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Successive Bifurcations at Infinity for Second Order O.D.E.'s

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We show that the classical procedure of succesive bifurcations at the origin can be applied also at infinity, both for polynomial and non-polynomial systems.

Key Words: Bifurcation from infinity, limit cycles, 16^{th} Hilbert's problem.

1. INTRODUCTION

In this paper we are concerned with large amplitude limit cycles of differential systems arising from second order scalar differential equations. The study of limit cycles of planar differential systems goes back to Poincaré, and had an enormous development since the appearance of the papers by Van der Pol [**?**] and Liénard [**?**], who lead the basis for the subsequent work. Proving the existence of a limit cycle means both finding a solution with a clear physical meaning, and giving an insight into the behaviour of the system in an open subset of the phase space.

The quest for limit cycles is also the subject of one of the most resistant problems of the list Hilbert presented at the International Congress of Mathematicians in 1900. Hilbert's 16^{th} problem asked for an upper bound to the number of limit cycles of a polynomial system of given degree. Recently, Smale proposed a modern version of Hilbert's 16^{th} problem, asking for un upper bound of the form d^q, where d is the degree of the polynomial system and q is a constant independent of d [**?**]. So far, both problems are still unsolved.

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The most relevant result related to such problems is the proof that the number of limit cycles of a polynomial system is finite, as shown independently in [**?**] and [**?**]. Unfortunately, those results did not lead to give estimates about the maximum number of limit cycles of polynomial systems. Even for quadratic systems we do not know an upper bound.

In this field, the most studied case is that of Liénard systems

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x), \tag{1}$$

equivalent to the second order differential equations

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{2}$$

where $F'(x) = f(x)$, $F(0) = 0$. This is due both to their relevance in applications, and to the possibility to reduce the study of several classes of systems to that of Liénard systems, by means of suitable transformations.

Most of the classical results about limit cycles of (**??**) were proved by showing, under suitable hypotheses, the existence of a positively compact solution. If this occurs, and some simple additional conditions are satisfied, by Poincaré-Bendixson theorem a limit cycle exists.

Results about the existence of several limit cycles are less frequent. The first paper showing the existence of polynomial Liénard systems with several limit cycles is probably [**?**]. In [**?**] Lins, de Melo and Pugh showed that for every integer k, there exists a $(2k+1)$-degree polynomial $F(x)$ such that the system

$$\dot{x} = y - F(x), \qquad \dot{y} = -x \ . \tag{3}$$

has exactly k limit cycles. They conjectured in [**?**] that k is the maximum number of limit cycles of a $(2k+1)$-degree polynomial system of type (**??**).

Let us observe that Lins-de Melo-Pugh's result allows to prove the existence of Liénard systems with k small amplitude (large amplitude) limit cycles. In fact, it is sufficient to apply to Lins-de Melo-Pugh's system the ε-dilatation

$$u = \varepsilon x, \qquad v = \varepsilon y.$$

The original system is taken into another Liénard system. Cycles are taken into small (large) cycles, according to the value chosen for ε. In this way one can show the existence of Liénard systems of degree $(2k+1)$ with k small (large) amplitude limit cycles. On the other hand, such a trick is not suitable to study bifurcation phenomena occurring when one of the cycles collapses into a critical point O, while the other ones remain out of a compact neighbourhood of O, or when a cycle expands to infinity, while the other ones remain inside a compact subset of the plane.

Zuppa [?] and Blows and Lloyd [?], studied bifurcations at the origin for the system (??). They showed that, by means of successive bifurcations, it is possible to construct polynomial systems of degree (2k+1) with exactly k small amplitude limit cycles. The procedure they followed in order to construct systems with at least k limit cycles consisted in a sequence of perturbations of a Liénard system with an asymptotically stable critical point. Let us consider a system of the form

$$\dot{x} = y - a_n x^n, \qquad \dot{y} = -x. \tag{4}$$

By adding a suitable $(n-2)$-degree perturbation, the stability of the origin changes, generating a limit cycle bifurcation. This can be replied up to get k small amplitude cycles. The methods they applied allowed also to show that no more than k limit cycles can bifurcate at the origin by perturbing (??). Their results were generalized in different ways. The recent paper [?] contains some new results about the system (??) and several references to previous results.

It is natural to think that, if lower degree perturbations generate small amplitude bifurcations, then higher degree perturbations should generate large amplitude bifurcations. On the other hand, in general it is not possible to attack the problem by the usual bifurcation techniques, due the difficulty to represent infinity as a singular point in such a way to preserve the good properties of the original system. For instance, if one performs the transformation $(x, y) \mapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$, that leads the point at infinity into the origin, the resulting system has a high degree of degeneracy at the origin. Some recent results about bifurcation at infinity are not applicable to second order differential equations, because they require the absence of singular points on the equator of Poincaré's sphere [?]. In fact, systems arising from nonlinear second order differential equations always have singular points on the equator. A bifurcation theorem dealing with systems with singular points at the equator is contained in [?], but deals with bifurcation from a separatrix.

The aim of this paper is to show that the first part of Zuppa, Blows and Lloyd procedure can be replied at infinity by applying a bifurcation theorem proved in [?]. Such a theorem is based on a purely topological approach. The local flow generated by (??) is studied on the one-point compactification of the plane, that allows to consider both polynomial and non-polynomial systems. Inversions of stability of the point at infinity are obtained as a consequence of inversions of boundedness properties of (??). Then, if the equilibrium points of (??) are uniformly bounded away from infinity, an inversion of the boundedness properties of the solutions generates an asymptotically stable invariant annulus bifurcating from infinity, containing at least a limit cycle. For instance, by applying Graef' theorem

[?], one can show that, starting from the system

$$\dot{x} = y + x, \qquad \dot{y} = -x,$$

and adding a suitable cubic perturbation, a large amplitude limit cycle bifurcates from infinity. This can be replied several times by adding higher and higher degree terms, up to get a $(2k+1)$-degree system with k large amplitude limit cycles. This proves that acting on higher degree terms one can actually reproduce at infinity the sequence of successive bifurcations studied in [?] and [?] at the origin.

Unfortunately, the method we present does not seem to be useful to give upper estimates to the number of limit cycles bifurcating at infinity. On the other hand, it allows to produce an example of analytic Liénard equation with infinitely many limit cycles, obtained by means of an infinite sequence of higher and higher degree perturbations (see theorem **??**).

Since the boundedness properties of second order O.D.E.'s have been widely studied (see [?], [?] and references therein), such a procedure can be easily applied to other classes of second order O.D.E.'s. This is the case, for instance, of Rayleigh equations,

$$\ddot{x} + f(\dot{x}) + g(x) = 0,$$

as shown in theorem **??**.

We may also combine successive bifurcations at infinity and at O, in order to get systems with assigned numbers of small and large amplitude limit cycles. In this paper we also show how to get simultaneous bifurcations at O and infinity. Moreover, by taking a nonlinear $g(x)$, we can produce simultaneous bifurcations at infinity and at several critical points. Also, we give an example of bifurcation at infinity for a non-polynomial Liénard equation.

For the reader's convenience, we describe some results that lead to prove the main bifurcation theorem. Several definitions and previous results are recalled in next section.

The theorems **??**, **??**, **??**, **??** are new.

2. RESULTS

We start reporting Lins, de Melo and Pugh's results about (**??**). Let us denote by $[x]$ the integer part of a real number x, that is $[x] = \max\{m \in N : m \leq x\}$.

THEOREM 1. *(Lins, de Melo, Pugh [?]). For every k, n integer, $n > 2$, $0 \leq k \leq [\frac{n-1}{2}]$, there exists an n-degree polynomial $F(x)$ such that (**??**) has exactly k limit cycles.*

Lins, de Melo and Pugh also formulated a conjecture about the maximum number of limit cycles that a system of the form (**??**) may have.

Conjecture. If (**??**) has degree n, then it cannot have more then $[\frac{n-1}{2}]$ limit cycles.

Such a conjecture is still unsolved.

Lins, de Melo and Pugh studied in more detail the system

$$\dot{x} = y - (a_1 x + a_2 x^2 + a_3 x^3), \qquad \dot{y} = -x \ . \tag{5}$$

about which they made the following remarks.

- if $a_1 a_3 > 0$, then (**??**) has no limit cycles;
- if $a_1 a_3 < 0$, then (**??**) has exactly one limit cycle;
- if $a_1 = 0$, $a_3 \neq 0$, then there are no limit cycles, and the origin is a weak attractor (repellor);
- if $a_1 \neq 0$, $a_3 = 0$, then there are no limit cycles, and the origin is a hyperbolic attractor (repellor);
- if $a_3 \neq 0$ and a_1 changes sign, then there is a Hopf bifurcation at the origin. For example, if $a_3 < 0$, then a limit cycle appears for small positive values of a_1.
- if $a_1 \neq 0$ and a_3 changes sign, then there is a Hopf-like bifurcation at infinity. For example, if $a_1 > 0$, then a limit cycle appears for small negative values of a_3.

In what above we can see a similarity between bifurcations at the origin and at infinity. The appearance of a low degree term, $a_1 x$, generates bifurcation of a limit cycle at the origin O. The appearance of a high degree term, $a_3 x^3$, generates bifurcation of a limit cycle at infinity.

This phenomenon is not bounded to cubic Liénard systems, as we show in the following.

Bifurcations of limit cycles at the origin for the n-degree system (**??**) were studied in detail in [**?**] and [**?**]. Here is the main result.

THEOREM 2. *(Zuppa [**?**], Blows and Lloyd [**?**]). For every k, n integer, $n > 2$, $0 \leq k \leq [\frac{n-1}{2}]$, there exists an n-degree polynomial $F(x)$ such that (**??**) has k small amplitude limit cycles.*

Proof. (sketch of proof, for odd n). For $n = 3$ the result comes from theorem (**??**). If $n > 3$, let us start with a system of the form

$$\dot{x} = y - a_n x^n, \qquad \dot{y} = -x, \qquad a_n > 0. \tag{6}$$

The origin is globally asympotically stable. Let us add a perturbation,

$$\dot{x} = y + a_{n-2} x^{n-2} - a_n x^n, \qquad \dot{y} = -x \qquad a_{n-2} > 0. \tag{7}$$

The stability of the origin changes, and a small ampitude limit cycle appears for small values of a_{n-2}. Then the same procedure can be applied again, reversing again the origin's stability,

$$\dot{x} = y - a_{n-4}x^{n-4} + a_{n-2}x^{n-2} - a_n x^n, \qquad \dot{y} = -x \qquad a_{n-4} > 0. \tag{8}$$

This procedure can be applied up to $[\frac{n-1}{2}]$ times, generating up to $[\frac{n-1}{2}]$ small amplitude limit cycles. ▮

Remark 3. The proof for even n requires only a minor modification. Initially one "forgets" the term $a_{2k}x^{2k}$, constructing the $n-1$-degree system

$$\dot{x} = y - (\mp a_1 x \pm \ldots - a_{n-1}x^{n-1}), \qquad \dot{y} = -x \ . \tag{9}$$

Such a system has $[\frac{n-2}{2}] = [\frac{n-1}{2}]$ small amplitude limit cycles. Then one adds the term $a_n x^n$. For small values of a_n, the new system still has $[\frac{n-1}{2}]$ limit cycles.

Both Zuppa and Blows and Lloyd applied variants of Poincaré-Liapunov method to study the stability of the origin. Both proved that $[\frac{n-1}{2}]$ is the maximum number of limit cycles that can bifurcate from O.

The method we present here in order to study bifurcation at infinity is an adaptation of that one applied in theorem **??**. In fact, we essentially follow the same procedure, considering the point at infinity as a fixed point of a suitable family of flows, defined on the one-point compactification of the plane.

For basic definitions about dynamical systems we refer to [**?**]. If $\pi(t, x)$ is a flow, we refer to $\pi(-t, x)$ as its negative flow. Properties of $\pi(-t, x)$ are referred to as negative properties of $\pi(t, x)$. For instance, an equilibrium point O is said to be negatively asymptotically stable w. r. to $\pi(t, x)$ if it is asymptotically stable for $\pi(-t, x)$.

Definition 4. Let X be a locally compact metric space with distance d. Let us consider $\mu^\sharp > 0$ and a continuous map $\pi : [0, \mu^\sharp) \times R \times X \to X$, $(\mu, t, x) \mapsto \pi_\mu(t, x)$. We say that π is a continuous family of flows if $\forall \mu \in [0, \mu^\sharp)$, $\pi_\mu(t, x)$ is a flow on X.

The following definition was given in [**?**].

Definition 5. Let X be a locally compact metric space with distance d, and C be the set of all proper, non-empty, compact subsets of X. Let us consider a map $K : [0, \mu^\sharp) \to C$, $\mu \mapsto K_\mu$, such that:

- $\forall \mu \in [0, \mu^\sharp)$, K_μ is π_μ-invariant,
- $\max\{d(x, K_0), x \in K_\mu\} \to 0$ as $\mu \to 0$,

then $\mu = 0$ is said to be a bifurcation point for the map K if there exists $\mu^* \in (0, \mu^\sharp)$, and a second map $M : (0, \mu^*) \to C, \mu \mapsto M_\mu$, satisfying the conditions:

- $\forall \mu \in (0, \mu^*)$, M_μ is π_μ-invariant and $K_\mu \cap M_\mu = \emptyset$,
- $\max\{d(x, K_0), x \in M_\mu\} \to 0$ as $\mu \to 0$.

Theorem 6. *(Marchetti, Negrini, Salvadori, Scalia [?]) Let X be connected and π be a continuous family of flows on X. Let $\mu^\sharp > 0$ and $K : [0, \mu^\sharp) \to C$ be a map as in definition 5. If K_0 is π_0-asymptotically stable and K_μ is π_μ-negatively asymptotically unstable for $\mu \in (0, \mu^\sharp)$, then $\mu = 0$ is a bifurcation point for K. Furthermore, the map M and μ^* can be chosen so that $\forall \mu \in (0, \mu^*)$, M_μ is π_μ-asymptotically stable.*

The proof of the above theorem is heavily based on topological properties of flows and level sets of Liapunov functions. With respect to analytic bifurcation theorems, theorem **??** has the advantage be applicable to a wider range of situations, being topological in nature. On the other hand, when it is possible to apply one of the usual analytic bifurcation theorems, one gets more information on the local properties of the flow.

In the special case of planar flows, theorem **??** allows to prove the existence of bifurcating limit cycles.

Corollary 7. *([?]) Let $X = R^2$ and π be a continuous family of flows on X. Assume that there exist $\mu^\sharp > 0$ and a neighbourhood U of O, such that for $\mu \in [0, \mu^\sharp)$, O is the only fixed point of π_μ belonging to U. If O is π_0-asymptotically stable and π_μ-negatively asymptotically stable for $\mu \in (0, \mu^\sharp)$, then a family of asymptotically stable invariant annuli M_μ bifurcate out of O as μ becomes positive. The inner and outer components of the boundary ∂M_μ of the bifurcating annuli M_μ are limit cycles of π_μ.*

In the following system one can apply corollary **??**, while Hopf' theorem and bifurcation theorems derived from Poincaré-Liapunov method cannot be used.

$$\dot{x} = \mu x + (y - x)(x^2 + y^2), \qquad \dot{y} = \mu y - (x + y)(x^2 + y^2). \tag{10}$$

In fact, for every value of μ, **??** has exactly one critical point at the origin O. The eigenvalues of the linearized system at O vanish for $\mu = 0$. This prevents to apply Hopf' theorem or Poincaré-Liapunov-like theorems. On the other hand, by using the Liapunov function $V(x, y) = x^2 + y^2$ one can easily prove that the origin O changes stability as μ becomes positive, hence, by the above corollary, a family of asympotically stable annuli M_μ

bifurcate out of O. The inner and outer components of ∂M_μ are limit cycles of (**??**), possibly coinciding.

We find a more interesting application if we consider $X = R^2 \cup \{\infty\}$, the one-point compactification of the plane. The topology of X is obtained from that one of R^2 by adding as new open sets the complements of compact subsets of R^2. Such new open sets are neighbourhoods of the point at infinity. Every flow $\pi(t,x)$ defined on R^2 admits a unique extension to a flow $\tilde{\pi}(t,x)$ on X, obtained by setting $\tilde{\pi}(t,\infty) \equiv \infty$. Such extensions were studied in [**?**], where several similarities between stability properties and boundedness properties of flows were analized.

Definition 8. A flow $\pi(t,x)$ is said to be ultimately bounded (UB) if there exists a compact subset K that is globally asymptotically stable. A flow is said to be negatively ultimately bounded (NUB) if the negative flow $\pi(-t,x)$ is ultimately bounded.

Theorem 9. *(Auslander and Seibert [**?**]) Let $\pi(t,x)$ be a flow on R^2. Then ∞ is asymptotically stable for $\tilde{\pi}(t,x)$ if and only if $\pi(t,x)$ is negatively ultimately bounded.*

Theorem **??**, in connection to theorem **??**, allows to show that bifurcation from infinity can be the consequence of changes in the boundedness properties of a family of flows. That was proved in [**?**]. The result holds in second countable, locally compact, non compact metric spaces. We recall here its two-dimensional version, that will be applied in order to count large amplitude limit cycles.

Theorem 10. *(Sabatini [**?**]) Let $X = R^2$ and π be a continuous family of flows on X. Assume that there exist $\mu^\sharp > 0$ and a compact set $N \subset R^2$, such that for $\mu \in [0,\mu^\sharp)$, there are no fixed points of π_μ out of N. Let π_0 be NUB and π_μ be UB for $\mu \in (0,\mu^\sharp)$. Then a family of asymptotically stable invariant annuli M_μ bifurcates from infinity as μ becomes positive. The inner and outer components of the boundary of the bifurcating annuli M_μ are limit cycles for π_μ.*

Remark 11. In the above theorem, as in the case of corollary 7, the inner and outer components of ∂M_μ can coincide. In this case there exists an aymptotically stable limit cycle.

Theorem **??** has a wide applicability, because there exist several boundedness theorems for second order differential equations. In fact, several results about the existence of limit cycles of second order equations come from a study of the boundedness properties of their solutions. Moreover, often a boundedness theorem can also be used to prove the unboundedness of solutions. Consider a second order scalar differential equation

$$\ddot{x} + h(x, \dot{x}) = 0. \tag{11}$$

If $x(t)$ is a solution, then $z(t) = x(-t)$ is a solution to the equation

$$\ddot{z} + h(z, -\dot{z}) = 0. \tag{12}$$

If the equation (**??**) satisfies the hypotheses of a boundedness theorem, then the solutions of (**??**) one are negatively bounded. In particular, the solutions of (**??**) are bounded if and only if those ones of

$$\ddot{x} - f(x)\dot{x} + g(x) = 0. \tag{13}$$

are negatively bounded, and vice-versa. Similarly, the solutions of system (**??**) are NUB if and only if the solutions of the system

$$\dot{x} = y + F(x), \qquad \dot{y} = -g(x) \tag{14}$$

are UB. We emphasize that (**??**) has not been obtained by multiplying the vector field by -1, but by using the equivalence of (**??**) and (**??**), of (**??**) and (**??**).

We report here the boundedness result that will be applied to study the number of large amplitude limit cycles of Liénard polynomial equations. We state it in a simpler form.

Theorem 12. *(Graef [**?**]). Let $f, g : R \to R$ be continuous, lipschitzian real functions. Let $F(x)$ satisfy $F(0) = 0$, $F'(x) = f(x)$. Assume there exists $c, r \in R$ such that*

- *$xF(x) > 0$, for $|x| > r$;*
- *$xg(x) > 0$, for $|x| > r$;*
- *either $F(x) \geq c > 0$ for $x > r$, or $F(x) \leq c < 0$ for $x < -r$.*

If additionally

$$\int_0^{\pm\infty} f(x) + |g(x)| = \pm\infty \tag{15}$$

*then the solutions of (**??**) are ultimately bounded.*

Graef' conditions are satisfied by (**??**) if F and g are odd-degree polynomials with positive leading coefficients. Graef' conditions are satisfied by (**??**) if F and g are odd-degree polynomials, g with positive leading coefficient, F with negative leading coefficient.

Remark 13. The solutions of the systems we consider are not always defined for all real values. In this case, we can work on the systems obtained by a suitable reparametrization of the time. For instance, we can replace the systems

$$\dot{x} = P_\mu(x,y), \qquad \dot{y} = Q_\mu(x,y) \tag{16}$$

with the systems

$$\dot{x} = \frac{P_\mu(x,y)}{1+P_\mu(x,y)^2+Q_\mu(x,y)^2}, \qquad \dot{y} = \frac{Q_\mu(x,y)}{1+P_\mu(x,y)^2+Q_\mu(x,y)^2}. \tag{17}$$

The new vector fields are bounded, hence every solution exists for all real values, so defining a family of flows on R^2. Moreover, the orbits of the new systems coincide with the orbits of the old ones, so that the topological properties of the systems coincide. In particular, their boundedness properties coincide. If (**??**) is a continuous family of vector fields, then also (**??**) is a continuous family of vector fields, generating a continuous family of flows.

We can now describe the procedure of generating several large amplitude limit cycles of Liénard equations by means of successive bifurcations at infinity.

THEOREM 14. *For every k, n integer, $n > 2$, $0 \le k \le [\frac{n-1}{2}]$, there exists an n-degree Liénard equation of type (**??**), having k large amplitude limit cycles obtained by successive bifurcations at infinity.*

Proof. Let us first prove the statement in the case of odd n. Also, we first show how to get n bifurcating limit cycles. Then, in order to get k limit cycles, $0 \le k \le [\frac{n-1}{2}]$, a minor change will be sufficient.

Let us start with a system of the form

$$\dot{x} = y + x, \qquad \dot{y} = -x. \tag{18}$$

This system, like all the systems that will be considered in this proof, has exactly one equilibrium point at the origin O. This ensures that all the bifurcating sets will be annuli and will contain limit cycles.

The point at infinity is globally asympotically stable w. r. to (**??**). Let us add a third degree perturbation.

$$\dot{x} = y + x - a_3x^3, \qquad \dot{y} = -x, \qquad a_3 > 0. \tag{19}$$

The leading coefficient of $F(x) = -x + a_3x^3$ is positive, hence the solutions of the above system are UB. Let us consider a_3 as a bifurcation parameter.

The stability of the point at infinity changes as a_3 becomes positive. By theorem **??**, a family of asymptotically stable invariant annuli M_{a_3} bifurcates from infinity as a_3 becomes positive. Every M_{a_3} contains at least a limit cycle γ_1.

Then the same procedure can be applied again, reversing again the stability of the point at infinity. Let us consider the system

$$\dot{x} = y + x - a_3x^3 + a_5x^5, \qquad \dot{y} = -x, \qquad a_5 > 0. \tag{20}$$

The leading coefficient of $F(x) = -x + a_3x^3 - a_5x^5$ is negative, so that the solutions of the new system are NUB. A new family of asymptotically stable invariant annuli M_{a_5} bifurcates from infinity as a_5 becomes positive. Each of them contains at least a limit cycle γ_2. Moreover, for small values of a_5, the total stability of M_{a_3} ensures that the system (**??**) still has a positively invariant compact set P_{a_3} in a neighbourhood of M_{a_3}. As a_3 tends to zero, the distance of M_{a_3} and P_{a_3} tends to zero (see [**?**], thm. (1.2) for details). Hence, for small values of a_3 and a_5, the system (**??**) has two large amplitude limit cycles.

Since n is odd, this procedure can be applied up to $[\frac{n-1}{2}]$ times, so generating up to $[\frac{n-1}{2}]$ large amplitude limit cycles.

If n is even, let us apply the above procedure up to degree $n-1$. This produces $[\frac{n-2}{2}] = \frac{n}{2} - 1$ limit cycles. Then we add the last perturbation

$$\dot{x} = y + x - a_3x^3 + \ldots + a_nx^n, \qquad \dot{y} = -x. \tag{21}$$

By the total stability of all the bifurcating annuli, for small values of a_n the last system still has $[\frac{n-2}{2}] = [\frac{n-1}{2}]$ large amplitude limit cycles.

In order to get an n-degree equation with k limit cycles bifurcating from infinity, $0 \leq k \leq [\frac{n-1}{2}]$, it is sufficient to stop the above procedure at the k-th step, and then add a perturbation of degree n, as done above in the case of even n. ▮

The method applied in the above theorem can be adapted to prove the existence of systems with both small amplitude and large amplitude limit cycles. All the cycles that appear in this way are concentric.

Theorem 15. *For every k, n integer, $n > 2$, $0 \leq k \leq [\frac{n-1}{2}]$, there exists an n-degree Liénard equation of type (**??**), having k small amplitude limit cycles obtained by successive bifurcations at the origin, and $[\frac{n-1}{2}] - k$ large amplitude limit cycles, obtained by successive bifurcations at infinity.*

Proof. We present the proof for odd n. The proof for even n can be obtained as in theorem **??**.

We combine the procedures of theorems **??** and **??**.

Let us start with a system of the form

$$\dot{x} = y + x^{2k+1}, \qquad \dot{y} = -x. \tag{22}$$

The point at infinity is globally asympotically stable. We add a suitable perturbation, in order to invert its stability.

$$\dot{x} = y + x^{2k+1} - a_{2k+3}x^{2k+3}, \qquad \dot{y} = -x, \qquad a_{2k+3} > 0. \tag{23}$$

We have obtained a family of large amplitude limit cycles bifurcating from infinity. We repeat the procedure until we reach degree n.

$$\dot{x} = y + x^{2k+1} - a_{2k+3}x^{2k+3} + \ldots \pm a_n x^n, \qquad \dot{y} = -x, \qquad a_n > 0 \tag{24}$$

Then we start generating bifurcations at the origin. We add perturbations of lower and lower degree, in order to have small amplitude limit cycles.

$$\begin{aligned} \dot{x} &= y \mp a_1 x \pm \ldots - a_{2k-1}x^{2k-1} + x^{2k+1} - \ldots \pm a_n x^n, \\ \dot{y} &= -x. \end{aligned}$$

Here all the a_j's are positive. After both steps, we have obtained k small amplitude limit cycles, and $[\frac{n-1}{2}] - k$ large amplitude limit cycles. ∎

One can also produce simultaneous bifurcations at infinity and at an equilibrium point. Consider the system

$$\dot{x} = y + \mu x - x^3 + \mu x^5, \qquad \dot{y} = -x. \tag{25}$$

The origin is the only equilibrium point of (**??**). As μ becomes positive, both O and the point at infinity change stability, so that there are simultaneous bifurcations at O and at infinity.

For higher degree systems one can produce more and more simultaneous bifurcations. In the following example, we first perturb by acting on μ_1, then by acting on μ_2. In general, we do not need a nondegenerate linear part at singular points.

$$\dot{x} = y + \mu_2 x - \mu_1 x^3 + x^5 - \mu_1 x^7 + \mu_2 x^9, \qquad \dot{y} = -x^{2h+1}. \tag{26}$$

It is possible to construct examples of simultaneous bifurcations at several equilibrium points and at infinity. Let us consider the following systems, where $g(x) = x(x-1)(x-2)(x-3)\ldots(x-2k)$, with k positive integer.

$$\dot{x} = y + \mu g(x) - g(x)^3 + \mu g(x)^5, \qquad \dot{y} = -g(x). \tag{27}$$

The points $(j, 0)$, $j = 0 \dots 2k$ are equilibrium points of (**??**). The stability of the points $(2i, 0)$, $i = 0 \dots k$, can be studied by means of the Liapunov functions $V_{2i}(x, y) = y^2 + 2G_{2i}(x)$, $i = 0 \dots k$, where $G'_{2i}(x) = g(x)$, $G_{2i}(2i) = 0$. Since $xg(x) > 0$ in a neighbourhood of $x = 2i$, $i = 0 \dots k$, the function $V_{2i}(x, y)$ is positive definite at the point $(2i, 0)$, $i = 0 \dots k$. The derivative $\dot{V}_{2i}(x, y)$ of $V_{2i}(x, y)$ along the solutions of (**??**) is $2(\mu g(x)^2 - g(x)^4 + \mu g(x)^6)$. For $\mu = 0$, we have $\dot{V}_{2i}(x, y) = -2g(x)^4$, hence the points $(2i, 0)$, $i = 0 \dots k$ are asymptotically stable. For $\mu > 0$, the dominant terms in $\dot{V}_{2i}(x, y)$ at $(2i, 0)$ are $y^2 + \mu g(x)^2$, hence the points $(2i, 0)$, $i = 0 \dots k$, are negatively asymptotically stable. As μ becomes positive, every such point changes stability, generating a small amplitude limit cycle. Similarly, the point at infinity changes stability as μ becomes positive, generating a large amplitude limit cycle. Then we get the simultaneous bifurcation of a large amplitude limit cycle and k small amplitude limit cycles.

There exist other classes of second order polynomial equations that can produce examples of n-degree systems with $[\frac{n-1}{2}]$ large amplitude limit cycles. This is the case of Rayleigh equation, which was widely studied for its relevance in applications (see, for instance, [**?**]).

$$\ddot{x} + f(\dot{x}) + g(x) = 0. \tag{28}$$

THEOREM 16. *For every k, n integer, $n > 2$, $0 \le k \le [\frac{n-1}{2}]$, there exists an n-degree Rayleigh equation, having k small amplitude limit cycles obtained by successive bifurcations at the origin, and $[\frac{n-1}{2}] - k$ large amplitude limit cycles, obtained by successive bifurcations at infinity.*

Proof. The scheme of the proof is the same as in theorem **??**. In this case we apply a different boundedness theorem, presented in [**?**] (see also [**?**] for the statement of a simplified version). Such a theorem shows that the solutions of the system

$$\dot{x} = y, \qquad \dot{y} = -g(x) - f(y),$$

equivalent to (**??**), are UB when f and g are odd-degree polynomials with positive leading coefficients, while they are NUB when they are odd-degree polynomials, g with positive leading coefficient, f with negative leading coefficient. We can take $g(x) = x$, or any other polynomial with $g'(x) > 0$, and work with $f(x)$ as done with $F(x)$ in theorem **??**.

Bifurcations at the origin can be easily generated by means of successive inversions of stability, by adding odd lower degree terms of $f(y)$. ▮

Successive bifurcations from infinity can be produced also in non-polynomial systems. Working as in theorem **??**, one can prove that there exist values of the parameters $a_3, \dots, a_{2k+1}$, such that the system

$$\dot{x} = y + x - a_3(\sinh x)^3 + \ldots + a_{2k+1}(\sinh x)^{2k+1}, \qquad \dot{y} = -x,$$

has k large amplitude limit cycles. This is due to the fact that the function $\sinh x$ is odd, increasing and superlinear, so that Graef' theorem can be applied in order to study the boundedness properties of the involved systems. In fact, in the above example $\sinh x$ could be replaced by any odd, increasing and superlinear function of class C^1.

So far, we have constructed examples of successive bifurcations at infinity similar to examples of successive bifurcations at the origin. In next theorem we construct a bifurcation procedure for which there is not a corresponding one at a critical point. The idea consist in performing infinitely many perturbations of higher and higher degrees, in order to get an analytic Liénard equation with infinitely many concentric limit cycles. This cannot be done by successive bifurcations at a critical point, because in that case perturbations' degrees decrease.

We first prove a technical lemma. If $v = (v_1, v_2)$, $w = (w_1, w_2)$ are two-dimensional vectors, we set $v \wedge w = v_1 w_2 - v_2 w_1$. If δ is a C^1 curve, we denote by δ' its tangent vector.

Lemma 17. *Let M be an asymptotically stable annulus of the differential system*

$$\dot{z} = v(z),$$

$z = (x, y) \in R^2$, $v \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, *having nontrivial cycles γ^i, γ^e as inner and outer components of its boundary. Then there exist C^1 curves δ^i enclosed by γ^i, δ^e enclosing γ^e, such that $\delta^{i\prime} \wedge v \neq 0$ on δ^i, $\delta^{e\prime} \wedge v \neq 0$ on δ^e.*

Proof. Let us set $v^\perp = (-v_2, v_1)$. The family of vector fields

$$v_\theta = v \cos\theta + v^\perp \sin\theta, \qquad \theta \in [0, 2\pi)$$

is a complete family of rotated vector fields, in the sense of Duff [**?**].

Without loss of generality, we can assume γ^e to be negatively oriented. Since M is asymptotically stable, γ^e is externally stable. By theorem 6 in [**?**], there exists an outer neighbourhood U^e of γ^e such that every $z \in U^e$ belongs to a cycle γ_θ of $\dot{z} = v_\theta(z)$, with $\theta > 0$, small enough. By the transversality of v and v_θ, we have $\gamma'_\theta \wedge v \neq 0$ on γ_θ. Then we can set $\delta^e \equiv \gamma_\theta$, for an arbitrary $\theta > 0$, small enough.

We can operate similarly for γ^i and δ^i, applying again theorem 6 of [**?**]. ∎

Remark 18. Due to the asymptotic stability of M, the curves δ^i, δ^e of the lemma can be chosen so that the annulus N having δ^i as inner boundary,

δ^e as outer boundary, is positively invariant. A similar statemente holds for negatively asymptotically stable sets M, which are contained in negatively invariant annuli. In next theorem, limit cycles will be negatively oriented, because for the involved systems, $\dot{y} = -x$. In this case external asymptotic stability implies the existence of a δ^e such that $\delta^{e\prime} \wedge v < 0$ on δ^e, and of a δ^i such that $\delta^{i\prime} \wedge v > 0$ on δ^i.

We say that a cycle is contained in a *period annulus* if it has a neighbourhood filled with nontrivial cycles.

THEOREM 19. *There exists an analytic Liénard equation with a unique singular point and infinitely many concentric limit cycles.*

Proof. We start as in the proof of theorem **??**, considering

$$\dot{x} = y + x, \qquad \dot{y} = -x.$$

By adding a cubic perturbation,

$$\dot{x} = y + x - a_3 x^3, \qquad \dot{y} = -x, \qquad a_3 > 0. \tag{29}$$

we produce a family of asymptotically stable invariant annuli M_3 bifurcating from infinity. Let us denote by v_3 the vector field associated to the system (**??**). By the above lemma, there exist C^1 curves δ_3^i, δ_3^e such that $\delta_3^{e\prime} \wedge v_3 \neq 0$, $\delta_3^{i\prime} \wedge v_3 \neq 0$, resp. on δ_3^e, δ_3^i. Due to the form of Liénard system, since M_3 is asymptotically stable, we have $\delta_3^{e\prime} \wedge v_3 < 0$ on δ_3^e, $\delta_3^{i\prime} \wedge v_3 > 0$, on δ_3^i. Let us denote by N_3 the annulus having δ_3^i as inner boundary, and δ_3^e as outer boundary.

Now let us apply a second perturbation,

$$\dot{x} = y + x - a_3 x^3 + a_5 x^5, \qquad \dot{y} = -x, \qquad a_5 > 0. \tag{30}$$

Let us denote by v_5 the corresponding vector field. A family of negatively asymptotically stable invariant annuli M_5, bifurcates from infinity as a_5 becomes positive. Let us choose a_5 small enough to have $\delta_3^{e\prime} \wedge v_5 < 0$, $\delta_3^{i\prime} \wedge v_5 > 0$, resp. on δ_3^e, δ_3^i. By the previous lemma and the negative asymptotic stability of M_5, there exist also C^1 curves δ_5^e, δ_5^i such that $\delta_5^{e\prime} \wedge v_5 > 0$ on δ_5^e, $\delta_5^{i\prime} \wedge v_5 < 0$ on δ_5^i.

By adding perturbations of higher and higher order, we construct a sequence of (negatively) asymptotically stable invariant annuli M_{2k+1}, with C^1 curves δ_{2k+1}^e, δ_{2k+1}^i, defining annuli N_{2k+1} such that

(i) $N_{2k+1} \cap N_{2h+1} = \emptyset$, for $k \neq h$; N_{2k+1} positively (negatively) invariant w. r. to v_{2k+1}, if M_{2k+1} is (negatively) asymptotically stable;

(ii) for $2h+1 \geq 2k+1$: ${\delta_{2k+1}^e}' \wedge v_{2h+1} \neq 0$, ${\delta_{2k+1}^i}' \wedge v_{2h+1} \neq 0$, resp. on δ_{2k+1}^e, δ_{2k+1}^i.

Moreover, we can choose the parameters a_{2k+3} small enough to satisfy
(iii) $\frac{|a_{2k+3}|}{|a_{2k+1}|} < \frac{1}{2k+3}$.
The power series

$$\sum_{k=1}^{\infty}(-1)^k a_{2k+1}x^{2k+1}$$

has radius of convergence ∞, because of condition (iii). Let us set $F(x) = x + \sum_{k=1}^{\infty}(-1)^k a_{2k+1}x^{2k+1}$. $F(x)$ is an analytic function defined on all of $\mathbb{R}$. We claim that the system

$$\dot{x} = y + F(x), \qquad \dot{y} = -x \tag{31}$$

has infinitely many limit cycles. Let us denote by v_∞ the corresponding vector field.

Assume M_{2k+1}, for some $k > 0$, to be asymptotically stable for v_{2k+1}, hence N_{2k+1} positively invariant w. r. to v_{2k+1}. Let $z = (x, y)$ be a point of δ^e_{2k+1}. Since ${\delta^e_{2k+1}}' \wedge v_{2k+1} < 0$ on δ^e_{2k+1}, for $h > k$ we have ${\delta^e_{2k+1}}' \wedge v_{2h+1} < 0$ on δ^e_{2k+1}, so that

$${\delta^e_{2k+1}}' \wedge v_\infty = {\delta^e_{2k+1}}' \wedge \left(\lim_{h\to\infty} v_{2h+1}(z)\right) = \lim_{h\to\infty} {\delta^e_{2k+1}}' \wedge v_{2h+1}(z) \le 0.$$

We can work similarly on δ^i_{2k+1}, proving that ${\delta^i_{2k+1}}' \wedge v_\infty \ge 0$. This proves that N_{2k+1} is positively invariant for v_∞.

Similarly, we can prove that if M_{2k+1} is negatively asymptotically stable for v_{2k+1}, then N_{2k+1} is negatively invariant for v_∞. Since the only critical point of v_∞ is the origin, in both cases we have proved that N_{2k+1} contains at least a cycle γ_{2k+1}.

It remains to prove that γ_{2k+1} is not contained in a period annulus. By absurd, let us assume that γ_{2k+1} is contained in a period annulus P, with inner boundary $\partial_i P$. Since there exists a unique critical point, either $\partial_i P$ is a singular point, or it is a cycle, or it is a generalized cycle containing a unique critical point. The origin is negatively asymptotically stable, hence neither the origin is a center, nor it can belong to a generalized cycle. The only remaining possibility is that $\partial_i P$ be a cycle. In this case we can consider its Poincaré's map, which is analytic and constant on one side of $\partial_i P$. Hence it has to be constant also on the other side of $\partial_i P$, so that $\partial_i P$ should have a neighbourhood of cycles, contradicting the fact that it is the inner boundary of P. This proves that γ_{2k+1} is a limit cycle of v_∞, completing the proof. ∎

Such a procedure, with minor changes, applies as well to Rayleigh equation.

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