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Geometry of Cycles In Quadratic Systems

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This paper is a study of the affine and euclidean differential geometry of cycles of quadratic systems. While the euclidean curvature must always be strictly positive, the affine curvature can take either sign, although with certain restrictions. Every quadratic cycle has exactly six affine vertices, but the number of euclidean vertices can vary, not just from cycle to cycle, but for the same cycle under a linear coordinate transformation. We prove that an upper bound on the number of euclidean vertices over all non-circular quadratic cycles is twelve, and provide evidence that a sharp upper bound is six.

Key Words: affine, curvature, cycle, quadratic system, sextactic, vertex

1. INTRODUCTION

Let X denote the quadratic system of differential equations

$$\begin{aligned} \dot{x} = P(x,y) &= \sum_{0\le i+j\le 2} a_{ij}x^iy^j \\ \dot{y} = Q(x,y) &= \sum_{0\le i+j\le 2} b_{ij}x^iy^j \end{aligned} \tag{1}$$

or equivalently the vector field (P, Q) on $\mathbb{R}^2$, where P and Q are relatively prime, and at least one is of degree two. This paper is devoted to a study of the affine and euclidean differential geometry of cycles appearing in the phase portrait of X. Specifically, we are interested in understanding the possibilities for the curvature, affine and euclidean, of a quadratic cycle, and the number, nature, and relative positions of affine and euclidean vertices, which are the local extrema of the corresponding curvatures. (Definitions appear in the next section. Henceforth in this paper "curvature" and "vertex" will mean "euclidean curvature" and "euclidean vertex;" we will always use the adjective "affine" explicitly when speaking of their affine differential counterparts, and use the adjective "euclidean" only for contrast.)

It is well known that a cycle or polycycle appearing in the phase portrait of a quadratic system bounds a convex subset of the plane, hence its curvature is always of one sign. In fact the curvature is never zero (Proposition **??**). We find that the affine curvature, whose sign is significant, can take either sign on a quadratic cycle. If not always positive, it can assume negative values on one, two, or three subarcs, each one of which contains exactly one affine vertex, and between any two of which lie an odd number of affine vertices (Theorem **??**).

Affine vertices prove to be the most amenable to study. We discover that every non-circular quadratic cycle contains exactly six of them, the minimum allowed by the Six Affine Vertex Theorem, all of them simple (Theorem **??**). The affine curvature can be negative at up to three of them, no two adjacent.

Vertices prove more difficult to work with. A local analysis shows that a vertex on a quadratic cycle has multiplicity at most three. Any triple vertex is simultaneously an affine vertex, at which the affine curvature is positive (Theorem **??**). By the Four Vertex Theorem every cycle has at least four vertices, a lower bound which is attained for example by an ellipse. It would seem that, in a sense, this lower bound should be difficult for an arbitrary smooth Jordan curve γ to attain, since to do so requires that the curve γ be tangent to its circumscribed circle at precisely two antipodal points (see [**?**]). Yet we shall see that sufficiently small cycles about centers in quadratic systems generically have four vertices, and that there is evidence to suggest that for any cycle in a quadratic system, there is always a linear change of variables so that in the new quadratic system the corresponding cycle has exactly four vertices (see Conjecture **??**). We also find that sufficiently small cycles in period annuli have at most six vertices (Proposition **??**), and that typically Hopf bifurcation produces cycles with at most six vertices (Proposition **??**; see also Remark **??**). An extensive examination of phase portraits, together with the results just mentioned, suggests that *every* cycle in a quadratic system has at most six vertices (Conjecture **??**). A proof of this conjecture, to the best of our knowledge originally propounded by Paul de Jager, has so far eluded us, however. The best we can do so far is to prove a universal upper bound of twelve, and this by a rather indirect method. Knowing that a nonsingular linear transformation μ can change the number of vertices of a cycle γ, we identify the possible bifurcation set for vertices of $\mu\langle\gamma\rangle$ in the space of linear transformations, and show how to assign a number, counting vertices of $\mu\langle\gamma\rangle$, to each component of its complement. Knowing all possible configurations of the bifurcation set then yields an upper bound of twelve for any cycle in the phase portrait of a quadratic system (Theorem **??**).

We emphasize that the results of this paper depend heavily on the fact that the degree of the system is two. At the end of the paper we include a general argument that yields the uniform bounds of $2(6n-2)(7n-3)$ on the number of vertices and $2(21n-9)(22n-10)$ on the number of affine vertices of any cycle occurring in the phase plane of a polynomial system of degree n. For quadratic systems this gives the far worse values of at most 220 vertices and at most 2244 affine vertices.

2. PRELIMINARIES

Throughout the paper we will use without comment a number of facts about quadratic systems. Complete discussion and proofs may be found in [**?**], [**?**], [**?**], and [**?**].

Euclidean arclength and curvature of a plane curve, as well as their formulas, are widely familiar. From the point of view of transformation groups, affine geometry is the study of invariants under equiaffine transformations, that is, linear transformations with determinant $+1$.

Invariants are those geometric properties preserved under parallel projection from one plane to another, hence angles and non-parallel lengths are not compared. Nevertheless, for curves for which $\det(\underline{\dot{x}}, \underline{\ddot{x}}) \neq 0$, an arclength s_A and curvature κ_A are defined. The actual formulas will not be important for us; see [**?**] for the derivations. The curves of constant affine curvature are the conic sections: ellipses (including circles) for $\kappa_A > 0$, parabolas for $\kappa_A = 0$, and hyperbolas for $\kappa_A < 0$.

For a smooth curve γ and a point A on γ at which the curvature is non-zero, the *osculating circle* C is the unique curve of constant curvature (here, of course, a circle) which best fits the curve γ at A; it agrees with γ out to second order, and has "three-point contact" with γ at A in the sense that γ and C share three infinitely close points at A. The point A is a *vertex* of γ if the curvature has a local maximum or minimum at A; this implies that C agrees with γ to at least one higher order, hence has at least four-point contact with γ at A. The same ideas apply in affine differential geometry, where now the auxiliary curve is the *osculating conic* Γ, which agrees with γ through fourth order, and has five-point contact with γ at A. The point A is an *affine vertex* of γ if the affine curvature has a local maximum or minimum at A; this implies that Γ agrees with γ to at least one higher order, hence has at least six-point contact with γ at A. For this reason an affine vertex is also called a *sextactic point*.

We say that γ is *hyperbolically curved* at A (resp., *parabolically curved*, *elliptically curved*) if the osculating conic Γ at A is a hyperbola (resp., a parabola, an ellipse (or circle)).

The Four Vertex Theorem, which dates back to Mukhopadhaya ([?], [?], [?], [?], [?]), asserts that any sufficiently smooth convex plane oval has at least four vertices. The Six Affine Vertex Theorem, also dating back to Mukhopadhaya ([?], [?], [?], [?]), asserts that any such curve has at least six *affine* vertices. (Allowance must be made in these theorems for circles and ellipses, as appropriate.)

3. LOCAL ANALYSIS

If $\gamma = (x(t), y(t))$ is a C^r curve in the plane, $r \geq 2$, then the curvature of γ at a regular point is given by the expression

$$\kappa = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}} . \tag{2}$$

Applying (??) to just the numerator in (??) determines a fifth degree polynomial function $\kappa(x, y)$ whose non-zero values have no particular meaning, but whose zero set picks out those points A such that the trajectory A_t of X through A has zero curvature at A. (We will always let A_t denote the trajectory through A which is at A when $t = 0$.)

We will need the following slight strengthening of the fact mentioned in the introduction. It is this result that guarantees that the affine arclength and affine curvature are defined for any quadratic cycle.

PROPOSITION 1. *At no point of a cycle γ of a quadratic system does the curvature vanish.*

Proof. Let any regular point A at which $\kappa(x, y) = 0$ be given. By a rigid motion we may place A at the origin, incorporating a rotation that yields $Q(0, 0) = 0$.

A further rescaling, if necessary, places the system in the form

$$\begin{aligned} \dot{x} &= 1 + ax + by + ex^2 + fxy + gy^2 \\ \dot{y} &= \qquad cx + dy + hx^2 + jxy + ky^2 . \end{aligned}$$

The curvature vanishes at $(0, 0)$ only if $\ddot{y} = 0$ there, hence only if $Q_x(0, 0) = c = 0$. But then either the x-axis is invariant (when $h = 0$), or

$\dot{y}$ is of one sign along it, vanishing only at A ($h \neq 0$), so that in either case the trajectory A_t through A does not close. ▮

We note in passing that the same result is true for any arc of a polycycle which does not lie in a segment of a straight line.

Now let us turn to the question of the order of vanishing of the derivative of the curvature along a trajectory of X. There is no question that the order of vanishing is independent of the parametrization of the trajectory, but since solutions are not parametrized by arclength, we must include the denominator in (**??**) in the computation of the first derivative. On the other hand, since we are concerned about the vanishing of a particular derivative only at a point at which all the previous derivatives are zero, we may apply the following simple fact:

PROPOSITION 2. *If $f(x) = n(t)/d(t)$ is a C^k function on an open interval containing t_0, then for any $m \leq k$, $f^{(j)}(t_0) = 0$ for $j = 0, 1, \ldots, m$ iff $n^{(j)}(t_0) = 0$ for $j = 0, 1, \ldots, m$.*

Thus we drop the denominator in the derivative of κ, and in all subsequent computations use the resulting expression

$$\kappa_1 = (\dot{x}^2 + \dot{y}^2)(\dot{x}y^{(3)} - x^{(3)}\dot{y}) - 3(\dot{x}\ddot{x} + \dot{y}\ddot{y})(\dot{x}\ddot{y} - \ddot{x}\dot{y})\,. \tag{3}$$

Substituting in for $\dot{x}$, $\dot{y}$, and so on using (**??**) we obtain a tenth order polynomial expression in x and y. The corresponding algebraic curve $\kappa_1(x, y) = 0$ picks out those points A in the phase plane at which the local trajectory A_t has a vertex (at least if the next derivative is non-zero). Differentiating (**??**) and applying (**??**) yields an eleventh order polynomial function κ_2. Points on the algebraic curve $\kappa_2(x, y) = 0$ are undistinguished, except where they also lie on $\kappa_1(x, y) = 0$, in which case they pick out a point A where the local trajectory A_t has a higher vertex. The process continues for all orders.

PROPOSITION 3. *Let A be a regular point of a quadratic system X, at which the curvature does not vanish.*

(1)If at A $\dot{\kappa} = \ddot{\kappa} = \kappa^{(3)} = \kappa^{(4)} = 0$ but $\kappa^{(5)} \neq 0$, then the osculating circle C at A to the trajectory A_t through A is without contact to X except at A, and A_t crosses into or out of the interior of C at A. Thus A does not lie on a cycle (or polycycle).

(2)If at A $\kappa^{(j)} = 0$ for $j = 1, \ldots, 5$, then A lies in an invariant circle.

Proof. This proposition depends on the fact that a non-invariant conic section has at most six points of contact with the quadratic system X. If the hypotheses in point (1) hold then the vector field X has sixth order contact with C at A, hence either points into C at every other point, or out of C at every other point. Since A is not a vertex, the trajectory A_t crosses C at A (Corollary 2.1.2 of [**?**]), hence cannot close. If the hypotheses in point (2) hold then the vector field X has seventh order contact with C at A, hence C must be invariant. ∎

Remark 4. If a real linear transformation μ of the plane is made, a cycle γ of X is transformed to a cycle $\bar{\gamma} := \mu\langle\gamma\rangle$ of the quadratic vector field $\mu_* X$, and the conclusion of Proposition **??** still holds for $\bar{\gamma}$. Now suppose μ is a linear transformation of $\mathbb{C}^2$, and make the identification of $\mathbb{C}^2$ with $\mathbb{R}^4$ defined by $(z_1, z_2) \leftrightarrow (\operatorname{Re} z_1, \operatorname{Im} z_1, \operatorname{Re} z_2, \operatorname{Im} z_2)$.

Then μ induces a linear transformation η of $\mathbb{R}^4$ which carries the 2-plane $\Pi := \{(x_1, y_1, x_2, y_2) \mid y_1 = y_2 = 0\}$ to a 2-plane $\tilde{\Pi} := \eta\langle\Pi\rangle \subset \mathbb{R}^4$. Regarding X as a vector field on $\mathbb{R}^4$, X leaves Π invariant, $\eta_* X$ leaves $\tilde{\Pi}$ invariant, and in fact in the coordinates naturally induced on $\tilde{\Pi}$ by η, is exactly the same quadratic vector field as X on $\mathbb{R}^2$. Thus the conclusion of Proposition **??** holds for the image curve $\eta\langle\gamma\rangle$.

Now we turn to the number of vertices on "small" cycles. Any cycle in a quadratic system surrounds a unique singularity, which is an antisaddle. By a rigid motion we may place an antisaddle A of the system X given by (**??**) at the origin, simultanesously making $b_{01} = a_{10}$, so that X has the form

$$\begin{aligned} \dot{x} &= ax + by + ex^2 + fxy + gy^2 \\ \dot{y} &= cx + ay + hx^2 + jxy + ky^2 \,. \end{aligned} \tag{4}$$

If $A : (0,0)$ is not a center when $a = 0$, then a unique limit cycle is created or destroyed as a is made to cross zero (while the other coefficients remain fixed). As long as the cycle is small, a local analysis suffices to count the number of its vertices.

Proposition 5. *Suppose that for $a = 0$ the system* (**??**) *does not have a center at $A : (0,0)$.*

(1) The unique cycle created or destroyed as a in (**??**) *crosses zero is either a circle, or has exactly four or exactly six vertices.*

(2) The set of quadratic systems for which the Hopf cycle so created has other than four vertices is an algebraic subset of codimension one.

Before giving the proof, we note that if A is a center for a quadratic system X, then by a rigid motion X can be placed in the form (**??**), with $a = 0$. Thus in the course of proving Proposition **??** we will establish the following result.

Proposition 6.

(1)Sufficiently small cycles surrounding a quadratic center are either circles (concentric or non-concentric), or have either exactly four or exactly six vertices.

(2)The set of quadratic centers not surrounded by cycles having exactly four vertices (in a small enough neighborhood of the center) is an algebraic subset of the set of systems containing a center, of codimension one.

Proof. (Proof of Propositions **??** and **??**.) Let σ and Δ denote the trace and determinant of the linear part of X at $A:(0,0)$. We then find that $\kappa_1(x,y) = \Delta q_1(x,y)q_2(x,y) + O(5)$, where

$$\begin{aligned} q_1(x,y) &= -cx^2 + by^2 \\ q_2(x,y) &= 3bc(b+c)xy + h(x,y,a)a\,, \end{aligned}$$

where h is a homogeneous quadratic polynomial function of x and y. The discriminant of q_1 is $4bc = \sigma^2 - 4\Delta < 0$, so $q_1(x,y) = 0$ has no real lines of zeros. Thus in the generic situation $b + c \neq 0$, quartic terms are present in $\kappa_1(x,y)$ when $a = 0$, and on a fixed neighborhood of A that is independent of a, $\kappa_1(x,y) = 0$ has two branches passing through A; sufficiently small cycles surrounding A have exactly four vertices.

If $b + c = 0$, so that q_2 is identically zero when $a = 0$, then applying a uniform rescaling, if necessary, to make $c = 1$, we find that

$$\begin{aligned} \kappa_1(x,y) = &- a(a^2+1)^2(x^2+y^2)^2 \\ &+ 2(a^2+1)(x^2+y^2)(K(x,y) + K_1(x,y)a + K_2(x,y)a^2) \\ &+ O(6)\,, \end{aligned} \tag{5}$$

where K, K_1, and K_2 are homogeneous cubic polynomial functions in x and y, and specifically $K(x,y) = \alpha x^3 + 3\beta x^2 y - 3\alpha xy^2 - \beta y^3$, for which $\alpha = 2(e-g-j)$ and $\beta = 2(f+h-k)$. $K(x,y)$ has discriminant $108(\alpha^2+\beta^2) \geq 0$, so that if the fifth order terms are present in $\kappa_1(x,y)$ when $a = 0$, then there are three branches of $\kappa_1(x,y) = 0$ passing through A. Thus if A is a center when $a = 0$, then sufficiently small cycles in the period annulus have six vertices. If A is not a center at $a = 0$, then a Hopf cycle appears and grows as a crosses zero, but now the algebraic curves $\kappa_1(x,y) = 0$ bifurcate as well.

To treat this case we observe that a unique Hopf cycle is created or destroyed as a crosses 0, and has radius of order $\sqrt{a}$, $\sqrt[4]{a}$, or $\sqrt[6]{a}$, depending on the order of the weak focus at $a = 0$. (We mean that, in the first case, there exist constants R_1 and R_2 such that the Hopf cycle lies in the annulus of radii $R_i\sqrt{a} + O(a)$, $i = 1, 2$, and similarly in the latter two cases. Because the real part of the complex eigenvalues of $dX(A)$ crosses the imaginary axis with non-zero speed, the usual proof for the case of a first order fine focus, as presented for example in Chapter 8 of [**?**], may be repeated verbatim in the two other cases, the key point being that the Implicit Function Theorem still applies in Proposition 8.2 of that text.)

For p in any closed interval in $(0,1)$ containing $\frac{1}{6}$, $\frac{1}{4}$, and $\frac{1}{2}$, evaluating κ_1 along the circle of radius $r = Ca^p$ we obtain from (**??**)

$$\kappa_1(Ca^p\cos\theta, Ca^p\sin\theta) = \\ C^2(a^2+1)a^{5p}[2K(\cos\theta,\sin\theta) + R(\cos\theta,\sin\theta,a)a^{\tilde{p}}]\,,$$

for some positive number $\tilde{p}$. We conclude that the Hopf cycle has exactly six vertices for $|a|$ sufficiently small.

If on the other hand $\alpha = \beta = 0$, then a rotation, which preserves the linear part and the condition $\alpha = \beta = 0$, will exist to make $k = -h$, and a computer algebra computation using Maple gives

$$\begin{aligned}\kappa_1(x,y) = &- a(a^2+1)(x^2+y^2)^2 \\ &+ 2a(a^2+1)(x^2+y^2)^2[-2hx + (3g+e)y + L(x,y)a] \\ &+ (x^2+y^2)^2[(e+g)Q(x,y) + M(x,y,a)a] \\ &+ O(7)\,,\end{aligned} \tag{6}$$

where $Q(x,y) = 2hx^2 + 3exy - hy^2$ and L and M are homogeneous polynomial functions of x and y of degrees one and two, respectively, the latter containing a as a parameter. A computation of the discriminant quantities of Li Chengzhi ([**?**], Theorem II.5.2) shows that (for $a = 0$, of course) A is a center if $(e+g)h = 0$, and is a first order fine focus otherwise.

When A is a fine focus for $a = 0$, then the Hopf cycle has radius of order $\sqrt{a}$. From (**??**) we have

$$\kappa_1(C\sqrt{a}\cos\theta, C\sqrt{a}\sin\theta) = \\ -a^3[1 + (e+g)C^2Q(\cos\theta,\sin\theta) + R(\cos\theta,\sin\theta,a)a^{\tilde{p}}]\,,$$

for some positive number $\tilde{p}$, so that there are at most four vertices on the Hopf cycle, hence by the Four Vertex Theorem, exactly four.

When A is a center for $a = 0$, if $(e+g)e \neq 0$, then the discriminant of Q is positive, and small cycles in the period annulus have four vertices. If $(e+g)e = 0$, then $\kappa_1 \equiv 0$, and in fact the plane is filled with one or two nests of (possibly non-concentric) circles. ▮

Remark 7. (1) Let X be a quadratic system having a limit cycle γ, and by a rigid motion place X in the form (**??**). The theorem of Cherkas ([**?**]; cf.[**?**]) states that the limit cycle γ can be connected to a Hopf bifurcation of the special type described in Proposition **??** simply by variation of a.

(2) For X in the form (**??**), the linear part of X at A is a composition $D \circ R$ of a dilatation $D = \operatorname{diag}(k\ k)$ and a rotation R if and only if $a = 0$ and $c = -b$. The first paragraph of the proof thus shows that when $dX(A)$ is not of this form, all the cycles produced from A in any Hopf bifurcation (including multi-parameter, however degenerate), have exactly four vertices. The degree of X is not important.

4. GLOBAL ANALYSIS

To examine the geometry of quadratic cycles in the large, we review the theory of bifurcation of vertices of smooth Jordan curves under the action of nonsingular linear transformation, developed in [**?**]. Let a smooth Jordan curve γ whose curvature is nowhere zero be given, say with parametrization $(x, y) = (u(t), v(t))$, $t \in J \subset \mathbb{R}$. A real linear transformation μ of the plane carries γ to a new curve $\bar{\gamma} := \mu\langle\gamma\rangle$, which receives a naturally induced parametrization. If we take μ to be nonsingular, then regular points on γ correspond with regular points on $\bar{\gamma}$ under the parametrizations, hence so do points at which the affine and euclidean curvatures are defined. If the matrix representative of μ in (x, y)-coordinates is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then using (**??**) the curvature $\bar{\kappa}$ along $\bar{\gamma}$ is expressed as

$$\bar{\kappa} = \omega(t) = \det \mu \, (u'v'' - u''v')D^{-3/2} \, ,$$

for

$$D(t) = \mu_0 u'^2 + 2\mu_1 u'v' + \mu_2 v'^2$$

where

$$\mu_0 = a^2 + c^2 \qquad \mu_1 = ab + cd \qquad \mu_2 = b^2 + d^2 \, . \tag{7}$$

Differentiating and factoring out the positive term $\det \mu \, D(t)^{-5/2}$,

$$\omega'(t) \sim \zeta_1 := \mu_0 f_0(t) + \mu_1 f_1(t) + \mu_2 f_2(t)$$

where

$$\begin{aligned} f_0(t) &= u'^2(u'v''' - u'''v') - 3u'u''(u'v'' - u''v') \\ f_1(t) &= 2u'v'(u'v''' - u'''v') - 3(u''v' + u'v'')(u'v'' - u''v') \\ f_2(t) &= v'^2(u'v''' - u'''v') - 3v'v''(u'v'' - u''v') \, . \end{aligned} \tag{8}$$

Identifying the set of linear transformations of $\mathbb{R}^2$ with $\mathbb{R}^4$, we are led to define an analytic map $L : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$L : \mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (\mu_0, \mu_1, \mu_2) \, ,$$

for μ_i as in (**??**). L maps $\mathbb{R}^4$ onto the set $F := \{(\mu_0, \mu_1, \mu_2) \mid \mu_0\mu_2 \geq {\mu_1}^2\}$, carrying the set of nonsingular linear transformations onto $\mathrm{Int}(F)$ and carrying the set of singular transformations to the boundary $\partial F = \{(\mu_0, \mu_1, \mu_2) \mid \mu_0\mu_2 = {\mu_1}^2\}$.

The transformation μ makes $\omega'(t_0) = 0$ iff $L(\mu)$ lies in the plane $\zeta_1 = 0$ in $\mathbb{R}^3$ arising when $t = t_0$. Using Proposition **??**, the simultaneous vanishing of the first k derivatives of ω at t_0 corresponds to $L(\mu)$ lying in each surface $Z_j : \zeta_j = 0$ for $j = 1, 2, \ldots, k$, where

$$\zeta_j := \mu_0 f_0^{(j-1)}(t) + \mu_1 f_1^{(j-1)}(t) + \mu_2 f_2^{(j-1)}(t)\,. \tag{9}$$

A bifurcation of the number of vertices on $\bar{\gamma} = \mu\langle\gamma\rangle$ should occur at a transformation μ which produces a point at which both ω' and ω'' vanish. It is shown in [**?**] that under the hypothesis that the curvature is nowhere zero along γ, which is true by Proposition **??**, this is correct: the two sets $\zeta_1 = 0$ and $\zeta_2 = 0$ are distinct planes in $\mathbb{R}^3$ which intersect along a line ℓ_{12} through the origin whose direction vector is $\vec{v} = (\mu_0^*, \mu_1^*, \mu_2^*)$, where

$$\mu_0^* = \det\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \qquad \mu_1^* = -\det\begin{pmatrix} f_0 & f_2 \\ f_0' & f_2' \end{pmatrix} \qquad \mu_2^* = \det\begin{pmatrix} f_0 & f_1 \\ f_0' & f_1' \end{pmatrix}\,.$$

As t runs through J a ruled surface in $\mathbb{R}^3$ is swept out, which is the bifurcation surface for the number of vertices on $\bar{\gamma}$. Exploiting the homogeneity of all the quantities involved, we apply the radial projection $\pi : \mathbb{R}^3 \to S^2$ to reduce the bifurcation surface to a bifurcation curve γ^* (and a second branch $-\gamma^*$), called the *star curve* for γ, in the two-dimensional parameter space S^2. Simple computations show that a point of γ^* lies in F if and only if the quantity

$$\mathsf{D} := 3\kappa\kappa'' - 5\kappa'^2 + 9\kappa^4$$

is positive. Clearly $\dot{\kappa}$ and its first *two* derivatives vanish at a point on γ if and only if the line ℓ_{12} lies in the plane Z_3, hence if and only if the determinant of the matrix whose ith row is $(f_0^{(i-1)}, f_1^{(i-1)}, f_2^{(i-1)})$, $i = 1, 2, 3$, for the quantities of (**??**) above, vanishes. When γ is parametrized by arclength, this corresponds to vanishing of the quantity

$$\mathsf{T} := 40\kappa'^3 + 9\kappa^2\kappa''' + 36\kappa^4\kappa' - 45\kappa\kappa'\kappa''\,.$$

(Here and in the displayed formula for D, derivatives are taken with respect to arclength. At some points it will be convenient to also use $\mathsf{T}_{\mathrm{det}}$, the full determinant, from which T is obtained by omission of a non-zero factor.)

It is convenient to choose as coordinate charts on S^2 the central projections $\phi_\pm$ onto the tangent plane Π_0 at $(1/\sqrt{2}, 0, 1/\sqrt{2})$, the image of the identity transformation under $\pi\circ L$, of the hemispheres $U_{\pm} := \{(\mu_0, \mu_1, \mu_2) \mid \pm(\mu_0 + \mu_2) > 0\}$,

$$(\lambda_1, \lambda_2) = \phi_{\pm}(\mu_0, \mu_1, \mu_2) = (\sqrt{2}\mu_1/(\mu_0+\mu_2), (-\mu_0+\mu_2)/(\mu_0+\mu_2))\,. \quad (10)$$

Note that both hemispheres $U_{\pm}$ map onto Π_0 under the projection, carrying both branches of γ^* (points and antipodal points) onto the same image in Π_0. Thus the entire star curve, except those points lying in the great circle $\mu_0 + \mu_2 = 0$, is captured by the one coordinate chart.

For simplicity of notation we will use the same name for objects in S^2 and their representations in Π_0.

To summarize, for each $t \in J$ a great circle in S^2 is specified, picking out those linear transformations which make the derivative of the curvature at the corresponding point $A_t \in \gamma$ vanish. As t runs through J, $Z_1(t)$ moves about the sphere, determining the envelope γ^*; $\gamma^*(t)$ is simultaneously the unique class of linear transformations, when they exist, making at least the first and second derivatives of the curvature vanish at A_t. As long as $\gamma^*(t)$ lies in F, there is such a real transformation, but for $\gamma^*(t)$ outside F a family of linear transformations with complex entries is picked out. To each point B in $S^2 \setminus \gamma^*$ a number $V(B)$ is associated: the number of points on γ^* (counting A and $-A$ only once) whose tangent great circles pass through B. For $B \in \mathrm{Int}(F)$, this is the same as the number of vertices on $\mu\langle\gamma\rangle$ for μ corresponding to B, and is given by the formula

$$V(B) = 4 + 2W(\gamma^*, B)\,, \quad (11)$$

where $W(\gamma^*, B)$ is the winding number of γ^* with respect to B. Even if γ^* does not lie in the single chart U_+, in which case some components of γ^* in Π_0 are in reality images of $-\gamma^*$ under ϕ_-, there is never any confusion as to the proper value to assign $W(\gamma^*, B)$ for $B \in F \subset \Pi_0$. More formally, by Proposition **??** below, stereographic projection ψ of S^2 onto Π_0 from $-(1/\sqrt{2}, 0, 1/\sqrt{2})$ will produce from γ^* (ignoring $-\gamma^*$) a closed curve γ^{**} in Π_0, and for $B \in F$ (but not for a general point) we can take $W(\gamma^*, B) := W(\gamma^{**}, \psi \circ \phi_+^{-1}(B))$.

There is an intimate connection of this construction with the affine differential geometry of γ. Consider the osculating conic Γ to γ at a point A. The linear transformation μ which makes A into a multiple vertex (we include a point at which the curvature vanishes to even or to infinite order under the term "vertex") is precisely that linear transformation (unique up to scaling) which simultaneously makes every point of Γ into a multiple vertex. When γ is elliptically curved at A, $A^* \in \gamma^*$ is the linear transformation making the ellipse Γ into a circle. This corresponds to $A^* \in \mathrm{Int}(F)$, and indeed D is simply the negative of the discriminant of Γ. Thus $A^* \in \partial F$ iff Γ is a parabola iff $\mathsf{D} = 0$ (A^* maps the entire plane to a straight line), and $A^* \in S^2 \setminus (F \cup -F)$ iff Γ is a hyperbola iff $\mathsf{D} < 0$ (the complex entries in A^* change the minus sign in Γ to map it to a circle).

Thus in fact to each conic section there is associated a point of S^2.

In the local coordinates $(U_\pm, \phi_\pm)$, given by (**??**) above, conics of the same eccentricity map to ellipses $2\lambda_1^2 + \lambda_2^2 = k$. Discounting dilatations of $\mathbb{R}^2$, this gives a one-to-one mapping of circles, ellipses, and parabolas to $Cl(F)$; hyperbolas like $x^2/a^2 - y^2/b^2 = 1$ and $-x^2/a^2 + y^2/b^2 = 1$ map to the same curve in S^2.

The vanishing of the quantity $\mathsf{T}(t_0)$ is precisely the condition that the osculating conic Γ to γ at $A : \gamma^*(t_0)$ have fifth order contact with γ at A (share six infinitely close points with γ), i.e., be an affine vertex. This explains point (2) in the following theorem, which collects relevant facts about the star curve from [**?**]. (Point (4) is an adaptation of a similar statement in [**?**], based on the fact that $\mathsf{T} = 3\kappa\mathsf{D}' - 8\kappa'\mathsf{D}$.)

THEOREM 8. (See [**?**].) *Let γ be a smooth Jordan curve whose curvature is nowhere zero.*

(1)The image of γ^* is a single point iff γ lies in a conic section iff $\mathsf{T} \equiv 0$.

(2)The curve γ^* has a non-regular point at a parameter value t_0 iff $\mathsf{T}(t_0) = 0$.

(3)The curve γ^* has a cusp at any non-regular point at which T vanishes with odd multiplicity, and has behavior geometrically indistinguishable from that at a regular point at which T vanishes with even multiplicity.

(4)If T has finitely many zeros, and if $t_a < t_b$ are such that $\mathsf{D}(t_a) = 0$ and $\mathsf{D}(t_b) = 0$ but D is non-zero on (t_a, t_b), then there are an odd number of zeros of T on (t_a, t_b).

(5)No regular point of γ^* in Π_0 is an inflection point.

The discussion so far in this section has concerned general smooth Jordan curves. Now we begin our consideration of curves which are cycles of quadratic systems.

LEMMA 9. *Let $\{\gamma_s \mid s \in S \subset \mathbb{R}\}$ be a family of Jordan curves satisfying (1) for each s, γ_s is a cycle of a quadratic system X_s, containing $(0,0)$ in its interior, and (2) each γ_s is parametrized as $x_s = r_s(\theta)\cos(\theta)$, $y_s(\theta) = r_s(\theta)\sin(\theta)$, and $r : \mathbb{R} \times S \to \mathbb{R}$ is a C^∞ function. Suppose that for $s = s_0$, γ_{s_0} is such that T_{s_0} has a simple zero at $\theta = \theta_0$. Then there are neighborhoods $\hat{S}$ of s_0 and $\hat{\Theta}$ of θ_0 so that for each s in $\hat{S}$, T_s has a unique zero in $\hat{\Theta}$, and it is simple.*

LEMMA 10. *Let γ be a cycle in a quadratic system X which is neither a circle nor an ellipse. Then at no parameter value t_0 is $\mathsf{T}(t_0) = \mathsf{T}'(t_0) = 0$.*

Proof. Let $n_j(t)$ denote the vector $(f_0^{(j-1)}(t), f_1^{(j-1)}(t), f_2^{(j-1)}(t))$, the normal vector to the plane $Z_j(t)$ determined by the vanishing of the quantity $\zeta_j(t)$ of (**??**). Recall that the intersection of the planes $Z_1(t)$ through $Z_k(t)$ picks out those linear transformations μ making the first k derivatives of the curvature vanish at $\mu\langle\gamma\rangle(t)$, and that $\mathsf{T}(t)$ is a non-zero multiple of $\mathsf{T}_{\det}(t) := \det[n_1(t)\, n_2(t)\, n_3(t)]$. Thus $\mathsf{T}(t_0) = 0$ if and only if $\mu\langle\gamma\rangle(t_0)$ is a triple vertex. By the Leibniz rule for differentiating the determinant of a matrix, $\mathsf{T}'(t) = \det[n_1(t)\, n_2(t)\, n_4(t)]$, so $\mathsf{T}'(t_0) = 0$ as well if and only if $n_4(t_0) \in \operatorname{span}\{n_1(t_0), n_2(t_0)\}$, i.e., if and only if the first four derivatives of the curvature vanish at $\mu\langle\gamma\rangle(t_0)$. By Proposition **??**(1) and Remark **??**, this is impossible, no matter where $\gamma^*(t_0)$ may lie in S^2. ▌

PROPOSITION 11. *Let $\{\gamma_s \mid s \in S \subset \mathbb{R}\}$ be a family of Jordan curves satisfying hypotheses (1) and (2) of Lemma* **??**, *and additionally that no cycle γ_s lies in a conic section. Then for each $s \in S$, T_s has only simple zeros. In particular, the number of cusps on γ_s^* is the same for all $s \in S$.*

Proof. Theorem **??**(1) implies that T_s is non-trivial, and by Lemma **??** T_s has only simple zeros. Thus the result follows directly from Lemma **??** and Theorem **??**(3). ▌

The following theorem was proved in [**?**].

THEOREM 12. (See [**?**].) *Let γ_A and γ_B be cycles of quadratic systems X_A and X_B, respectively, having the same orientation. Then there is a C^∞ arc α from $[0,1]$ into the space of quadratic vector fields (identified by their coefficients with $\mathbb{R}^{12}$) such that $\alpha(0) = X_A$, $\alpha(1) = X_B$, and such that for each $s \in [0,1]$ the vector field $\alpha(s)$ has a cycle γ_s so that the family $\{\gamma_s \mid 0 \le s \le 1\}$ satisfies (1) $\gamma_0 = \gamma_A$ and $\gamma_1 = \gamma_B$, (2) the family is C^∞ in the sense that there is a C^∞ arc $\beta : [0,1] \to \mathbb{R}^2$ so that for each $s \in [0,1]$, $\beta(s) \in \gamma_s$, and (3) for every $s \in (0,1)$ γ_s is a limit cycle which does not lie in a conic section.*

To *properly connect* two quadratic cycles will mean to connect them by an arc provided by Theorem **??** in such a way that no intermediate cycle lies in a conic section, hence so that none of the corresponding intermediate star curves reduces to a point.

We easily obtain the following fact of independent interest concerning star curves of quadratic cycles.

THEOREM 13. *The star curve of a cycle of a quadratic system, unless it reduces to a point, has exactly six cusps.*

Proof. For ϵ sufficiently close to zero, the star curve of any cycle γ_1 sufficiently close to the circle of radius 1 in the period annulus surrounding the center at $(0,0)$ in the quadratic Hamiltonian system $\dot{x} = -y$, $\dot{y} = x + \epsilon x^2$

has six cusps. (Computational details are given in Example 3.8 of [**?**].) Given a cycle γ of a quadratic system, use Theorem **??** to properly connect it to γ_1. With obvious modification Proposition **??** applies, from which we conclude that the number of cusps on the corresponding star curves does not change along the arc, giving the result. ▮

The star curve γ_1^* corresponding to the curve γ_1 introduced in the proof of Theorem **??** is a simple closed curve lying wholly within $\mathrm{Int}(F)$, hence projects to a simple closed curve in Π_0, for which a positive orientation is defined. It has six approximately equally spaced cusps; the tangent vector to each smooth arc turns about $1/3$ revolution clockwise running from one cusp to the next. For this reason, γ_1^* (positively oriented) will be referred to as the *snowflake*.

The work done so far combines to yield the following description of the affine differential geometry of quadratic cycles. We will show in Section **??** that all possibilities in Theorem **??**(2) are realized.

THEOREM 14. *Let γ be a cycle of a quadratic system which is not a circle or an ellipse.*

(1) There are exactly six affine vertices on γ, all simple.

(2) There is an arc (possibly all of γ) along which γ is elliptically curved.

(3) There are at most three disjoint arcs along which γ is hyperbolically curved.

(4) Each arc in (3) contains a single affine vertex; each pair is separated by an odd number of affine vertices.

Proof. By the discussion in the paragraph immediately preceding Theorem **??**, zeros of T correspond to affine vertices. By Lemma **??** and point (3) of Theorem **??**, they also correspond to cusps of the star curve. Point (1) now follows from Theorem **??**.

Since cusps of γ^* correspond to zeros of T, and intersections of γ^* with ∂F correspond to zeros of D, by Theorem **??**(4) up to three of the six cusps on γ^* can lie outside F. By the discussion in the second paragraph preceding Theorem **??**, points of γ^* outside F correspond to points at which γ is hyperbolically curved, so that point (2) and the first part of (3) are established. The second part of point (3) follows by Theorem **??**(4). ▮

Along similar lines we have the following result.

THEOREM 15. *Let γ be a cycle of a quadratic system.*

(1) Any vertex of γ has order at most three.

(2) A multiple vertex of order three is an affine vertex, at which γ is elliptically curved.

Proof. The first point is a consequence of Proposition **??**(1).

If A is a vertex of order three, then $\dot{\kappa} = \ddot{\kappa} = \kappa^{(3)} = 0$ but $\kappa^{(4)} \neq 0$. A simple computation shows that the osculating circle C to γ at A has 6-point contact with γ at A, hence is the osculating conic Γ. Thus (2) follows. ▮

In order to bound the number of vertices of quadratic cycles, we must understand how the star curves vary during the homotopy of one quadratic cycle into another provided by Theorem **??**. We begin with three preliminary propositions, the first of which applies to general curves, not just quadratic cycles.

PROPOSITION 16. *Let A be a regular point of a smooth planar curve γ whose curvature never vanishes. Then the tangent great circle to γ^* at the corresponding point A^* contains a point in* $\mathrm{Int}(F)$*, hence has exactly two points of intersection with ∂F.*

Proof. If we parametrize γ by arclength and rotate to make the tangent line at A horizontal, then from (**??**) and (**??**) we obtain $\zeta_1(t) = \kappa'\mu_0 + (-3\kappa^2)\mu_1 + 0 \cdot \mu_2$. Thus the equation of the tangent great circle (actually of the corresponding plane Z_1) is

$$(\kappa')\mu_0 + (-3\kappa^2)\mu_1 = 0\,. \tag{12}$$

Taking the (μ_0, μ_1)-plane to be horizontal, Z_1 is a vertical plane determined by the line (**??**) in the (μ_0, μ_1)-plane, and because $\kappa \neq 0$, that line is not the μ_1-axis, $\mu_0 = 0$. Thus the tangent great circle passes through the north and south poles $(0, 0, \pm 1)$, but is not tangent to ∂F. Thus besides the crossing at $N : (0, 0, 1)$, it has a non-antipodal crossing of ∂F as well, which implies the result. ▮

PROPOSITION 17. *Let γ be a cycle of a quadratic system. Then $\gamma^* \subset S^2 \setminus (-F)$.*

Proof. If γ is such that $\gamma^* \nsubseteq F$, properly connect γ_1 to γ (Theorem **??**), inducing a homotopy from γ_1^* to γ^* for which γ_s^* never reduces to a point. By parts (3) and (4) of Theorem **??** the mechanism by which γ_s^* first fails to lie in F is that a cusp C on γ_s^* crosses ∂F. Similarly, γ_s^* could first fail to lie in $S^2 \setminus (-F)$ only by means of a cusp crossing $\partial(-F)$. It is clear that since zeros of T correspond to cusps on γ_s^*, zeros of D to intersections of γ_s^* with ∂F and $\partial(-F) = -\partial(F)$, and by Proposition **??** no bifurcation of C into additional cusps is possible, Theorem **??**(4) would be violated if C were to enter $-F$. ▮

The ideas involved in the proof just given show that if C is a cusp on a positively oriented star curve γ^* of a quadratic cycle γ, and if C lies outside F, then there are unique points O (for "Out") and I (for "In") on

$\gamma^* \cap \partial F$ such that proceeding from O to I in the positive direction along γ^*, C is the unique cusp encountered. The tangents to γ^* at O and I are significant.

PROPOSITION 18. *Let γ be a cycle of a quadratic system. Let C be a cusp on γ^* which lies outside F, and let ℓ and m be the tangent great circles to γ^* at its points of intersection O and I with ∂F, respectively, where the parametrization of γ^* induces the ordering O–C–I, and let $\tilde{O}$ and $\tilde{I}$ denote the second intersections of ℓ and m with ∂F, which exist by Proposition* **??**. *Let W and $-W$ be the points of intersection of ℓ and m in S^2.*

(1)C lies in the curvilinear spherical triangle bounded by the arcs $(-\tilde{O})W$ and $(-\tilde{I})W$ of ℓ and m, and the arc $(-\tilde{O})(-\tilde{I}) \subset \partial(-F)$

(2)The arcs OC and CI of γ^* lie in the curvilinear triangle bounded by the great circle arcs OC and CI, and the arc $OI \subset \partial F$.

Proof. This is an immediate consequence of Proposition **??** and part (5) of Theorem **??**. ▌

COROLLARY 19. *In Π_0, the lines ℓ and m of Proposition* **??** *are either parallel or intersect at a point $W \in \Pi_0 \setminus F$.*

Now consider a positively oriented, piecewise smooth simple closed planar curve δ, consisting of K smooth arcs δ_k, $k = 1, \dots, K$, with tangent vector fields τ_k. At the kth corner let ρ_k denote the positive angle from τ_k^-, the limiting position of the tangent vector approaching the corner along the arc preceding it, to τ_k^+, the limiting position of the tangent vector approaching the corner along the arc following it, in the ordering induced by positive orientation. Then the index $\iota(\delta)$ of the curve (or the tangent vector field to the curve) may be defined as

$$\iota(\delta) := \sum_{k=1}^{K} \iota(\delta_k) + \frac{1}{2\pi} \sum_{k=1}^{K} \rho_k \,, \tag{13}$$

where $\iota(\delta_k)$ is the usual index of the tangent vector along the smooth arc δ_k. (Compare with Corollary VII.2.1 of [**?**].)

By Theorem **??** an arbitrary quadratic cycle γ can be connected to the quadratic cycle γ_1 which gives rise to the snowflake γ_1^*, for which a positive orientation is defined. Thus γ^*, however complicated it may be, receives an orientation by the homotopy of the correctly oriented γ_1 to γ. This is what will be meant by the *positive orientation* of γ^*.

In [**?**] it is shown that for a general smooth near-circle curve γ, under suitable parametrization the corresponding local coordinate parametrization $(\lambda_1^*(t), \lambda_2^*(t))$ of the star curve γ^* satisfies

$$\begin{aligned}\lambda_1^{*\prime}(t,\epsilon) &\sim (-\sin 2t + \epsilon h_1(t,\epsilon))\,\mathsf{T}_{\det}(t,\epsilon)\\ \lambda_2^{*\prime}(t,\epsilon) &\sim (\cos 2t + \epsilon h_2(t,\epsilon))\,\mathsf{T}_{\det}(t,\epsilon)\,,\end{aligned}$$

where $\epsilon > 0$ is a small parameter measuring closeness to the circle.

The following proposition is an immediate consequence.

PROPOSITION 20. *Let δ^* be the star curve of a near-circle cycle in the phase portrait of a smooth planar vector field, having K corners and K smooth arcs δ_k^*, and suppose that δ^* is simple. Let ρ_k be the limiting rotation angle at the kth corner, as defined above. If δ^* is positively oriented (in local coordinates), then:*

(1)$\sum_{k=1}^{K} \iota(\delta_k^*) = -2$
(2)$\rho_k = \pi$ *for all* k .

It is possible that the star curve of a quadratic cycle have non-empty intersection with the great circle $\mu_0 + \mu_2 = 0$ in S^2, so that in local coordinates, as the point A_t traces out γ in $\mathbb{R}^2$, the corresponding point $A_t^* \in \gamma^* \subset \Pi_0$ crosses the line at infinity, returns to the finite plane, and exits and returns again. Nevertheless, a well-defined index can still be assigned to the corresponding local coordinate representation $\gamma^* \subset \Pi_0$, which properly extends (**??**).

It is this index which is constructed and evaluated in the following proposition.

PROPOSITION 21. *Let γ be a quadratic cycle, oriented in such a way that the star curve γ^* is positively oriented. Then $\iota(\gamma^*) = 1 - I$, where I is the number of cusps of γ^* that lie in the hemisphere $U_- : \mu_0 + \mu_2 < 0$.*

Proof. For the (positively oriented) snowflake γ_1^* (from the proof of Theorem **??**), by (**??**) and Proposition **??** the index is $\iota(\gamma_1^*) = -2 + 6\pi/2\pi = +1$.

For a general quadratic cycle γ, properly connect γ to γ_1 (Theorem **??**). If $\gamma^* \subset S^2$ lies wholly within $U_+ : \mu_0 + \mu_2 > 0$, then $\iota(\gamma^*)$ varies continuously with the homotopy, hence has constant value $+1$.

If γ^* has a cusp lying in $\mu_0 + \mu_2 \leq 0$, it is easiest to think in terms of the homotopy of the star curves within Π_0 as a cusp C crosses the line at infinity, i.e., as the corresponding cusp in S^2 crosses the great circle $\mu_0 + \mu_2 = 0$. By a rotation we can arrange that the tangent to γ^* at the cusp C which crosses the line at infinity be horizontal. Just after the cusp has crossed infinity, there are two asymptotes ℓ and m, which are the coordinate representations of the great circles tangent to $\gamma^* \subset S^2$ at its crossings with $\mu_0 + \mu_2 = 0$, call them C_ℓ and C_m. There are points B and D on γ^* so that as $\gamma^* \subset \Pi_0$ is traced out with positive orientation, we first encounter B, at which the tangent line is horizontal, then C, then D, at which again the tangent line is horizontal. On the original $\gamma^* \subset S^2$ we encounter B, then C_ℓ, then C, then C_m, and finally D, but C_ℓ and C_m have no image in Π_0. The contribution to $\iota(\gamma^*)$ of the subarc of γ^* from D to B (which contains all the cusps but C) is the same as before C crossed infinity. The tangent vector to γ^* is turning counterclockwise between C_ℓ and C_m. Thus the contribution from B to C_ℓ, along which the tangent is still turning to the right, exactly cancels with the contribution from C_ℓ to C, and the contribution from C to C_m exactly cancels with that from C to D. Thus the contribution to the index from the smooth arcs and all cusps but C is unchanged. At C the tangent vector now jumps π radians clockwise, rather than counterclockwise, as was previously the case, so its contribution to the index drops by 1. ∎

A *loop* in the star curve γ^* occurs if there exist parameter values t_1 and t_2 for which $\gamma^*(t_1) = \gamma^*(t_2)$, and there is no cusp $\gamma^*(t)$ for $t_1 < t < t_2$. Two loops, on parameter intervals $[t_1, t_2]$ and $[t_3, t_4]$ respectively, are *non-overlapping* if $t_2 \leq t_3$ or if $t_4 \leq t_1$. It is apparent from the discussion in the proof of Proposition **??** that any loop must lie wholly within F.

PROPOSITION 22. *Let γ be a cycle of a quadratic system. Then the star curve corresponding to γ has at most three non-overlapping loops.*

Proof. Assuming that γ^* is positively oriented, each smooth arc makes a negative contribution to the index. (By "smooth arc" we mean an arc joining one cusp to the next, even if one of the cusps lies in U_-, so that in Π_0 the arc is actually composed of two components.) Since each loop must lie wholly within F, where the tangent vector to γ^* is turning clockwise, each loop that overlaps with no other loop contributes $\frac{1}{2\pi}(-\pi - \epsilon_1) < -\frac{1}{2} - \epsilon$ to the index, for some $\epsilon_1 > 0$, since the tangent vector must turn at least $\pi + \epsilon_1$ radians clockwise as the loop is traced out. Recall from the proof of Proposition **??** that each cusp in U_+ (respectively, in U_-) contributes $+\frac{1}{2}$ (respectively, $-\frac{1}{2}$) to the index. Thus when I cusps lie in U_-, if there were $N \geq 4$ non-overlapping loops, then by (**??**)

$$\iota(\gamma^*) = \sum_{k=1}^{6} \iota(\alpha_k) + (3 - I) < N(-\frac{1}{2} - \epsilon) + (3 - I) \leq (1 - I) - 4\epsilon\,,$$

contradicting Proposition **??**. ∎

Restrictions on the star curve of a quadratic cycle combine to give the following result.

THEOREM 23. *Let γ be a cycle of a quadratic system. Then the image of γ under nonsingular linear transformation has at most twelve vertices, counting multiplicity.*

Proof. By (**??**) we must show that for $B \in \mathrm{Int}(F)$, $W(\gamma^*, B) \leq 4$. Winding numbers are most readily computed using coordinates arising from stereographic projection, but in order to exploit Theorem **??**(5) so as to relate $W(\gamma^*, B)$ to the number of loops and cusps on γ^*, we must work in the local coordinates arising from central projection. Thus we first explain how $W(\gamma^*, B)$ can be computed in Π_0 in the coordinates of (**??**), even when γ^* has more than one component. We will continue to use the same notation for objects in S^2 and their images in Π_0 under central projection, but for clarity we will place tildes over points and rays in Π_0 arising from stereographic projection, and as before let γ^{**} denote the image of $\gamma^* \subset S^2$ under stereographic projection.

Fix the positive orientation of γ^*, fix $B \in \mathrm{Int}(F) \setminus \gamma^*$, and let $\tilde{B} := \psi \circ \phi_+^{-1}(B)$. Choose any ray $\tilde{\rho}$ in Π_0 with endpoint $\tilde{B}$, nowhere tangent to γ^{**}, and not passing through either a cusp on γ^{**} or a point of intersection of γ^* with the great circle $\mu_0 + \mu_2 = 0$. Let R denote the great circle which projects to the line containing $\tilde{\rho}$.

$W(\gamma^{**}, \tilde{B})$ is the difference, with the correct choice of sign, between the number of left-to-right and right-to-left crossings of $\tilde{\rho}$ by γ^{**} as $\tilde{\rho}$ is traced out, starting at $\tilde{B}$. By Proposition **??** there will be no crossing beyond the finite initial segment of $\tilde{\rho}$ which is the projection of the arc in R running from B to $\partial(-F)$.

Suppose that the opposite ray to $\tilde{\rho}$ intersects ∂F at $\tilde{A}$, and let $A = \phi_+ \psi^{-1}(\tilde{A})$. In the usual coordinates arising from central projection the initial segment of $\tilde{\rho}$ corresponds to the union ρ of disjoint rays ρ_+, with endpoint B, and ρ_-, with endpoint A, in the line $\overleftrightarrow{AB}$; the induced direction is from B to infinity in ρ_+ followed by infinity to A in ρ_-.

Under stereographic projection the two curves $\gamma^* \subset S^2$ and $-\gamma^* \subset S^2$ project to distinct curves, the curve γ^{**} and a curve that we will denote $-\gamma^{**}$. In the computation of $W(\gamma^{**}, \tilde{B})$ crossings of $\tilde{\rho}$ with $-\gamma^{**}$ are ignored. In central projection both γ^* and $-\gamma^*$ map to the single curve that we have also denoted γ^*. For the remainder of this proof it will be convenient to denote by γ_+^* (respectively, γ_-^*) the image under central projection of the branch of γ^* in S^2 having non-empty intersection with F (respectively, with $-F$). The direction of the turning of the tangent reveals which branch is the pre-image. In computing $W(\gamma^*, B)$ in Π_0 using central projection, crossings of γ_-^* with ρ_+ (respectively, of γ_+^* with ρ_-) are ignored. The remaining intersection points will be termed *relevant*.

Turning now to the proof of the theorem, for fixed $B \in \mathrm{Int}(F) \setminus \gamma^*$ choose a ray $\tilde{\rho}$ as described above, and let ρ_+ and ρ_- be the two corresponding rays under central projection, and ρ their union, directed as described earlier. By a rotation we may assume that line $\overleftrightarrow{AB}$ is horizontal and that A is to the right of B. We will write the rays as $\rho_+ = (-\infty, B]$ and $\rho_- = [A, \infty)$.

Tracing out ρ starting at B, at any intersection with γ^* at which γ^* is locally concave right (resp., concave left), i.e., locally like the graph of $x = y^2$ (resp., $x = -y^2$), the winding number increases (decreases) by one in crossing γ^*. By Proposition **??**, the winding number is zero at any point of ρ_- sufficiently close to A. Thus if N (resp., $N + S$) is the number of crossings at which γ^* is locally concave right (left), then N and $N + S$ are in $\mathbb{Z}^+ \cup \{0\}$ and $S = W(\gamma^*, B)$.

The remainder of the argument is a modification of that given for the similar theorem (Theorem 3.21) of [**?**]. Fix a relevant point $p \in \rho \cap \gamma^*$ at which γ^* is locally concave left. We claim: letting q denote the next point of intersection of γ^* with the line $\overleftrightarrow{AB}$ when proceeding along γ^* in either direction from p, either the arc pq contains a loop or a cusp, or q is a relevant point in ρ at which γ^* is locally concave right.

Given the truth of the claim, we have associated to each relevant point p of $\gamma^* \cap \rho$ at which γ^* is locally concave left a pair from among the set of loops, cusps, and relevant points at which γ^* is locally concave right. A loop or a cusp is associated to a unique such point p, but a point q may be associated with up to two such points p. Thus $2(S+N) \leq 2N + \#(\text{loops}) + \#(\text{cusps})$.

Applying Theorem **??** and Proposition **??** we obtain $W(\gamma^*, B) = S \leq 4$, as required.

To prove the claim, first suppose that p lies in $(-\infty, B]$, and proceed along γ^* from p. If the arc pq is connected, then the claim is immediate. See Figure 4 of [**?**] for an illustration. If proceeding from p the arc crosses the line at infinity before q is reached, then we may assume that the arc pq contains no loop, else there is nothing to prove. If the remote component of the arc pq intersects $\overleftrightarrow{AB}$ at all, by Proposition **??** it does so at a point of $[A, \infty)$. If there is no such intersection point, then the remote component contains the cusp that has crossed the line at infinity. If the remote component intersects $[A, \infty)$, the intersection point is q. The reversal of the direction of turning of the tangents in γ^*_- implies that γ^* is locally concave right at q, as required.

Now suppose that p lies in $[A, \infty)$. If in moving along γ^* from p the arc pq is connected, it lies wholly in γ^*_-. If it does not contain a cusp (it cannot contain a loop), q lies to the left of p and γ^* is locally concave right at q. By Proposition **??** again q cannot lie to the left of A, so $q \in \rho_- \subset \rho$.

If the arc pq is not connected, and does not contain the unique cusp C on the component of γ^*_- containing p, then the arc obtained in proceeding along γ^* from p in the opposite direction does contain C, and need not be considered further. Let O, I, and W be as in Proposition **??**. We may assume without loss of generality that the arc under consideration is obtained by travelling downward from p along γ^*, and that the first intersection with ∂F is O rather than I. There is no loop or cusp on the subarc of γ^* in question between p and O.

By Proposition **??**(1) the subarc of γ^* near O lies on the opposite side of $\overleftrightarrow{OW}$ from p. If $\overleftrightarrow{OW}$ is either horizontal, or crosses $\overleftrightarrow{AB}$ to the right of p, then $(-\infty, A]$ lies on the same side of $\overleftrightarrow{OW}$ as p. The portion of pO in γ_+^* lies on the opposite side of $\overleftrightarrow{OW}$ from p, hence the subarc pO of γ^* cannot intersect $(-\infty, B]$. Since γ^* is tangent to $\overleftrightarrow{OW}$ at O and lies to one side of it, there must be a loop or a cusp in the arc Oq in order for γ^* to cross $\overleftrightarrow{OW}$ and intersect $(-\infty, A]$ at q.

If $\overleftrightarrow{OW}$ crosses $\overleftrightarrow{AB}$ to the left of p, then the point K of intersection lies outside F and to the left of B. For by Proposition **??** $\overleftrightarrow{AB}$ must be obtainable (before the rotation that makes it horizontal) by rotating $\overleftrightarrow{OC}$ at C in the direction that gives an inclination equal to that of some ray in the angle $\angle WOC$. K is outside F, and the ordering along $\overleftrightarrow{AB}$ is $K{-}B{-}A{-}C$; parallel translation (to produce p) yields the order $K - B - A - p$; thus rotating $\overleftrightarrow{AB}$ to be horizontal puts K to the left of B. Tracing γ^* from p, when the line at infinity is crossed, γ_+^* lies on the opposite side of $\overleftrightarrow{OW}$ from p hence from B, since K is to the left of B. If the subarc pO of γ^* intersects $\overleftrightarrow{AB}$, q lies outside F, hence in $(-\infty, B]$, so that it is relevant (and γ^* is locally concave right there). If it does not, then either Oq lies in F, and since q is on the same side of $\overleftrightarrow{OW}$ as B, there must be a loop or a cusp in Oq in order to cross $\overleftrightarrow{OW}$, or Oq does not lie in F, in which case it contains a cusp (by the proof of Theorem **??**). ▮

We note that when there exists a point $z \in F$ for which $W(\gamma^*, z) = 0$, then the proof of Theorem 3.21 in [**?**] can be copied practically verbatim to establish the conclusion. (We believe that such a point exists; see Conjecture **??** below.) When no such point z exists, somewhat more involved arguments show that the constraints on γ^* arising from Theorems **??** and **??** imply that there are only three essentially different configurations possible, so an independent proof of Theorem **??** can be obtained by deriving and inspecting these configurations.

A bound on the number of vertices of a quadratic cycle is now immediate.

THEOREM 24. *Any non-circular cycle in a quadratic system has at most twelve vertices.*

Proof. Given a cycle γ of a quadratic system, we have just established that for any nonsingular real linear transformation of the plane, the image curve $\bar{\gamma}$ of γ under the linear transformation has at most twelve vertices. Since the transformation could have been the identity, γ itself had at most twelve vertices. ∎

To contrast the estimate which can be made using specific properties of quadratics with what can be done (or at least, has been done so far) in the general case, we present the following more general theorem.

THEOREM 25. *Let X be a polynomial vector field of degree n, and γ a cycle appearing in the phase plane of X. Then γ has at most $2(6n-2)(7n-3)$ vertices, and at most $2(21n-9)(22n-10)$ affine vertices.*

Proof. Let $X = (P, Q)$, and let γ be a cycle of X. A vertex of γ corresponds to a crossing of γ and the algebraic curve $\kappa_1(x, y) = 0$, so to bound the number of crossings we bound the number of contacts of X with $\kappa_1(x, y) = 0$.

Such a contact corresponds to an intersection of the algebraic curves $\kappa_1 = 0$ and $\kappa_{1x}P + \kappa_{1y}Q = 0$. Since the former has degree $6n - 2$ and the latter has degree $7n - 3$, by Bézout's Theorem ([**?**]) there are at most $(6n - 2)(7n - 3)$ such intersection points.

The worst conceivable case is that in which each intersection point lies on a distinct branch of the algebraic curve $\kappa_1 = 0$. Then each of the two topological rays of the branch as determined by the contact with X is an arc without contact for X, hence γ can intersect each ray at most once. Since the vertices lie in $\kappa_1 = 0$, there can be therefore at most $2(6n - 2)(7n - 3)$ vertices on γ.

The same argument applies in the case of affine vertices, but with $\kappa_1(x, y)$ replaced by $\mathsf{T}(x, y)$. In fact we may use $\mathsf{T}_{\det}$ from the proof of Lemma **??**; by formulas (**??**) it has degree $21n - 9$, which leads to the stated bound on the number of affine vertices. ∎

5. EXAMPLES AND CONJECTURES

Computational expressions for the affine and euclidean curvature of a plane curve are notoriously complicated. As noted before, applying (**??**) to the expression (**??**) yields a tenth order polynomial function in x and y. Typically period annuli about centers in Hamiltonian systems are easiest to analyze.

Suppose that the saddle point on a polycycle of a Hamiltonian system is hyperbolic. Linearization of the saddle suggests that cycles in the period annulus that are sufficiently close to the polycycle will be hyperbolically curved near the saddle point, and this is indeed the case. Since there exist quadratic Hamiltonian systems which have polycycles with one, two, and three saddle points, all hyperbolic, we obtain examples of quadratic cycles having exactly one, two, or three hyperbolically curved subarcs, as allowed by Theorem **??**. The specific example $\dot{x} = -x(x+2y-1)$, $\dot{y} = y(2x+y-1)$, which has an invariant triangle bounded by the two coordinate axes and the line with equation $x + y = 1$, is instructive. Small cycles about the center have four vertices, but those near the triangle have six.

When the curve $\kappa_1 = 0$ is plotted inside the period annuli of concrete examples, we always find either two or three branches (sometimes crossing at points other than the center, as in the example just cited), so that cycles in the period annuli have either four or six vertices.

Although there exists a near-circle closed curve γ in the plane, with nowhere vanishing curvature, for which γ^* has as many as three loops, for every quadratic cycle γ for which the star curve has been computed, γ^* has turned out to be essentially the snowflake: always a simple closed curve, up to three of whose six cusps can be located outside the region F, but such that $\mathrm{Int}(\gamma^*)$ does not completely cover F. Combined with the first parts of Propositions **??** and **??** and Remark **??**, this suggests the following conjecture.

Conjecture 26. A cycle γ of a quadratic system has at most six vertices, counting multiplicity.

In terms of the role that star curves play for the effect of linear transformations of the plane, this also suggests:

Conjecture 27. Given cycle γ of a quadratic system, there is an open set of nonsingular linear transformations of the plane changing the number of vertices from four to six if γ does not lie in a conic section, and from six to four in any case.

We remark that it is a consequence of a theorem in [**?**] that if a quadratic cycle does not lie in a circle or an ellipse, then there is an open set of nonsingular linear transformations such that for μ in this set, $\mu\langle\gamma\rangle$ has at least six simple vertices.

The star curve construction shows that Conjectures **??** and **??** are implied by the following conjecture.

Conjecture 28. A cycle γ of a quadratic system has at most one multiple vertex (i.e., point at which $\dot{\kappa} = \ddot{\kappa} = 0$).

We close with the following observation. If γ is *cubic oval*, that is, an oval lying in the algebraic set $H(x,y) = 0$ for some cubic polynomial function $H(x,y)$, then γ is a cycle for the quadratic system $X = (-H_y, H_x)$. Thus the geometric statements in Proposition **??** and Theorems **??**, **??**, and **??** are also statements about cubic ovals.

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