QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **3**, 181–?? (2002)
ARTICLE NO. 36

Balanced Coordinates for Spiraling Dynamics

Fernando Sanz

Departamento de Álgebra, Geometría y Topología. Facultad de Ciencias. Prado de la Magdalena, s/n. Universidad de Valladolid. E-47005, Valladolid. SPAIN
E-mail: fsanz@agt.uva.es

We study some aspects of the dynamics of an analytic vector field X in a neighbourhood of an invariant non-singular curve Γ in $\mathbb{R}^3$. Namely, the spiraling behaviour: any trajectory of X *spirals* asymptotically around Γ. This is measured by means of the angle, θ, and the distance, r, functions of the trajectories with respect to cylindrical coordinates around Γ. Coordinates are called balanced if these functions are monotone, which is not an intrinsic property. Balanced coordinates always exist in the case of *elementary singularity* (non-nilpotent linear part) and we show, in the general case when Γ is not contained in the singular locus of X, the existence of coordinates for which the angle is monotone. These are obtained as *maximal contact* coordinates for the reduction of the singularity. The results can be viewed as generalizations of the corresponding results in dimension two which we study first as a motivation.

Key Words: vector field, singularities, trajectory, oscillation, spiraling, reduction of singularities, maximal contact

1. INTRODUCTION

Let X be an analytic vector field on a manifold M and let γ be a trajectory of X defined in a maximal interval $[0, b) \subset \mathbb{R}$. In general, γ can have complicated asymptotic behaviour: accumulation to every point in M, chaotic dynamics, etc. Only in particular situations we have a satisfactory description; the classical case is when M is two-dimensional and has genus zero, by Poincaré-Bendixon's Theorem.

We concentrate our attention to the simplest case from the dynamic point of view: γ has a single ω-limit point

$$\omega(\gamma) = \lim_{t\to b} \gamma(t) = p \in M.$$

In this situation, $b = \infty$ and p is an equilibrium point. This is the case for the important class of analytic gradient vector fields, for which trajectories have finite length and accumulate to single points [**?**]. We want to study geometric asymptotic properties of γ while it approaches p. For instance, one important property is the existence of a *well defined tangent* at p, which is also true for gradients [**?**]. In this paper, we investigate some aspects of another property which is the *spiraling* behaviour, in case $n = \dim M$ is 2 and, mostly, 3. Since our study is local, we can put $M = \mathbb{R}^n$ and $p = \mathbf{0}$ using an analytic chart.

In dimension two, to say that γ spirals around a point means that the angle $\theta(t)$ of points $\gamma(t)$ of the trajectory diverges to $+\infty$ or $-\infty$. This angle, obtained from polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ is not known explicitly and depends on the coordinates (x, y). The aim of this work is to find systems of coordinates that provide monotonicity of $\theta(t)$ as a function of t. If this is the case, then $\theta = \theta(t)$ can be consider as a parameter and the single function $\theta \mapsto r(\theta)$ determines γ. Theorem **??** solves this question for all trajectories spiraling around a singular point of order one.

In section 3, we investigate the same question in dimension three. First, we want to give a meaning to the phrase "γ is a spiraling trajectory". A priori, it makes no sense to say that γ spirals around a point, as in dimension two. What we have in mind is the *axial spiraling*: roughly speaking, γ makes spirals around an analytic curve Γ through $\mathbf{0}$. This is the subject of the work [**?**] in a general setting. In this paper, we only consider the case where Γ is a regular curve, for which the notion of axial spiraling is more "visual" and represents a good generalization of spiraling in dimension two.

More precisely, let Γ be an analytic non-singular curve given in some coordinates (x, y, z) at $\mathbf{0} \in \mathbb{R}^3$ by $\Gamma = \{x = y = 0\}$ and write $\gamma(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t), z(t))$ in associated cylindric coordinates. Then we say that γ *spirals around* Γ or that Γ is a *spiraling axis* for γ if the angle $\theta(t)$ diverges to $+\infty$ or $-\infty$ while $z(t)$ and $r(t)$ go to zero as $t \to \infty$. An additional condition also considered is that $r(t)$ goes to zero faster than any power of $z(t)$. This means that γ and Γ are "infinitely close" and so γ can have a single spiraling axis. It implies also the property of *existence of iterated tangents* for γ, which we recall in paragraph 3.1, and that Γ is invariant for X. We suppose that Γ is not contained in the singular locus of X (*non-degenerated*). In this case there is an open domain composed of trajectories spiraling around Γ ([**?**]). This situation has already been studied in [**?**] for $\mathcal{C}^\infty$ vector fields, where non-degenerated means not infinitely flat along Γ.

Functions $r(t), z(t)$ and $\theta(t)$ above determine the trajectory γ and it is natural to ask for coordinates in which they have good properties. For example, if they are monotone functions of t then coordinates (x, y, z) are called *balanced*. If only $\theta(t)$ is monotone, we call them *monotone spiraling coordinates*. In paragraph 3.3, we propose some examples for which balanced coordinates are not always possible. A general result, proved in [**?**], is Theorem **??**, which asserts that balanced coordinates exist at $\mathbf{0}$ if it is an elementary singularity, that is, the linear part of the vector field is not nilpotent.

The main result of this work is Theorem **??**, which generalizes the corresponding result in dimension two. It asserts that, even if the singularity is not elementary, there always exists a system of monotone spiraling coordinates for a non-degenerated spiraling axis Γ. The proof of this theorem reduces to the precedent one after *Reduction of Singularities* of X along Γ, stated in Theorem **??**, by means of blowing-ups of points and of curves and some ramifications. In this reduction process, we need to choose coordinates at each step in a controlled way, in order to assure that final monotone spiraling coordinates are obtained from a suitable initial system. This control is given by a *Maximal Contact Surface*, as in [**?**], which is the subject of paragraph 3.6.3. The existence of maximal contact, stated in Theorem **??**, is important by itself.

The application of Theorem **??** to Theorem **??** is not automatic after reducing the singularity to an elementary one; we need the technical version of Proposition **??**. The proof of this proposition uses the *Newton Polygon* of a vector field (see [**?**]) and needs basically the whole proof of the most important part of Theorem **??**. For this reason and, in order to have a self-contained work, we include paragraph 3.5 with the proof of Theorem **??**. It contains also a useful condition for an invariant regular curve, at an elementary singular point, in order to be an spiraling axis. This condition reduce the spiraling behaviour in dimension three to the spiraling in the two-dimensional divisor after the blowing-up of the axis. In summary, the whole section 3 can be viewed by itself as an independent study of axial spiraling in dimension three, with emphasis to balanced coordinates.

2. SPIRALING TRAJECTORIES IN DIMENSION TWO

In this section X is a real analytic vector field defined in a neighbourhood V of the origin of $\mathbf{0} \in \mathbb{R}^2$, which we suppose to be a singular point; that is, $X(\mathbf{0}) = 0$. Our study depends only in the germ of X at $\mathbf{0}$, so we do not mind about the size of V. Also, by means of analytic coordinates, we can treat the case where X is a vector field in a real analytic manifold. We summarize first some classical well known results [**?**, **?**], following [**?**], where elementary proofs can be found.

Let $\gamma : \mathbb{R}_{\geq 0} \to V$ be an asymptotically stable trajectory of X; that is, such that it has a single ω-limit point

$$\omega(\gamma) = \lim_{t\to\infty} \gamma(t) = \mathbf{0},$$

(or, by reversing the time, an asymptotically unstable trajectory). Denote by $\mid \gamma \mid = \{\gamma(t) \,/\, t \geq 0\}$ the (germ of the) trace of γ. There are only two possible asymptotic behaviour for γ:

(a) It has a *well defined tangent* at $\mathbf{0}$: there exists the limit of secants

$$\lim_{t\to\infty} \frac{\gamma(t)}{\parallel \gamma(t) \parallel} \in \mathbb{S}^1 = \{w \in \mathbb{R}^2 \,/\, \parallel w \parallel = 1\},$$

where $\parallel \cdot \parallel$ is an euclidean norm in $\mathbb{R}^2$. We call γ a *characteristic trajectory*.

(b) It has no tangent and then it *spirals* around the origin. This means that γ cuts transversally any analytic semi-curve T at $\mathbf{0}$ infinitely many times and, eventually for time t big enough, γ crosses T always in the same sense, from one to another of the two sides locally determined by T. We call γ a *spiraling trajectory*.

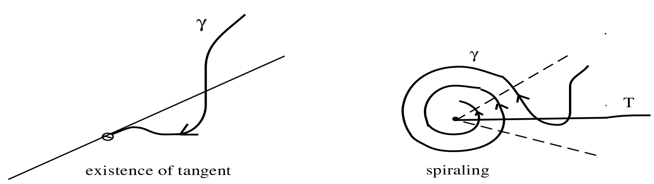

Figure 1

In the former case, γ has a good finiteness property: for any analytic curve T at the origin, either $|\gamma|$ is contained in T or γ cuts T only finitely many times. A trajectory with this property is called *non-oscillatory*.

On the other hand, if γ is a spiraling trajectory then it oscillates with respect to any analytic curve at $\mathbf{0}$. Moreover, the existence of a spiraling trajectory determines the local dynamics of X: the origin is an attracting point and any trajectory in a neighourhood accumulates to $\mathbf{0}$ an it is a spiraling trajectory. The singular point $\mathbf{0}$ is then called a *monodromic focus*

of X. Proof of this fact follows from the existence of a first returning map $P : T \to T$ over an analytic semi-curve T using Poincaré and Bendixon's arguments (see [**?**]). Alternatively, we can use *Reduction of Singularities* in dimension two [**?**, **?**] that means the following: there exists an analytic map $\pi : M \to V \subset \mathbb{R}^2$, which is a diffeomorphism outside the total divisor $D = \pi^{-1}(\mathbf{0})$ onto $V \setminus \{\mathbf{0}\}$, and there exists an analytic vector field $\tilde{X}$ over M with properties:
(1) Singularities of $\tilde{X}$ are isolated and the linear part of $\tilde{X}$ at each of them is non nilpotent (*elementary* singularities).
(2) The push-forward vector field $\pi_*(\tilde{X} \mid_{M\setminus D})$ is orbitally equivalent to $X \mid_{V\setminus\{\mathbf{0}\}}$ (they have the same trajectories up to parameterization).
The vector field $\tilde{X}$ is called the strict transform of X. The map π is obtained as a composition of blowing-ups.

If we know already that there is a trajectory γ of X that accumulates to the origin, then we can determine if it is or not a monodromic focus of X by means of a resolution of singularities π. More precisely, γ spirals around $\mathbf{0}$ if and only if D is invariant by $\tilde{X}$, the singularities of $\tilde{X}$ are contained in D and they are either hyperbolic saddle points with both the stable and the unstable manifold also contained in D or saddle-nods with the same topological saddle behaviour.

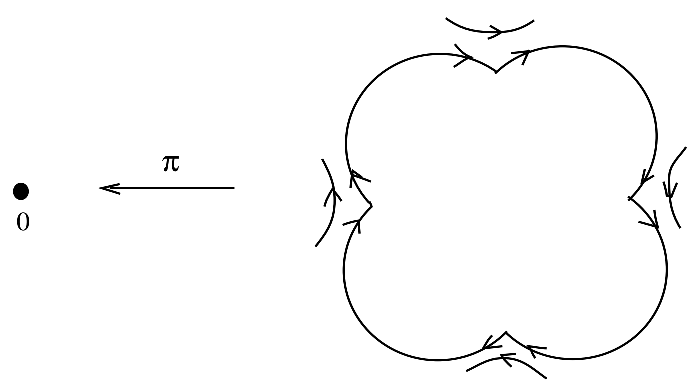

Figure 2

Unfortunately, to know if such a γ exists is not an easy question. Its answer contains in particular the solubility of the *center-focus* problem. For instance, as a consequence of Dulac's theorem [**?**, **?**], we can say that if there are no asymptotically stable or unstable trajectories then $\mathbf{0}$ is a center for X. In this case, the configuration of the strict transform $\tilde{X}$ in a resolution of singularities will be also as in the monodromic focus case.

2.1. Monotone spiraling coordinates

Take some analytic coordinates (x, y) in V and write in these coordinates $t \mapsto \gamma(t) = (x(t), y(t))$. Consider the polar coordinates

$$\begin{aligned} x(t) &= r(t) \cos \theta(t) \\ y(t) &= r(t) \sin \theta(t) \end{aligned}$$

where $r(t) = (x(t)^2 + y(t)^2)^{1/2} > 0$ is the modulus of the vector $\gamma(t) \in \mathbb{R}^2$ and $\theta(t) \in \mathbb{R}$ is its angle in a universal covering $\mathbb{R} \to \mathbb{S}^1$. Possibilities (a) and (b) above for γ can be red as

(a) γ is a characteristic orbit if and only if $\lim_{t\to\infty} r(t) = 0$ and $\lim_{t\to\infty} \theta(t) \in \mathbb{R}$ exists.

(b) γ is a spiraling trajectory if and only if $\lim_{t\to\infty} r(t) = 0$ and $\lim_{t\to\infty} \theta(t)$ is either $+\infty$ or $-\infty$.

Thus, these asymptotics of the modulus $t \mapsto r(t)$ and the angle function $t \mapsto \theta(t)$ are independent of coordinates (x, y).

Suppose, for the rest of this paragraph, that γ is a spiraling trajectory. The fact that the angle diverges does not imply that this function is monotone.

Example 1. *Let X be the vector field written in coordinates (x, y) in $\mathbb{R}^2$:*

$$X = (y - x^3)(y - 2x^3)T - (x^2 + y^2)^2 R$$

where $T = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ are respectively the tangential and the radial vector field in the given coordinates. The origin $\mathbf{0} \in \mathbb{R}^2$ is the only singular point and, at other points, the coefficient of R is negative. This means that X is transversal to any circle centered at $\mathbf{0}$ and X "points" to the interior of such circles. Thus, any trajectory γ of X is defined for all $t \geq 0$ and satisfies $\omega(\gamma) = \mathbf{0}$. Moreover, after reduction of singularities, it can be seen that all singular points are hyperbolic saddles whose stable and unstable manifold are contained in the divisor. Then the origin is a monodromic focus of X. Given such a trajectory

$$\gamma : t \mapsto \gamma(t) = (r(t) \cos \theta(t), r(t) \sin \theta(t))$$

in polar coordinates we see that $\theta(t)$ is not monotone. In fact, values t_0 for which the derivative $\frac{d\theta}{dt}(t_0)$ vanishes are geometrically determined by the condition that $X(\gamma(t_0))$ and $R(\gamma(t_0))$ are collinear. The set of points where X and R are collinear vectors is given by the union of the cubics $\{y - x^3 = 0\} \cup \{y - 2x^3 = 0\}$ and this set is intersected infinitely many times by γ, a spiraling trajectory. The angle of γ increases outside these cubics but it decreases inside them (it goes "back" in this region, see Figure 3).

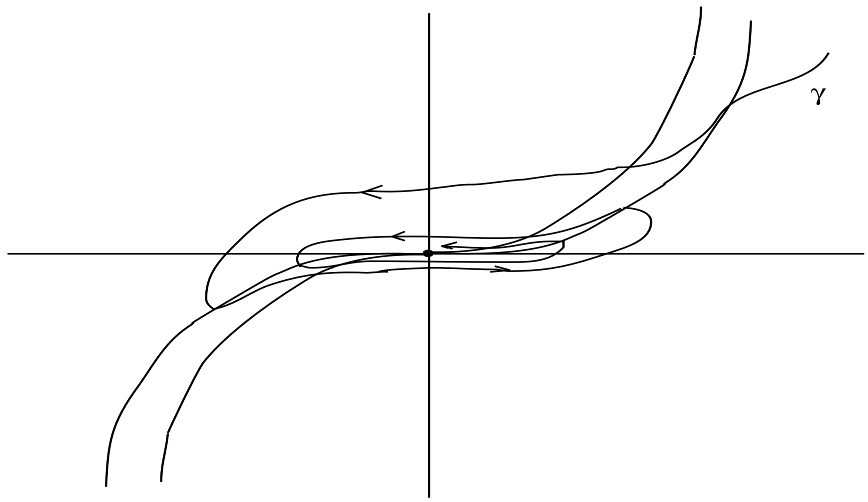

Figure 3

We remark in this example that the lack of monotonicity of $\theta(t)$ does not depend on γ. It is a consequence of the existence of a curve of tangencies between X and the radial vector field R; this last one depending on the coordinates. We summarize this general fact in the following proposition that provides a definition.

Proposition 2. *Let X be an analytic vector field defined in a neighbourhood of $\mathbf{0} \in \mathbb{R}^2$ and suppose that $\mathbf{0}$ is a monodromic focus of X. Given a system of analytic coordinates (x, y) at the origin, we denote by $r = (x^2 + y^2)^{1/2}$, $\theta = \arctan y/x$ the associated polar coordinates. We have exactly one of the following possibilities:*

(i) The angle $\theta(t)$ of any trajectory γ of X accumulating to the origin is ultimately monotone.

(ii) The vector field X is tangent to the radial vector field $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ along an analytic curve (of dimension 1) and, in this case, the angle $\theta(t)$ is not ultimately monotone for any such trajectory γ.

Definition 3. We say that (x, y) is a *monotone spiraling* system of coordinates for X if and only if case (i) in the proposition above is satisfied.

Equivalently, (x, y) are monotone spiraling coordinates if there exists a neighbourhood V of the origin such that X is transversal to $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ in $V \setminus \{\mathbf{0}\}$. In other terms, if we consider the analytic 1-form $\omega = \omega_{(x,y)} = -ydx + xdy$, the function $\omega(X)$ does not change sign in $V \setminus \{\mathbf{0}\}$. In a more geometric way, X is transversal in $V \setminus \{\mathbf{0}\}$ to the foliation $\mathcal{F}_\theta = \{\omega = 0\}$ whose leaves are the punctured straight lines through the origin.

Remark 4. Given coordinate systems (x, y) and (x', y') at the origin related by a linear change of coordinates, the radial vector fields $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $R' = x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'}$ are the same. Hence, the concept of monotone spiraling system is preserved by linear changes of coordinates.

The same concept of monotone spiraling coordinates can be applied for a vector field with a center at $\mathbf{0} \in \mathbb{R}^2$. We have the same proposition **??**, this time for closed trajectories γ surrounding the origin. We propose the following example of a hamiltonian center written in non-monotone coordinates.

EXAMPLE 5. *Consider coordinates (x, y) in $\mathbb{R}^2$ and the polynomial function*

$$H = (y + 3x^2)^2 + x^4$$

and let X be the hamiltonian (nilpotent) vector field $X = -\frac{\partial H}{\partial y}\frac{\partial}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial}{\partial y}$. Level curves of H in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, which are traces of trajectories of X, are closed and smooth. Hence, X has a global center at the origin. All these trajectories are tangent to the radial vector field $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ at points of the analytic curve

$$\{\omega(X) = x\frac{\partial H}{\partial x} + y\frac{\partial H}{\partial y} = 0\} = \{y + 4x^2 = 0\} \cup \{y + 5x^2 = 0\}.$$

On the other hand, if we consider new coordinates

$$\begin{aligned} x' &= x \\ y' &= y + 3x^2 \end{aligned}$$

then level curves of H are given by quartics $\{{y'}^2 + {x'}^4 = cst\}$. They are transversal to the radial vector field $R' = x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'}$ and then (x', y') is a monotone spiraling system of coordinates for X.

We prove the existence of a monotone spiraling system of coordinates in general for a vector field whose singularity has order one.

THEOREM 6. *Let X be an analytic vector field in a neighbourhood of $\mathbf{0} \in \mathbb{R}^2$, which is a monodromic focus or a center of X. Suppose that the linear part $DX(\mathbf{0})$ is not identically zero. Then there exists a monotone spiraling system of coordinates for X.*

Proof. Suppose first that $\mathbf{0}$ is an elementary singularity, that is, the linear part $DX(\mathbf{0})$ is not nilpotent. Then it is a center or a monodromic focus if and only if $DX(\mathbf{0})$ has complex conjugate eigenvalues with non-zero imaginary part. Take analytic coordinates (x, y) such that the linear part is written in canonical form as

$$DX(\mathbf{0}) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

with $b \neq 0$. The function $(-ydx + xdy)(X) = b(x^2 + y^2) + \cdots$ has the same sign as b and does not vanish in a punctured neighbourhood of the origin. Thus (x, y) is a monotone spiraling system of coordinates. In fact, any analytic system of coordinates at $\mathbf{0}$ is a monotone spiraling one.

Suppose now that the linear part $DX(\mathbf{0})$ is nilpotent but not zero. In some coordinates (x, y) we can write

$$X = -y\frac{\partial}{\partial x} + \cdots$$

where $\cdots$ means terms of order ≥ 2. Takens' formal normal form [**?**] of such a vector field is

$$\hat{X} = -(y + \sum_{k=m}^{\infty} a_k x^k)\frac{\partial}{\partial x} + \varepsilon x^{l-1}\frac{\partial}{\partial y},$$

with $l \geq 3$, $\varepsilon \neq 0$ (or $l = +\infty$, in which case $\varepsilon = 0$) and $m \geq 2$ the first integer for which $a_m \neq 0$ or $m = +\infty$. This form is obtained after a formal change of coordinates (see [**?**]). Nevertheless, the used method permits to show that, for any $q \geq 1$, there are analytic coordinates (x, y) for which the first q-jet of the expression of X coincides with the first q-jet of $\hat{X}$. In [**?**] it is proved that X has a center or a monodromic focus at the origin if and only if one of the following conditions for the normal form $\hat{X}$ is satisfied:

(1) $l = 2n < \infty$, $\varepsilon > 0$ and $m > n$, or

(2) $l = 2n < \infty$, $\varepsilon > 0$, $m = n$ and $a_m^2 < 4\varepsilon$.

(See also [**?**] about conditions that distinguish between center and focus). Conditions (1) or (2) only concern with a finite jet $j^q(\hat{X})$ of the normal form. In other words, having a center or a monodromic focus is a finitely determined problem for nilpotent vector fields. Consider some analytic coordinates (x, y) at $\mathbf{0}$ such that the q-jet of X in these coordinates coincide with $j^q(\hat{X})$. Let's proof that (x, y) is a monotone spiraling system of coordinates for X. Write

$$X = -(y + a_m x^m + \cdots)\frac{\partial}{\partial x} + (\varepsilon x^{l-1} + \cdots)\frac{\partial}{\partial y},$$

where condition (1) or (2) holds and $\cdots$ denotes terms with higher order than the explicit order in the expression. Consider $\omega = -ydx + xdy$ and write

$$\omega(X) = y^2(1 + A(x,y)) + yB(x) + C(x).$$

By condition (1) or (2) we have $C(x) = x^l(\varepsilon + O(x))$ and $m' = ord(B) \geq m$. Furthermore, if $m \leq l$ (for instance in case of condition (2)), then $m = m'$ and $B(x) = x^m(a_m + O(x))$. In any case, either $B \equiv 0$ or we write $B(x) = x^{m'}(a' + O(x))$, where $m' < \infty$ and $a' = a_m$ if $m' = m$.
Consider *quasi-homogeneous* coordinates $x = r\bar{x}$, $y = r^n\bar{y}$ for $(\bar{x}, \bar{y}) \in \mathbb{S}^1$ and write

$$\begin{aligned} \omega(X)(x,y) &= r^l(\bar{y}^2 + a' r^{m'-n}\bar{x}^{m'}\bar{y} + \varepsilon\bar{x}^l + O(r)) \\ &= r^l(\phi(\bar{x},\bar{y}) + O(r)), \end{aligned}$$

where $\phi \in \mathbb{R}[\bar{x}][\bar{y}]$ is a quadratic polynomial and $O(r)$ is analytic in $(r, \bar{x}, \bar{y})$ and vanishes for $r = 0$. The discriminant Δ of ϕ is given by

$$\Delta = \begin{cases} -4\varepsilon\bar{x}^l & \text{in case (1)} \\ (a_m^2 - 4\varepsilon)\bar{x}^l & \text{in case (2)}\ . \end{cases}$$

In both cases, ϕ does never vanish as a function on $\mathbb{S}^1$ and then, there exists a constant $C > 0$ such that $|\ \phi(\bar{x},\bar{y})\ | \geq C$, $\forall (\bar{x},\bar{y}) \in \mathbb{S}^1$. This implies that $\omega(X)$ has a constant sign in a punctured neighbourhood of the origin of the form $\{(x,y) = (r\bar{x}, r^n\bar{y})\ /\ 0 < r < \delta\}$ for δ sufficiently small. This ends the proof. ∎

The vector field in example **??** has order three at the origin and the theorem above does not apply. Nevertheless, the reader can check that new coordinates (x', y') with $x' = x,\ y' = y - x^3$ form a monotone spiraling system of coordinates.

2.2. Monotonicity of the modulus

Most of the time in this work we are interested in monotonicity of the angle $\theta(t)$ of trajectories γ for some polar coordinates. However, we can also look for monotonicity of the modulus function $r(t)$. This is equivalent for γ to be transversal to circles $x^2+y^2 = cst$ in coordinates $x = r\cos\theta, y = r\sin\theta$.

Monotonicity of $r(t)$ is not possible for a center since any closed analytic curve around the origin must be tangent to the circles.

Proposition **??** is also true for the modulus $r(t)$ with the corresponding slight modifications. In particular, $r(t)$ is monotone if the vector field itself is transversal to the foliation $\mathcal{F}_r = \{x^2 + y^2 = cst\} = \{xdx + ydy = 0\}$, independently of γ. But Theorem **??** has no counterpart:

EXAMPLE 7. *Let X be a vector field in a neighbourhood of $\mathbf{0} \in \mathbb{R}^2$ written in some coordinates (x, y) as*

$$X = (y + f_1(x, y))\frac{\partial}{\partial x} + f_2(x, y)\frac{\partial}{\partial y},$$

with f_1, f_2 analytic functions of order ≥ 2. Suppose that $\mathbf{0}$ is a monodromic focus. Put, in cylindric coordinates

$$(xdx + ydy)(X) = xy + xf_1 + yf_2 = r^2(\frac{\sin 2\theta}{2} + O(r)),$$

where $O(r)$ vanishes at $\mathbf{0}$. We see that this function changes sign in $V \setminus \{\mathbf{0}\}$ for any neighbourhood V of the origin. Thus, X is not transversal to $\mathcal{F}_r$ in $V \setminus \{\mathbf{0}\}$. The reader can check that the same thing happens for any system of coordinates.

On the other hand, we remark that transversality with $\mathcal{F}_r$ implies some topological property of the dynamics of X without assuming a priori that the origin is or not a monodromic focus: all trajectories in a neighbourhood accumulate to the origin and X is locally topologically equivalent to the radial vector field $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. A somewhat analogous result for the angle is false:

EXAMPLE 8. *Consider in $\mathbb{R}^2$ the vector field*

$$X = (x^2 + y^2)(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) - (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}).$$

All trajectories of X accumulates to the origin and they are transversal to the lines $\{y/x = cst\}$; but the origin is not a center or a monodromic focus.

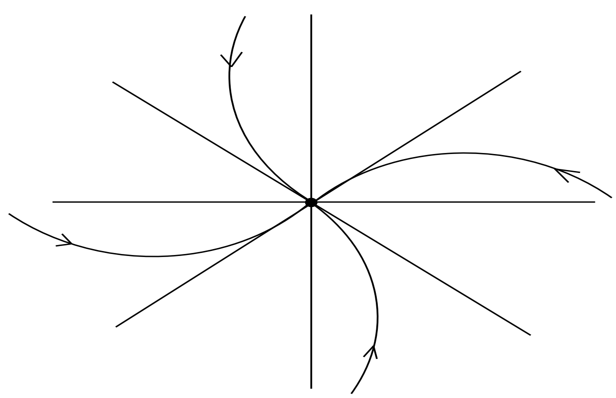

Figure 4

3. AXIAL SPIRALING IN DIMENSION THREE

In this section, X is an analytic vector field in a neighbourhood of $\mathbf{0} \in \mathbb{R}^3$ and $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ is an integral curve of X such that $\gamma(t) \neq \mathbf{0}, \forall t \geq 0$ and

$$\omega(\gamma) = \lim_{t \to \infty} \gamma(t) = \mathbf{0}.$$

3.1. Blowing-up and iterated tangents

As in dimension two (and in general in any dimension), the existence of well defined tangent for γ at the origin is defined as the existence of the limit of secants

$$\lim_{t \to \infty} \frac{\gamma(t)}{\| \gamma(t) \|} \in \mathbb{S}^2.$$

The language of blowing-ups is well adapted to this definition. Recall that the *blowing-up* of $\mathbb{R}^3$ at the origin is an onto analytic proper map $\pi : M \to \mathbb{R}^3$, M a 3-dimensional analytic manifold, such that its restriction to $M \setminus \pi^{-1}(\mathbf{0})$ is an isomorphism onto $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ and the fiber $D = \pi^{-1}(\mathbf{0}) \simeq \mathbb{P}^2$ is identified with the set of directions of $\mathbb{R}^3$ at $\mathbf{0}$. This fiber D is called the *exceptional divisor* of π. Let $\gamma_1 = \pi^{-1} \circ \gamma : \mathbb{R}_{\geq 0} \to M$ be the lifted parameterized curve of γ in M. It is a trajectory of a vector field X_1 in M called the *(total) transform* of X, which is uniquely characterized by the property $\pi_* X_1 = X$. Then, naturally, γ has a well defined tangent at the origin if and only if γ_1 accumulates to a single point

$$\omega(\gamma_1) = \lim_{t \to \infty} \gamma_1(t) = p_1 \in D.$$

We also say that p_1 is the *tangent* of γ at $\mathbf{0}$.

The blowing-up construction is local. It can be made in an analytic manifold at any point considering this point as the origin of an euclidean space by means of a chart. In particular, once γ has p_1 as a tangent, we can ask if its lifted trajectory γ_1 has or not a tangent at p_1, and so on.

Definition 9. Put $M_0 = \mathbb{R}^3$, $\gamma_0 = \gamma$, $\pi_1 = \pi$ and $p_0 = \mathbf{0}$. We say that γ has the property of *existence of all iterated tangents* if there exists a sequence of maps

$$M_0 \xleftarrow{\pi_1} M_1 \xleftarrow{\pi_2} M_2 \cdots \xleftarrow{\pi_n} M_n \longleftarrow \cdots$$

where, inductively for $n \geq 1$, π_n is the blowing-up with center p_{n-1} and such that the lifted trajectory $\gamma_n = \pi_n^{-1} \circ \gamma_{n-1}$ has a single ω-limit point $p_n = \omega(\gamma_n)$. The sequence of points $TI(\gamma) = \{p_n\}_{n \geq 0}$ so constructed is called the *sequence of iterated tangents* of γ.

Notice (see [**?**]) that, in dimension two, γ has all iterated tangents at its limit point once it has the first tangent. This is not the case for higher dimension. So, in a natural way, if we study some asymptotic properties of γ and we want these properties to be preserved by blowing-ups, we must assume the existence of iterated tangents.

The first example of a trajectory γ with all iterated tangents is the case where $|\gamma|$ is contained in an irreducible analytic curve Γ through $\mathbf{0}$. In this case, we denote $TI(\gamma) = TI(\Gamma)$. Notice that an analytic curve Γ is characterized by its sequence of iterated tangents $TI(\Gamma)$. On the other hand, examples of non-analytic (even non-$\mathcal{C}^\infty$) trajectories with all iterated tangents can occur: consider the vector field $X = -x\frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial y}$ in $\mathbb{R}^2$ with $\lambda > 0$ and take γ the integral curve of X with initial condition $\gamma(0) = (1,1)$. Then $|\gamma|$ is contained in the graph of the function $x \mapsto x^\lambda$, which is not analytic at the origin if λ is irrational. The existence of iterated tangents is already treated in [**?**], where the squence $TI(\gamma)$ corresponds to a $\mathcal{C}^\infty$-curve at $\mathbf{0}$ ("generalized direction") or to a formal curve $\hat{\Gamma}$.

Blowing-up a curve. Let $C \subset U$ be an analytic non-singular curve through $\mathbf{0} \in \mathbb{R}^3$ in an open set U. Recall that the (local) blowing-up with center C is an analytic map

$$\pi : M \to U \hookrightarrow \mathbb{R}^3$$

such that $\pi|_{M\setminus\pi^{-1}(C)}$ is an isomorphism onto $U \setminus C$ and the divisor $\pi^{-1}(C)$ is identified with the cylinder $C \times \mathbb{P}^1$ of normal to C directions in $\mathbb{R}^3$. As in the case of the blowing-up of a point, if X is a vector field in U and C is invariant for X then there exists the *transformed* vector field X' in M such that $\pi_* X' = X$. Let Γ be an analytic regular curve transversal to C at $\mathbf{0}$. Then the *strict transform* $\Gamma' = \overline{\pi^{-1}(\Gamma \setminus \{\mathbf{0}\})}$ of Γ by π is also regular in M and $\Gamma' \cap \pi^{-1}(\mathbf{0})$ is a single point p, called the *(generalized) tangent* of Γ (or simply the tangent, with an abuse of terminology).

3.2. Notion of axial spiraling. Twister axis

We recall here some definitions and results from [**?**].

Let Γ be an analytic non-singular curve through $\mathbf{0}$ which does not intersect $|\gamma|$. Take some analytic coordinates (x, y, z) at the origin such that we have (as germs)

$$\Gamma = \{x = y = 0\}.$$

We call such coordinates *adapted* to Γ. Suppose that the trace $|\gamma|$ is contained in the half space $\{z > 0\}$. Take the associated cylindric coordinates $x = r\cos\theta, y = r\sin\theta, z = z$ and write

$$\gamma(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t), z(t)).$$

Call functions $r(t)$ and $\theta(t)$ the *modulus* and the *angle* of γ around Γ, respectively.

DEFINITION 10. We say that γ *spirals around* Γ or that Γ is a *spiraling axis* for γ if the following asymptotic behaviour holds:
(1) $\lim_{t\to\infty} z(t) = 0$ (since $\omega(\gamma) = \mathbf{0}$).
(2) $\lim_{t\to\infty} \frac{r(t)}{z(t)^n} = 0, \ \ \forall n \in \mathbb{Z}_{\geq 0}$.
(3) $\lim_{t\to\infty} \theta(t) = +\infty$ or $-\infty$.
We say that a non-singular analytic curve Γ is an spiraling axis *for* X if it is so for some trajectory γ of X.

EXAMPLE 11. *Consider the vector field in* $\mathbb{R}^3$

$$X = (-x-y)\frac{\partial}{\partial x} + (-y+x)\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z}.$$

for which the vertical axis $\Gamma = \{x = y = 0\}$ *is invariant. Solutions of* X *can be explicitly obtained as*

$$\gamma(t) = (r_0e^{-t}\cos(t+t_0), r_0e^{-t}\sin(t+t_0), \frac{z_0}{z_0t+1}), \ t \geq 0.$$

We see that those trajectories γ *for which* $z_0 > 0$ *are contained in* $\{z > 0\}$ *and spiral around* Γ.

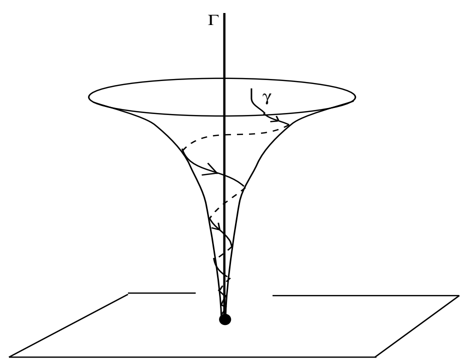

Figure 5

Remark 12. Suppose that γ spirals around an axis $\Gamma = \{x = y = 0\}$ and let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear projection given by $p(x,y,z) = (x,y)$. Then the fact that the angle $\theta(t)$ of γ diverges is equivalent to the fact that the projection $\bar{\gamma} = p \circ \gamma$ spirals around the origin in $\mathbb{R}^2$. We do not adopt this last property as a definition of axial spiraling for γ because, otherwise, the spiraling axis Γ is not unique.

Example 13. *Consider the vector field*

$$X = (x^2+y^2)[(-x-y)\frac{\partial}{\partial x} + (-y+x)\frac{\partial}{\partial y}] - z\frac{\partial}{\partial z}.$$

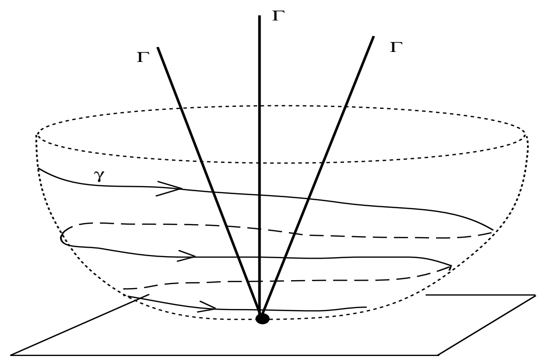

Figure 6

Any trajectory γ outside the z-axis makes "great turns" around any analytic curve Γ which is transversal to the plane $\{z=0\}$.

Condition (2) in the definition of spiraling axis means that γ and Γ are "infinitely close" and thus, γ can have at most a single spiraling axis. In more precise terms, γ has the property of existence of iterated tangents and $TI(\gamma) = TI(\Gamma)$. We say that γ has *flat contact* with Γ. Since γ and Γ have the same tangent at $\mathbf{0}$, given a local coordinate z such that the plane $\{z=0\}$ is transversal to Γ, either $|\gamma|$ is contained in $\{z>0\}$ or in $\{z<0\}$. By a change of sign, we can always suppose that $|\gamma| \subset \{z>0\}$. Thus, our assumption about γ holds for almost all system of adapted coordinates to Γ. The (germ of the) semi-curve

$$\Gamma^+ = \Gamma \cap \{z>0\}$$

does not depend on the transversal coordinate z such that $|\gamma| \subset \{z>0\}$. It is called the *spiraling semi-axis* for γ when we need to specify in which side of a transversal plane we have the spiraling behaviour.

Remark 14. As in the example **??**, if a trajectory γ spirals around Γ then it is *oscillatory*. In fact, γ cuts infinitely many times any analytic surface that contains the axis Γ, even any semi-analytic surface whose closure contains Γ. This last condition is essentially the general notion of spiraling axis for γ treated in [**?**]. It is equivalent to the given definition here for the case Γ is regular and it generalizes the corresponding notion in dimension two. Nevertheless, one important difference arrives: in $\mathbb{R}^2$, if γ has a tangent then it can not spiral and it is not oscillatory; in $\mathbb{R}^3$, a spiraling trajectory γ has all iterated tangents and it is oscillatory. The main result in [**?**] is that if a trajectory γ of a three-dimensional vector field has all iterated tangents and it is oscillatory then there exists an analytic curve Γ which is an spiraling axis for γ. As a consequence, we have that the definition of spiraling axis does not depend on the chosen adapted coordinates.

We recall the following result from [**?**]:

PROPOSITION 15. *Let Γ be a spiraling axis at $\mathbf{0} \in \mathbb{R}^3$ for an analytic vector field X. Then Γ is invariant by X.*

A spiraling axis Γ for a trajectory γ can be either composed of singular points of X or not. We say that Γ is a *degenerated* or *non-degenerated* axis according to this distinction. If Γ is non-degenerated then $\mathbf{0}$ is an isolated singularity of the restriction of X to Γ and $\Gamma \setminus \{\mathbf{0}\}$ is composed of two analytic non singular semi-curves Γ^+, Γ^- which are trajectories of X. If Γ^+ is the spiraling semi-axis then it accumulates to the origin.

In dimension two, we know already that the existence of a single trajectory that spirals around a singularity implies that any other trajectory in a neighbourhood also spirals. In view of example **??**, we can not expect that if Γ is a spiraling axis for any trajectory in a full neighbourhood of the origin. A desirable result will be that this is true for all trajectories in a "good" neighbourhood of the corresponding spiraling semi-axis Γ^+. Unfortunately, we do not know if such a result is true in general, except for the case of non-degenerated axis. (See also [**?**], where the existence of the neighbourhood corresponds to case II.B of that work).

THEOREM 16 ([**?**]). *Let Γ be a non-degenerated spiraling axis at $\mathbf{0} \in \mathbb{R}^3$ for a trajectory γ of an analytic vector field X. Let $\Gamma^+ = \Gamma \cap \{z > 0\}$ be the spiraling semi-axis, with z some transversal coordinate. Then there exists an open subanalytic neighbourhood V of Γ^+, positively invariant by X, such that for any $q \in V \setminus \Gamma$, the trajectory γ_q with initial condition $\gamma_q(0) = q$ accumulates to the origin and spirals around Γ.*

If such a neighbourhood V exists for some spiraling axis Γ then we say that Γ is a *twister axis* of X and that V is a *twister domain*.

Remark 17. It is a consequence of the proof of theorem **??** that if Γ is a non-degenerated twister axis then there exists a fundamental system of neighbourhoods of the germ of Γ^+ which are twister domains. In particular, we can always suppose that twister domains are contained in the same half-space as Γ^+. Another (technical) remark is that these twister domains can be chosen semi-analytic and not only subanalytic.

3.3. Balanced coordinates. Examples and results

We want to study the monotonicity of cylindric coordinates along spiraling trajectories. We restrict to twister axis since we are interested in coordinates giving such a monotonicity for all trajectories at the same time.

Definition 18. Let Γ be a (non-singular) twister axis for an analytic vector field X at $\mathbf{0} \in \mathbb{R}^3$. Let $w = (x, y, z)$ be an analytic system of coordinates at the origin, adapted to Γ, and consider (r, θ, z) the corresponding cylindric coordinates. We say that w is a system of *balanced coordinates* for the pair (X, Γ) if there exists a twister domain V of Γ such that, for any trajectory γ of X in V written as

$$\gamma(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t), z(t)),$$

we have that $t \mapsto r(t)$, $t \mapsto \theta(t)$ and $t \mapsto z(t)$ are eventually monotone functions, for big enough t. If the angle function $t \mapsto \theta(t)$ is eventually monotone we say that w is a system of *monotone spiraling coordinates* for (X, Γ).

Balanced or monotone spiraling coordinates can be tested in terms of transversality of X with some analytic foliations, as in the two-dimensional case: a system of adapted coordinates $w = (x, y, z)$ is monotone spiraling if there exists a twister domain V of Γ such that the vector field X is transversal to the analytic foliation

$$\mathcal{F}_\theta = \{\omega_\theta = -ydx + xdy = 0\} = \{\frac{y}{x} = cst\}$$

in $V \setminus \Gamma$. The system is balanced if X is also transversal in $V \setminus \Gamma$ to the foliations

$$\mathcal{F}_z = \{\omega_z = dz = 0\} = \{z = cst\},$$
$$\mathcal{F}_r = \{\omega_r = xdx + ydy = 0\} = \{x^2 + y^2 = cst\}.$$

In other terms, consider the analytic functions $\omega_\theta(X), \omega_r(X), \omega_z(X)$ (which depend on the coordinates w) over a twister domain V. We have to test if they vanish only along $\Gamma \cap V$.

In this paper, we deal only with non-degenerated twister axis. Thus, using the remark after theorem **??**, in order to prove that a system of coordinates (x, y, z) is balanced, we only have to look for transversality of X with $\mathcal{F}_\theta, \mathcal{F}_r, \mathcal{F}_z$ in $V \setminus \Gamma$ where $V \subset \{z > 0\}$ is a neighbourhood of the semi-axis $\Gamma^+ = \Gamma \cap \{z > 0\}$. We do not mind if V is or not positively invariant by X.

We give first some examples with "bad" systems of coordinates.

EXAMPLE 19. [Non monotone spiraling system] *A slight modification of the two-dimensional example* **??** *permits to show an example of non monotone spiraling coordinates for a twister axis. Consider the vector field X in $\mathbb{R}^3$ given in coordinates (x, y, z) by*

$$X = (y - x^3)(y - 2x^3)T - (x^2 + y^2)^2 \tilde{R},$$

where $T = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $\tilde{R} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z}$. The z-axis $\Gamma = \{x = y = 0\}$ and the z-plane $\{z = 0\}$ are invariant by X. Also, X is tangent to all "flat" cones

$$C_K = \{(x, y, z) \ / \sqrt{x^2 + y^2} = K \exp -\frac{1}{z},\ x^2 + y^2 \neq 0,\ z > 0\},$$

for $K > 0$ a constant. Thus, any trajectory γ of X in $\{z > 0\}$, except for those in Γ, is contained in some C_K and has flat contact with Γ. On the other hand, the canonical projection of γ to $\{z = 0\}$ is a trajectory of the two-dimensional vector field in example **??**. *This shows that γ spirals around Γ and that Γ is a (degenerated) twister axis for X. The given coordinates (x, y, z) are not monotone spiraling since X is tangent to $\mathcal{F}_\theta$ at points in the surface*

$$\{\omega_\theta(X) = 0\} = \{(y - x^3)(y - 2x^3) = 0\}.$$

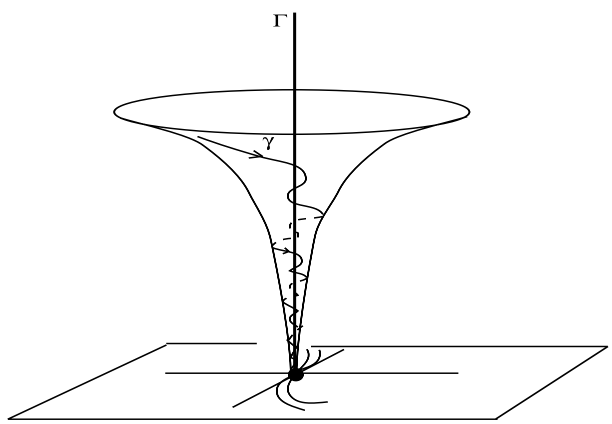

Figure 7

EXAMPLE 20. [Non monotone system for non-degenerated axis] *The existence of non monotone spiraling systems of coordinates is not exclusive* *of degenerated axis. Consider the algebraic vector field in* $\mathbb{R}^3$

$$X = -(x+y)\frac{\partial}{\partial x} + (-y - 3yz + 2xz^2)\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z}.$$

The z*-axis* $\Gamma = \{x = y = 0\}$ *is a non-degenerated twister axis for* X *(see Proposition* **??** *below) and coordinates* (x, y, z) *are not monotone spiraling. In fact,* X *is tangent to* $\mathcal{F}_\theta$ *at points in the ruled surface* $\{(y-xz)(y-2xz) = 0\}$*. However, the reader can check that a change of variables of the form* $y \to y + \lambda xz$ *for some* $\lambda \in \mathbb{R}$*, gives us new coordinates which are monotone spiraling.*

Example 21. [No transversality with $\mathcal{F}_z$] *Consider the algebraic vector field in* $\mathbb{R}^3$

$$X = (x^2+y^2)\{(-x-y)\frac{\partial}{\partial x} + (-y+x)\frac{\partial}{\partial y}\} - (\lambda(x^2+y^2) + xy)z^2\frac{\partial}{\partial z}$$

where $0 < \lambda < 1$*. The* z*-axis* $\Gamma = \{x = y = 0\}$ *and the plane* $\{z = 0\}$ *are invariant by* X*. Let* γ *be any trajectory of* X *contained in* $\{z > 0\}$ *and write in cylindric coordinates* $\gamma(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t), z(t))$*. By means of a reparameterization of time* t*, we have*

$$r(t) = r(0)e^{-t},\ \ \theta(t) = t,\ \ z(t) = \frac{z(0)}{z(0)\int_0^t [\sin\theta(s)\cos\theta(s) + \lambda]ds + 1}.$$

Then, γ *is defined for all* $t \geq 0$ *and* $\omega(\gamma) = \mathbf{0}$*, since* $\lambda > 0$ *and the integral in the denominator of* $z(t)$ *diverges. Also, we have that* $\lim_{t\to\infty} r(t)/z(t)^n = 0$ *for any* n *and* $\lim_{t\to\infty}\theta(t) = \infty$*. Then* Γ *is a twister axis and* $\{z > 0\}$ *is a twister domain. On the other hand,* X *is tangent to the foliation* $\mathcal{F}_z = \{dz = 0\}$ *at points in the two planes defined by*

$$\{\lambda(x^2+y^2) + xy = 0\},$$

and coordinates (x, y, z) *are not balanced. Another property of this example that the reader can check is that for any adapted system of coordinates* (x', y', z')*, the vector field* X *is not everywhere transversal to* $\mathcal{F}_{z'} = \{dz' = 0\}$ *in any neighbourhood of* $\Gamma \cap \{z > 0\}$*. This shows that there are no balanced coordinates at all. Notice that* $\Gamma \subset Sing(X)$ *and that* Γ *is a degenerated axis.*

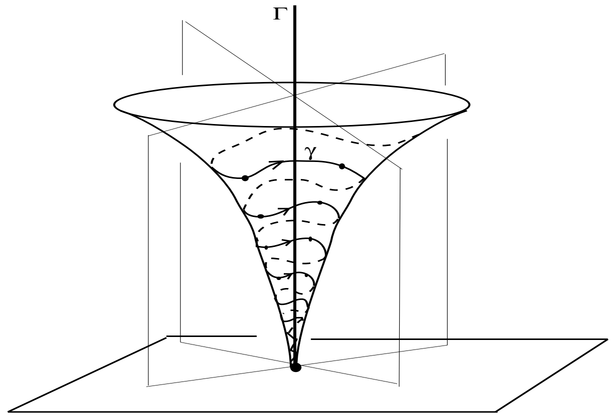

Figure 8

If Γ is non-degenerated, the precedent example can not occur and we do not have to worry about transversality with $\mathcal{F}_z$:

PROPOSITION 22. *Let Γ be a non degenerated twister axis for X at $\mathbf{0} \in \mathbb{R}^3$ and let $w = (x, y, z)$ be a system of adapted coordinates. Suppose that $\Gamma^+ = \Gamma \cap \{z > 0\}$ is the corresponding spiraling semi-axis. Then there exists an open semi-analytic neighbourhood V of Γ^+ where X is transversal to $\mathcal{F}_z = \{dz = 0\}$.*

Proof. Consider the analytic function $f = dz(X)$ defined in the domain of the coordinate system w. Zeroes of f are either singular points of the vector field X or points p where $X(p)$ is a horizontal vector, tangent to $\mathcal{F}_z$. Since Γ is vertical and contains no singular points but $\mathbf{0}$, f does not vanish at any point of $\Gamma^+ \cap U$ for U a sufficiently small neighbourhood of the origin. We can consider U semi-analytic and take $V = U \setminus \{f = 0\}$. ▌

Finally, we wonder if transversality with $\mathcal{F}_r$ is possible for some coordinates, at least for non-degenerated twister axis. The answer is that it is not always true. Examples can be constructed in the same way as Example **??** with a nilpotent singularity. These examples suggest that, perhaps, monotonicity of the modulus function $r(t)$ of a trajectory γ is not a very natural question, even if γ has flat contact with the axis $\Gamma = \{r = 0\}$.

EXAMPLE 23. *Consider*

$$X = y\frac{\partial}{\partial x} - xz\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z}$$

in $\mathbb{R}^3$*, with a nilpotent singularity at the origin. By means of a reduction of singularities (see Theorem* **??** *below), we can see that* $\Gamma^+ = \{x = y = 0, z > 0\}$ *is a non-degenerated twister semi-axis. As in example* **??***, the function* $\omega_r(X) = (xdx+ydy)(X)$ *changes sign in any neighbourhood of* Γ^+*. Moreover, this property is independent of the system of adapted coordinates for* $\Gamma = \{x = y = 0\}$*.*

We state the results.

Theorem 24 (Balanced coordinates for elementary singularity). *Let* Γ *be a non-degenerated twister axis for a vector field* X *at* $\mathbf{0} \in \mathbb{R}^3$ *and suppose that the linear part* $DX(\mathbf{0})$ *is not nilpotent. Then there exists a balanced system of coordinates for* Γ*.*

This theorem is already proved in [**?**]. The most difficult and important part is the existence of a monotone spiraling system for Γ. Transversality with $\mathcal{F}_z$ is guaranteed by Proposition **??** and transversality with $\mathcal{F}_r$ will be easy to prove. In view of the last example above, theorem **??** can not be improved to nilpotent singularities. Nevertheless, we look for monotone spiraling coordinates, the main objective in this work:

Theorem 25 (Monotone spiraling coordinates for twister axis). *Let* Γ *be a regular non-degenerated twister axis for a vector field* X *at* $\mathbf{0} \in \mathbb{R}^3$*. Then there exists a monotone spiraling system of coordinates for* Γ*.*

This result gives a generalization in dimension 3 of Theorem **??**. In order to give a self-contained proof of it, and to be able to recognize some of the used arguments, we are going to reprise the proof of Theorem **??** in paragraph 3.5. We introduce first some normal forms along an invariant curve, as well as a general condition for finding monotone spiraling coordinates.

3.4. Non-degenerate invariant curves

3.4.1. Some normal forms

Let X be a vector field in $\mathbb{R}^3$ an let Γ be a regular analytic curve at the origin, invariant for X. Suppose that Γ is non-degenerated, that is, not contained in the singular locus of X.

Take analytic coordinates (x, y, z), adapted to Γ. Since Γ is invariant, we can write X in the following way

$$X = \sum_{i \geq 0} z^i L_i(x, y) + f(x, y, z)\frac{\partial}{\partial x} + g(x, y, z)\frac{\partial}{\partial y} + c(x, y, z)\frac{\partial}{\partial z},$$

where $L_i(x, y)$ is a linear vector field and f, g have order ≥ 2 in the variables (x, y). The summand $\sum_{i\geq 0} z^i L_i(x, y)$ is called the *normal linear part of* X *along* Γ and denoted by $N X = N_\Gamma X$. On the other hand, since Γ is non-degenerated, the restriction $X\mid_\Gamma$ is not identically zero and it can be written as

$$X\mid_\Gamma = c(0,0,z)\frac{\partial}{\partial z} = z^{q+1}(\alpha + O(z))\frac{\partial}{\partial z},$$

with $q+1<\infty$ and $\alpha\neq 0$. Notice that, if $\Gamma^+=\Gamma\cap\{z>0\}$ is the trace of an asymptotically stable trajectory, then $\alpha<0$.

The number q is called the *(adapted) order* of X along Γ and it is well defined intrinsically for X and Γ. In case of a twister axis, we can be more precise:

LEMMA 26 (see [**?**] and [**?**]). *If Γ is a twister axis for X (in fact it suffices that there exists a trajectory γ not contained in Γ with flat contact with Γ, then the adapted order q of X along Γ is greater or equal than* 1.

Proof. Assume that $q=0$ and write $X=a\partial/\partial x+b\partial/\partial y+c\partial/\partial z$, with $c(0,0,z)=\alpha z+\cdots$. Then α is an eigenvalue of the linear part $DX(\mathbf{0})$ and the others are given by the eigenvalues of L_0. Suppose that $\Gamma^+=\Gamma\cap\{z>0\}$ is the corresponding spiraling semi-axis. Hence, $\alpha<0$. Let $\pi:M\to\mathbb{R}^3$ be the blowing-up with center $\mathbf{0}$ and denote by Γ' the strict transform of Γ by π, a non-degenerated invariant curve for the transformed vector field X' at the tangent point $p'\in\pi^{-1}(\mathbf{0})$ of Γ. Take usual coordinates $w'=(x',y',z')$ at p' such that $\pi(x',y',z')=(x'z',y'z',z')$. We have

$$X'=\{\frac{(a\circ\pi-x'c\circ\pi)}{z'}\frac{\partial}{\partial x'}\}+\{\frac{(b\circ\pi-y'c\circ\pi)}{z'}\frac{\partial}{\partial y'}\}+c\circ\pi\frac{\partial}{\partial z'}.$$

and $Spec(DX'(p'))=\{\alpha\}\cup Spec(L_0-\alpha I_2)$, where I_2 is the identity matrix. Repeating this process, we find $n>>0$ such that the eigenvalues of $L_0-n\alpha I_2$ have positive real part. Then, we can suppose that $\mathbf{0}$ is a hyperbolic singular point of saddle type, where Γ is the stable manifold. This implies that the only trajectories accumulating to $\mathbf{0}$ are those contained in Γ, which contradicts the fact that Γ is a spiraling axis. (This is case II.A in [**?**]).

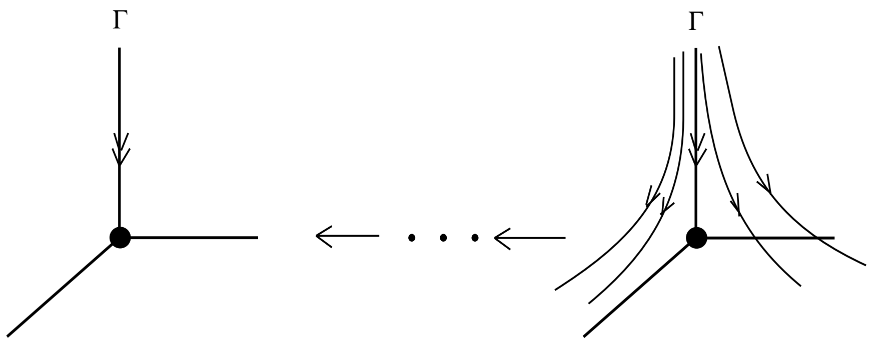

Figure 9

∎

The normal linear part $N\,X$, viewed as a family of linear vector fields in the plane, depends on the coordinates. However, its $(q+1)$-jet can be considered as a linear endomorphism of the normal bundle of Γ in $\mathbb{R}^3$ and it is invariant. More particularly, we have the following lemma, proved by a simple calculation:

Lemma 27. *Write in matricial notation*

$$N\,X = (x \;\; y)\; P(z) \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}, \quad P(z) = P_0 + zP_1 + z^2 P_2 + \cdots ,$$

where P_i is a 2×2 real matrix. Consider new coordinates $(\bar{x}, \bar{y}, \bar{z})$ such that

$$\begin{aligned} (\bar{x} \;\; \bar{y}) &= (x \;\; y)\, T(z) \\ \bar{z} &= z, \end{aligned} \tag{1}$$

where $T(z) = T_0 + zT_1 + \cdots$ is a matrix series, with T_0 invertible. Then $\Gamma = \{\bar{x} = \bar{y} = 0\}$ and the normal linear part in coordinates $(\bar{x}, \bar{y}, \bar{z})$ is given by

$$N\,X = (\bar{x} \;\; \bar{y})\{T(z)^{-1}P(z)T(z) + c(0,0,z)T(z)^{-1}\frac{d}{dz}(T(z))\} \begin{pmatrix} \frac{\partial}{\partial \bar{x}} \\ \frac{\partial}{\partial \bar{y}} \end{pmatrix}. \tag{2}$$

Using this lemma, Taken's normal forms for matrices P_i can be found, depending on the linear term P_0 (see [**?**]). We recall here a single case to be used later.

LEMMA 28. *Suppose that L_0 has a complex eigenvalue with non-zero imaginary part b_0. Then, up to a change of coordinates of type (??), the matrix P_i is in the normal form*

$$\begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}, \quad a_i, b_i \in \mathbb{R}, \tag{3}$$

for $i = 0, \ldots, q+1$.

3.4.2. The discriminant. Condition for monotone spiraling

Now we give sufficient conditions, depending only on $N_\Gamma X$, to get monotone spiraling coordinates. Consider the 1-form $\omega_\theta = -ydx + xdy$ in analytic coordinates $w = (x, y, z)$, adapted to Γ and let

$$\phi(x, y, z) = \omega_\theta(X)(x, y, z) = \omega_\theta(N\,X) + (-yf(x, y, z) + xg(x, y, z)).$$

The first summand, denoted by $Q(x, y, z) = Q^w(x, y, z)$, written as

$$Q(x, y, z) = \omega_\theta(\sum_{n=0}^{\infty} z^n L_n(x, y)) = A(z)y^2 + B(z)xy + C(z)x^2,$$

is an analytic family of quadratic forms in variables (x, y) with parameter z. Consider

$$\Delta(z) = B(z)^2 - 4A(z)C(z)$$

its discriminant.

LEMMA 29. *Suppose that $\Gamma^+ = \{x = y = 0, z > 0\}$ is a spiraling semi-axis for X. Then (x, y, z) is a monotone spiraling system of coordinates if there exists some $l < \infty$ and some unity $u(z)$ with $u(0) < 0$ such that*

$$\Delta(z) = z^l u(z).$$

Proof. We have to show that ϕ has a constant sign in $V \setminus \Gamma$ for V a (semi-analytic) neighbourhood of the germ of Γ^+ at $\mathbf{0}$. Since $-yf + xg$ has order ≥ 3 in (x, y), we can write, in cylindric coordinates ($x = r\cos\theta, y = r\sin\theta, z$),

$$\phi(r, \theta, z) = r^2[Q(\cos\theta, \sin\theta, z) + r\phi_1(r, \theta, z)],$$

where ϕ_1 is bounded by some constant K in $\{|r|, |z| \leq \varepsilon\}$, for some $\varepsilon > 0$. The hypothesis about $\Delta(z)$ says that $Q(\cos\theta, \sin\theta, z)$ has a constant sign in $\{0 < z \leq \varepsilon\}$, independently of θ. Consider the semi-analytic function

$$m(z) = \min\{Q(\cos\theta, \sin\theta, z) \;/\; \theta \in \mathbb{R}\}, \quad 0 < z \leq \varepsilon$$

and let $V = \{0 \leq r < \frac{|m(z)|}{K},\; 0 < z < \varepsilon\}$. Then V is a semi-analytic open set which contains $\Gamma^+ \cap \{0 < z < \varepsilon\}$ and the sign of ϕ is constant in $V \setminus \Gamma$. ▮

3.5. Balanced coordinates for an elementary singularity

Now we suppose that $\mathbf{0} \in \mathbb{R}^3$ is an elementary singularity of X.

3.5.1. Blowing-up the axis. Conditions for spiraling

In example **??**, the fact that $L_0 = DX(\mathbf{0})$ has complex non-real eigenvalues implies the existence of trajectories spiraling around Γ. However, in the general case, twister axis are far to be determined by the linear part of the vector field. As a counterpart, we are going to see that, for an elementary singularity, the problem of determining if Γ is a twister axis reduces to a planar spiraling behaviour over the divisor of the blowing-up of Γ.

More precisely, let $\Gamma = \{x = y = 0\}$ be an analytic regular curve in $\mathbb{R}^3$. Take the associated cylindric coordinates (r, θ, z) and consider the *(cylindric) blowing-up of* Γ:

$$\begin{array}{rccl} \pi \;: & \mathbb{R} \times \mathbb{S}^1 \times \mathbb{R} & \longrightarrow & \mathbb{R}^3 \\ & (r, \Theta = (\cos\theta, \sin\theta), z) & \mapsto & (r\Theta, z). \end{array}$$

The map π is a local diffeomorphism outside the cylinder $D = \{0\} \times \mathbb{S}^1 \times \mathbb{R} = \{r = 0\} = \pi^{-1}(\Gamma)$. Moreover, there exists a unique vector field X' in $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}$ such that $d\pi(X') = X$, called the transform of X. The following proposition is an avatar of the Theorem of Reduction to the Center Manifold, in an analogous version as the one in [**?**]. The proof can be seen in [**?**].

Proposition 30. *Suppose that* $\mathbf{0}$ *is an elementary singularity, that* 0 *is an eigenvalue of* $DX(\mathbf{0})$ *and that* Γ *is tangent to the eigenvector associated to* 0. *Then* Γ *is a twister axis for* X *if and only if (up to sign), trajectories of* $X' |_D$ *in* $\mathbb{S}^1 \times \mathbb{R}^+$ *accumulate to the circle* $\mathbb{S}^1 \times \{0\}$ *spiraling around it.*

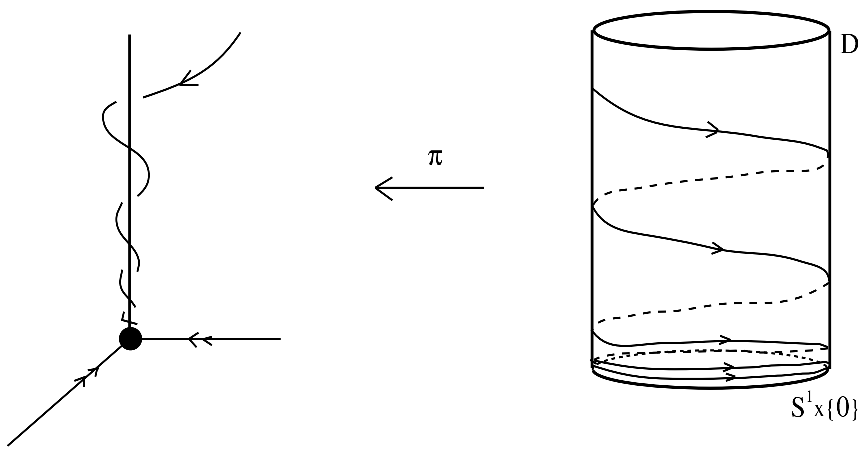

Figure 10

We can consider $D = \mathbb{S}^1 \times \mathbb{R}$ as the resulting space after the (polar) blowing-up of the origin in $\mathbb{R}^2$ $D \longrightarrow \mathbb{R}^2, \quad (\Theta, z) \mapsto z\Theta$. Then, we can check if trajectories of $X' \mid_D$ spiral around $\mathbb{S}^1 \times \{0\}$ using resolution of singularities, as in section 2. We remark also that the result in Proposition **??** does not depend neither on the coordinates or in the particular type of blowing-up. In practice, what we use is the usual algebraic coordinates.

To be more precise, if $\Gamma = \{x = y = 0\}$ for some coordinates (x, y, z) in a neighbourhood U of the origin, the blowing-up $\pi : M \to U$ with center $\Gamma \cap U$ is defined in two charts $(U', (x', y', z'))$ and $(U'', (x'', y'', z''))$ of the variety M by

$$\pi(x', y', z') = (x', x'y', z')$$
$$\pi(x'', y'', z'') = (y''x'', y'', z'').$$

The exceptional divisor $D = \pi^{-1}(\Gamma)$ is given by $\{x' = 0\}$ and $\{y'' = 0\}$ in the corresponding charts. Writing $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$, we compute the transform X' of X as

$$X' = a(x', x'y', z')\frac{\partial}{\partial x'} + \left(\frac{b(x', x'y', z')}{x'} - \frac{y'}{x'}a(x', x'y', z')\right)\frac{\partial}{\partial y'} + c(x', x'y', z')\frac{\partial}{\partial z'}.$$

Let $N\,X$ be the normal linear part of X along Γ in coordinates (x, y, z) and put $Q(x, y, z) = \omega_\theta(N\,X) = (-ydx + xdy)(N\,X)$. Then we have $X' = Q(1, y', z')\frac{\partial}{\partial y'} + c(0, 0, z')\frac{\partial}{\partial z'} + Z$, where Z is an analytic vector field that vanishes over $D \cap U' = \{x' = 0\}$. As a consequence, D is invariant for X' and its restriction in U' is

$$X' \mid_D = Q(1, y', z') \frac{\partial}{\partial y'} + z'^{q+1}(\alpha + O(z')) \frac{\partial}{\partial z'}. \tag{4}$$

Since $\alpha \neq 0$, the fiber $F = \{x' = z' = 0\} = \pi^{-1}(\mathbf{0})$ is also invariant for $X' \mid_D$. It is composed of singular points if and only if z'^l divides $Q(1, y', z')$ for some $l \geq 1$. Let l be the maximum of such exponents and consider the vector field Y in D which is given in chart U' (and analogously in U'') by

$$Y = \frac{X' \mid_D}{z'^l}.$$

Trajectories of Y in $\{z' > 0\}$ coincide with those of $X' \mid_D$ up to parameterization. Also, Y has isolated singularities in F. In this situation, Proposition **??** can be stated as in section 2 in terms of resolution of singularities as follows:

($*$) Let $\pi_1 : M_1 \to D$ be a reduction of singularities of Y at its singular points in F. Then, Γ is a twister axis for X if and only if the (elementary) singular points of the strict transform Y_1 of Y by π_1 are contained in the corners of the divisor $\pi_1^{-1}(F)$ and they are saddle points (see Figure 11).

Or, equivalently,

($**$) There are no characteristic orbits of Y accumulating to points of F except for those contained in F.

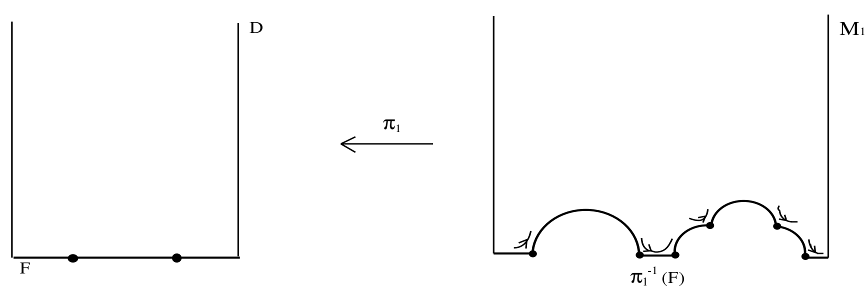

Figure 11

3.5.2. Conjugate eigenvalues. Monodromic and non-monodromic cases

Let Γ be a twister axis through $\mathbf{0} \in \mathbb{R}^3$, an elementary singularity of X. We want to find monotone spiraling coordinates for Γ, as part of the proof of Theorem **??**. For that, we need a system of adapted coordinates (x, y, z) such that the discriminant $\Delta(z)$ of $Q(x, y, z) = \omega_\theta(N\,X)$ satisfies the hypothesis in Lemma **??**. Since the vector field $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is annihilated by ω_θ, we can ignore the radial terms in the normal linear part $N\,X$. More precisely, write $N\,X = \sum_{i \geq 0} z^i L_i(x, y)$ and let

$$k = \min\,\{i \geq 0 \;\,/\;\, L_i \text{ is not a multiple of } R\},$$

called the *radiality index* (of X along Γ). We have $\omega_\theta(L_i) = 0$ for $i < k$ and

$$Q(x, y, z) = z^k \tilde{Q}(x, y, z) \tag{5}$$

for a family of quadratic forms $\tilde{Q}$.

The index k depends on the chosen coordinates. However, by Lemma **??**, the condition $k \leq q$ and, in this case, k itself, are invariant. This is the case for twister axis. Furthermore, we have:

PROPOSITION 31 (see [**?**]). *If Γ is a twister axis of X then $k \leq q$ and L_k has complex conjugate eigenvalues.*

Proof. Let X' be the transform of X by the blowing-up π with center Γ and consider its restriction to the divisor $X' \mid_D$ as in (**??**). Let $Y = X' \mid_D /z'^l$ be the strict transform where l is the maximum exponent such that z'^l divides $X' \mid_D$. By (**??**), we have $l \geq k$. If $k > q$ then $X' \mid_D$ can be divided by z'^{q+1} and then Y is transversal to the fiber $F = \pi^{-1}(\mathbf{0})$. This contradicts property $(**)$. Hence $l = k$ and

$$Y = \tilde{Q}(1, y', z')\frac{\partial}{\partial y'} + z'^{q-k+1}(\alpha + O(z'))\frac{\partial}{\partial z'}$$

satisfies $Y \mid_{F=\{z'=0\}} = \omega_\theta(L_k)(1, y')\frac{\partial}{\partial y'}$. Suppose that L_k has two different real eigenvalues. Then Y has two elementary singular points over F, which contradicts property $(*)$. ▌

In particular, $Spec(L_0) = \{\lambda, \bar{\lambda}\}$, $\lambda \in \mathbb{C}$, and $Spec(DX(\mathbf{0})) = \{0, \lambda, \bar{\lambda}\}$, since $q \geq 1$. Notice that $\lambda \neq 0$ because we assumed $\mathbf{0}$ to be elementary. We distinguish two cases.

Asymptotically monodromic: $Im(\lambda) = -Im(\bar{\lambda}) \neq 0$ $(\Rightarrow k = 0)$.

In this case, Γ is automatically a twister axis, even if we only suppose that Γ is a non-degenerated invariant curve of X. And, in fact, any adapted system of coordinates is a monotone spiraling system: let $N\,X = \sum_{i=0}^{\infty} z^i L_i(x,y)$ be the normal linear part in coordinates (x,y,z) such that $\Gamma = \{x = y = 0\}$, and consider the family of quadratic forms $Q(x,y,z) = \omega_\theta(N\,X)$. Its discriminant is given by

$$\Delta(z) = \Delta(L_0) + O(z),$$

where $\Delta(L_0)$ is the discriminant of the linear vector field L_0 considered as a linear map. Since $Im(\lambda) \neq 0$, $\Delta(L_0) < 0$ and we Lemma **??** applies.

So, to find balanced coordinates, we need only to find coordinates (x,y,z) such that X is transversal to $\{\omega_r = xdx + ydy = 0\}$ in $V \setminus \Gamma$, for a neighbourhood V of Γ^+. Consider coordinates $w = (x,y,z)$ given by lemma **??** such that the matrix of the term L_i is as in (**??**), for $i = 0,\ldots,q+1$. If $a_0 \neq 0$ then we have, in corresponding cylindric coordinates,

$$\omega_r(X) = r^2[2a_0 + O(z) + O(r)],$$

which does not vanish in some neighbourhood of $\mathbf{0}$ except at points of $\Gamma = \{r = 0\}$. The case $a_0 = 0$ is already studied in [**?**]. Alternatively, we can use the following proposition, which is generally true if we suppose that there exists a trajectory γ with flat contact with Γ:

Proposition 32. *If Γ is a twister axis, we have*

$$h = min\{i \leq q+1 \;/\; a_i \neq 0\} < q$$

and moreover, a_h is negative.

Then, the function

$$\omega_r(X) = r^2[2a_l z^l + z^{l+1}\varphi_1 + r\varphi_2], \quad \varphi_1, \varphi_2 \text{ bounded}$$

does not change sign in $V \setminus \Gamma$ for a domain of the type $V = \{r < Kz^{l+1}\}$, K a constant. Our coordinates are then balanced and the asymptotically monodromic case is finished.

Asymptotically non-monodromic: $\lambda = \bar{\lambda} \in \mathbb{R}$.

Contrary to the monodromic case, monotone spiraling coordinates are not automatically given (see example **??**). We dedicate the rest of this section to obtain them. On the other hand, transversality with $\mathcal{F}_r$ is easier to get and we do it first.

Consider coordinates $w = (x, y, z)$ such that, in the normal linear part $N\,X$, the first term L_0 is written with a normalized matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ \varepsilon & \lambda \end{pmatrix},$$

where either $\varepsilon = 0$ or ε is any prescribed non zero real value (depending if $k > 0$ or $k = 0$). Consider the 1-form $\omega_r = xdx + ydy$. We have in cylindric coordinates

$$\omega_r(X) = r^2[2\lambda + \varepsilon\frac{\sin 2\theta}{2} + O(z) + O(r)],$$

and this function does not change sign in $V \setminus \Gamma$ for a neighbourhood V of $\mathbf{0}$ if $\varepsilon = 0$ or if $\mid \varepsilon \mid$ is small compared to $\mid a_0 \mid$. In both cases, this property only depends on the expression of the linear part $DX(\mathbf{0})$, and we suppose that it is satisfied for our initial coordinates w. Then we look for a monotone spiraling system obtained from w by a change of coordinates tangent to the identity. This change preserves transversality with $\mathcal{F}_r$ and will give balanced coordinates. Also, for simplicity of notation, we suppose $\varepsilon = 1$ in case $\varepsilon \neq 0$.

Write $N\,X = \sum_{i\geq 0} z^i L_i(x, y)$ and consider $Q(x, y, z) = \omega_\theta(N\,X)$ the corresponding family of quadratic forms. Let k be the radiality index along Γ and put $Q = z^k\tilde{Q}$ as in (**??**). Then

$$\Delta(z) = z^{2k}\tilde{\Delta}(z),$$

where $\Delta, \tilde{\Delta}$ are the discriminants of the family Q and $\tilde{Q}$ respectively. If L_k has complex eigenvalues with non zero imaginary part then $\tilde{\Delta}(z) = \Delta(L_k) + O(z)$ with $\Delta(L_k) < 0$. Then $w = (x, y, z)$ is a monotone spiraling system of coordinates by Lemma **??**. In the other case, L_k has a double real eigenvalue and, since it is not radial, L_k is not diagonalizable. By means of a change of coordinates of the form (**??**), we write L_k as the Jordan block matrix

$$\begin{pmatrix} \lambda_k & 0 \\ 1 & \lambda_k \end{pmatrix}.$$

(Notice that this is our initial hypothesis if $k = 0$).

In this situation, the following proposition finishes the proof.

Proposition 33. *Consider a vector field Y in $\mathbb{R}^2$ written in coordinates (y, z) as*

$$Y = (A(z)y^2 + B(z)y + C(z))\frac{\partial}{\partial y} + z^{q+1}(\alpha + O(z))\frac{\partial}{\partial z}, \qquad (6)$$

where A, B, C are power series in z, $A(0) = -1$ and $\alpha < 0$. Suppose that Y satisfies property ($*$) or ($**$) above for $F = \{z = 0\}$. Then there are new coordinates $(\bar{y}, \bar{z})$, given by

$$\begin{array}{rcl} \bar{y} & = & y + \mu_1 z + \cdots + \mu_n z^n \\ \bar{z} & = & z \end{array} \tag{7}$$

with $\mu_1, \ldots, \mu_n \in \mathbb{R}$, such that, in the expression

$$Y = (\bar{A}(\bar{z})\bar{y}^2 + \bar{B}(\bar{z})\bar{y} + \bar{C}(\bar{z}))\frac{\partial}{\partial \bar{y}} + \bar{z}^{q+1}(\alpha + O(\bar{z}))\frac{\partial}{\partial \bar{z}},$$

the discriminant $\bar{\Delta}(\bar{z}) = \bar{B}^2(\bar{z}) - 4\bar{A}(\bar{z})\bar{C}(\bar{z})$ satisfies $\bar{\Delta}(\bar{z}) = \bar{z}^l u(\bar{z})$ with $l < \infty$ and $u(0) < 0$.

End of the proof of Theorem **??**: Let $\pi : M \to \mathbb{R}^3$ be the blowing-up with center Γ and consider the restriction $X' \mid_D$ of the transform of X to the exceptional divisor. Consider the chart $(U', (x', y', z'))$ in M for which π is written as $\pi(x', y', z') = (x', x'y', z')$ and let

$$Y = \frac{X' \mid_D}{{z'}^k} = \tilde{Q}(1, y', z')\frac{\partial}{\partial y'} + {z'}^{q-k+1}(\alpha + O(z'))\frac{\partial}{\partial z'}.$$

A simple computation shows that the origin $(y', z') = (0, 0)$ of $D \cap U'$ is the unique singular point of Y over the fiber $F = \pi^{-1}(\mathbf{0})$ and that the expression of Y in coordinates (y', z') is as in (**??**). Moreover, it satisfies property ($*$) or ($**$) by Proposition **??**. Find numbers $\mu_1, \ldots, \mu_n$ for which the conclusion of Proposition **??** holds and consider new coordinates $\bar{w} = (\bar{x}, \bar{y}, \bar{z})$ such that

$$\begin{array}{rcl} \bar{x} & = & x \\ \bar{y} & = & y + x(\mu_1 z + \cdots + \mu_n z^n) \\ \bar{z} & = & z. \end{array} \tag{8}$$

(A tangent to the identity change of coordinates). Then Lemma **??** implies that $\bar{w}$ is a monotone spiraling system for (X, Γ). □

3.5.3. The Newton Polygon

In this paragraph we prove Proposition **??**, where we suppose, for simplicity, that $\alpha = -1$. We use the so-called Newton Polygon of the vector field Y, as can be found for example in [**?**].

In general, let $Y = a(y,z)\frac{\partial}{\partial y} + b(y,z)\frac{\partial}{\partial z}$ be a planar vector field and write the Taylor expansions

$$a(y,z) = \sum_{u+v\geq 1} a_{uv} y^u z^v, \quad b(y,z) = \sum_{u+v\geq 1} b_{uv} y^u z^v.$$

Consider the set of points in the plane $\mathbb{R}^2_{(u,v)}$

$$\Omega = \{(u,v) \ / \ a_{uv} y^u z^v \frac{\partial}{\partial y} + b_{u-1,v+1} y^{u-1} z^{v+1} \frac{\partial}{\partial z} \neq 0\}.$$

The *Newton Polygon* $N = N_Y(z;y)$ of Y (*in the coordinates* (y,z) *with respect to* y) is defined to be the bounded boundary of the convex hull of

$$\bigcup_{p\in\Omega} p + (\mathbb{R}_{\geq 0})^2 \subset \mathbb{R}^2.$$

It is composed of *vertices* and segments (*sides*) of negative slope joining them. Order vertices of N from up to bottom and from left to right and call the first vertex the *pivot point*.

Remark 34. Notice that if $(0,1) \in N$ then it comes from one or both of the linear terms $y\frac{\partial}{\partial y}$, $z\frac{\partial}{\partial z}$. In this case, the singularity is elementary.

Let Y be as in (**??**). Then the pivot point of its Newton polygon N is the point $(0,2)$, obtained from the term $-y^2\frac{\partial}{\partial y}$. Denote $s = ord(B(z))$, $t = ord(C(z))$. We have $t < \infty$ and $(t,0)$ is the last vertex of N; otherwise, $C(z) \equiv 0$ and the line $\{y = 0\}$ is invariant by Y and transversal to F, contradicting property $(**)$. On the other hand, N can have, a priori, one or two sides, depending on the relative position of the numbers q, s and t.

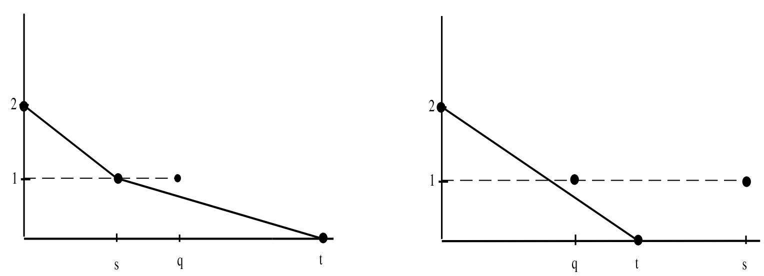

Figure 12

Assertion: The Newton Polygon $N = N_Y(z;y)$ has only one side.

To see this, we control the behaviour of the Newton polygon by blowing-ups. Let $\pi_1 : M \to \mathbb{R}^2$ be the blowing-up at the origin and consider the point $p_1 \in \pi_1^{-1}(\mathbf{0})$ which corresponds to the tangent of the axis $\{y = 0\}$. Let (y_1, z_1) be coordinates at p_1 in a usual chart of M such that $\pi(y_1, z_1) = (y_1 z_1, z_1)$. The transformed vector field Y_1 of Y is given by

$$\begin{aligned} Y =&\{A(z_1)y_1^2 z_1 + B(z_1)y_1 + \frac{C(z_1)}{z_1} - z_1^q y_1(-1 + O(z_1))\}\frac{\partial}{\partial y_1} \\ &+ z_1^{q+1}(-1 + O(z_1))\frac{\partial}{\partial z_1} \end{aligned}$$

The Newton polygon $N_1 = N_{Y_1}(z_1; y_1)$ is obtained from N by the affine transformation in the plane

$$(u, v) \longmapsto (u + v - 1, v).$$

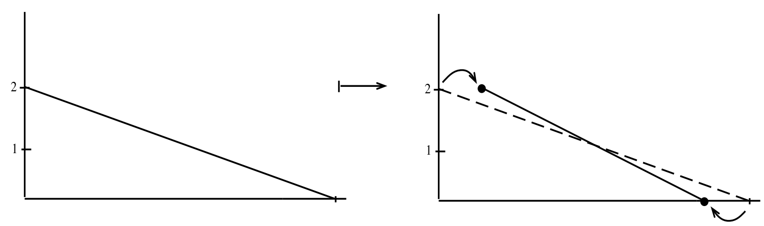

Figure 13

By blowing-up points $p_0 = \mathbf{0}, p_1, p_2, \ldots$ in the sequence of iterated tangents of $\{y = 0\}$, construct $Y, Y_1, Y_2, \ldots$ and $N, N_1, N_2 = N_{Y_2}(z_2; y_2), \ldots$ inductively in this way, with coordinates (y_i, z_i) at p_i (while p_i is a singular point of Y_i). Suppose that N has two sides and let $(l, 1)$ be their common vertex. Notice that $l \leq \min\{s, q\}$ and $l < t$. Then $N_1, \ldots, N_{l-1}$ has also two sides and N_l has a single side with $(l, 1)$ as the pivot point. We have that the transformed vector field Y_l by the composition $\pi = \pi_l \circ \cdots \circ \pi_1$ can be divided by $(z_l)^l$. Consider the strict transform $\tilde{Y}_l = \frac{Y_l}{(z_l)^l}$. Its Newton polygon $\tilde{N}_l = N_{\tilde{Y}_l}(z_l; y_l)$ is obtained from $N_l = N_{Y_l}(z_l; y_l)$ by translation $(u, v) \mapsto (u - l, v)$. Thus, $(0, 1)$ is the pivot point of $\tilde{N}_l$. Thus, p_l is an elementary singularity of $\tilde{Y}_l$. But p_l is not a corner of $\pi^{-1}(F)$, a contradiction with property $(*)$. Notice that this proof is independent of the chosen coordinates (y, z) to construct the Newton polygon, once the invariant line F is given by $\{z = 0\}$. In particular, N has a single side for any coordinates obtained from the initial ones by a change of type (**??**).

Let $-\frac{1}{d}$ be the slope of the side of N, where $d = d(N) = \frac{t}{2}$ is called *inclination*. We have $d \leq \min\{s, q\}$. Performing a change of coordinates of the type

$$\bar{y} = y + \mu z^n, \quad \bar{z} = z, \quad \mu \in \mathbb{R}, n \geq 1, \tag{9}$$

we have a new coefficient of $\frac{\partial}{\partial \bar{y}}$ for which (with evident notations):

$$\begin{array}{rcl} \bar{A}(z) & = & A(z) \\ \bar{B}(z) & = & B(z) + 2\mu z^n A(z) \\ \bar{C}(z) & = & C(z) + \mu^2 z^{2n} A(z) + \mu z^n B(z) + \mu n z^{q+n}(-1 + O(z)). \end{array} \tag{10}$$

Since the inclination d is bounded by the invariant number q, we can suppose, up to a finite number of such changes of coordinates, that the Newton polygon for our coordinates (y, z) has maximal inclination. To finish, let's proof that for these coordinates, the discriminant has a first negative coefficient. Write $A(z) = -1 + \sum_{i \geq 1} A_i z^i$, $B(z) = \sum_{i \geq 0} B_i z^i$, $C(z) = \sum_{i \geq 0} C_i z^i$ and compute

$$\Delta(z) = B(z)^2 - 4A(z)C(z) = \begin{cases} z^t(4C_t + O(z)) & \text{if} \quad d < s \\ z^t(B_s^2 + 4C_t + O(z)) & \text{if} \quad d = s. \end{cases}$$

First, we can suppose that $d < s$. In fact, if $s = d$ then we consider the change (**??**) with $n = d$ and $\mu \in \mathbb{R}$ such that $B_s - 2\mu = 0$. By (**??**), the coefficient of $\bar{B}(z)$ of degree s is 0 and then the new inclination $\bar{d}$ for coordinates $(\bar{y}, \bar{z})$ will be greater or equal than d. By our assumption, $\bar{d} = d$ and $\bar{s} > s = d = \bar{d}$. It suffices then to proof that $C_t < 0$. Distinguish two cases:

(i) d is not an integer. Make blowing-ups at points in the sequence of iterated tangents of the axis $\{y = 0\}$. Looking at the evolution of the Newton polygon by these blowing-ups, we can suppose that $t = 1$ and that Y has the linear term $C_1 z \frac{\partial}{\partial y}$. Suppose that $C_1 > 0$. Let $p' \in \pi^{-1}(\mathbf{0})$ be the corresponding point of the tangent direction of $\{z = 0\}$ by the blowing-up π of the origin $(y, z) = \mathbf{0}$. Then p' is an elementary singularity of the strict transform of Y by π which is not a saddle point. This contradicts property $(*)$.

(ii) d is integer. We distinguish again two cases:

(ii-a) If $d < q$ then $C_t < 0$. Otherwise, we would consider the change (**??**) with $n = d$ and $\mu \in \mathbb{R}$ such that $C_t - \mu^2 = 0$. Using formulas (**??**), these new coordinates give a Newton polygon with two sides.

(ii-b) If $d = q$ then $t = 2q$ and the discriminant of the equation $-\mu^2 + \mu q + C_t = 0$ must be negative just by the same reason as in the precedent case. Thus $C_t < 0$. □

We finish this paragraph with the following technical variation of existence of monotone spiraling coordinates needed later

PROPOSITION 35. *With the same notations as above, suppose that L_0 has a double eigenvalue and it is not radial. Consider coordinates $w = (x, y, z)$ adapted to Γ in which we have*

$$L_0(x,y) = (\lambda x + y)\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}.$$

*Let $m \in \mathbb{Z}_{>0}$. Then, up to a change of coordinates of the form (**??**), we can suppose that that X is transversal to the foliation*

$$\tilde{\mathcal{F}}_{\theta,m} = \{\tilde{\omega}^w_{\theta,m} = -ydx + xdy + m\frac{xy}{z}dz = 0\}$$

in $V \setminus \Gamma$, for some neighbourhood $V \subset \{z > 0\}$ of $\Gamma \cap \{z > 0\}$.

Proof. Consider the functions

$$\begin{array}{rcccl} f(x,y,z) &=& \tilde{\omega}^w_{\theta,0}(X) = \omega_\theta(X) &=& Q^w(x,y,z) + \cdots \\ \tilde{f}(x,y,z) &=& \tilde{\omega}^w_{\theta,m}(X) &=& \tilde{Q}^w(x,y,z) + \cdots \end{array}$$

where $Q^w = A(z)y^2 + B(z)xy + C(z)x^2$, $\tilde{Q}^w = \tilde{A}(z)y^2 + \tilde{B}(z)xy + \tilde{C}(z)x^2$ are families of quadratic forms and $\cdots$ means terms whose order in (x, y) is greater or equal than 3. The relation in these two families is that

$$\tilde{B}(z) = B(z) + m\frac{dz(X)}{z} \mid_\Gamma = B(z) + mz^q(-1 + O(z)).$$

Let $\Delta(z)$, $\tilde{\Delta}(z)$ the corresponding discriminant of $Q^w, \tilde{Q}^w$. We must proof that the coordinates w can be chosen in such a way that

$$\tilde{\Delta}(z) = \tilde{B}(z)^2 - 4\tilde{A}(z)\tilde{C}(z) = z^{\tilde{l}}\tilde{u}(z)$$

with $\tilde{u}(0) < 0$ and $l < \infty$. In this situation, Lemma **??** will apply in the same way and will give the result. Compare this condition with the one for $\Delta(z)$, which we have already proved. We see that the two conditions are the same in cases (i) and (ii-a) treated above. In fact, in these cases,

$$l = ord_z(\tilde{\Delta}) = ord_z(\Delta) < 2q \text{ and } \tilde{u}(0) = u(0) = C_t < 0.$$

In case (ii-b), we consider a new change of variables

$$\bar{w} : \bar{x} = x, \bar{y} = y + \mu x z^q, \bar{z} = z.$$

By formulas (**??**), we obtain a new family of quadratic forms $\tilde{Q}^{\bar{w}}$ with discriminant $\tilde{\Delta}(\bar{z}) = \bar{z}^{2q}\tilde{u}(\bar{z})$ and

$$\tilde{u}(0) = (-m - 2\mu)^2 + 4(C_{t=2q} - \mu^2 + \mu(q - m)) = 4c_{2q} + m^2 + 4\mu q.$$

A convenient choice of $\mu \neq 0$ gives $\tilde{u}(0) < 0$, as wanted. ∎

3.6. Monotone spiraling coordinates for non-degenerated axis

This paragraph is devoted to the proof of Theorem **??**. We first reduce the singularity $\mathbf{0}$ of X to an elementary singularity by means of certain transformations. After, we apply Theorem **??**. These transformations consist of blowing-ups with center points and invariant curves and, eventually, a ramification. In this process, the fact that Γ is a twister axis is preserved. However, we must control the chosen coordinates at each step in order to be sure that the system of monotone spiraling coordinates at the end point comes from spiraling coordinates at the initial point. This control is assured by Theorem **??** below which gives the existence of a *Maximal Contact Surface* that polarizes the reduction process.

Let Γ be a spiraling axis for some trajectory γ of a vector field X at $\mathbf{0} \in \mathbb{R}^3$.

3.6.1. Stability of spiraling axis by blowing-ups and ramifications

PROPOSITION 36. *Let $\pi : M \to \mathbb{R}^3$ be either the blowing-up with center $\mathbf{0}$ or the local blowing-up with center an analytic non-singular curve C transversal to Γ at $\mathbf{0}$ and invariant for X. Let $\Gamma' = \overline{\pi^{-1}(\Gamma \setminus \{\mathbf{0}\})}$ be the strict transform of Γ by π. Then Γ' is an spiraling axis for the transformed vector field X'. Moreover, Γ' is non-degenerated if and only if Γ is non-degenerated.*

Proof. Take coordinates $w = (x, y, z)$ adapted to Γ and suppose that the trace $|\gamma|$ is contained in $\{z > 0\}$. Write in polar coordinates

$$\gamma(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t), z(t))$$

and assume that (1), (2) and (3) in the definition of spiraling trajectories hold. Suppose also that coordinates are chosen in such a way that C is locally given by $\{y = z = 0\}$, in case π is the blowing-up with center a curve. Let $p \in \pi^{-1}(\mathbf{0})$ be the generalized tangent of Γ. We can take local coordinates $w' = (x', y', z')$ at p such that π is written as

$$\begin{array}{lll} T1(w',w): & x = x'z', y = y'z', z = z' & \text{if the center is } \mathbf{0}. \\ T2(w',w): & x = x', y = y'z', z = z' & \text{if the center is } C. \end{array}$$

Then $\Gamma' = \{x' = y' = 0\}$ and the lifted trajectory $\gamma' = \pi^{-1} \circ \gamma$ satisfies $|\,\gamma'\,| \subset \{z' > 0\}$. Consider the cylindric coordinates (r_1, θ_1, z') associated to w' and write $\gamma'(t) = (r_1(t)\cos\theta_1(t), r'(t)\sin\theta_1(t), z'(t))$. We have $\lim z'(t) = \lim z(t) = 0$ and

If π is the blowing-up at $\mathbf{0}$ then $r_1(t) = \frac{r(t)}{z(t)}$ and $\theta_1(t) = \theta(t)$.

If π is the blowing-up with center C then $r_1(t) \leq \frac{r(t)}{z(t)}$ and $\tan\theta(t) = z(t)\tan\theta_1(t)$.

In both cases, we see that γ' spirals around Γ'. Furthermore, since π is an isomorphism outside the divisor $\{z' = 0\}$, Γ is composed of singular points of X iff Γ' is composed of singular points of X'. ▮

Notice that we have proved in fact that if w, w' are adapted coordinates to Γ and Γ' respectively and related by the transformation of coordinates $T1(w', w)$ then w is a monotone spiraling system for (X, Γ) if and only if w' is so for (X', Γ').

We will need to consider also ramifications with branched points along an invariant surface transversal to Γ at $\mathbf{0}$. More precisely, we need the following result, whose proof is similar to the one in the precedent proposition.

Proposition 37. *Let $w = (x, y, z)$ be adapted coordinates to Γ such that $|\,\gamma\,| \subset \{z > 0\}$ and suppose that the plane $\{z = 0\}$ is invariant by X. Consider coordinates $w' = (x', y', z')$ in $\mathbb{R}^3$ and let $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ be the double ramified cover written as*

$$T3(w', w): \quad x = x', y = y', z = {z'}^2.$$

Then, there exists a vector field X' in $\mathbb{R}^3$ which satisfies $d\pi(X') = X$ and the curve $\Gamma' = \pi^{-1}(\Gamma) = \{x' = y' = 0\}$ is a spiraling axis for X'. Moreover, Γ' is non-degenerated iff Γ is non-degenerated. X' and Γ' are called the transform and the strict transform of X and Γ by π.

The coordinate transformations $T1$, $T2$ and $T3$ will be used in the sequel. To simplify our sentences, if $\pi : M' \to M$ is a map and $\pi(p') = p$, we will say that π *satisfies* $T1(w', w)$, etc. *at* p' if there are charts (U, w) and

(U', w') at p and p' respectively, such that $\pi\mid_{U'}$ is written as $T1(w', w)$ in the corresponding domains.

3.6.2. Reduction to elementary singularity

Following [**?**], we develop here a procedure to reduce any singularity to an elementary one *along* a twister axis Γ. In fact, we only use the assumption that Γ is a non-degenerated invariant curve of X. (See also [**?**], where a similar procedure is given to reduce the singularity alonag a "generalized direction").

Suppose that we have initially a surface D transversal to Γ and invariant by X. (This does not impose any restriction since we can always start performing a blowing-up at $\mathbf{0}$ and consider D as the divisor). Put $X_0 = X$, $\Gamma_0 = \Gamma$, $D_0 = D$ and consider a sequence of maps

$$\mathbb{R}^3 = M_0 \overset{\pi_1}{\longleftarrow} M_1 \overset{\pi_2}{\longleftarrow} M_2 \cdots \overset{\pi_n}{\longleftarrow} M_n \longleftarrow \cdots \tag{11}$$

constructed inductively as follows: for $1 \le i \le n$, either π_i is the blowing-up at $p_{i-1} = \Gamma_{i-1} \cap D_{i-1}$, or the local blowing-up of an analytic non-singular curve $C_{i-1} \subset D_{i-1}$ through p_{i-1}, invariant by X_{i-1} and transversal to Γ_{i-1}, or π_i is a double cover ramified along D_{i-1}; Γ_i, X_i are respectively the strict and the total transform of Γ_{i-1}, X_{i-1} by π_i and $D_i = \pi_{i-1}^{-1}(D_{i-1})$.

Definition 38. A finite composition $\pi = \pi_1 \circ \cdots \circ \cdots \pi_n : M' = M_n \to \mathbb{R}^3$ of a sequence (**??**) is called a *resolution of singularities of X along* Γ if there exists a vector field $\tilde{X}$ defined in a neighbourhood of $p_n \in M'$ and there exists some $s \ge 0$ such that $D\tilde{X}(p_n)$ is not nilpotent and

$$X_n = (z_n)^s \tilde{X},$$

where $\{z_n = 0\}$ is a local equation of D_n at p_n. The vector field $\tilde{X}$ is called the strict transform of X by π at p_n.

Theorem 39 (Resolution of singularities). *Let Γ be a non-degenerated regular invariant curve of a vector field X at $\mathbf{0} \in \mathbb{R}^3$. Then there exists a resolution of singularities of X along Γ.*

The first step of this theorem is easy

Lemma 40 (Reduction to non-zero linear part). *Let $TI(\Gamma) = \{p_i\}_{i \ge 0}$ be the sequence of iterated tangents of Γ, obtained by a sequence of blowing-ups of points as in (**??**). Let $w = (x, y, z)$ be a system of adapted coordinates to Γ at $\mathbf{0}$. Then, there exists some $l \ge 0$ and a system of coordinates*

$w_l = (x_l, y_l, z_l)$ at p_l, adapted to Γ_l, such that

$$X_l = z_l^s \tilde{X}_l \quad \textit{and} \quad \begin{cases} \tilde{X}_l(p_l) \neq 0 \ \textit{or} \\ D\tilde{X}_l(p_l) \not\equiv 0, \end{cases}$$

for some $s \in \mathbb{Z}_{\geq 0}$. Furthermore, coordinates w_l are obtained as a composite of coordinate transformations of type T1 from the initial coordinates w.

Proof. Let q be the adapted order of X along Γ. Then $1 \leq q < \infty$. Write $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$, where $a(0,0,z) \equiv b(0,0,z) \equiv 0$ and $ord_z(c(0,0,z)) = q+1$. Consider coordinates $w_1 = (x_1, y_1, z_1)$ at p_1 such that the blowing-up π_1 at $\mathbf{0}$ satisfies T1(w_1, w). A computation shows that X_1 can be divided by $z_1^{s_1}$ with

$$s_1 \geq \min\{ord(a), ord(b), ord(c)\} - 1.$$

Write $X_1 = z_1^{s_1}\tilde{X}_1$ and $\tilde{X}_1 = a_1\frac{\partial}{\partial x_1} + b_1\frac{\partial}{\partial y_1} + c_1\frac{\partial}{\partial z_1}$. Then we have

$$c_1(x_1, y_1, z_1) = \frac{1}{z_1^{s_1}} c(x_1 z_1, y_1 z_1, z_1).$$

If $DX(\mathbf{0}) \equiv 0$ then $s_1 > 0$ and the adapted order $q_1 = ord_{z_1}(c_1(0,0,z_1))$ of $\tilde{X}_1$ along Γ_1 is strictly smaller than q. Repeating this discussion, the case $D\tilde{X}_i(p_i) \equiv 0$ can not be verified indefinitely for $i = 0, 1, 2, \ldots$ if p_i is a singular point of $\tilde{X}_i$. ▌

Proof of Theorem **??**.- Up to some initial blowing-ups at points in the sequence of iterated tangents of Γ, we can suppose, by Lemma **??**, that the linear part $DX(\mathbf{0})$ is not zero, but nilpotent. In this situation, the procedure that follows is, essentially, a particular case of the standard reduction of Ordinary Differential Singular Equations (see [**?**]).

Let q be the adapted order of X along Γ. We proceed by induction in the number q or, in some cases, by a direct proof. If $q = 0$ then the tangent of Γ at $\mathbf{0}$ is a proper direction of $DX(\mathbf{0})$ with non zero eigenvalue. So, the singularity is elementary and we are done. Suppose that $q \geq 1$ (as for the case of a twister axis). Let $w = (x, y, z)$ be a system of coordinates adapted to Γ, consider $N_\Gamma X = \sum_{i \geq 0} z^i L_i(x, y)$ the normal linear part of X along Γ. Write

$$L_i(x, y) = (a_i x + b_i y)\frac{\partial}{\partial x} + (c_i + d_i)\frac{\partial}{\partial y}.$$

Since $DX(\mathbf{0}) = L_0 \not\equiv 0$, we can suppose our coordinates such that

$$L_0 = y\frac{\partial}{\partial x}.$$

Write $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$. Considering the blowing-up at $\mathbf{0}$, if necessary, we can also assume that $c(x,y,0) \equiv 0$ and thus, $D = \{z = 0\}$ is invariant by X and $X\mid_D = L_0(x,y)$. We have two cases:

Case 1.- $c_1 = 0$. Let $\pi : M' \to \mathbb{R}^3$ be the blowing-up with center the curve $C = \{y = z = 0\} \subset D$, invariant for X. Let Γ' be the strict transform of Γ and $D' = \pi^{-1}(D)$. Consider coordinates $w' = (x', y', z')$ adapted to Γ' at $p' \in \pi^{-1}(\mathbf{0})$, the tangent of Γ such that π satisfies $\mathrm{T2}(w', w)$. The transform X' of X is written as

$$X' = a \circ \pi \frac{\partial}{\partial x'} + (\frac{b \circ \pi}{z'} - \frac{y'}{z'} c \circ \pi) \frac{\partial}{\partial y'} + c \circ \pi \frac{\partial}{\partial z'}.$$

and $X' = z'\tilde{X}$, where $\tilde{X}$ is the strict transform. The order q' of $\tilde{X}$ along $\Gamma' = \{x' = y' = 0\}$ is $q - 1$. Moreover, if we write $N_{\Gamma'}\tilde{X} = \sum_{i \geq 0} {z'}^i L_i^*(x', y')$ then

$$L_0^* = (a_1 x' + y') \frac{\partial}{\partial x'} + (c_2 x' + d_1 y') \frac{\partial}{\partial y'}. \tag{12}$$

Apply induction on q if L_0^* is nilpotent to finish.

Case 2.- $c_1 \neq 0$. In this case, $C = \{y = z = 0\}$ is again invariant, but the blowing-up with center C does not improve the expression: the transformed vector field X' has a non zero nilpotent linear part at p' and it is not divisible by z'. We proceed in an alternative way. Let $\pi_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be the double cover ramified over D that satisfies $\mathrm{T3}(w_1, w)$. The transform X_1 of X by π_1 is given by

$$X_1 = a(x_1, y_1, z_1^2) \frac{\partial}{\partial x_1} + b(x_1, y_1, z_1^2) \frac{\partial}{\partial y_1} + \frac{c(x_1, y_1, z_1^2)}{2z_1} \frac{\partial}{\partial z_1}.$$

Write $N_{\Gamma_1} X_1 = \sum_{i \geq 0} z_1^i L_i^{(1)}(x_1, y_1)$, where $\Gamma_1 = \{x_1 = y_1 = 0\}$ is the transform of Γ. Then, with the identification $(x, y) = (x_1, y_1)$, we have for $l \geq 0$

$$\begin{array}{rcl} L_{2l}^{(1)} & = & L_l \\ L_{2l+1}^{(1)} & = & 0. \end{array} \tag{13}$$

Thus, X_1 is in the situation of the precedent case. Nevertheless, its adapted order along Γ_1 is $2q$, greater than the initial one. Anyhow, there is a direct proof in this case. Let $\pi : M' \to \mathbb{R}^3$ be the blowing-up of the curve $C_1 = \{y_1 = z_1 = 0\}$ and let $\tilde{X}$ be the strict transform of X_1 by π. Consider coordinates $w' = (x', y', z')$ such that π satisfies $\mathrm{T2}(w', w_1)$ at $p' \in M'$, the tangent of Γ_1. By equations (**??**) and (**??**), the linear part L_0^* of $\tilde{X}_1$ is

$$L_0^*(x', y') = y' \frac{\partial}{\partial x'} + c_1 x' \frac{\partial}{\partial y'},$$

which is not nilpotent since $c_1 \neq 0$. □

Remark 41. Suppose that Γ is a non-degenerated twister axis of X at $\mathbf{0}$ and let $\pi : M' \to \mathbb{R}^3$ be a resolution of X along Γ and let $\tilde{X}$ be the strict transform of X by π at the elementary singularity $p' \in M'$. We have two remarks:
1) Assume that $DX(\mathbf{0})$ is nilpotent but not identically zero. In view of the expressions (**??**) and (**??**), p' is asymptotically non-monodromic and $D\tilde{X}(p')$ is not diagonalizable.
2) π is composed of at most a ramification map. If it exists one, the ramification is made at the final step and p' is asymptotically monodromic.

3.6.3. Maximal Contact

Theorem **??** shows a constructive process which can be implemented in an algorithmic way. In this paragraph we see that, in fact, this process is governed by a *Maximal Contact Surface*. This notion is studied in [**?**]. In our context, this means the following:

Assume that $D = D_0$ is an invariant surface at $\mathbf{0}$, transversal to Γ. Let $\pi : M' \to \mathbb{R}^3$ be a resolution of singularities of X along Γ, obtained by the composition $\pi = \pi_1 \circ \cdots \circ \pi_n$. Then there exists an analytic non-singular surface $W = W_0$ at $\mathbf{0} \in \mathbb{R}^3$ such that, for $i = 1, \ldots, n-1$, if π_i is the blowing-up with center a curve C_{i-1}, then $C_{i-1} = W_{i-1} \cap D_{i-1}$, where, inductively, $W_i = \overline{\pi_i^{-1}(W_{i-1} \setminus C_{i-1})}$ is the strict transform of W_{i-1} by π_i and $D_i = \pi_i^{-1}(D_{i-1})$. (See Figure 14).

In practice, we use this result in terms of coordinates, in order to find good expressions for the resolution π. We consider the case of a nilpotent singularity with non-zero linear part. The transition from the general case to this case is controlled by Lemma **??** and presents no problem.

Let $w = (x, y, z)$ be a system of adapted coordinates to Γ and write $NX = \sum_{i \geq 0} z^i L_i(x, y)$. We say, to simplify, that these coordinates are *normalized* if the term L_0 is written as a matrix P_0 with equal entries at the principal diagonal. Also, a change of coordinates of the type (**??**)

$$(\bar{x}, \bar{y}, \bar{z}) = ((x, y)\, T(z), z)$$

with $T(z)$ a polynomial 2×2 matrix will be called a *(polynomial) adapted change*. It will be tangent to the identity if $T(0)$ is the identity matrix I_2. This last change preserves the normalized character for the coordinates.

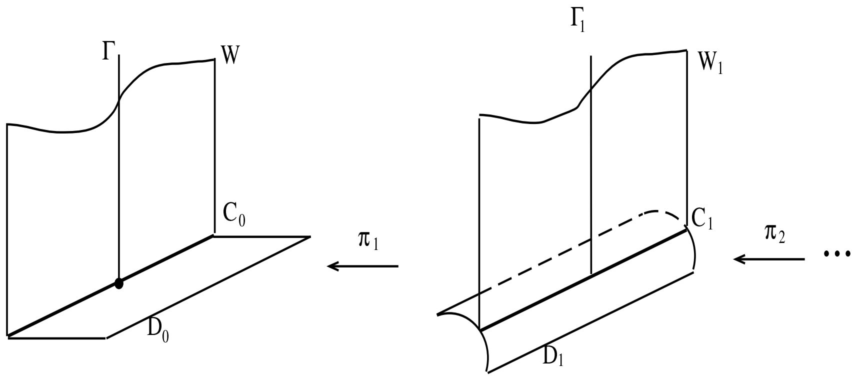

Figure 14

Theorem 42 (Maximal Contact). *Assume that $DX(\mathbf{0})$ is nilpotent but not identically zero. Let $\pi = \pi_1 \circ \cdots \circ \pi_n : M' \to \mathbb{R}^3$ be a resolution of singularities of X along Γ and denote by $\tilde{X}$, Γ' the strict transforms of X, Γ by π at $p' \in \pi^{-1}(\mathbf{0})$. Then, up to a polynomial adapted and tangent to the identity change of coordinates at $\mathbf{0}$, there exist systems of normalized coordinates $w = (x, y, z)$ at $\mathbf{0}$ and $w' = (x', y', z')$ at p', adapted to Γ and Γ', respectively, such that π is written in coordinates w', w as*

$$T^{m,\epsilon}(w', w) : \begin{cases} x &= x' \\ y &= y' {z'}^m \\ z &= {z'}^\epsilon \end{cases}$$

where m is an integer ≥ 1 and $\epsilon = 1$ or 2. In particular, $W = \{y = 0\}$ is a Maximal Contact Surface for the resolution π.

Proof. Fix initial adapted normalized coordinates $w = (x, y, z)$ such that $L_0 = y\frac{\partial}{\partial x}$. Write $L_i = (a_i x + b_i y)\frac{\partial}{\partial x} + (c_i x + d_i y)\frac{\partial}{\partial y}$ for the terms in $N X = \sum_{i \geq 0} z^i L_i$. The proof is by induction in the number of steps n (≥ 1) of the resolution.

If $n = 1$ then $c_1 = 0$ and π_1 is necessarily the blowing-up of the curve $C = \{y = z = 0\}$. The result is a consequence of the following lemma, which we separate by clarity and to be used again later:

Lemma 43. *Assume that* $c_1 = 0$ *and let* $\pi : M' \to \mathbb{R}^3$ *be the blowing-up with center* $C = \{y = z = 0\}$*. Denote by* $\tilde{X}$*,* Γ' *the strict transforms of* X *and* Γ *by* π *at* $p' = \Gamma' \cap \pi^{-1}(C)$*. Then there exists a system of coordinates* $\bar{w} = (\bar{x}, \bar{y}, \bar{z})$ *obtained from* w *by an adapted change, tangent to the identity, such that* π *satisfies* $T2(\bar{w}', \bar{w})$ *for normalized and adapted coordinates* $\bar{w}'$ *at* p'*.*

Proof. We use the same notations as in Theorem **??**. The linear term L_0^* of $N\tilde{X}$ has the expression (**??**) in coordinates w' such that π satisfies $T2(w', w)$. Coordinates w' at p' are normalized if the entries a_1 and d_1 of the matrix P_1 of L_1 are equal. With an initial adapted change of coordinates of the type $(\bar{x}, \bar{y}, \bar{z}) = ((x, y)(I_2 + zT_1), z)$ we will have, using (**??**), a new matrix $\bar{P}_1 = P_1 + P_0T_1 - T_1P_0$, where $P_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We have the desired condition with a convenient choice of matrix T_1. ∎

If $c_1 \neq 0$ then π_1 is a ramification. Let $\tilde{X}_1, \Gamma_1$ be the strict transforms of X and Γ by π_1. The usual coordinates $w_1 = (x_1, y_1, z_1)$ such that π_1 satisfies $T3(w_1, w)$ are normalized and adapted to Γ_1. Apply induction to $\pi_2 \circ \cdots \circ \pi_n$, a resolution of $\tilde{X}_1$ along Γ_1. (In fact, in this case, $n = 2$ and a blowing-up with center a curve reduces the singularity).

Suppose, finally, that $c_1 = 0$ and that $n > 1$. The first step π_1 is the blowing-up with center $C = \{y = z = 0\}$. Let $\tilde{X}_1$, Γ_1 be the strict transforms of X, Γ at $p_1 \in \pi_1^{-1}(\mathbf{0})$, the tangent of Γ. By the lemma above, we can suppose that π_1 satisfies $T2(w_1, w)$ for a system of adapted and normalized coordinates $w_1 = (x_1, y_1, z_1)$ at p_1. By induction, there exists an adapted change of coordinates, tangent to the identity, $\phi_1 : w_1 \mapsto \bar{w}_1$ and adapted normalized coordinates w_n for $\tilde{X}$ at p' such that $\pi_2 \circ \cdots \circ \pi_l$ is written as $T^{m,\epsilon}(w_n, \bar{w}_1)$. Write explicitly

$$\phi_1 : \begin{cases} \bar{x}_1 &= x_1 + z_1\varphi(x_1, y_1, z_1) \\ \bar{y}_1 &= y_1 + z_1\psi(x_1, y_1, z_1) \\ \bar{x}_1 &= z_1 \end{cases}$$

where φ, ψ are polynomials in z_1 whose coefficients are linear homogeneous functions of (x_1, y_1). Consider new coordinates w^* at $\mathbf{0}$ by means of the adapted change of coordinates

$$\phi^* : \begin{cases} x^* &= x + z\varphi(x, \frac{y}{z}, z) \\ y^* &= y + z^2\psi(x, \frac{y}{z}, z) \\ z^* &= z \end{cases}$$

We have $C = \{y^* = z^* = 0\}$ and π_1 satisfies $T_2(w^*, w_1)$. Thus $\pi = \pi_1 \circ \cdots \circ \pi_l$ is written in coordinates w_n and w^* as $T^{m+\epsilon,\epsilon}(w_n, w^*)$. We will finish if ϕ^* is tangent to the identity. But this is not necessarily true if $\varphi(0,1,0) = a \neq 0$. In this case, we consider again new coordinates $\bar{w} = (\bar{x}, \bar{y}, \bar{z})$ at $\mathbf{0}$ with $\bar{x} = x^* - ay^*, \bar{y} = y^*, \bar{z} = z^*$ and new coordinates $\bar{w}' = (\bar{x}', \bar{y}', \bar{z}')$ at p' with $\bar{x}' = x_n - ay_n(z_n)^{m+\epsilon}, \bar{y}' = y_n, \bar{z}' = z_n$. In these coordinates, π is written as $T^{m+\epsilon,\epsilon}(\bar{w}', \bar{w})$ and the changes of coordinates $\phi : w \mapsto \bar{w}$, $\phi_n : w_n \mapsto \bar{w}'$ are polynomial adapted and tangent to the identity. This finishes the proof. ▌

3.6.4. End of the proof

We finish now the proof of Theorem **??**. Let Γ be a non-degenerated axis for X at $\mathbf{0} \in \mathbb{R}^3$. Let $w = (x, y, z)$ be an initial system of coordinates such that $\Gamma = \{x = y = 0\}$ and $\Gamma^+ = \Gamma \cap \{z > 0\}$ is the corresponding semi-axis.

The elementary case. If $DX(\mathbf{0})$ is not nilpotent then Theorem **??** shows the existence of a monotone spiraling system of coordinates. Furthermore, as follows from the proof in 3.5, these system can be obtained from the initial one by means of a polynomial adapted change.

The nilpotent case. Suppose that $DX(\mathbf{0})$ is nilpotent but not identically zero. Assume that the linear term in $N X$ is $L_0 = y\frac{\partial}{\partial x}$. Let $\pi : M' \to \mathbb{R}^3$ be a resolution of singularities of X along Γ and denote by $\tilde{X}$ and Γ' the corresponding strict transforms at the elementary singularity $p' \in M'$. By the Maximal Contact Theorem **??**, up to a polynomial and adapted change of coordinates, we can assume that there exists a chart $(U', w' = (x', y', z'))$, adapted to Γ' at p' such that $\pi\mid_{U'}$ is written as $T^{m,\epsilon}(w', w)$, for some $m \geq 0$ and $\epsilon = 1$ or 2. Consider the 1-form $\omega_\theta = -ydx + xdy$ in $\mathbb{R}^3$. Its pull-back by π is given in U' by

$$\pi^*(\omega_\theta) = (z')^m \tilde{\omega}_{\theta,m} = (z')^m(-y'dx' + x'dy' + m\frac{x'y'}{z'}dz').$$

So, the foliations $\mathcal{F}_\theta = \{\omega_\theta = 0\}$ and $\tilde{\mathcal{F}}_{\theta,m} = \{\tilde{\omega}_{\theta,m} = 0\}$ are isomorphically related by π if we restrict to respective domains $\{z > 0\}$ and $\{z' > 0\}$. In particular, if we prove that $\tilde{X}$ is transversal to $\tilde{\mathcal{F}}_{\theta,m}$ in $V' \setminus \Gamma'$, for some neighbourhood V' of $\Gamma' \cap \{z' > 0\}$, then X will be transversal to $\mathcal{F}_\theta$ in $V \setminus \Gamma$, for $V = \pi(V')$ a neighbourhood of Γ^+. This will imply that w is a monotone spiraling system of coordinates for (X, Γ).

We distinguish two cases for the elementary singularity p' of $\tilde{X}$:

(i) p' is asymptotically monodromic. Then the linear term $L_0(x', y')$ of the normal linear part $N_{\Gamma'}\tilde{X}$ has eigenvalues with non-zero imaginary part. On the other hand, the order q' of $\tilde{X}$ along Γ' is ≥ 1; that is, $dz'(\tilde{X})(0,0,z')$ has order ≥ 2. Then we have

$$\tilde{\omega}_{\theta,m}(\tilde{X}) = (-y'dx' + x'dy')(L_0) + {r'}^2[z'\varphi_1 + r'\varphi_2]$$

with φ_1 and φ_2 bounded functions and ${r'}^2 = {x'}^2 + {y'}^2$. The sign of this function is constant in $V' \setminus \Gamma'$ for some neighbourhood V' of p'.

(ii) p' is asymptotically non-monodromic. In this case, L_0 has a double real eigenvalue and, moreover, by part 1) of Remark **??**, L_0 is not diagonalizable. Also, by part 2) of the same remark, we have $\epsilon = 1$. Apply Proposition **??** to find a new system of coordinates $\bar{w}' = (\bar{x}', \bar{y}', \bar{z}')$ at p' obtained from w' by a polynomial adapted change of the form

$$\begin{aligned} \bar{x}' &= x' \\ \bar{y}' &= y' + x'(\mu_1 z' + \cdots + \mu_n {z'}^n) \\ \bar{z}' &= z' \end{aligned}$$

such that $\tilde{X}$ is transversal to the foliation given by

$$\tilde{\omega}^{\bar{w}'}_{\theta,m} = -\bar{y}'d\bar{x}' + \bar{x}'d\bar{y}' + m\frac{\bar{x}'\bar{y}'}{\bar{z}'}d\bar{z}'.$$

Consider new coordinates $\bar{w} = (\bar{x}, \bar{y}, \bar{z})$ at $\mathbf{0}$ obtained from w by the same change as above without primes. Then the expression of π in the corresponding systems of coordinates $\bar{w}'$ and $\bar{w}$ is also $T^{m,1}(\bar{w}', \bar{w})$ and $\bar{w}$ is a monotone spiraling system for (X, Γ).

The general case. We can reduce the situation to one of the precedent cases by using Lemma **??**. In fact, with the same notations as in this lemma, let $l \geq 0$ be such that the strict transform $\tilde{X}_l$ by the point blowing-ups $\pi = \pi_1 \circ \cdots \circ \pi_l$ satisfies $D\tilde{X}_l(p_l) \not\equiv 0$. Consider coordinates $w_l = (x_l, y_l, z_l)$ at p_l such that $\pi = \pi_1 \circ \cdots \circ \pi_l$ is written in coordinates w_l, w as an l times composition $(T1)^l$ of transformation T1. The cases studied above show that there exists a monotone spiraling system of coordinates for Γ_l at p_l, obtained by means of a polynomial adapted change:

$$\bar{w}_l = (\bar{x}_l, \bar{y}_l, \bar{z}_l) = ((x_l, y_l)\ T(z_l), z_l).$$

The same change at $\mathbf{0}$ $\bar{w} = (\bar{x}, \bar{y}, \bar{z}) = ((x, y)\ T(z), z)$, gives coordinates such that the expression of π in coordinates $\bar{w}_l, \bar{w}$ is also $(T1)^l$. Thus, $\bar{w}$ is a monotone spiraling system of coordinates for (X, Γ).

REFERENCES

1. J. M. Aroca, H. Hironaka and J. L. Vicente, *The theory of the maximal contact*, Mem. Mat. Inst. Jorge Juan, **29**, Madrid. (1975).
2. I. Bendixon, *Sur les points singulières deséquations différentielles* , Ofv. Kongl. Vent. Ak.Forhan. ,**9**,186 (1898), 635-658.
3. P. Bonckaert, *Smooth invariant curves of singularities of vector fields on* $\mathbb{R}^3$, Ann. Inst. Henri Poincaré, Analyse non-linéaire **3** (1986), 111-183.
4. P. Bonckaert and F. Dumortier, *Smooth invariant curves for germs of vector fields in* $\mathbb{R}^3$ *whose linear part generates rotations*, Jour. Diff. Eq., **62** (1986), 95-116.
5. F. Cano, R. Moussu and F. Sanz, *Oscillation, spiralement, tourbillonnement*, Comm. Math. Helv., **75** (2000), 284-318.
6. J. Cano, *An extension of the Newton- Puiseux polygon construction to give solutions of Pfaffian forms*, Ann. de L'Institut Fourier, **43** (1993), 125-142.
7. F. Dumortier, *Singularities of vector fields on the plane*, Jour. Diff. Eq., **23** (1977), 53-106.
8. J. Ecalle, J. Martinet, R. Moussu and J. P. Ramis, J. P., *Non- accumulation des cycles-limites. II.*, C.R.A.S. **304** (1987), no. 14, 431-434.
9. P. Fortuny and F. Sanz, *Gradient Vector Fields do not generate Twister Dynamics*, Jour. Diff. Eq., **174** (2001), 91-100.
10. M. W. Hirsch, C. C. Pugh, C. C and M. Shub, *Invariant Manifolds*, Lect. Notes in Math., **583**, Springer-Verlag. (1977).
11. Y. S. Ilyashenko, *Finiteness theorems for limit cycles*, Translations of Mathematical Monographs, **94**. American Mathematical Society, Providence, RI (1991).
12. A. Kelley, *Stability of the Center-Stable Manifold*, Jour. of Math. Anal. and Appl., **18**, 336-344. (1967).
13. K. Kurdyka, T. Mostowski and A. Parusinski, *The gradient conjecture of R. Thom*, Ann. of Math. (2) **152** (2000), no. 3, 763-792.
14. S. Lojasiewicz, *Sur les trajectoires du gradient d'une fonction analytique*, Seminari di Geometria, Bologna (1983), 115-117.
15. A. M. Lyapunov, *Stability of Motion*, Ann. of Math. Studies, Princeton Univ. Press, **17** (1947).
16. R. Moussu, *Symétrie et forme normale des centres et foyers dégénérés*, Erg. Th. and Dyn. Sys., **2** (1982), 241-251.
17. J. Palis, F. Takens, *Topological equivalence of normally hyperbolic dynamical systems*, Topology, **16**, 335-345. (1977).
18. H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle*, J. Mathématiques, **7** (1881), 375-422.
19. A. Seidenberg, *Reduction of the singularities of the differential equation* $Ady = Bdx$, Am. J. of Math., **90** (1968), 248-269.
20. F. Takens, *Singularities of vector fields*, IHES, **43** (1974), 47-100.
21. W. Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, Intersciencie, New York, 1965.