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On the Geometric Structure of the Class of Planar Quadratic Differential Systems

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In this work we are interested in the global theory of planar quadratic differential systems and more precisely in the geometry of this whole class. We want to clarify some results and methods such as the isocline method or the role of rotation parameters. To this end, we recall how to associate a pencil of isoclines to each quadratic differential equation. We discuss the parameterization of the space of regular pencils of isoclines by the space of its multiple base points and the equivariant action of the affine group on the fibration of the space of regular quadratic differential equations over the space of regular pencils of isoclines. This fibration is principal, with a projective group as structural group, and we prove that there exits an open cone in its Lie algebra whose elements generate rotation parameter families. Finally we use this geometric approach to construct specific families of quadratic differential equations depending in a nonlinear way of parameters wh! ich have a geometric meaning : they parameterize the set of singular points or are rotation parameters leaving fixed this set.

Key Words: Quadratic systems, pencil of isoclines, rotation parameters

1. INTRODUCTION

We consider real planar polynomial differential systems, i.e., systems of the form

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$$D = Q(x,y)\ \frac{\partial}{\partial x}\ - P(x,y)\ \frac{\partial}{\partial y} \tag{1}$$

$$\frac{dx}{dt} = Q(x,y), \quad \frac{dy}{dt} = -P(x,y) \tag{2}$$

where P and Q are polynomial with real coefficients. In this work we are interested in the global theory of systems (**??**) with max $(\deg P, \deg Q) \leq 2$. Such systems will be called here quadratic systems. There are a number of long-standing open problems about polynomial differential systems, the most famous one being Hilbert's 16th problem, formulated by Hilbert in his address to the International Congress of Mathematicians in Paris in 1900. The second part of this problem asks to determine the maximum $I\!H(n)$ of the number of limit cycles which appear in systems of the form (**??**) with $n = \max\ (\deg\ P, \deg\ Q)$, and also their possible relative positions in $I\!R^2$. This problem is still unsolved even for quadratic systems. The difficulty of this problem is due to its twice global nature: analysis of the systems in their whole domain of existence, including the points at infinity and in the whole parameter space.

We know (cf. [**?**],[**?**]) that for a given system (**??**) the number of limit cycles is finite. For fixed n, if the coefficients of P and Q vary while max (deg P, deg Q) $\leq n$, we therefore have $I\!H(n) \leq \aleph_0$. Is $I\!H(n)$ finite ? The answer is not known, even for the quadratic case. In over 100 years since the statement of the problem, no example of a quadratic system was found for which we can prove that we have more than four limit cycles. Due to this, not only is it conjectured that $I\!H(2)$ is finite but also that $I\!H(2) = 4$.

At the moment, work on quadratic systems proceeds in two different directions: first, a program is under way to prove that $I\!H(2)$ is finite (cf. [**?**], [**?**]) and secondly, attempts are made to gain insight into the class of quadratic systems by studying specific subclasses or by attempting to understand more of the geometry of this whole class (cf. [**?**], [**?**]). Our work goes in this last direction. Work on specific classes of quadratic systems usually imply tedious calculations. All too often one becomes aware that there is just not enough mathematical structure around these calculations to make them more transparent.

Classification works on special classes of quadratic systems are done in terms of the coefficients appearing in the specific normal forms chosen for the equations (**??**) which are studied. Since coefficients change with coordinate changes and since in general the coefficients have no geometric meaning, arises the problem of classifying systems in more intrinsic ways and attaching geometric meaning to the parameters.

The first goal of this work is to introduce some geometrical structure which could help to make calculations more transparent and clarify a number of results and methods such as the *isocline method* or the role of the *rotation parameters.*

The article is written so as to be as self contained as possible. In Section 2 we define the space of quadratic differential equations and consider the action on the affine group on this space. In Section 3 we discuss the notion of isocline and associate a geometric object: a pencil of conics, to a quadratic system. In Section 4 we construct a principal fiber bundle associated to the space of quadratic equations. This fibration will be used to define some natural rotation parameter families and also to obtain parameterization of all vector fields with limit cycle, in which the parameters have a geometrical meaning : they move the singular points or they are rotation parameters.

2. THE SPACE OF QUADRATIC DIFFERENTIAL EQUATIONS AND THE ACTION OF THE AFFINE GROUP ON THIS SPACE

We are interested in the class of systems (**??**) with max (deg P,deg Q)=2, over the field $I\!\!K = I\!\!R$ or $\mathbb{C}$. The linear differential systems with constant coefficients are thus included since, understanding how systems (**??**) with max (deg P, deg Q)= 2 change with respect to parameters, also imply the limiting case when the systems become linear. A system (**??**) is also called vector field. When discussing issues not involving the time variable, we associate to the vector field (**??**) its dual differential form $\omega = P(x,y)dx + Q(x,y)dy$ and its associated differential Pfaff equation :

$$\omega = P(x,y)dx + Q(x,y)dy = 0 \tag{3}$$

The integral manifolds of the Pfaff equation (**??**) coincide with the phase curves of the vector field (**??**).

Two equations (**??**) differing from one another by multiplication with a non-zero constant, have identical integral manifolds. Then, leaving aside the trivial case when both P and Q are identically zero, we regard a Pfaff equation as a point in the projective space associated to the vector space $I\!\!K_2[x,y] \times I\!\!K_2[x,y]$ (isomorphic to $I\!\!K^{12}$), where $I\!\!K_2[x,y]$ is the

6-dimensional vector space of polynomials of degree than or equal to 2.

We denote by $F(\mathbb{K})$ this projective space, which is isomorphic to $P^{11}(\mathbb{K})$. If ω is a non zero Pfaff form, we write $[\omega]$ the class of the equation $\{\omega = 0\}$ in $F(\mathbb{K})$.

We consider affine transformations, i.e. maps $f : \mathbb{K}^2 \longrightarrow \mathbb{K}^2$, where $f(x, y) = (a_{11}x + a_{12}y + b_1,\ a_{21}\ x + a_{22}y + b_2)$, for a_{ij} and b_{ij} in $\mathbb{K}$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Let $A(2, \mathbb{K})$ be the affine group of transformations of $\mathbb{K}^2$:

$$A(2, \mathbb{K}) = \{f : \mathbb{K}^2 \longrightarrow \mathbb{K}^2 \ \mid \ f \text{ is an affine transformation}\}. \tag{4}$$

The affine group $A(2, \mathbb{K})$ acts on $F(\mathbb{K})$, the space of Pfaffian equations, by :

$$f \cdot [\omega] = [f^*\omega]. \tag{5}$$

This is a right action, induced by the action f^* on the 1-forms, which is given explicitely by :

$$f^*(Pdx + Qdy) = P_f dx + Q_f dy \tag{6}$$

with :

$$(P_f(x, y), Q_f(x, y)) = (P \circ f(x, y), Q \circ f(x, y))M_f \tag{7}$$

where $M_f = ((a_{ij}))$ is the matrix associated with the transformation f.

The foliation with singularities defined by a Pfaff equation $[\omega]$ is sent onto the foliation of $f \cdot [\omega]$ by the action on an element of $f \in A(2, \mathbb{K})$. Then, it is natural to consider the orbit space $F(\mathbb{K})/A(2, \mathbb{K})$. Since $\dim_{\mathbb{K}}(A(2, \mathbb{K})) = 6$ and $\dim_K(F(\mathbb{K})) = 11$, this orbit space has dimension 5. We are interested in the classification of the phase portraits of systems (**??**) under orbital equivalence, i.e. equivalence under homeomorphisms which preserve orientation of orbits (it is equivalent to consider the topological equivalence of Pfaff forms). For this classification, it will be sufficient to consider moduli for $A(2, \mathbb{K})$-action, i.e., families induced by a map $\Lambda \xrightarrow{\varphi} F(\mathbb{K})$ with the property that the image of φ contains at least one point on each $A(2, \mathbb{K})$-orbit. In the case $\mathbb{K} = \mathbb{R}$ and if we are just interested in the limit cycles, it will suffice that the image of φ contains one representative in each $A(2, I\!R)$-orbit of the Pfaffian equation with at least one anti-saddle singular point. There are many examples of such families with 6 and even 5 parameters (See [?],[?] for instance). Every one is defined by some linear map φ. In the final paragraph, we shall deduce from our study new examples of families with 5 and 4 parameters, whose interest is link! ed to the geometric interpretation of the parameter space. On the other side, the maps φ will be not linear, but polynomial.

It is a classical method to extend a Pfaffian equation (??) on $I\!K^2$ as a cubic homogeneous Pfaff equation on $I\!K^3$ [?]. Let Ω be defined by :

$$\Omega = \widetilde{P}(X,Y,Z)dX + \widetilde{Q}(X,Y,Z)dY + \widetilde{R}(X,Y,Z)dZ = 0 \tag{8}$$

where $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{R}$ are cubic homogeneous polynomials over $I\!K$ such that :

$$X\widetilde{P} + Y\widetilde{Q} + Z\widetilde{R} \equiv 0, \quad \text{in} \quad I\!K[X,Y,Z]. \tag{9}$$

The Pfaff equation (??), subject to the condition (??), induces a *singular foliation* on the projective space $P^2(I\!K)$.

Now, any quadratic Pfaffian equation (??) can be extended as an equation (??), subject to the condition (??), by taking :

$$\begin{array}{l} \widetilde{P} = Z^3 P\Big(\frac{X}{Z}\ ,\ \frac{Y}{Z}\Big) \ \ \widetilde{Q} = Z^3 Q\Big(\frac{X}{Z}, \frac{Y}{Z}\Big) \\ \widetilde{R} = -Z^2\Big(XP\Big(\frac{X}{Z}, \frac{Y}{Z}\Big) + YQ\Big(\frac{X}{Z}, \frac{Y}{Z}\Big)\Big). \end{array} \tag{10}$$

The Pfaffian equations (??) can be characterized among the equations (??) , as the ones defining foliations with singularities which have the line at infinity : $\ell_\infty = \{Z = 0\}$ as an invariant line. The projective group $P(2, I\!K) = PGL(3, I\!K)$ acts on the space $F_{proj}(I\!K)$ of all cubic Pfaff equations (??), subject to the condition (??), and $A(2, I\!K)$ is the subgroup leaving invariant the line at infinity ℓ_∞. But in the following paragraph we shall restrict ourself essentially to the study of the space $F(I\!K)$. Nevertheless, in order to treat the general situation, pencils of conics which will be associated to $[\omega] \in F(I\!K)$, will be considered in $P^2(I\!K)$, which is the natural ambient space for the conics.

3. THE METHOD OF ISOCLINES. PENCILS OF CONICS ASSOCIATED TO QUADRATIC SYSTEMS

In this paragraph, we associate to any quadratic differential Pfaffian equation, a geometric object : the set of its *isoclines.* First of all, recall the concept of isocline.

We shall denote by $[x_1, \cdots, x_{n+1}]$ the point of $P^n(I\!K)$ associated to the vector $(x_1, \cdots, x_{n+1})$ in $I\!K^{n+1} - \{0\}$. Consider a quadratic Pfaff equation $[\omega] \in F(I\!K)$, represented by the differential form $\omega = Pdx + Qdy$. Let $\Sigma[\omega]$ be the set of its singular points in $I\!K^2$. We define a map $I_{[\omega]} : I\!K^2 \backslash \Sigma[\omega] \longrightarrow P^1(I\!K)$ by :

$$I_{[\omega]}(m) = [Q(m), -P(m)]. \tag{11}$$

It is the line in $I\!K^2$ defined by the kernel of ω at m, or equivalently, directed by the dual vector field $-Q \dfrac{\partial}{\partial x} + P \dfrac{\partial}{\partial y}$ at the point m. This map is algebraic. Given any $\ell = [u, v] \in P^1(I\!K)$, $I^{-1}_{[\omega]}(\ell)$ extends into the algebraic curve with equation : $\{uP + vQ = 0\}$.

Definition 1. Let $\omega = Pdx + Qdy$ be a quadratic differential 1-form which represents an element $[\omega] \in F(I\!K)$. Let $\ell = [u, v]$ be a point of $P^1(I\!K)$. The *isocline* of $[\omega]$ *with slope* ℓ, is the algebraic curve C_ℓ : $\{uP + vQ = 0\}$. The curve C_ℓ is equal to $I^{-1}_{[\omega]}(\ell) \cup \Sigma[\omega]$.

The isoclines are introduced in elementary courses as a convenient method to tracing direction fields. The school of Erugin ([**?**], [**?**], [**?**]) used isoclines portraits for gaining insight into quadratic systems, as V. Gaiko indicates in [**?**] where he refers to [**?**], [**?**]. This is the so-called "isoclinic method", a term which according to V. Gaiko was introduced by Nemytskii and Stepanov. In this paper, we also intend to give a more precise meaning to this term by introducing more mathematical structure in the context of quadratic systems.

First, we shall introduce the space of conics : $\mathcal{C}(I\!K)$, because it is the space where the isoclines live. In general the set of isoclines of a given Pfaff equation $[\omega]$ is a line in $\mathcal{C}(I\!K)$, called the *pencil of conics* associated to $[\omega]$. We shall consider in detail these pencils, how they are determined by their singular sets and the different possible situations which may occur.

3.1. The projective space of conics : $\mathcal{C}(I\!\!K)$.

Let $P(x,y) \in I\!\!K[x,y]\backslash\{0\}$ with $\deg(P) \leq 2$. Associated to P is the algebraic set defined by the equation :

$$P(x,y) = a_0 + a_1\ x + a_2\ y + a_{20}\ x^2 + a_{11}\ xy + a_{02}\ y^2 = 0. \qquad (12)$$

This set could be empty or it could be a line or a conic in the affine plane $I\!\!K^2$. This set does not change if we multiply the equation (**??**) by a non-zero factor. So the sets defined by (**??**) in $I\!\!K^2$ correspond biunivocally to points $[P] = [a_0, a_1, a_2, a_{20}, a_{11}, a_{02}] \in P^5(I\!\!K)$. To each such point (or equation (**??**)) we can associate a conic in $P^2(I\!\!K)$ defined by the equation :

$$P^*(X,Y,Z) = 0. \qquad (13)$$

where $P^* = a_0\ Z^2 + q_1\ XZ + a_2\ YZ + a_{20}\ X^2 + a_{11}\ XY + a_{02}\ Y^2$.

In particular if $a_0 \neq 0$, to the empty set defined by the equation $\{a_0 = 0\}$ in $I\!\!K^2$, we associate the conic $\{a_0 Z^2 = 0\}$ in the projective space; to a line of $\mathbb{C}^2$ we associate a reducible conic in $P^2(I\!\!K)$, formed by the line and the line at infinity $\{Z = 0\}$.

NOTATION 2. *We shall call $\mathcal{C}(I\!\!K)$ the projective space of all conics* (**??**) *in $P^2(I\!\!K)$. This space is identified with $P^5(I\!\!K)$. We shall denote by $[P^*]$ (or simply by $[P]$) the conic defined by the equation* (**??**) *in $P^2(I\!\!K)$.*

DEFINITION 3. We say that a conic $[P]$ is *irreducible* if $P^*(X,Y,Z)$ is an irreducible polynomial of degree 2 over $I\!\!K$. A *reducible* (or *singular*) conic is given over $I\!\!K$ by a reducible polynomial $P^*(X,Y,Z) = \ell_1(X,Y,Z).\ell_2(X,Y,Z)$ where ℓ_1, ℓ_2 are linear factors. The conic $[P]$ is then the union of two lines (which may be at infinity if $\deg P \leq 1$). We shall say that the conic is *non degenerate* if it does not reduce to a double-line.

NOTATION 4. *Let us denote by $L(I\!\!K)$ the space of lines in $P^2(I\!\!K)$ and by $S\mathcal{C}(I\!\!K)$ the set of reducible conics.*

The set of reducible conics $S\mathcal{C}(I\!\!K)$ is clearly a closed algebraic subset of $\mathcal{C}(I\!\!K)$, which is parameterized by $L(I\!\!K) \times L(I\!\!K)$. The space $L(I\!\!K)$ is the same as the Grassmannian space of 2-dimensional subspaces in $I\!\!K^3$ and therefore its dimension is equal to 2. The dimension of the set $S\mathcal{C}(I\!\!K)$ is equal to 4.

The group $A(2, I\!\!K)$ of all affine transformations acts naturally on the space $\mathcal{C}(I\!\!K)$, on the right :

$$(a,[P]) \in A(2,\mathbb{K}) \times \mathcal{C}(\mathbb{K}) \longrightarrow a \cdot [P] = [P \circ a] \in \mathcal{C}(\mathbb{K}). \qquad (14)$$

Also, the projective group $P(2,\mathbb{K})$ acts on $\mathcal{C}(\mathbb{K})$. Even if we are principally interested in the study of Pfaff equations modulo the action of $A(2,\mathbb{K})$, it will be convenient to consider the action of the projective group on conics, to simplify the proofs, by simplifying the classification. For instance if $\mathbb{K} = \mathbb{C}$, the action of $P(2,\mathbb{C})$ on $\mathcal{C}(\mathbb{C})$ has just three orbits : the orbit of the "double-line" $[x^2]$, the orbit of the singular conic $[x^2+y^2]$ and of the circle $[x^2+y^2-1]$. On the other hand, the classification of the orbits of $\mathcal{C}(\mathbb{R})$ under the action of $A(2,\mathbb{R})$ is more complicated. But in any case we have just a finite number of orbits (this is related to the fact that the dimension of each group we consider is greater than or equal to 6 and the dimension of the space $\mathcal{C}(\mathbb{K})$ is equal to 5).

We shall now prove the following basic result :

Lemma 5. *Given five distinct points in $P^2(\mathbb{K})$, any four of them not on a same line, there exists one and only one conic over $\mathbb{K}$ passing through them. This conic is non degenerate (but it may be reducible).*

Proof. First, let us suppose that three of the points are on a same line ℓ. Any conic containing these points must contain ℓ, and then it is reducible. The two other points define a line ℓ', *distinct from* ℓ, and the only conic through the five points is the reducible one : $\ell \cup \ell'$.

Let $\{p_1, \ldots, p_5\}$ be the subset of these five points and suppose now that no line in $P^2(\mathbb{K})$ contains three of them. The condition that these five points belong to a same conic define a system of five linear equations on the coefficients of the conic. If the rank of the system matrix is strictly less than 5, this means that one of these equations is a linear combination of the other ones. Let us suppose that it is the last one. Let ℓ and ℓ' the lines passing through the points p_1, p_2 and p_4, p_5 respectively. Then, the coefficients of the reducible conic $\ell \cup \ell'$ satisfy the last equation It follows that the point p_5 is on ℓ and ℓ'. This contradicts the hypothesis that no line contains three of the points p_i.

The above proof is valid in any field. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, it is possible to give a more geometric proof. Let us suppose first that $\mathbb{K} = \mathbb{C}$. Let us suppose again that no line contains three of the points. By the action of an element of $P(2,\mathbb{C})$ we can bring two of the points on $\ell_\infty = \{Z = 0\}$ in position $(\pm i, 1, 0)$. Any conic of $\mathbb{C}^2$ whose completion in $P^2(\mathbb{C})$ passes through these two points is a complex circle (which may be reducible). Here, by a complex circle we mean a conic defined by a polynomial with quadratic part equal to $x^2 + y^2$. The three other points are in $\mathbb{C}^2$, not on a same line. Then the problem reduces to the classical one :

Finding a circle passing through three points of $\mathbb{C}^2$, not on a same line.

This problem is equivalent to the system of the 3 linear equations in $(a, b, c) \in \mathbb{C}^2$: $ax_i + by_i + c = -(x_i^2 + y_i^2)$ for $i = 1, 2, 3$, where (x_i, y_i) are the given points. Clearly this system has a unique solution.

Let now $\mathbb{K} = \mathbb{R}$ and five points in $P^2(\mathbb{R})$. The above proof gives us a complex conic through these points. The real or the imaginary part of the equation of this conic is the equation of a real conic through the given set of points. More generally, given a set of five points in $P^2(\mathbb{C})$, globally invariant by conjugacy, it is possible to find a real conic containing it. For instance if three of the points are real in $\mathbb{R}^2$ and the two other ones are complex and conjugate on the line at infinity, we can bring these two last points in position $(\pm i, 1, 0)$. Then the problem reduces to find a real circle passing through 3 points of $\mathbb{R}^2$, not on a same line. ▮

Remark 6. If four points were on a same line ℓ and the fifth one e not on ℓ, any reducible conic $\ell \cup \ell'$ where ℓ' is a line through e, would be a solution. If the five points were on the same line ℓ, then all reducible conics $\ell \cup \ell'$, where ℓ' is arbitrary, would be a solution.

We would like to generalize lemma **??** to include the case when two or more of the five points could coalesce. Thinking that the line through the two points m_1, m_2 will have a limiting position δ, a *double* point may be thought as an contact element, i.e. an ordered couple (m, δ) with $m = m_1 = m_2$. Analogously a *triple* point may be thought as (m, δ^2) with $m = m_1 = m_2 = m_3$ and δ the limiting position of the lines through the couple of points $(m_1, m_2), (m_1, m_3)$ respectively. More generally we have the following definition :

Definition 7. A *point of multiplicity* $n + 1$ with $n \geq 0$ is defined as follows :
(i) If $n = 0$, a point of multiplicity 1 or a simple point is just a point m of $P^2(\mathbb{K})$.
(ii) If $n > 0$, a point of multiplicity $n+1$ is an ordered couple (m, δ^n) where $m \in P^2(\mathbb{K})$ and δ is a line in $P^2(\mathbb{K})$ such that $m \in \delta$.
A point with some multiplicity (even equal to one) will be also called a *multiple point.*

We want to recall now what is the intersection number of a line with an algebraic curve in $P^2(\mathbb{K})$:

Definition 8. Let δ be a line, γ be an algebraic curve defined by the equation $\{f = 0\}$ and m a point in $\gamma \cap \delta$. Let $\varphi(t) = m + tb, \quad t \in \mathbb{K}$ be a parameterization of δ, with $b \in P^2(I\!K), b \neq m$. The intersection number $I_m(\delta, \gamma)$ is the order of the function $\Phi(t) = f \circ \varphi(t)$ at $t = 0$. In other terms, $I_m(\delta, \gamma)$ is equal to $\inf\{k \mid \Phi^{(k)}(0) \neq 0\}$. For instance $I_m(\delta, \gamma)$ is infinite if and only if $\delta \subset \gamma$.

Remark 9. We point out that the above definition is a particular case for the definition of the intersection number $I_m(\gamma_1, \gamma_2)$ of two algebraic curves $\gamma_1 = \{f = 0\}, \gamma_2 = \{g = 0\}$ at a point m of $P^2(I\!K)$ (cf. [**?**] for the axiomatic definition of $I_m(\gamma_1, \gamma_2)$ and [**?**] for the definition using the resultant of the polynomials f, g).

Now, in terms of intersection number we can say what means that an algebraic curve passes through some multiple point :

Definition 10. We shall say that *an algebraic curve γ passes through the multiple point* (m, δ^n), for $m \in \gamma$, if and only if the intersection number $I_m(\gamma, \delta)$ is greater than or equal to $n + 1$.

Hereafter, we just consider conics. It is possible to list all the possible cases for a conic γ passing through a multiple point (m, δ^n) :

(i) If $n = 0$, i.e. if m is a simple point, this just means that m belongs to γ,

(ii) if $n = 1$, i.e. if (m, δ) is a double point, then δ is the tangent line at a regular point m of γ or δ is any line through a singular point m of a reducible conic γ,

(iii) if $n > 1$, the conic γ must be reducible and $\delta \subset \gamma$.

Definition 11. By a *subset σ of ℓ points counted with multipliplicity in* $P^2(I\!K)$, we shall mean a subset of k distinct multiple points (cf. definition **??**) : $\{(m_1, {\delta_1}^{n_1}), \ldots, (m_k, {\delta_k}^{n_k})\}$, with $k \leq \ell$, such that $m_i \neq m_j$ if $i \neq j$ and $\sum_i (n_i + 1) = \ell$.

Some of the points m_i may be simple, i.e. of multiplicity 1. In which case the point is of the form (m_i, δ_i^0) for *any* line δ_i containing m_i, *but we shall write it simply : m_i*. We call *support* of σ the set $\mid \sigma \mid = \{m_1, \ldots, m_k\} \subset P^2(I\!K)$.

We shall say that a conic γ passes through σ or contains σ (or that σ is contained in γ) if γ passes through each multiple point $(m, \delta^n) \in \sigma$, in the sense of the definition **??**.

Now, the lemma **??** has a direct generalization for subsets of five points counted with multiplicity :

Lemma 12. *Let us consider a subset σ of five points counted with multiplicity in $P^2(I\!K)$, as it is defined above. Suppose that the multiplicity of* *each of them is less than or equal to 3. Moreover suppose that σ does not contain one of the following configurations :*

(i) four simple points on a same line of $P^2(I\!K)$,

(ii) two (multiple) points on the line of a double point,

(iii) a pair of double points with a common line,

(iv) a simple point on the line δ of a triple point. Then there exists one and only one conic passing through these points.

Proof. As in the previous lemma, it is possible to give an algebraic proof which would be valid in any field (see the remark **??** below). We prefer to give a geometric proof in $\mathbb{C}$. As above, the real case follows easily.

First, let us suppose that three points are on a same line ℓ, or that a simple or double point belongs to the line ℓ of another double point or that one has a triple point (m, ℓ^2) in σ. In these cases, any conic through the points must be reducible and contains ℓ. Moreover an investigation of all possible cases shows that the conic solution exists and is unique. It would be fastidious to list all these cases. We just mention one of them: σ is formed by two double points $(m_1, \delta_1), (m_2, \delta_2)$ and a simple point m_3, such that the points m_1, m_2, m_3 are on a same line ℓ, with of course $\delta_1, \delta_2 \neq \ell$. In this case, the unique solution is the degenerate conic ℓ^2.

Let us suppose now that σ is generic, i.e. not as above. In comparison with the previous lemma, we have just two new possibilities :

(i) σ contains just one double point (m, δ) and three simple points, not on a same line and not on δ. We can bring two of them at the positions $(\pm i, 1, 0)$. Then the problem reduces to the construction of a (complex) circle tangent to a line δ at a point $m \in \delta$ and passing through another point of $\mathbb{C}^2$, not on δ.

(ii) σ contains two double points $(m_1, \delta_1), (m_2, \delta_2)$ and a simple point m_3 not on the line ℓ through m_1, m_2, with of course δ_1 and δ_2 different from ℓ. We can bring the line δ_2 to the line at infinity. Then the problem reduces to the construction of a parabola tangent to a line δ_1 at a point $m_1 \in \delta_1$, $m_1 \in \mathbb{C}^2$, and passing through a point $m_3 \in \mathbb{C}^2$ not on δ_1. We can also bring the two points m_1, m_2 at the positions $(\pm i, 1, 0)$ and the intersection point $\delta_1 \cap \delta_2$ at the origin of $\mathbb{C}^2$. Then the problem reduces to find a complex circle centered at the origin and passing through a point of $\mathbb{C}^2$. ▮

Remark 13. (a) The conditions (ii),(iii) and (iv) on multiple points in the lemma **??** are just the limiting situations of the condition (i) on simple points. This condition is the condition which is given in the lemma **??**.

(b) Of course, the discussion made in the remark after the lemma **??** can be extended to multiple points.

3.2. Pencils of conics.

DEFINITION 14. A *pencil of conics* is a line, i.e. a 1-dimensional projective subspace of the projective space $\mathcal{C}(I\!K)$. Let $[P]$, $[Q]$ be two *distinct* conics in $\mathcal{C}(I\!K)$. The pencil of conics generated by them is the line passing through them :

$$\alpha([P],[Q]) = \Big\{[uP + vQ] \mid [u,v] \in P^1(I\!K)\Big\} \subset \mathcal{C}(I\!K). \tag{15}$$

Remark 15. In agreement with the notation $\alpha([P],[Q])$ we shall often designate a pencil by α, when we shall not make reference to a special choice of two conics $[P],[Q]$ generating it.

The set of all pencils of conics is then the set of all projective lines in $\mathcal{C}(I\!K) \simeq P^5(I\!K)$. It coincides with the Grassmannian manifold of 2-dimensional subspaces in $I\!K^6$.

NOTATION 16. *We shall denote by* $G(I\!K)$, *this set of all pencils of conics, with the induced algebraic structure of Grassmannian manifold.* $G(I\!K)$ *is a manifold of dimension 8 over* $I\!K$.*(For a general introduction to linear systems of algebraic curves in* $P^2(I\!K)$, *one may see [**?**]).*

In order that the set of isoclines associated to a given Pfaff equation is a pencil of conics, we have to restrict to *non trivial Pfaff equations.*

DEFINITION 17. We say that $[\omega]$ is a non trivial Pfaff equation if it is represented by $\{\omega = Pdx + Qdy = 0\}$ with $P, Q \neq 0$ and $[P] \neq [Q]$. We shall denote by $E(I\!K) \subset F(I\!K)$ the set of all non trivial Pfaff equations.

Remark 18. $E(I\!K)$ is an Zariski open set of $F(I\!K)$. Its complement $S(I\!K)$ is made by the equations $[\omega]$ such that $\omega = P(adx + bdy)$, $[a,b] \in P^1(I\!K)$ and $[P] \in \mathcal{C}(I\!K)$. Then, $S(I\!K) \simeq P^5(I\!K) \times P^1(I\!K)$ is a closed algebraic subset of dimension 6 in $E(I\!K)$.

DEFINITION 19. The *set of isoclines* associated to a non trivial Pfaff equation $\{\omega = Pdx + Qdy = 0\}$ is the pencil of conics : $\{[uP + vQ] \mid [u,v] \in P^1(I\!K)\}$.

This defines a map from $E(I\!K)$ to $G(I\!K)$, we denote again by α,

$$\text{given by } \ \alpha([\omega]) = \alpha([P],[Q]) \ \text{ where } \ \omega = Pdx + Qdy. \tag{16}$$

The action of the group $A(2, I\!K)$ on $\mathcal{C}(I\!K)$ induces an action on $G(I\!K)$, the space of lines in $\mathcal{C}(I\!K)$:

$$g \cdot \alpha([P],[Q]) = \alpha([P \circ g],\ [Q \circ g]). \tag{17}$$

for $g \in A(2, I\!\!K)$ and $[P], [Q] \neq 0, [P] \neq [Q]$.

This action is well-defined because the right hand term in (**??**) is independent of the choice of the conics $[P], [Q]$ defining the line $\alpha([P], [Q])$.

We can also define an action of $P(2, I\!\!K)$ on $G(I\!\!K)$ by taking $g^* \in P^2(I\!\!K)$ and replacing P, Q by the polynomials P^*, Q^* as defined in (**??**).

Now, $A(2, I\!\!K)$ acts on $F(I\!\!K)$, as we have seen above, leaving $E(I\!\!K)$ invariant, and the group actions commutes with the map $[\omega] \longrightarrow \alpha([\omega])$:

LEMMA 20. *Let $[\omega]$ be a point of $E(I\!\!K)$ and $g \in A(2, I\!\!K)$. Then $g \cdot (\alpha([\omega])) = \alpha([g^*\omega])$.*

Proof. Let us consider a non trivial Pfaff form

$$\omega = Pdx + Qdy, \quad \text{and} \quad g(x, y) = (ax + by + \gamma_1,\ cx + dy + \gamma_2)$$

an element of $A(2, I\!\!K)$. We have that :

$$g^*\omega = P'dx + Q'dy \quad \text{with} \ :$$

$$P' = a\ P \circ g + c\ Q \circ g\ , \quad Q' = b\ P \circ g + d\ Q \circ g.$$

Clearly :

$$\alpha([P'], [Q']) = \alpha([P \circ g], [Q \circ g])$$

because the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, and this gives :

$$\alpha([g^*\omega]) = \alpha([P \circ g], [Q \circ g]) = g \cdot (\alpha[\omega]).$$

∎

Remark 21. In the lemma, it is not possible to consider the action of $P(2, I\!\!K)$ because the map α does not extend to the projective extension of $E(I\!\!K)$ in $F_{proj}(I\!\!K)$.

In the next paragraph, we shall study the map $[\omega] \mapsto \alpha[\omega]$. In the remaining part of the present section we just concentrate on the space $G(I\!\!K)$ of pencils of conics, considering the geometry of possible pencils.

Remark 22. We can use the notion of pencil of conics to construct conics with specified properties. For instance let us give an alternative proof of the lemma **??**. If we consider, as in lemma **??**, a set σ of five simple points in $P^2(I\!\!K)$, not four of them on a same line, it is easy to find a subset $\tilde{\sigma} \subset \sigma$

which is a projective basis : this means that no three points in $\tilde{\sigma}$ are on a same line. By the action of $P(2,I\!\!K)$ we can send $\tilde{\sigma}$ on the quadruple $\{[0,0,1],[1,0,1],[0,1,1],[1,1,1]\}$ which we shall continue to designate by $\tilde{\sigma}$. The two conics $\{P(x,y)=x(x-1)=0\}$ and $\{Q(x,y)=y(y-1)=0\}$ contain $\tilde{\sigma}$. Let $m=[a,b,c]\in P^2(I\!\!K)$ be the point such that $\sigma=\{m,\tilde{\sigma}\}$. Each conic in the pencil $\alpha([P],[Q])=\{[uP+vQ]\mid[u,v]\in P^1(I\!\!K)\}$ contains $\tilde{\sigma}$.Then the lemma **??** reduces to prove that there is one and only one conic in the pencil which passes through the point m. This is equivalent to solve the following equation on $[u,v]\in P^1(I\!\!K)$:

$$ua(a-c)+vb(b-c)=0. \tag{18}$$

As $m\notin\tilde{\sigma}$, one has that $a(a-c)$ or $b(b-c)$ is different from 0 and then the equation (**??**) has a unique solution.

It is easy to give a similar proof of the lemma **??**. For instance, if the set σ contains a subset $\tilde{\sigma}=\{(m_1,\delta_1),(m_2,\delta_2)\}$ such that the line ℓ containing m_1,m_2 is different from δ_1 and δ_2, we can bring m_1 at $[0,0,1]$, m_2 at $[0,1,1]$ and δ_1,δ_2 to be parallel to the Ox-axis. To finish the proof, one considers the pencil generated by the conics $\{P(x,y)=x^2=0\}$ and $\{Q(x,y)=y(y-1)=0\}$.

From now on, we suppose that $I\!\!K=\mathbb{C}$. Of course, it will be always possible to consider a real object (conic or pencil) as a complex object.

Definition 23. A *non degenerate* pencil is a pencil which contains at least one irreducible conic. We shall say that a pencil is *regular* if it is non degenerate and contains at least two distinct reducible (i.e. singular) conics.

Remark 24.

(a) The set of degenerate pencils $D(\mathbb{C})$ is the union of two non-disjoint subsets :

$D_1(\mathbb{C})$, the subset of pencils $\alpha=\alpha([P],\,[Q])$ generated by two distinct reducible conics $[P],[Q]$ with a common singular point. If the singular point is at finite distance, we can bring it to the origin by an element of $A(2,\mathbb{C})$. Then, the pencil just contains homogeneous conics of degree 2.

$D_2(\mathbb{C})$, the subset of pencils $\alpha=\alpha([P],\,[Q])$ generated by two reducible conics $[P],[Q]$ with exactly one common linear factor. We write these conics $P=\ell.p$, $Q=\ell.q$, with $[p]\neq[q]$, where ℓ is the common factor. Using the $P(2,\mathbb{C})$-action on the pencils in $P^2(\mathbb{C})$, we can bring the line $[\ell]$ to infinity $\{\ell=0\}=\ell_\infty$. Then, as a pencil in $P^2(\mathbb{C})$ (i.e. up to the action of $P(2,I\!\!K)$), the pencil reduces to a pencil of lines, passing through a common point $a\in P^2(\mathbb{C})$. We can suppose that $a=(0,0)\in\mathbb{C}^2$ (if $a\notin[\ell]$) or $a=[1,0,0]\in P^2(\mathbb{C})$ (if $a\in[\ell]$). In $\mathbb{C}^2$ (up to the action of $A(2,\mathbb{C})$), we have more possible cases and the pencil is no longer always equivalent to a pencil of lines.

(b) We shall write $O(\mathbb{C})$ the set of non degenerate and non regular pencils. Each of them contains just one reducible conic which is non-degenerate in one case (a pair of distinct lines) and degenerate in the other case (a double line). Any pair of distinct conics generating a pencil in $O(\mathbb{C})$ intersects in a point with intersection number equal to 3 or 4.

NOTATION 25. *We shall denote by* $G_0(\mathbb{C}) = G(\mathbb{C})\backslash(D(\mathbb{C}) \cup O(\mathbb{C}))$, *the open algebraic subset of regular pencils.*

We shall see that the study of a regular pencil reduces to the study of its multiple base points in a sense to be made precise below.

DEFINITION 26. Let α be a pencil of conics which contains at least one irreducible conic $[P]$ (this means that $[P] \in G_0(\mathbb{C}) \cup O(\mathbb{C})$). Let $[Q] \in \alpha$ a second conic such that $[P]$ and $[Q]$ generate α. These two conics intersect just at a finite number of points in $P^2(\mathbb{C})$. We associate to α a *subset* $\sigma(\alpha)$ *of points counted with multiplicity* in $P^2(\mathbb{C})$, as follows :

Let $|\sigma(\alpha)| = [P] \cap [Q]$. Let m be a point in $|\sigma(\alpha)|$. If $I_m([P],[Q]) = 1$, i.e. the two conics have a transversal intersection at m, then this point is contained in $\sigma(\alpha)$ as a simple point. If $I_m([P],[Q]) = n+1 > 1$, and δ is the tangent line to $[P]$ at m, then the multiple point (m, δ^n) is contained in $\sigma(\alpha)$.

The subset $\sigma(\alpha)$ of points counted with multiplicity just depends on the pencil α and not on the choice of the conics $[P],[Q]$ as above. $|\sigma(\alpha)|$ will be called the set of *base points* of α and $\sigma(\alpha)$ will be called the set of *multiple base points* of α.

LEMMA 27. *If α is a regular pencil then $\sigma(\alpha)$ is defined. It a subset of four points counted with multiplicity in $P^2(\mathbb{C})$. The multiplicity of each of these points is less or equal than* 2.

Proof. Let us consider $\alpha \in G_0(\mathbb{C})$. First, $\sigma(\alpha)$ is defined because in the definition **??** we have just supposed that the pencil α contains at least one irreducible conic, which is the case for pencils belonging to $G_0(\mathbb{C})$. We can generate α by an irreducible conic $[P]$, and a reducible one $[Q]$. We know that these two conics have just a finite number of intersection points and by the Bezout's theorem the sum of the intersection numbers at these different intersection points is equal to four. Now, if m is any intersection point, the multiplicity that we have given at the corresponding multiple point in $\sigma(\alpha)$ is precisely equal to the intersection number $I_m([P],[Q])$. This intersection number is less or equal than 2, if not α would belong to $D(\mathbb{C})$ or $O(\mathbb{C})$. It follows that $\sigma(\alpha)$ is a subset of four points counted with multiplicity in $P^2(\mathbb{C})$, and that the multiplicities are less or equal than 2. ▮

Remark 28. If $\alpha \in O(\mathbb{C})$, the set $\sigma(\alpha)$ is made of one single quadruple point or a pair of one triple point and one simple point. If α is a degenerate pencil, we can just consider the set $\Sigma(\alpha) = [P] \cap [Q] \subset P^2(\mathbb{C})$ (when $\alpha \in G_0(\mathbb{C}) \cup O(\mathbb{C})$ as in definition **??**, we can take $\Sigma(\alpha) = |\sigma(\alpha)|$). If $\alpha \in D_2(\mathbb{C})$, the set $\Sigma(\alpha)$ contains a line $[\ell]$. If $\alpha \in D_1(\mathbb{C}) \backslash D_2(\mathbb{C})$ then $\Sigma(\alpha)$ is reduced to one point.

We want to be more precise, concerning the possible sets $\sigma(\alpha)$, when α in a regular pencil and concerning the map σ defined on $G_0(\mathbb{C})$ the space of regular pencil.

Definition 29. Let $S^4(\mathbb{C})$ be the set of subsets σ of four points counted with multiplicity in $P^2(\mathbb{C})$, with the following properties :

(a) The multiplicity of each point is less than or equal to two.

(b) No three multiple points $(m_i, \delta_i^{n_i})$, $i = 1, 2, 3$, are on a same line. This means precisely that no line $\ell \in P^2(\mathbb{C})$ contains m_1, m_2, m_3 and that $m_i \notin \delta_j$ for $i \neq j$.

It is possible to give an algebraic structure to $S^4(\mathbb{C})$. Let us call $\widetilde{S}^4(\mathbb{C}) \subset P^2(\mathbb{C}) \times \cdots \times P^2(\mathbb{C})$ (four times) the Zariski open set, complement of the union of triple-diagonals $\Delta_{ijk} = \{(z_1, z_2, z_3, z_4) \mid z_i = z_j = z_k)$, where $\{i, j, k\}$ is any subset of 3 elements in $\{1, 2, 3, 4\}$. Let $\widehat{S}^4(\mathbb{C})$, be the set which is obtained by blowing up $\widetilde{S}^4(\mathbb{C})$ along the union of the double-diagonals $\Delta_{ij} = \{(z_1, \cdots, z_4) \mid z_i = z_j\}$ for $i \neq j$ (these submanifolds are 2 by 2 distinct in $\widetilde{S}^4(\mathbb{C})$). Let us consider the quotient of $\widehat{S}^4(\mathbb{C})$ by the action of the permutation group on the coordinates $(z_1, \cdots, z_4)$. Then $S^4(\mathbb{C})$ may be identified with the subset of elements in this quotient which verify the condition (b) in the definition **??** : a point $z \in \widetilde{S}^4(\mathbb{C})$ with $z_1 = z_2$ for instance, is replaced by the subset of points $((z_1, \delta), z_3, z_4) \in \widehat{S}^4(\mathbb{C})$, where δ is any line of $P^2(\mathbb{K})$ through the point $z_1 \in P^2(\mathbb{K})$; then, passing to the quotient space allows to forget the order of points.

We can stratify the space $S^4(\mathbb{C})$:

$$S^4(\mathbb{C}) = S_0^4(\mathbb{C}) \cup S_1^4(\mathbb{C}) \cup S_2^4(\mathbb{C}) \ , \quad \text{where, in } S^4(\mathbb{C}) :$$

i) $S_0^4(\mathbb{C})$ is the set of quadruples of simple distinct points, no three of them on a same line. It is an open set of dimension 8 in $S^4(\mathbb{C})$.
ii) $S_1^4(\mathbb{C})$ is the set of subsets of points counted with multiplicity in $P^2(\mathbb{C})$, made by a double point (m, δ) and two simple points, none of them on δ. It is a manifold of dim $S_1^4(\mathbb{C}) = 7$.
iii) $S_2^4(\mathbb{C})$ is the set of pairs of double points $\{(m_1, \delta_1), (m_2, \delta_2)\}$ with $m_1 \neq m_2$, such that the two lines δ_1, δ_2 are distinct from the line joining m_1 and m_2. It is a manifold of dim $S_2^4(\mathbb{C}) = 6$.

The structure of strata incidence : $S^4_1(\mathbb{C}) \subset \overline{S^4_0(\mathbb{C})}$ and $S^4_2(\mathbb{C}) \subset \overline{S^4_1(\mathbb{C})}$, is related to the above construction by blowing-up. Indeed, the groups $A(2,\mathbb{C})$, $P(2,\mathbb{C})$ act on $S^4(\mathbb{C})$ and preserve the stratification. In fact each stratum $S^4_1(\mathbb{C})$, $S^4_2(\mathbb{C})$ is just one orbit of the $P(2,\mathbb{C})$-action.

Now, in a precise sense, a non-degenerate pencil is characterized by its set of multiple base points :

THEOREM 30. *The map σ, restricted to $G_0(\mathbb{C})$, takes values in $S^4(\mathbb{C})$. This map $\sigma : G_0(\mathbb{C}) \longrightarrow S^4(\mathbb{C})$ is a bijective, birationnal map.*

This result is a consequence of some elementary properties of pencils :

LEMMA 31. *Let α be any pencil in $G_0(\mathbb{C})$. Then, through each $m \in P^2(\mathbb{C}) \setminus \mid \sigma(\alpha) \mid$ there passes one and only one conic of the pencil α.*

Proof. Let us consider a point $m \in P^2(\mathbb{C}) \setminus \mid \sigma(\alpha) \mid$. The set $\{m, \sigma(\alpha)\}$ is a subset of five points counted with multiplicity in $P^2(\mathbb{C})$, which verifies the hypothesis of lemma **??** and the result follows from that lemma. ▮

Remark 32. With a little more work, it is easy to prove the lemma **??** for any pencil in $G(\mathbb{C})$.

LEMMA 33. *Let α be any pencil in $G_0(\mathbb{C})$. Then the pencil α contains at least two different reducible conics and no more than three. This last case happens if and only if $\sigma(\alpha) \in S^4_0(\mathbb{C})$.*

Proof. Let us consider for instance that $\sigma(\alpha) \in S^4_0(\mathbb{C})$, i.e. that $\sigma(\alpha)$ is a subset of four simple points $\{a, b, c, d\}$ in $P^2(\mathbb{K})$. There are three ways to make a partition of $\sigma(\alpha)$ by subsets of two elements. Each of these partitions gives a reducible conic of the pencil. For instance consider the partition $\{\{ab\}, \{c, d\}\}$, and take $m = (a, b) \cap (c, d) \in P^2(\mathbb{C})$ (here (a, b) designates the line of $P^2(\mathbb{C})$ through the points a, b). The conic of α through m is precisely the reducible conic $(a, b) \cup (c, d)$. Clearly, any pair of reducible conics obtained in this way, generates the pencil α. It is possible to treat similarly the more degenerate cases $\sigma(\alpha) \in S^4(\mathbb{C}) \setminus S^4_0(\mathbb{C})$. Notice that it is in these cases, that we can find just two different reducible conics (and not three of them, like in the previous case). ▮

Remark 34. If $\alpha \in D(\mathbb{C})$ every conic in the pencil is reducible. On the other hand, if $\alpha \in O(\mathbb{C})$, the pencil contains just one reducible conic.

We can now return to the proof of theorem **??**.

Proof of theorem ??

Proof. We first prove that for $\alpha \in G_0(\mathbb{C})$ we have $\sigma(\alpha) \in S^4(\mathbb{C})$. In the lemma **??** we have proved that $\sigma(\alpha)$ is a subset of four points counted with multiplicity in $P^2(\mathbb{C})$ and that each point has a multiplicity less than or equal to two: this is the condition (a) in the definition **??**. To verify the condition (b) of the definition **??** we have just to notice that if three points of $\sigma(\alpha)$ belong to a same line (in the precise sense given here), then each conic in the pencil α would be reducible and the pencil could not be regular.

We now prove that σ is surjective. Let be s an element of $S^4(\mathbb{C})$. Then, using the lemma **??**, we know that a unique conic passes through $\{m, s\}$ where $m \in P^2(\mathbb{C})\backslash|s|$. If we construct two distinct conics $[P]$, $[Q]$ in this way, it is clear that $s = \sigma(\alpha([P],\, [Q]))$.

The above argument implies that the conics of the pencil $\alpha([P],\, [Q])$ are determined by s, and then, the map σ is injective.

To prove that σ is a rational map, it suffices to notice that $G(\mathbb{C})$ (and then $G_0(\mathbb{C})$) can be parameterized locally by pairs of distinct conics : taking any $\alpha_0 \in G(\mathbb{C})$, choose two hyperplanes $\mathcal{P}$, $\mathcal{Q}$ in $\mathcal{C}(\mathbb{C})$ cutting transversally the line α_0 at points $[P_0]$, $[Q_0]$ respectively ; then the $\alpha \in G(\mathbb{C})$ near α_0 are parameterized by the pairs $([P],\, [Q])$, $[P]$ near $[P_0]$ on $\mathcal{P}$ and $[Q]$ near $[Q_0]$ on $\mathcal{Q}$. This defines an algebraic open chart U containing α_0 in $G(\mathbb{C})$ and on this chart $\mid \sigma(\alpha) \mid= [P] \cap [Q]$ (where the points are equipped with their multiplicity in $\sigma(\alpha)$). This is a rational map. Conversely, taking $s_0 \in S^4(\mathbb{C})$, we can choose two distinct points $m_1,\, m_2 \in P^2(\mathbb{C})\backslash \mid s_0 \mid$, such that the conics through $\{m_1, s_0\}$ and $\{m_2, s_0\}$ are distinct. Then, we can define a local section of σ in a neighborhood of s_0, in the following way : taking s near s_0 in $S^4(\mathbb{C})$, two conics $[P(s)]$ and $[Q(s)]$ pass through $\{m_1, s\}$ and $\{m_2, s\}$ respectively. The map $s \longrightarrow \alpha([P(s)],\, [Q(s)]) \in G_0(\mathbb{C})$ gives a local algebraic section of σ. ▮

To the stratification of $S^4(\mathbb{C})$ corresponds via the map σ, a stratification of $G_0(\mathbb{C})$:

NOTATION 35. *Let us define :* $G_0^i(\mathbb{C}) = \sigma^{-1}(S_i^4(\mathbb{C})\}$ *for* $i = 0, 1, 2$.

It is easy to describe the different strata of $G_0(\mathbb{C})$. Recall that each element $\alpha \in G_0(\mathbb{C})$ may be written $\alpha([P], [Q])$ where $[P]$ and $[Q]$ are two reducible conics. Moreover the two linear factors of $[P]$ are transversal to the two linear factors of $[Q]$.

(i) If the two conics intersect at regular points, we have that $\alpha \in G_0^0(\mathbb{C})$ and $\sigma(\alpha)$ is the set of four simple points : $[P] \cap [Q]$.

(ii) If the singular point m of one conic (let us say $[P]$) coincides with a regular point of the other conic (let us say Q), situated on the linear factor δ of Q, we have that $\alpha \in G_0^1(\mathbb{C})$ and $\sigma(\alpha) = \{(m, \delta), p, q\}$ where $[P] \cap [Q] = \ \mid \sigma(\alpha) \mid= \{m, p, q\}$.

(iii) If one conic is degenerate (let us say $[Q] = \ell^2$), we have that $\alpha \in G_0^2(\mathbb{C})$. If $[P]$ is the product of two distinct lines δ_1 and δ_2 such that $m_1 = \delta_1 \cap \ell$ and $m_2 = \delta_2 \cap \ell$, $m_1, m_2 \neq \delta_1 \cap \delta_2$, we have that $\sigma(\alpha) = \{(m_1, \delta_1), (m_2, \delta_2)\}$.

Let us notice that each stratum $G_0^i(\mathbb{C})$ is just one orbit of the action of the projective group $P(2,\mathbb{C})$. Then, up to a projective transformation, each pencil in $G_0^0(\mathbb{C})$ is equivalent to the pencil of circles passing through the points $(-1, 0)$, $(1, 0) \in \mathbb{C}^2$; each pencil in $G_0^1(\mathbb{C})$ is equivalent with the pencil of circles tangent at $(0, 0)$ with the vertical axis; each pencil in $G_0^2(\mathbb{C})$ is equivalent with the pencil of concentric circles at the origin.

The classification of pencils up to the action of $A(2,\mathbb{C})$ is more complicated, and we encounter in this case moduli of orbits. Let us consider for instance the classification of $S_0^4(\mathbb{C})$, modulo the action of $A(2,\mathbb{C})$, in the case of *four simple points at finite distance.* We can bring three of them at the positions $a = (0, 0)$, $b = (1, 0)$ and $c = (0, 1)$. The last point d could be any point in $\mathbb{C}^2 \backslash (a, b) \cup (b, c) \cup (a, c)$. This space parameterizes the orbit space (space of moduli). Of course we have other strata in $S_0^4(\mathbb{C})$ corresponding to the possibility of points at infinity.

If we consider $\mathbb{K} = \mathbb{R}$, the classification is even more complicated. For the classification up the $A(2, \mathbb{R})$-action, the simpler possibility in $S_0^4(\mathbb{R})$ corresponds to a pencil given by four real simple points at finite distance. This set of pencils splits into two connected components : one of them corresponding to the subsets $s \in S_0^4(\mathbb{R})$ having a triangular convex hull and the other one corresponding to the subsets s which have a quadrilateral convex shape. (This new phenomenon comes from the fact that any line disconnects $\mathbb{R}^2$). Of course, for a complete classification, we have to take into account the points at infinity of $s \in S^4(\mathbb{R})$ and the partition between real or complex points (0,2 or 4 points may be complex).

To finish this paragraph, we return to the general pencils of conics to formulate the following result (recall that $S\mathcal{C}(\mathbb{C})$ is the space of all reducible or singular conics).

NOTATION 36. *Let $G_1(\mathbb{C})$ be equal to $G(\mathbb{C}) \backslash O(\mathbb{C})$.*

We have the following result :

PROPOSITION 37. *Let Δ be the diagonal in $S\mathcal{C}(\mathbb{C}) \times S\mathcal{C}(\mathbb{C})$. The map*

$$\alpha : S\mathcal{C}(\mathbb{C}) \times S\mathcal{C}(\mathbb{C}) \backslash \Delta \longrightarrow G_1(\mathbb{C})$$

is an algebraic ramified covering, regular above G_0^0 and ramified along the singular set $G_1(\mathbb{C}) \backslash G_0^0(\mathbb{C}) = D(\mathbb{C}) \cup G_0^1(\mathbb{C}) \cup G_0^2(\mathbb{C})$.

Proof. Let $S = S\mathcal{C}(\mathbb{C}) \times S\mathcal{C}(\mathbb{C}) \backslash \Delta$. The map $\alpha \ : ([P], [Q]) \in S \longrightarrow \alpha([P], [Q])$ is of course algebraic. If $s \in G_0(\mathbb{C})$, there exists $([P], [Q]) \in S$

such that $\alpha([P],[Q]) = s$ as a consequence of lemma **??**. If $s \in D(\mathbb{C})$, there exists such a $([P],[Q]) \in S$ as above, because any element of s is a reducible conic. Then, the map α is surjective over $G_1(\mathbb{C})$.

Moreover, in lemma **??**, we have shown that on any $s \in G_0^0(\mathbb{C})$ there exist 3 and only 3 distinct reducible conics in s. This proves that the map α is a regular covering with 3 sheets above $G_0^0(\mathbb{C})$. Each sheet corresponds to a choice of 2 reducible conics among 3. This map is ramified along the algebraic closed subset $D(\mathbb{C}) \cup G_0^1(\mathbb{C}) \cup G_0^2(\mathbb{C})$. ∎

4. QUADRATIC EQUATIONS STRUCTURED AS A BUNDLE OVER PENCILS OF CONICS.

In the previous paragraph, we have introduced the space $E(\mathbb{K})$ of non trivial Pfaff equations and its mapping $\alpha : E(\mathbb{K}) \longrightarrow G(\mathbb{K})$ onto the space of pencils of conics. Moreover, we introduce :

Notation 38. *The set of regular Pfaff equations $E_0(\mathbb{K})$, will be the set of Pfaff equations having a regular pencil of conics: $E_0(\mathbb{K}) = \alpha^{-1}(G_0(\mathbb{K}))$.*

The set of singular points of a given Pfaff equation $[\omega] \in E_0(\mathbb{K})$, equipped with their multiplicities, coincides with the set of multiple base points of its pencil $\alpha([\omega])$. We shall denote this set $\sigma([\omega])$, i.e $\sigma([\omega]) = \sigma \circ \alpha([\omega])$.

We can use the decomposition of the space $G(\mathbb{K})$ to obtain a decomposition of the space $E(\mathbb{K})$ and to derive informations about the singular points of the Pfaff equations :

(a) $\alpha^{-1}(G_0^0(\mathbb{K}))$ is the set of generic equations with four non degenerate singular points over $\mathbb{C}$. Non-degenerate means that the two eigenvalues are non zero. When $\mathbb{K} = \mathbb{R}$, the set of singular points splits into the set of saddle and the set of anti-saddle points of the equation.

(b) $\alpha^{-1}(G_0^1(\mathbb{K}))$ is the set of equations with two non degenerate singular point and one degenerate point (a generic semi-hyperbolic or nilpotent singular point).

(c) $\alpha^{-1}(G_0^2(\mathbb{K}))$ is the set of equations with two degenerate singular points.

(d) $\alpha^{-1}(O(\mathbb{K}))$ is a set of equations which have a semi-hyperbolic or a nilpotent point of codimension 3 or 4 (see [**?**]).

(e) $\alpha^{-1}(D_1(\mathbb{K}))$ is the set of equations with at most a unique singular point, having a trivial linear part (when the singular point exists and up some projective map, such an equation is an homogeneous one).

(f) $\alpha^{-1}(D_2(I\!\!K))$ is the set of equations having a line of non-isolated singular points.

4.1. The fiber bundle structure $\boldsymbol{\alpha} : \mathbf{E}(\mathbf{K}) \longrightarrow \mathbf{G}(\mathbf{K})$.

Recall that the right action of the affine group $A(2, I\!\!K)$ in the phase space $I\!\!K^2$ induces right actions on $E(I\!\!K)$ and $G(I\!\!K)$ which commute with α (we can see $E(I\!\!K)$, $G(I\!\!K)$ as $A(2, I\!\!K)$-spaces and α as a $A(2, I\!\!K)$-map).

We now want to introduce another group-action of $E(I\!\!K)$. It corresponds to the action of the group $P(1, I\!\!K) = PGL(2, I\!\!K)$ at each point $m \in I\!\!K^2$, on the direction of the vector $X(m) = Q(m)\,\dfrac{\partial}{\partial x} - P(m)\,\dfrac{\partial}{\partial y}$, dual to the exterior form $\omega(m)$. It is induced by: $(g \cdot X)(m) = g(X(m))$ for any $g \in GL(2, I\!\!K)$. In contrast with the action of $A(2, I\!\!K)$ on $E(I\!\!K)$ which is simply the change of coordinates, in general this action of $P(1, I\!\!K)$ does not preserve the phase portrait of the Pfaff form.

From now on, using the identification of $E(I\!\!K)$ with $P^{11}(I\!\!K)$, we shall write $[P, Q]$ to denote the Pfaff equation $[\omega] = \{Pdx + Qdy = 0\}$. For each $m \notin \Sigma([\omega])$ one can define the value of $[\omega]$ to be $[\omega](m) = [P(m), Q(m)] \in P^1(I\!\!K^*)$. The action of $P(1, I\!\!K)$ on $E(I\!\!K)$ is the mapping :

$$\rho : P(1, I\!\!K) \times E(I\!\!K) \longrightarrow E(I\!\!K) \tag{19}$$

given by $\rho([g], [P, Q])(m) = \Big[\big(P(m), Q(m)\big)g^{-1}\Big]$. where $[g] \in P(1, I\!\!K)$ is the class of $g \in GL(2, I\!\!K)$. Explicitely, if $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\rho([g], [P, Q]) = [P', Q']$ with :

$$P' = \delta P - \gamma Q \quad \text{and} \quad Q' = -\beta P + \alpha Q \tag{20}$$

Clearly, ρ is a right action which preserves each fiber of the map α, i.e. :

$$\alpha(\rho([g], [\omega])) = \alpha[\omega] \tag{21}$$

$$\text{for} \quad \forall [g] \in P(1, I\!\!K) \quad \text{and} \quad \forall \omega \in E(I\!\!K).$$

Two properties for α and ρ follow easily :

LEMMA 39. *Each fiber $\alpha^{-1}(p)$, for $p \in G(I\!\!K)$, is diffeomorphic with $P(1, I\!\!K)$. More precisely, given any pair of Pfaff equations $[\omega]$, $[\omega'] \in \alpha^{-1}(p)$, there exists one and only one $[g] \in P(1, I\!\!K)$ such that $\rho([g], [\omega]) = [\omega']$.*

Proof. The element $p \in G(I\!\!K)$ is a line in the projective space $\mathcal{C}(I\!\!K) \simeq P^5(I\!\!K)$, or equivalently, a 2-dimensional subspace in the vector space $I\!\!K^6$. The fiber $\alpha^{-1}(p)$ is just the collection of all base s in this 2-dimensional subspace, up to scalar multiplication. The group action of $P(1, I\!\!K)$ is clearly transitive, with a trivial isotropy group. ▮

LEMMA 40. *The map α admits local sections.*

Proof. We can use the same argument as in the proof of the theorem **??**. Given any $p_0 \in G(I\!\!K)$, we choose two hyperplanes $\mathcal{P}$, $\mathcal{Q}$ in $\mathcal{C}(I\!\!K)$, which cut transversally the line defined by p_0 in $\mathcal{C}(I\!\!K)$.
Let us consider $[P_0] = p_0 \cap \mathcal{P}$ and $[Q_0] = p_0 \cap \mathcal{Q}$.

Recall that $G(I\!\!K)$ can be locally parameterized by the pairs $([P], [Q])$ with $[P]$ near $[P_0]$ in $\mathcal{P}$ and $[Q]$ near $[Q_0]$ in $\mathcal{Q}$. This defines a neighborhood of p_0 in $G(I\!\!K)$ and a chart map on it: $p \to ([P], [Q])$. If this neighborhood is small enough, we can lift $[P]$, $[Q]$ into two polynomials P_p, Q_p in $I\!\!K_2[x, y]$, which define a local section $s(p) = [P_p, Q_p] \in E(I\!\!K)$. ▮

The two preceeding lemmas imply clearly the following :

THEOREM 41. *The map α is a principal fibration with total space $E(I\!\!K)$, base space $G(I\!\!K)$ and structural group $P(1, I\!\!K)$.*

Remark 42. An element $p \in G(I\!\!K)$ defines a line in $P^2(I\!\!K)$. Now, to take an element $[\omega] \in \alpha^{-1}(p)$ is the same as to take *three distinct points* on p, i.e. a projective basis on the line $p \simeq P^1(I\!\!K)$. The action of $P(1, I\!\!K)$ on $\alpha^{-1}(p)$ is just the action of $P(1, I\!\!K) = \text{Aut }(P^1(I\!\!K))$ on the projective bases of p.

4.2. Structure of principal bundle on the orbit spaces.

As the $A(2, I\!\!K)$-action on the phase space does not change the dynamical properties, it would be convenient to take the quotient by this action.

Denoting $A(2, I\!\!K)$ by A, $E(I\!\!K)$ by $E, \cdots$ to simplify the notations, the quotient spaces E/A, G/A are algebraic spaces of dimension 5 and 2 respectively and, as the map α commutes with the action, we have an induced map :

$$\dot{\alpha} = \alpha/A : E/A \longrightarrow G/A \tag{22}$$

We shall also denote by $\dot{p}$ the class of $p \in G$, by $[\dot{\omega}]$, the class of $[\omega]$. It is easy to see that the isotropy group of the A-action on G is trivial precisely at the points of G_0. As a consequence for $p \in G_0$, $\dot{\alpha}^{-1}(\dot{p})$ is isomorphic with $\alpha^{-1}(p) \simeq P(1, I\!\!K)$ and $\dot{\alpha}$ is a fibration of E_0/A over G_0/A (where $E_0 = \alpha^{-1}(G_0)$). The actions of $P(1, I\!\!K)$ on the two fibers of p and $g \cdot p$, where $g \begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} + b$ is an element of A, differ precisely by the conjugacy by the matrix M. It follows that there is no natural way to put a structure of principal $P(1, I\!\!K)$-bundle on the quotient by A. Nevertheless, it will be possible to define this structure on any part of G_0/A which can be identified with a subset of G_0 by the choice of a moduli space. As we have seen, it is the case for G_0^0/A which can be identify with an open subset of $!P^2(I\!\!K)$, and admits a section of the quotient map : $G_0^0 \longrightarrow G_0^0/A$. To put a structure of principal bundle on the quotient, it suffices to restrict the structure of principal bundle of $\alpha : E_0^0 = \alpha^{-1}(G_0^0) \longrightarrow G_0^0$.

Recall that $p \in G_0^0$ is a pencil of conics such that $\sigma(p)$ is a set of four simple points. In the lemma **??** , we have proved that such a pencil contains three and only three reducible conics. These three conics determine a projective basis of p. In this way, we define a global section of α above G_0^0, so that the bundle is trivial above this subset. This triviality passes to the quotient by A. Moreover, taking any base point a of the pencil p, we can identify each conic of p with its tangent line at a. In this way, we identify p with $P(T_a I\!\!K^2)$. The existence of three reducible conics in the pencil p gives a distinguished triple of element in $P(T_a I\!\!K^2)$ and defines a natural identification of $P(T_a I\!\!K^2)$ with $P^1(I\!\!K)$ (at least after a choice of an order between the three reducible conics). Finally we obtain a natural identification of p with $P^1(I\!\!K)$. Now, for each $[\omega] \in \alpha^{-1}(p)$, we have defined the isocline map $I_{[\omega]}$, which has as a factor a map we can denote again $I_{[\omega]}$, from p to $P^1(I\!\!K)$. Using ! the above identification between p and $P^1(I\!\!K)$, we have defined a map :

$$[\omega] \in \alpha^{-1}(p) \longrightarrow \Big\{ c \in p \longrightarrow I_{[\omega]}(c) \Big\} \in P(1, I\!\!K) \tag{23}$$

This map defines a trivialization of the fiber bundle $\alpha : E_0^0 \longrightarrow G_0^0$, which is compatible with its structure of principal bundle, and passes as well to the quotient by A. Each Pfaff equation $[\omega]$ with a given pencil $p = \alpha([\omega])$ is then characterized by its *isocline map* $c \in p \longrightarrow I_{[\omega]}(c)$ which gives the value of the constant slope along the isocline c.

Consider the case $I\!\!K = I\!\!R$. Let $[\omega] \in E_0^0$ be a Pfaff equation and any of its singular point $a \in \sigma[\omega]$. The degree of the map

$$I_{[\omega]} : p \simeq PT_a I\!R^2 \longrightarrow P^1(I\!R)$$

is the *index* of the singular point a. It is equal to ± 1 ($+1$ if a is an anti-saddle and -1 if a is a saddle point).

We have seen that each of the component of $G_0^0(I\!R)$ corresponds to a definite shape of the convex hull over the set of singular points : it may be a triangle or a quadrilateral. Using the above interpretation of the index in terms of the isocline map $I_{[\omega]}$, it is easy to establish the following well-known result for the repartition of index values :

(a) *For the quadrilateral case* : the indices are equal on each diagonal and opposite on each side.

(b) *For the triangular case* : the value of the index is the same at all the vertices of the triangle and opposite at the middle point in the interior of the triangle.

4.3. Rotation-parameter families associated to the $\mathbf{PL(1,R)}$-action.

In this part we shall suppose that $I\!K = I\!R$ and we want to consider limit cycles, i.e isolated closed orbits of a real vector field X on $I\!R^2$. Of course it is the same as to consider the isolated closed integral curves of the dual form ω of X, but for the properties we want to consider now, it is more convenient to consider vector fields. Questions about limit cycles are much more difficult in general that the ones related uniquely to singular points, which can be studied through the associated pencil of conics. Many of these questions are related to the behavior of limit cycles in function of the variations of parameters.

Definition 43. ([**?**]) Suppose that X_λ is a smooth 1-parameter family of vector fields on $I\!R^2$, with λ in some interval I of $I\!R$. We shall say that this family is a *rotation-parameter family* if the set of singular points of X_λ does not depend of λ and if, for all $m \in I\!R^2$ such that $X(m) \neq 0$ and $\forall \lambda \in I$, we have $\dfrac{d}{d\lambda}\,[X_\lambda(m)] \neq 0$ ($[X_\lambda(m)] \in P^1(I\!R)$ is the direction of the vector $X_\lambda(m)$).

Remark 44. The condition in the definition means that the direction of X_λ rotates regularly (with a non-zero speed) at each regular point of X_λ and for any $\lambda \in I$. It is equivalent to say that det $(X_\lambda(m), \dfrac{d}{d_\lambda}\, X_\lambda(m)) \neq 0$, for any regular point m and any $\lambda \in I$. Interest for this notion comes from the fact that one has a certain control on the behavior of the limit cycles of X_λ in function of a rotation parameter λ : for instance, an hyperbolic limit cycle must increase or decrease in function of λ, depending on its property of stability and on the sign of det $\left(X_\lambda(m), \frac{d}{d\lambda}X_\lambda(m)\right)$. (This sign is the same everywhere). This property and other properties of rotation parameter families are related to the following simple observation : the graph of any return map associated to X_λ is globally translated when λ varies.

The $P(1,I\!R)$-action on $E(I\!R)$ we have introduced above, allows us to construct 3-parameter families of quadratic vector fields. In fact, given any vector field X, with dual Pfaff equation $[\omega] \in E(I\!R)$, we can consider the 3-parameter family $X_M(m) = M.X(m)$, with parameter $M \in P(1,I\!R)$, space which may be identified with the 3-dimensional manifold $\{(a,b,c,d) \in I\!R^4 \mid ab - cd = \pm 1\}$.

From this, it follows that the Lie algebra of $P(1,I\!R)$ is $sl(2,I\!R)$, the Lie algebra of the group $SL(2,I\!R)$, connected component of the identity in $P(1,I\!R)$. For each $\mathcal{M} \in sl(2,I\!R) = \left\{\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \mid (\alpha,\beta,\gamma) \in I\!R^3\right\}$, we can associate a real one parameter family of vector fields :

$$X_\lambda = \exp\,(\lambda\mathcal{M}).X \tag{24}$$

where X is a given quadratic vector field.

It is easy to characterize the matrices $\mathcal{M}$ which generate a rotation parameter family :

Definition 45.

We say that $\mathcal{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ is *elliptic* if Spec $(\mathcal{M}) \subset iI\!R - \{0\}$, is *hyperbolic* if Spec $(\mathcal{M}) \subset I\!R - \{0\}$ and is *parabolic* if Spec $(\mathcal{M}) = \{0\}$.

Remark 46. The characteristic polynomial of $\mathcal{M}$ is equal to $\Delta(\xi) = \xi^2 - (\alpha^2 + \beta\gamma)$. Then, the matrix $\mathcal{M}$ is elliptic if $\alpha^2 + \beta\gamma < 0$, hyperbolic if $\alpha^2 + \beta\gamma > 0$ and parabolic if $\alpha^2 + \beta\gamma = 0$. The set of elliptic matrices is equal to the interior of the cone of parabolic matrices, in $I\!R^3$, and then, this set is not connected.

It is easy to characterize the matrices $\mathcal{M}$ which generate a rotation parameter family :

Proposition 47. *Any elliptic matrix $\mathcal{M}$ defines a rotation parameter family, $X_\lambda = \exp\,(\lambda\mathcal{M})X$ (where X is a vector field with a dual Pfaff form in $E(I\!R)$).*

Proof. One has

$$\frac{d}{d\lambda}\ \exp\,(\lambda\mathcal{M})X(m)_{|\lambda=\lambda_0}\ = \frac{d}{ds}\ \exp\,(s\mathcal{M})\ \exp\,(\lambda_0\mathcal{M})X(m)_{|s=0}.$$

Then :

$$\frac{d}{d\lambda}\ X_\lambda(m)_{|\lambda=\lambda_0}\ = \mathcal{M}.X_{\lambda_0}(m).$$

Now, if $\mathcal{M}$ is elliptic, the map $[u] \longrightarrow [\mathcal{M}.u]$ from $P^1(\mathbb{R})$ to $P^1(\mathbb{R})$ has no fixed point. It follows that det $(X_{\lambda_0}(m), \dfrac{d}{dX}\ X_{\lambda_0}(m)) \neq 0$ for any $\lambda_0 \in \mathbb{R}$ and any regular point of $X = X_{\lambda_0}$. Of course, the set of singular points of X_λ for $\forall\lambda \in \mathbb{R}$, is the same as the set of singular points of $X = X_{\lambda_0}$. ▮

Remark 48. It is easy to generalize the above result :

Let $\varphi(\lambda)$ be a smooth map from $\mathbb{R}$ to $SL(2,\mathbb{R})$. Suppose that $\dfrac{d\varphi}{d\lambda} \in T_{\varphi(\lambda)}\ SL(2,\mathbb{R})$ is the right translation by $\varphi(\lambda)$ of an elliptic matrix in $sl(2,\mathbb{R})$, for $\forall\lambda \in \mathbb{R}$. Then $X_\lambda = \varphi(\lambda).X$ is a rotation parameter family.

It is easy to find a set of three elliptic matrices which is a basis of $sl(2,\mathbb{R})$, and then to generate the connected component $SL(2,\mathbb{R})$ of the identity in $P(1,\mathbb{R})$ by three 1-parameter groups which defines for any X three rotation parameter families. This will define a surjective mapping $(\mu_1,\ \mu_2,\ \mu_3) \in \mathbb{R}^3 \longrightarrow SL(2,\mathbb{R})$ such that, if we maintain fixed two of the parameters μ_i, we obtain a rotation parameter family in function of the third parameter. Of course, it is difficult to control the effect of the simultaneous variations of two independent rotation parameters.

4.4. Constructing families.

Let us consider again the fibration $\alpha : E(\mathbb{K}) \longrightarrow G(\mathbb{K})$. As we have seen above, the singular points of the Pfaff form $[\omega]$ are defined in terms of the pencil $\alpha([\omega])$. On the other hand, the $P(1,\mathbb{K})$-action on the fibers, being generated by rotation parameters when $\mathbb{K} = \mathbb{R}$, has impact on limit cycles. Then, it would be interesting to work with families of Pfaff forms or vector fields, chosen in agreement with the fibration α. This is illustrated by the choice of the 5-parameter family used in [**?**] : two parameters act on the singular points, and define a subfamily of center type systems. The three other parameters are rotation parameters, at least on open regions where the possible limit cycles are known to be confined.

It is possible to construct in a more systematic way, families well-adapted to the fibration α. As an example , we shall construct a parameterization of $E_1(\mathbb{K}) = \alpha^{-1}(G_1)$ (we recall that $E_1(\mathbb{K}) = G(\mathbb{K})\backslash\alpha^{-1}(O(\mathbb{K}))$ where each Pfaff equation in $\alpha^{-1}(O(I\!\!K))$ has one semi-hyperbolic or one nilpotent point of codimension equal to 3 or 4).

For this construction, we choose a family $\varphi(\lambda) : \Lambda \longrightarrow E(I\!\!K)$ which covers $G_1(I\!\!K)$, i.e $G_1(I\!\!K) \subset \alpha \circ \varphi(\Lambda)$. Let us suppose that $I\!\!K = \mathbb{C}$. Using the same idea as in the proof of proposition **??** , we see that we can take the parameter space equal to $\Lambda = \mathcal{A}(\mathbb{C})^4$, where $\mathcal{A}(\mathbb{C})$ is the space of lines in $P^2(\mathbb{C})$, and φ defined by :

$$\varphi(\ell_1, \ell_2, \ell_3, \ell_4) = [\ell_1.\ell_2 dx + \ell_3.\ell_4.dy] \tag{25}$$

with $\ell_i = \alpha_i x + \beta_i y + \gamma_i$, $[\alpha_i, \beta_i, \gamma_i] \in P^2(\mathbb{C})$.

Next, we obtain a family which parameterizes the whole space $E_1(\mathbb{C})$ by taking $M.\varphi(\lambda)$ with $M \in P(1,\mathbb{C})$. In this family, the eight parameters $[\alpha_i,\, \beta_i,\, \gamma_i] \in P^2(\mathbb{C})$, $i = 1, \cdots, 4$, parameterized the pencils of conics. The parameters of M, which can be taken as rotation parameters when we restrict to $I\!\!R$ (see the section 4.3), keep fixed the pencils and parameterize the fibers of the map α.

Of course, the number of parameters is not optimal. It is possible to decrease it by taking moduli for the $A(2,\mathbb{C})$-action. We shall restrict to the open dense subset $\widetilde{G}_1(\mathbb{C}) \subset G_1(\mathbb{C})$ of Pfaff equations with at least one non degenerate point in $\mathbb{C}^2$. In this case it is possible to suppose that $[\ell_1]$, $[\ell_2]$ are transversal at some point of $\mathbb{C}^2$: up to the action of $A(2,\mathbb{C})$, we can suppose that $\ell_1\ell_2 = xy$.

Next, one of the two lines $\{\ell_3 = 0\}$ or $\{\ell_4 = 0\}$ is not the line at infinity (if not, $[\omega]$ has no singular point at finite distance). Then, we can suppose that the line $\{\ell_3 = 0\}$ cuts the axis $0y$ at a finite point, which can be supposed chosen at $\{y = 1\}$: the point $(0,1) \in \mathbb{C}^2$ is then a non degenerate point of the Pfaff equation. The line $\{\ell_3 = 0\}$ may be transversal or parallel to the axis Ox and we can cover the set $\widetilde{G}_1$ by two families :

(a) A 2-parameter family corresponding to a line $\{\ell_3 = 0\}$ transversal to the two axis. Up to the $A(2,\mathbb{C})$-action we can take :

$$\varphi_1(\lambda) = [xydx + (x + y - 1)(\alpha x + \beta y + \gamma)dy] \tag{26}$$

with $\lambda = (\alpha, \beta, \gamma) \in P^2(\mathbb{C})$

(b) A 1-parameter family corresponding to the lines $\{\ell_3 = 0\}$ and $\{\ell_4 = 0\}$, parallel to the axis $0x$:

$$\varphi_2(\lambda) = [xydx + (x - 1)(x - \lambda)dy] \tag{27}$$

with $\lambda \in \mathbb{C}$.

The image of the family (**??**) is a dense open subset in $\alpha^{-1}(G_1)$, and the image of the family (**??**) is an analytic subset of non zero codimension.

Of course, if we are interested to limit cycles, we have to restrict to real vector fields, in the above families, up to the $A(2, \mathbb{R})$-action. For instance, we have to replace (**??**) by two families which are $A(2,\mathbb{C})$ but not $A(2, \mathbb{R})$ equivalent : $(x^2 + y^2)dx + (x^2 + a)dy$ with $a \in \mathbb{R}$ aside the family (**??**) with $\lambda \in \mathbb{R}$.

Other choices of the map $\varphi(\lambda)$ are indeed possible. For instance, if we want just to attain the non-degenerate Pfaff equations in $E_0(\mathbb{C})$, we can take for $\varphi(\lambda)$ the parameterization of Hamiltonian equations $\{dP = 0\}$ by the coefficients λ of the cubic Hamiltonian functions P. The number of parameters can be reduced as above, using the $A(2,\mathbb{C})$-action.

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