\rho_0}) = \frac{1}{1-b_0}$. As $b_0 < 0$, then $0 < \varphi(\widehat{\rho_0}) < 1$. The periodic solution $\widehat{\rho_0} = \varphi^{-1}(\frac{1}{1-b_0})$ is unique since φ is monotone. For $\epsilon > 0$, we have $\rho_0 = \epsilon\widehat{\rho_0}$, so $\varphi'_\epsilon(\rho_0) = [\varphi(\frac{\rho_0}{\epsilon})]' = \frac{1}{\epsilon}\varphi'(\frac{\rho_0}{\epsilon}) = \frac{1}{\epsilon}\varphi'(\widehat{\rho_0})$. As ϵ decreases, $\varphi'_\epsilon(\rho_0)$ increases. We recall that $\varphi'_\epsilon(\widehat{\rho_0})$ is constant and strictly positive, and we can decrease ϵ as much as necessary to have the derivative small enough. So, we have that $\gamma_\epsilon = \{(\theta, \rho_0)\}$ is the periodic orbit of this simplified case. We assert that there exists $\delta > 0$ such that the second coordinate of the periodic orbit of the general case is in the interval $(\epsilon(\widehat{\rho_0} - \delta), \epsilon(\widehat{\rho_0} + \delta))$. Then, decreasing ϵ as much as necessary, we will have the derivative of φ big enough to guarantee that the periodic orbit is hyperbolic. Indeed, as the points of the poly-trajectory are regular, we have $det[X, Y](p) \neq 0$, for any $p \in D$. In these coordinates, $det[X,Y](p) = a(p).1 - b(p).0 = a(p) \neq 0$. We can consider that $a(p) > 0$. As D is compact, we have $k = min\{a(p); p \in D\} > 0$. The regularized vector field is given by

$$Z_{\epsilon(\theta,\rho)} = (\varphi_\epsilon(\rho)a(\theta,\rho), (1-\varphi_\epsilon(\rho)) + \varphi_\epsilon(\rho)b(\theta,\rho)).$$

Calculating the second component of Z_ϵ in $(\theta, \rho = \epsilon(\widehat{\rho_0} \pm \delta))$ we have

$$Z_\epsilon^2(\theta, \epsilon(\widehat{\rho_0} \pm \delta)) = 1 - \varphi_\epsilon \epsilon(\widehat{\rho_0} \pm \delta)) + \varphi_\epsilon(\rho)b(\theta, \epsilon(\widehat{\rho_0} \pm \delta)).$$

We will prove that $Z_\epsilon^2(\theta, \epsilon(\widehat{\rho_0} + \delta)) < 0$ and $Z_\epsilon^2(\theta, \epsilon(\widehat{\rho_0} - \delta)) > 0$, so we will have that the orbits are entering in the annulus $(\theta, \epsilon(\widehat{\rho_0} \pm \delta))$, and by the Poincaré-Bendixson Theorem, the periodic orbit will be in this annulus. The expansions of b and φ are as follows:

$$b(\theta, \epsilon(\widehat{\rho_0} \pm \delta)) = b_0 + b_1(\theta)\epsilon(\widehat{\rho_0} \pm \delta) + b_2(\theta, \epsilon(\widehat{\rho_0} \pm \delta))\epsilon^2(\widehat{\rho_0} \pm \delta))^2,$$

$$\varphi(\widehat{\rho_0} \pm \delta) = \varphi(\widehat{\rho_0}) \pm \varphi'(\widehat{\rho_0})\delta + \frac{\varphi''(\widehat{\rho_0})\delta^2}{2} + r_3(\delta),$$

where $r_3(\delta) = \varphi'''(\widehat{\rho_0} \pm m\delta)$, $0 < m < 1$. So,

$$Z_\epsilon^2(\theta, \epsilon(\widehat{\rho_0} + \delta)) = 1 - [\frac{1}{1-b_0} + \varphi'(\widehat{\rho_0})\delta + \frac{\varphi''(\widehat{\rho_0})\delta^2}{2} + r_3(\delta)]$$

$$(1 - b_0 - \epsilon(\widehat{\rho_0} + \delta)(b_1(\theta) + b_2(\theta,\rho)\epsilon(\widehat{\rho_0} + \delta))).$$

Fixed $\delta > 0$, we can diminish $\epsilon > 0$ such that the factor $b_0 - \epsilon(\widehat{\rho_0} + \delta)(b_1(\theta) + b_2(\theta,\rho)\epsilon(\widehat{\rho_0} + \delta))$ will not be relevant and

$$[\frac{1}{1-b_0} + \varphi'(\widehat{\rho_0})\delta + \frac{\varphi''(\widehat{\rho_0})\delta^2}{2} + r_3(\delta)].(1-b_0) =$$

$$= 1 + [\varphi'(\widehat{\rho_0})\delta + \frac{\varphi''(\widehat{\rho_0})\delta^2}{2} + r_3(\delta)].(1-b_0) > 1,$$

and $Z_\epsilon^2(\theta, \epsilon(\widehat{\rho_0} + \delta)) < 0$.

And now, calculating $Z_\epsilon^2(\theta, \epsilon(\widehat{\rho_0} - \delta))$, we have

$$Z_\epsilon^2(\theta, \epsilon(\widehat{\rho_0} - \delta)) = 1 - [\frac{1}{1-b_0} - \varphi'(\widehat{\rho_0})\delta + \frac{\varphi''(\widehat{\rho_0})\delta^2}{2} + r_3(\delta)]$$

$$(1 - b_0 - \epsilon(\widehat{\rho_0} - \delta)(b_1(\theta) + b_2(\theta,\rho)\epsilon(\widehat{\rho_0} - \delta))).$$

As $\delta^2 < \delta < 1$, we can diminish δ so we have $(1-b_0)(\varphi'(\widehat{\rho_0})\delta - \frac{\varphi''(\widehat{\rho_0})\delta^2}{2} > 0$.
After fixed δ we can diminish ϵ to have $Z_\epsilon^2(\theta, \epsilon(\widehat{\rho_0} - \delta)) > 0$. We can choose a unique δ and fix ϵ_0 such that $Z_\epsilon^2(\theta, \epsilon(\widehat{\rho_0} - \delta)) > 0$ and $Z_\epsilon^2(\theta, \epsilon(\widehat{\rho_0} + \delta)) < 0$ and $\varphi'(\widehat{\rho_0})$ be big enough for the periodic orbit be hyperbolic.

Case where γ is of type 3: It will be used, without lost of generality, that γ has a unique trajectory arc of X and a unique trajectory arc of F_Z, and this is a sliding arc. We will considered that $f(x,y) = y$. In this case, the proof is made using four transversal sections to γ, called L_1, L_2, L_3 and L_4 such that L_1 and L_2 cross D and L_3 and L_4 are out of the regularization zone for small ϵ, as can be seen in Figure 4. We assume that the transversal sections were chosen so that the orbits of X that cross L_3 enter in the sliding arc after crossing L_4 and then cross L_1, and the orbits that cross L_1 enter the sliding arc too. We define η_1, η_2, η_3 and η_4 to be the transition maps between L_1 and L_2, L_2 and L_3, L_3 and L_4, L_4 and L_1, respectively. It can be proved that there is a neighborhood V of γ such that the orbits of Z are entering in V, and for ϵ small enough, Z_ϵ has not singularities in V. So, by the Poincaré-Bendixson Theorem, there exists at least one periodic orbit γ_ϵ in V. We can consider that $\eta_\epsilon = \eta_{4\epsilon} \circ \eta_{3\epsilon} \circ \eta_{2\epsilon} \circ \eta_{1\epsilon}$ is the first return map of γ_ϵ. The derivative of the map η_ϵ will be calculated by calculation each derivative of the maps $\eta_{i\epsilon}$. Let $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (x_3, y_3)$ and $p_4 = (x_4, y_4)$ be the points that the orbit γ_ϵ cross the sections L_1, L_2, L_3 and L_4, respectively, and let θ_1, θ_2, θ_3 and θ_4 the angles formed by the sections and γ_ϵ. Next, we will study the transition map in each part:

Calculation of η_1': As an simplification which exhibits the essentials of the situation, we will consider that $X(x,y) = (-1,-1)$ and $Y(x,y) = (-1,1)$.
So, the regularized vector field can be reduced to $\frac{dy}{dx} = \frac{1-2\varphi_\epsilon}{-1}$.
Then, $\frac{dy}{1-2\varphi_\epsilon} = \frac{dx}{-1}$, which implies

$$\int_{y_1}^{y_2} \frac{1}{1-2\varphi_\epsilon}\, dy = \int_{x_1}^{x_2} -1\, dx = x_1 - x_2.$$

The values x_1 and x_2 correspond to the sections L_1 and L_2, respectively. So, given y_1, we can find y_2 such that the orbit that passes by $(x_1, y_1) \in L_1$ crosses L_2 in (x_2, y_2) using the integral equation.
We define

$$\Psi(y_1) = \int_{y_1}^{y_2} \frac{1}{1-2\varphi_\epsilon}\, dy = x_1 - x_2 = H(y_2) - H(y_1).$$

So,

$$\frac{\partial\Psi(y_1)}{\partial y_1} = H'(y_2)\frac{dy_2}{dy_1} - H'(y_1) = \frac{1}{1-2\varphi_\epsilon(y_2)}\frac{dy_2}{dy_1} - \frac{1}{1-2\varphi_\epsilon(y_1)} = 0,$$

then

$$0 < \frac{dy_2}{dy_1} = \frac{1}{1-2\varphi_\epsilon(y_2)} = \frac{1-2\varphi_\epsilon(y_2)}{1-2\varphi_\epsilon(y_1)} << 1.$$

It can be easily proved that y_2 goes to $\varphi^{-1}(\frac{1}{2})$ when ϵ goes to zero.

The derivative of the function $\eta_{1\epsilon}$ corresponding to the arc of orbit γ_ϵ between L_1 and L_2 is

$$\eta'_{1\epsilon} = \frac{\cos\theta_1|Z_\epsilon(p_1)|}{\cos\theta_2|Z_\epsilon(p_2)|}exp[\int_{t_1}^{t_2} divZ_\epsilon(\gamma_\epsilon(t)\,dt] = \frac{\cos\theta_1|Z_\epsilon(p_1)||1-2\varphi_\epsilon(y_2)|}{\cos\theta_2|Z_\epsilon(p_2)||1-2\varphi_\epsilon(y_1)|}$$

Calculation of η'_2: As an simplification which exhibits the essentials of the situation, we will consider that $X(x,y) = (-1,-x)$ and $Y(x,y) = (-1,1)$. In these coordinates, the fold point is $p = (0,0)$. So, the regularized vector field is given by $Z_\epsilon = (-1, 1-\varphi_\epsilon - x\varphi_\epsilon)$. We have $\frac{dy}{dt} = 1-\varphi_\epsilon - x\varphi_\epsilon$. Then, $\frac{dt}{dy} = \frac{1}{1-\varphi_\epsilon - x\varphi_\epsilon}$.

The derivative of the function $\eta_{2\epsilon}$ corresponding to the arc of orbit γ_ϵ between L_2 and L_3 is

$$\eta'_{2\epsilon} = \frac{\cos\theta_2|Z_\epsilon(p_2)|}{\cos\theta_3|Z_\epsilon(p_3)|}exp[\int_{t_2}^{t_3} divZ_\epsilon(\gamma_\epsilon(t)\,dt] =$$

$$= \frac{\cos\theta_2|Z_\epsilon(p_2)||1-\varphi_\epsilon(y_3)(1+x_3)|}{\cos\theta_3|Z_\epsilon(p_3)||1-\varphi_\epsilon(y_2)(1+x_2)|}$$

Calculation of η'_3: It is easy to see that for ϵ small enough, the orbit γ_ϵ is out of the regularization strip, so γ_ϵ only has points of X.

The derivative of the function $\eta_{3\epsilon}$ corresponding to the arc of orbit γ_ϵ between L_3 and L_4 is

$$\eta'_{2\epsilon} = \frac{\cos\theta_3|Z_\epsilon(p_3)|}{\cos\theta_4|Z_\epsilon(p_4)|}K,$$

where $K \in \mathbb{R}$ is constant.

Calculation of η'_4: As an simplification which exhibits the essentials of the situation, we will consider the same coordinates as those considered between L_1 and L_2, so

$$\eta'_{4\epsilon} = \frac{\cos\theta_4 |Z_\epsilon(p_4)||1 - 2\varphi_\epsilon(y_1)|}{\cos\theta_1 |Z_\epsilon(p_1)||1 - 2\varphi_\epsilon(y_4)|}.$$

composing the derivatives, we have

$$\eta'_\epsilon(p_1) = \eta'_{4\epsilon}(p_4)\eta'_{3\epsilon}(p_3)\eta'_{2\epsilon}(p_2)\eta'_{1\epsilon}(p_1) = K\frac{|1 - \varphi_\epsilon(y_3)(1 + x_3)||1 - 2\varphi_\epsilon(y_2)|}{|1 - 2\varphi_\epsilon(y_4)||1 - \varphi_\epsilon(y_2)(1 + x_2)|}.$$

As $\varphi_\epsilon(y_3) = \varphi_\epsilon(y_4) = 1$, we have

$$\eta'_\epsilon(p_1) = K\frac{|x_3||1 - 2\varphi_\epsilon(y_2)|}{|1 - \varphi_\epsilon(y_2)(1 + x_2)|}.$$

As y_2 goes to $\varphi^{-1}(\frac{1}{2})$ when ϵ goes to zero and from the fact that $x_2 < 0$, follows that, diminishing ϵ_0 if necessary, $\eta'_\epsilon(p_1) < 1$, for $\epsilon \in (0, \epsilon_0]$. This conclusion is true also in the general case, with considerable complication in the calculations. ▌

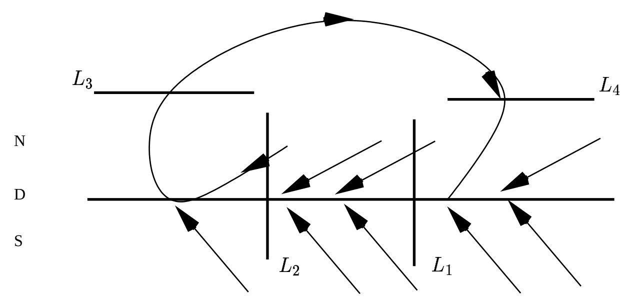

FIG. 4. Poly-trajectory of type 4

3. REGULARIZATION AT POINTS OF $D \cap \partial M$

In this section we provide conditions on $Z = (X, Y)$ for the regularized fields Z_ϵ have only orbits transversal or with quadratic contact to ∂M.

PROPOSITION 14. *Let $Z=(X,Y) \in \Omega^r$, $r \geq 1$, and $p \in D \cap \partial M$ with $X\beta(p)Y\beta(p) > 0$. So, given a transition function φ, there is a neighborhood V of p and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, Z_ϵ is transversal to ∂M in V.*

Proof. We can use coordinates such that $X(x,y) = (a(x,y), b(x,y))$, $Y(x,y) = (c(x,y), d(x,y))$, $p = (0,0)$, $\beta(x,y) = x$ and $f(x,y) = y$. So, $\partial M = \{x = 0\}$, $D = \{y = 0\}$, $X\beta(x,y) = a(x,y)$ and $Y\beta(x,y) = c(x,y)$. As $X\beta(p)Y\beta(p) = a(0)c(0) > 0$, we have that $a(0)$ and $c(0)$ have the same sign at p. By the continuity of these functions, this situation is holds in a neighborhood V of p. We can take $\epsilon_0 > 0$ such that, for $0 < \epsilon \leq \epsilon_0$, near p, the intersection of the regularized strip is contained in V. So, as $(1-\varphi_\epsilon) \geq 0$ and $\varphi_\epsilon \geq 0$, we have that $Z_\epsilon\beta(q) = (1-\varphi_\epsilon(q))Y\beta(q) + \varphi_\epsilon(q)X\beta(q)$ has the same sign that $X\beta(q)$ and $Y\beta(q)$, if $q \in V$. In particular, Z_ϵ is transversal to ∂M in V. ▌

PROPOSITION 15. *Assume that $Z = (X,Y) \in \Omega^r$, $r > 1$, and $p \in D \cap \partial M$ with $X\beta(p)Y\beta(p) < 0$ and $X\beta(p)Yf(p) - Xf(p)Y\beta(p) \neq 0$. So, given a transition function φ, the following possibilities hold:*

1.If $X\beta(p)Yf(p) - Xf(p)Y\beta(p) > 0$, there is a neighborhood V of p and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, Z_ϵ has a point of internal quadratic tangency in V.

2.If $X\beta(p)Yf(p) - Xf(p)Y\beta(p) < 0$, there is a neighborhood V of p and $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, Z_ϵ has a point of external quadratic tangency in V.

Proof. We can use coordinates such that $X(x,y) = (a(x,y), b(x,y))$, $Y(x,y) = (c(x,y), d(x,y))$, $p = (0,0)$, $\beta(x,y) = x$ and $f(x,y) = y$. So, $\partial M = \{x = 0\}$, $D = \{y = 0\}$, $X\beta(x,y) = a(x,y)$, $Xf(x,y) = b(x,y)$, $Y\beta(x,y) = c(x,y)$ and $Yf(x,y) = d(x,y)$. As $X\beta(p)Y\beta(p) = a(0)c(0) < 0$, we have that $a(0)$ and $c(0)$ are not zero and have different sign at p. By the continuity of these functions, this situation holds in a neighborhood V of p. Define $h(y) = \frac{c(0,y)}{c(0,y)-a(0,y)}$. By Lemma 7, there exists $\epsilon_0 > 0$ and constants $0 < h_0 < h_1 < 1$ such that $0 < h_0 < h(y) < h_1 < 1$, if $|y| \leq \epsilon_0$. If necessary, we can decrease ϵ_0 in order to have $(0,y) \in V$ if $|y| \leq \epsilon_0$. For a point $(0, y_\epsilon)$ to be a tangency of Z_ϵ and ∂M, it must be true that $Z_\epsilon\beta(0,y_\epsilon) = (1-\varphi_\epsilon(y_\epsilon))c(0,y_\epsilon) + \varphi_\epsilon(y_\epsilon)a(0,y_\epsilon) = 0$. Assume that $g(y) = (1-\varphi_\epsilon(y))c(0,y) + \varphi_\epsilon(y)a(0,y)$. We claim that there is $y_\epsilon \in (-\epsilon,\epsilon)$ such that $g(y_\epsilon) = 0$. Indeed, we have $g(-\epsilon) = c(0,-\epsilon)$ and $g(\epsilon) = a(0,\epsilon)$. By hypothesis, $a(0,\epsilon)$ and $c(0,-\epsilon)$ have different sign, so, by continuity, there is at least one point $y_\epsilon \in (-\epsilon,\epsilon)$ such that $g(y_\epsilon) = 0$, and the field Z_ϵ is tangent to ∂M.

It remains to show that this tangency is unique and to establish when it is internal or external. In fact, at the point $(0,y_\epsilon)$, we have $\varphi(y_\epsilon) = h(y_\epsilon)$, so $Z_\epsilon^2\beta = \frac{(ad-bc)(\varphi_\epsilon'(c-a)^2 + ac_y - a_y c)}{(c-a)^2}$. It is easy to show that this expression, if ϵ is enough small, has the same sign as $a(0)d(0) - b(0)c(0)$. So, Z_ϵ has a quadratic tangency at $(0,y_\epsilon)$, which is internal if $a(0)d(0) - b(0)c(0) > 0$ or external if $a(0)d(0) - b(0)c(0) < 0$. ▌

4. DISCONTINUOUS VECTOR FIELDS WITH STRUCTURALLY STABLE REGULARIZATION

In this section we define a set $G^r(M)$ of discontinuous vector fields whose elements, Z, have structurally stable regularizations. Z_ϵ, for any transition function φ and small ϵ.

DEFINITION 16. Write $G^r(M) = G^r(1) \cap G^r(2) \cap G^r(3) \cap G^r(4)$, where:

1. $G^r(1) = \{Z = (X,Y) \in \Omega^r; X|_N$ and $Y|_S \in \Sigma^r$; each D-singularity of Z is elementary and does not belong to $\partial M\}$.
2. $G^r(2) = \{Z = (X,Y) \in \Omega^r; X|_N$ and $Y|_S \in \Sigma^r$; each closed poly-trajectory of Z is elementary and does not have points of $\partial M\}$.
3. $G^r(3) = \{Z = (X,Y) \in \Omega^r; X|_N$ and $Y|_S \in \Sigma^r$; for each $p \in D \cap \partial M$, $X\beta(p)Y\beta(p) \neq 0$, and when this expression is negative, then $X\beta(p)Yf(p) - Xf(p)Y\beta(p) \neq 0\}$.
4. $G^r(4) = \{Z = (X,Y) \in \Omega^r; X|_N$ and $Y|_S \in \Sigma^r$; Z does not have saddle or tangency connections; if $p \in D \cap \partial M$, $X\beta(p)Y\beta(p) < 0$ and $X\beta(p)Yf(p) - Xf(p)Y\beta(p) > 0$, then the trajectory through p does not contain any separatrix or a point $p_1 \neq p$ in $D \cap \partial M$ with $X\beta(p_1)Y\beta(p_1) < 0$ and $X\beta(p_1)Yf(p_1) - Xf(p_1)Y\beta(p_1) > 0\}$.

Remark 17. The conditions $p \in D \cap \partial M$, $X\beta(p)Y\beta(p) < 0$ imply that the regularization Z_ϵ has a point of tangency with ∂M near p; the condition $X\beta(p)Yf(p) - Xf(p)Y\beta(p) \neq 0$ implies that this tangency point is quadratic. This will be proved in Proposition 20.

In what follows appear the sets $\Sigma^r(i)$ of Definition 2.

PROPOSITION 18. *Let $Z = (X,Y) \in G^r(1), r \geq 1$. Then, given a transition function φ, there is an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, $Z_\epsilon \in \Sigma^r(1)$.*

Proof. As $X|_N$ and $Y|_S \in \Sigma^r$, it remains to prove that the singularities that appear during the regularization process are hyperbolic and disjoint of ∂M. Let p be a D-singularity of Z. By definition, p is elementary, so it only can be a hyperbolic singularity of F_Z or a fold. By Proposition 8, if p is a hyperbolic singularity of F_Z, then there is an $\epsilon_0 > 0$ such that, for $0 < \epsilon \leq \epsilon_0$, Z_ϵ has a unique singularity near p, and this singularity is hyperbolic. By Propositions 8 and 9, if p is D-regular point or a fold, then there is an $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0]$, Z_ϵ does not have singularities near p. As F_Z does not have singularities in $D \cap \partial M$, there is $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0]$, Z_ϵ does not have singularities in ∂M.

It remains to show that we can choose an ϵ_0 that works for every case. Indeed, let p be a point of D, then p can be a D-regular point, a hyperbolic singularity of F_Z or a fold. For each case, there is a proposition that guarantees a number $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0]$, Z_ϵ has no singularities near p or has a unique hyperbolic singularity. The union of these neighborhoods cover D, and, as D is compact, there is a sub covering made by a finite number of these neighborhoods.

Then, we can chose ϵ_0 as the smallest ϵ_0 associated to these neighborhoods. To finish, we have to verify if this ϵ_0 is small enough so that there are no singularities on the border. ▮

PROPOSITION 19. *Let $Z = (X,Y) \in G^r(2) \cap G^r(4)$, $r \geq 1$. Then, given a transition function φ, there is an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, $Z_\epsilon \in \Sigma^r(2)$.*

Proof. As $X|_N$ and $Y|_S \in \Sigma^r$, all their periodic orbits are hyperbolic and disjoint of ∂M, so it remains to prove that the same occurs to the periodic orbits that appear during the regularization process. Let γ be an elementary closed poly-trajectory of Z. Then, by Proposition 13, there is an $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$, Z_ϵ has a hyperbolic closed orbit near γ. We can choose a unique positive ϵ_0 since the elementary poly-trajectories, are finite number. As X and Y are structurally stable, there is no possibility of a Hopf bifurcation. So, the case of periodic orbits emerging from singularities during the regularization process is excluded. As $Z \in G^r(4)$, Z does not have separatrix graphs, so there is not the possibility of appearance of a periodic orbit from such a graph. So, the periodic orbits emerging from the regularization of closed poly-trajectories are the only new periodic orbits of Z_ϵ. ▮

PROPOSITION 20. *Let $Z = (X,Y) \in G^r(3), r \geq 1$. Then, given a transition function φ, there is an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, $Z_\epsilon \in \Sigma^r(3)$.*

Proof. It follows from Propositions 14 and 15. ▮

PROPOSITION 21. *Let $Z = (X,Y) \in G^r(4) \cap G^r(2) \cap G^r(3), r \geq 1$. Then, given a transition function φ, there is $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, $Z_\epsilon \in \Sigma^r(4)$.*

Proof. We first analyze the case where $D \cap \partial M = \phi$. We claim that there is an $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$, then Z_ϵ does not have saddle or tangency connections. Indeed, as $X|_N$ and $Y|_S \in \Sigma^r$, and Z does not have separatrix connections, the only possibilities for Z_ϵ to have such separatrix are as follows:

1. passing through points of the curve D;
2. passing through a tangency of order three or bigger;
3. due to the presence of a semi-stable periodic orbit, which could disappear and allow a connection of two separatrices.

Possibility 2 is discarded, since $Z \in G^r(3)$; possibility 3 is also dicarded, since $Z \in G^r(2)$. We must analyze possibility 1. Let δ be the minimum of the set $\{dist(e_i, e_j); e_i$ is a separatrix of Z, and $i \neq j\}$. Of course, $\delta > 0$, since the number of separatrices is finite. Then, we diminish ϵ_0 so that the minimum distance of the separatrices for the regularized vector field never be less than $\frac{\delta}{2}$.

In the case that $D \cap \partial M \neq \phi$, it remains to analyze what happens when a point of internal quadratic tangency of Z_ϵ with ∂M appears. It follows from the last condition of the definition of $G^r(4)$, where we required that if two points of internal quadratic tangency appear, they are not connected. See Definition 16. ▮

Recall that $\Sigma^r(M)$ stands for structurally stable vector field on M.

THEOREM 22. *If $Z = (X, Y) \in G^r(M)$, $r \geq 1$, then, given a transition function φ, there is $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, then $Z_\epsilon \in \Sigma^r(M)$.*

Proof. It follows from Propositions 18, 19, 20 and 21. ▮

5. GENERICITY

In this section we prove that the set G^r is open and each discontinuous vector field Z of Ω^r can be approximated by fields of G^r, through translations and rotations. So, we prove the genericity of G^r.

THEOREM 23. *The set $G^r(M)$, $r \geq 1$, is open in Ω^r.*

Proof.
Let $Z = (X, Y)$ be a vector field in $G^r(M)$. It will proved that there is $\delta > 0$ such that if $\widehat{Z} = (\widehat{X}, \widehat{Y}) \in \Omega^r$ and $|Z - \widehat{Z}|_r = \max\ \{|X - \widehat{X}|_r, |Y - \widehat{Y}|_r\} < \delta$, then $\widehat{Z} \in G^r(M)$. For do this, we have to prove that $\widehat{Z} \in G^r(i), i = 1, \ldots, 4$.

• We claim that there is a $\delta_1 > 0$ such that if $|Z - \widehat{Z}|_r < \delta_1$, then $\widehat{Z} \in G^r(1)$.

Indeed, as $Z = (X, Y) \in G^r(1)$, we have $X|_N$ and $Y|_S \in \Sigma^r$, and from the openness of Σ^r, there is $\delta_1 > 0$ such that if $|Z - \widehat{Z}|_r < \delta_1$, then $\widehat{X}|_N$ and $\widehat{Y}|_S \in \Sigma^r$. Diminishing δ_1, if necessary, we have all D-singularities of $\widehat{Z}$ elementary and disjoint of ∂M. We can establish this claim proving that if p is an elementary D-singularity of Z and $\widehat{Z}$ is close to Z, then there is a point $\widehat{p}$ near p which is a elementary D-singularity of $\widehat{Z}$. If p is an elementary D-singularity of Z, then p is a fold or a hyperbolic singularity of F_Z.

Let p be a fold of Z. We can suppose that $Xf(p) = 0$, $X^2 f(p) \neq 0$ and $Yf(p) \neq 0$. As $Xf(p) = 0$ and $X^2 f(p) \neq 0$, the curve $\{Xf = 0\}$ crosses transversally the curve D on the point p, and, by continuity, the same occurs to the curve $\{\widehat{X}f = 0\}$, for $\widehat{Z}$ near Z. This means that there is $\widehat{p}$ near p such that $\widehat{X}f(\widehat{p}) = 0$ and $\widehat{X}^2f(\widehat{p}) \neq 0$. If δ_1 is enough small, we can assume that also is true that $\widehat{Y}f(\widehat{p}) \neq 0$. So, $\widehat{p}$ is a fold of $\widehat{Z}$.

Let p be a hyperbolic singularity of F_Z. We have $Xf(p)Yf(p) < 0$, $det[X,Y](p) = 0$ and $d(det[X,Y])|_D(p) = 0$. Analogous to the fold case, the curve $\{det[X,Y](p) = 0\}$ crosses transversally the curve D at the point p, and the same is true for $\widehat{Z}$ near Z. So, there is $\widehat{p}$ near p such that $\widehat{X}f(\widehat{p})\widehat{Y}f(\widehat{p}) < 0$, $det[\widehat{X},\widehat{Y}](\widehat{p}) = 0$ and $d(det[\widehat{X},\widehat{Y}])|_D(\widehat{p}) = 0$. This implies that $\widehat{p}$ is a hyperbolic singular point of $F_{\widehat{Z}}$. As δ_1 can be chosen so that no involved function have changed sign, and therefore $\widehat{p}$ is a singularity of the same kind that p. It is obvious that δ_1 can be chosen so that the singularities of $\widehat{Z}$ are not in ∂M. As the D-singularities are isolated, δ_1 can be chosen strictly positive. It is easy to show that $\widehat{Z}$ does not have others singularities, so $\widehat{Z} \in G^r(1)$.

- We claim that there is a $\delta_2 > 0$ such that if $|Z - \widehat{Z}|_r < \delta_2$, then $\widehat{Z} \in G^r(2)$.

As $Z = (X,Y) \in G^r(2)$, we have $X|_N$ and $Y|_S \in \Sigma^r$, and each closed poly-trajectory of Z is elementary and does not have points in ∂M.

Let γ be an elementary closed poly-trajectory of type 1 of Z. Associated to γ there is a first return map η, differentiable and such that $\eta'(p) \neq 1$, for $p \in \gamma$. This means that p is a hyperbolic fixed point of the diffeomorphism η. So, there is a number $k > 0$ such that if μ is a diffeomorphism with $|\eta - \mu| < k$, then μ have a hyperbolic fixed point p_μ near p. Then, it is enough to choose $\delta_2 > 0$ small as necessary for if $|Z - \widehat{Z}| < \delta_2$, the first return map $\widehat{\eta}$ associated to $\widehat{Z}$ satisfy $|\eta - \widehat{\eta}| < k$. So, $\widehat{\eta}$ has a hyperbolic fixed point $\widehat{p}$ which correspond to a elementary closed poly-trajectory of type 1 of $\widehat{Z}$.

Let γ be an elementary closed poly-trajectory of type 2 of Z. Then, $\gamma = D$ and, for each $p \in D$, $Xf(p)Yf(p) < 0$ and $det[X,Y](p) \neq 0$. As D is compact and the functions involved are continuous, there is $\delta_2 > 0$ such that if $|Z - \widehat{Z}|_r < \delta_2$, then $\widehat{X}f(p)\widehat{Y}f(p) < 0$ and $det[\widehat{X},\widehat{Y}](p) \neq 0$. By definition, $\widehat{\gamma} = D$ is a closed poly-trajectory of type 2 of $\widehat{Z}$.

Let γ be an elementary closed poly-trajectory of type 3 of Z. By the continuity of the functions involved, can be shown that there is $\delta_2 > 0$ such that if $|Z - \widehat{Z}|_r < \delta_2$, $\widehat{Z}$ has an elementary closed poly-trajectory $\widehat{\gamma}$ of type 3 near γ.

As the number of poly-trajectories is finite, we can choose $\delta_2 > 0$ small enough so that $\widehat{Z}$ has only elementary poly-trajectories, disjoint of ∂M. So, we have proved that $\widehat{Z} \in G^r(2)$.

• We assert that there is a $\delta_3 > 0$ such that if $|Z - \widehat{Z}|_r < \delta_3$, then $\widehat{Z} \in G^r(3)$.

Indeed, as $Z = (X,Y) \in G^r(3)$, we have $X|_N$ and $Y|_S \in \Sigma^r$ and for each $p \in D \cap \partial M$, $X\beta(p)Y\beta(p) \neq 0$, and case this expression is negative, then $X\beta(p)Yf(p) - Xf(p)Y\beta(p) \neq 0\}$. It is immediate, by continuity, that there is $\delta_3 > 0$ such that if $|Z - \widehat{Z}|_r < \delta_3$, then for $p \in D \cap \partial M$, $\widehat{X}\beta(p)\widehat{Y}\beta(p) \neq 0$, and case this expression is negative, then $\widehat{X}\beta(p)\widehat{Y}f(p) - \widehat{X}f(p)\widehat{Y}\beta(p) \neq 0\}$. So, we have prove that $\widehat{Z} \in G^r(3)$.

• We claim that there is a $\delta_4 > 0$ such that if $|Z - \widehat{Z}|_r < \delta_4$, then $\widehat{Z} \in G^r(4)$.

Indeed, as $Z = (X,Y) \in G^r(4)$, we have $X|_N$ and $Y|_S \in \Sigma^r$ and there is $\delta_4 > 0$ such that if $|Z - \widehat{Z}|_r < \delta_4$, then $\widehat{X}|_N$ and $\widehat{Y}|_S \in \Sigma^r$. So, $\widehat{X}$ and $\widehat{Y}$ does not have separatrix connections in N and S, respectively. It remains to analyze the appearance of a connection with at least one point in D. We know that the Z have only a finite number of separatrices and does not have a connection. As $\widehat{Z}$ has separatrix corresponding to the separatrix of Z, it is easy to show that $\delta_4 > 0$ can be chosen so that $\widehat{Z}$ does not have separatrix connections. In this way, we have established that $\widehat{Z} \in G^r(4)$.

To finish the proof, we can take $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$, then if $\widehat{Z} = (\widehat{X}, \widehat{Y}) \in \Omega^r$ and $|Z - \widehat{Z}|_r = \max\{|X - \widehat{X}|_r, |Y - \widehat{Y}|_r\} < \delta$, then $\widehat{Z} \in G^r(M)$. As a consequence, the set $G^r(M)$ is open in Ω^r. ∎

Definition 24. Assume that $Z = (X,Y) \in \Omega^r$, $r \geq 1$. For each pair $(\theta, v) \in \mathbb{R}^2 \times \mathbb{R}^2$, we call $Z_{\theta,v}$ the field Z translated by $v = (v_1, v_2)$ and rotated by $\theta = (\theta_1, \theta_2)$; this means that

$$Z_{\theta,v} = R_\theta(Z + v) = (R_{\theta_1}(X + v), R_{\theta_2}(Y + v)),$$

where

$$R_{\theta_1}(X + v) = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} X_1 + v_1 \\ X_2 + v_2 \end{pmatrix}.$$

Theorem 25. *Assume that $Z \in \Omega^r$, $r \geq 1$.*
The set $\{(\theta, v) \in \mathbb{R}^2 \times \mathbb{R}^2; Z_{\theta,v} \notin G^r\}$ has null Lebesgue measure in $\mathbb{R}^4$.

Proof. Let Z be a field of Ω^r. Let $crit(Z)$ be the set of critical values of Z, which means $crit(Z) = \{v \in \mathbb{R}^2 : \exists p \in N$ with $X(p) = v$ and $det(DX(p)) = 0$ or $\exists p \in S$ with $Y(p) = v$ and $det(DY(p)) = 0\}$. By Sard's Theorem, the set $crit(Z)$ has null Lebesgue measure in $\mathbb{R}^2$. Of course, the sets $X(\partial M)$ and $Y(\partial M)$ have null Lebesgue measure too. So, the set $C(Z) = crit(Z) \cup X(\partial M) \cup Y(\partial M)$ has null Lebesgue measure.

If $v \in \mathbb{R}^2$ satisfy $-v \notin crit(Z)$, then $X+v$ and $Y+v$ have all their singularities simple and contained inside the interior of N and S, respectively. It must be proved that if $-v \notin crit(Z)$, then the set $\Psi(Z,v) = \{\theta \in \mathbb{R}^2; R_\theta(Z+v) \notin G^r(M)\}$ has null Lebesgue measure in $\mathbb{R}^2$. As $\Psi(Z,v) = \cup\Psi^i(Z,v), i=1,\ldots,4$, where $\Psi^i(Z,v) = \{\theta \in \mathbb{R}^2; R_\theta(Z+v) \notin G^r(i)\}$, the proof will be carried out showing that each set $\Psi^i(Z,v)$ has null Lebesgue measure. Let v be a point not in $C(Z)$. We can suppose that $v=(0,0)$ and $X(x,y) = (a(x,y), b(x,y))$ and $Y(x,y) = (c(x,y), d(x,y))$. So $R_\theta(Z+v) = (R_{\theta_1}(X+v), R_{\theta_2}(Y+v)) = ((a\cos\theta_1 - b\sin\theta_1, a\sin\theta_1 + b\cos\theta_1), (c\cos\theta_2 - d\sin\theta_2, c\sin\theta_2 + d\cos\theta_2))$.

- $\Psi^1(Z,v) = \{\theta \in \mathbb{R}^2; R_\theta(Z+v) \notin G^r(1)\}$ has null Lebesgue measure in $\mathbb{R}^2$.

From the continuous case (see [9]), it follows that $\{\theta \in \mathbb{R}^2; R_\theta(X) \notin \Sigma^r(1)\}$ $\cup$ $\{\theta \in \mathbb{R}^2; R_\theta(Y) \notin \Sigma^r(1)\}$ has null Lebesgue measure. So, it remains to analyze the D-singularities. It is a calculation to show that if the field F_{Z_θ} associated to Z_θ has one non-hyperbolic singularity p, then

$$(a(p)c(p) + b(p)d(p))\sin(\theta_2 - \theta_1) = -(a(p)d(p) - b(p)c(p))\cos(\theta_2 - \theta_1)$$

and we can suppose that $(a(p)c(p) + b(p)d(p)) \neq 0$. So, at the point p, we have $\tan(\theta_2 - \theta_1) = -\frac{ad-bc}{ac+bd}$, which implies that $\theta_2 - \theta_1 - k\pi = \arctan(-\frac{ad-bc}{ac+bd})$.

Define

$$\theta_2(p,k,\theta_1) = \theta_0(p) + \theta_1 + k\pi,$$

where

$$\theta_0(p) = \arctan(-\frac{a(p)d(p) - b(p)c(p)}{a(p)c(p) + b(p)d(p)}).$$

The singularities of F_{Z_θ} will occur when $\theta_2 = \theta_2(p,k,\theta_1)$, for k even or odd. We can suppose that it is for k even. The critical values of the map θ_2 correspond to non-hyperbolic singularities of F_{Z_θ}. It is a calculation to show that this values form a null Lebesgue measure set.

It remains to show that the set of discontinuous fields such that X or Y have a tangency of order plus than two has null Lebesgue measure. Indeed, it is easy to show that for a point p be a point of tangency of X and D of order plus than two, we must have $\theta_1 = \arctan(\frac{-b}{a}) + k_1\pi = \arctan(\frac{-b_y}{a_y}) + k_2\pi$, which occurs only in isolated points. The other situations are similar.

So, $\Psi^1(Z,v) = \{\theta \in \mathbb{R}^2; R_\theta(Z+v) \notin G^r(1)\}$ has null Lebesgue measure in $\mathbb{R}^2$.

• $\Psi^2(Z,v) = \{\theta \in \mathbb{R}^2; R_\theta(Z+v) \notin G^r(2)\}$ has null Lebesgue measure in $\mathbb{R}^2$.

As in the case of Ψ^1, it remains to analyze the case of the closed poly-trajectories of Z_θ. Suppose that $\widehat{\theta} = (\widehat{\theta_1 \theta_2})$ is such that $R_{\widehat{\theta}}(Z+v)$ has a non-elementary closed poly-trajectory $\widehat{\gamma}$. Then, $\widehat{\gamma}$ is of type 1 and $\widehat{\eta'} = 1$ or $\widehat{\gamma}$ is of type 3 and has only trajectories arcs of $X+v$ and $Y+v$ and one fold of $Z+v$, or $\widehat{\gamma}$ is of type 3 and has points of the sliding and escaping arcs. It can be proved that each one of these cases only occurs for isolated $\widehat{\theta}$.

So, $\Psi^2(Z,v) = \{\theta \in \mathbb{R}^2; R_\theta(Z+v) \notin G^r(2)\}$ has null Lebesgue measure in $\mathbb{R}^2$.

• $\Psi^3(Z,v) = \{\theta \in \mathbb{R}^2; R_\theta(Z+v) \notin G^r(3)\}$ has null Lebesgue measure in $\mathbb{R}^2$.

As in the cases of Ψ^1 and Ψ^2, it remains to analyze the case of the tangencies that appears during the regularization process. It must be proved that if $X_{\theta_1}\beta(p)Y_{\theta_2}\beta(p) < 0$, then $X_{\theta_1}\beta(p)Y_{\theta_2}f(p) - X_{\theta_1}f(p)Y_{\theta_2}\beta(p) \neq 0$. Assume that $X_{\theta_1}\beta(p)Y_{\theta_2}\beta(p) < 0$. Then $X_{\theta_1}\beta(p)Y_{\theta_2}f(p) - X_{\theta_1}f(p)Y_{\theta_2}\beta(p) = (ac + bd)\sin(\theta_2 - \theta_1) + (ad - bc)\cos(\theta_2 - \theta_1)$. As $X(p) \neq 0$ and $Y(p) \neq 0$, $(ac+bd)$ and $(ad-bc)$ does not vanish simultaneously in p, so, for each θ_1 fixed, the last expression only vanish for isolated values of θ_2. The same occurs if we fix θ_2 and vary θ_1.

So, $\Psi^3(Z,v) = \{\theta \in \mathbb{R}^2; R_\theta(Z+v) \notin G^r(3)\}$ has null Lebesgue measure in $\mathbb{R}^2$.

• $\Psi^4(Z,v) = \{\theta \in \mathbb{R}^2; R_\theta(Z+v) \notin G^r(4)\}$ has null Lebesgue measure in $\mathbb{R}^2$.

As in the cases of Ψ^1, Ψ^2 and Ψ^3, it remains to analyze when γ is a separatrix connection that has at least one point in D. We can distinguish two cases: γ has at least one sewing point; γ does not have sewing points. It can be proved, in both cases, that the connection can be broken, except for isolated values of θ.

So, $\Psi^4(Z,v) = \{\theta \in \mathbb{R}^2; R_\theta(Z+v) \notin G^r(4)\}$ has null Lebesgue measure in $\mathbb{R}^2$.

We can conclude that the set $\Psi(Z,v) = \{\theta \in \mathbb{R}^2; R_\theta(Z+v) \notin G^r(M)\}$ has null Lebesgue measure in $\mathbb{R}^2$, since is a union of four sets of null Lebesgue measure. From Fubini's Theorem follows that the set $\{(\theta, v) \in \mathbb{R}^2 \times \mathbb{R}^2; Z_{\theta,v} \notin G^r\}$ has null Lebesgue measure in $\mathbb{R}^4$. ▮

THEOREM 26. *The set $G^r(M)$, $r \geq 1$, is open and dense in Ω^r.*

Proof. Immediate from the Theorems 23 and 25. ▮

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