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Periodic Perturbations of an Isochronous Center

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This paper discusses the possibility of producing resonance in a nonlinear isochronous center. In some cases it is shown that one can find periodic forcings (with the same period of the center) such that all the solutions of the perturbed equation are unbounded.

Key Words: Resonance, unbounded solution, periodic forcing

1. INTRODUCTION

Consider the differential equation

$$\ddot{x} + g(x) = 0 \qquad (1)$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. It is also assumed that the associated initial value problem has always a unique solution.

The origin $x = 0$ is an isochronous center if g satisfies

$$g(0) = 0, \quad xg(x) > 0 \text{ if } x \neq 0$$

and there is a fixed number $T > 0$ such that every solution of (**??**) is periodic with period T. The linear function $g(x) = \omega^2 x$ produces the simplest example of an isochronous center. In this case all non-trivial solutions have minimal period $\frac{2\pi}{\omega}$ and T can be any multiple of this number. We refer to [**?**, **?**, **?**] for the construction of nonlinear isochronous centers.

Let us now force the equation,

$$\ddot{x} + g(x) = f(t) \qquad (2)$$

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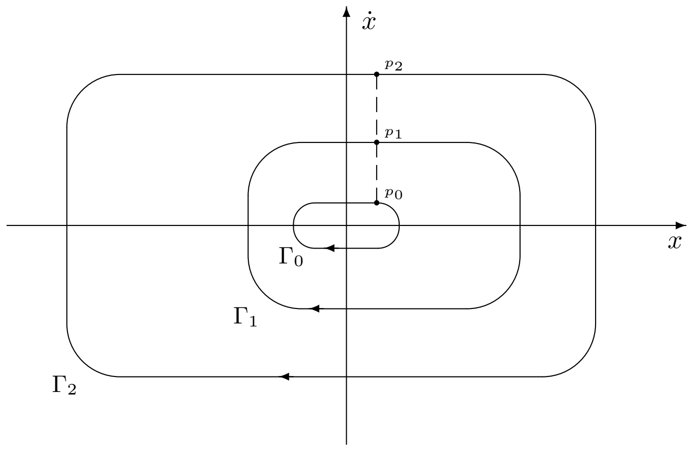
with f T-periodic. For the linear case we have the phenomenon of resonance: there are forcings f which are arbitrarily small and such that all the solutions of (**??**) are unbounded. This implies in particular that there are no periodic solutions. The main question in this paper is: can we find the phenomenon of resonance in any nonlinear isochronous center? This is part of a problem posed by Professor Roussarie at the Open Problems Section of the Lleida's conference 2000. We shall obtain some partial answers. First it will be shown that it is possible to produce resonance in any isochronous center if one interprets the notion of forcing in a liberal sense. The isochronous center will be arbitrary but our $f(t)$ will be a measure instead of a standard function. Thinking in mechanical terms this means that we allow f to act as an impulse at certain discrete times. These are the contents of Section 2. Later, in Section 3, we regularize f and produce examples of equations (**??**) with f analytic and small and such that all the solutions are unbounded. To do this we must assume that g is Lipschitz-continuous. Finally, in Section 4, we collect some remarks about local isochronous centers and also about a special global case which has been previously studied (the asymmetric oscillator).

2. A CONSTRUCTION LEADING TO RESONANCE

Let us consider a particle of unit mass moving on the real line. The position is denoted by $x = x(t)$, $-\infty < x < \infty$. We assume that this particle is subjected to the restoring force $-g(x)$. If the origin is an isochronous center the particle will oscillate around the origin with a fixed period (independent of the amplitude of the oscillation). Let us now add to the system an external force so that the particle moves according to (**??**). The force $f(t)$ only acts at the discrete times $t = nT$, $n = 0, \pm 1, \pm 2, \ldots$ and has the effect of increasing the velocity of the particle by a fixed amount ϵ. It seems clear that, after a finite number of periods, the particle will start gaining energy in each oscillation. To describe this situation in geometrical terms we look at the phase portrait. See the figure in the next page.

At time $t = 0^+$ we are at p_0 and, after one period, we return to this point by Γ_0. Then the impulse at $t = T$ makes the point to jump to p_1 and the process keeps repeating. The isochronicity is used to guarantee that all the points $p_0, p_1, p_2, \ldots$ lie on the same vertical line.

To make precise these intuitive ideas we shall employ the theory of distributions. We refer to [**?**] for the terminology.



Let $\mathcal{D}'(\mathbb{R})$ be the space of distributions over $\mathbb{R}$, where $\mathcal{D}(\mathbb{R}) = C_0^\infty(\mathbb{R})$ is taken as the class of test functions. The "periodic δ-function" is defined by

$$\delta_\sharp \in \mathcal{D}'(\mathbb{R}), \quad \langle \delta_\sharp, \phi \rangle = \sum_{n \in \mathbf{Z}} \phi(nT), \;\; \forall \phi \in \mathcal{D}(\mathbb{R}).$$

We consider the differential equation

$$\ddot{x} + g(x) = \epsilon \delta_\sharp(t), \quad \epsilon \neq 0, \tag{3}$$

where ϵ is a parameter. This equation is understood in the sense of distributions. By a solution (defined over $\mathbb{R}$) we mean a function $x \in C(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} \{x \ddot{\phi} + g(x)\phi\} = \epsilon \langle \delta_\sharp, \phi \rangle \quad \forall \phi \in \mathcal{D}(\mathbb{R}).$$

To make this definition compatible with our previous intuition we present another formulation of the notion of solution.

LEMMA 1. *A function $x \in C(\mathbb{R})$ is a solution of (**??**) if and only if x satisfies, for each integer n,*

$$x \in C^2[nT, (n+1)T], \quad \dot{x}(nT+) = \dot{x}(nT-) + \epsilon$$

and

$$\ddot{x}(t) + g(x(t)) = 0 \;\text{ if } t \in (nT, (n+1)T).$$

The proof of this result is based on standard arguments in the theory of distributions.

From this lemma we can show that the initial value problem for (**??**) is well posed. We can even integrate (**??**). Let $\psi(t; x_0, v_0)$ denote the solution of (**??**) for the initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$. Then

$$x(t) = \psi(t; x_0, v_0 + n\epsilon) \quad \text{if } t \in [nT, (n+1)T)$$

is a solution of (**??**). In fact it is clear that we can describe all the solutions in this way. From here the following result is obvious.

THEOREM 2. *Assume that $x = 0$ is an isochronous center for (**??**). Then every solution $x(t)$ of (**??**) satisfies*

$$|x(t)| + |\dot{x}(t)| \to \infty \quad \text{as } t \to +\infty.$$

3. RESONANCE FOR SMOOTH FORCINGS

In this section we shall assume that g is Lipschitz-continuous. This means that, for some $L > 0$,

$$|g(x_1) - g(x_2)| \leq L|x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{R}. \tag{4}$$

This is the case for the harmonic oscillator (g linear) but there are also many nonlinear functions g which define an isochronous center and satisfy (**??**). We will discuss this question in the next section. In this section we prove the following result.

THEOREM 3. *Assume that g satisfies (**??**) and $x = 0$ is an isochronous center. Then there exists a function $p \in C^\omega(\mathbb{R}/T\mathbf{Z})$ such that every solution $x(t)$ of*

$$\ddot{x} + g(x) = \epsilon p(t), \quad 0 < |\epsilon| \leq 1, \tag{5}$$

satisfies

$$|x(t)| + |\dot{x}(t)| \to \infty \quad \text{as } t \to +\infty.$$

As the reader probably expects, to prove this theorem we will consider forcings $p(t)$ which are close to the periodic δ-function. Before doing this it is convenient to change the system of reference in such a way that (**??**) becomes a standard differential equation.

Consider the periodic function

$$\mathcal{H}(t) = \frac{t(T-t)}{2T} \quad \text{if } t \in [0, T), \quad \mathcal{H}(t+T) = \mathcal{H}(t) \quad \forall t \in \mathbb{R}.$$

This function is continuous and satisfies

$$\ddot{\mathcal{H}} = -\frac{1}{T} + \delta_\sharp$$

in the sense of distributions. The change of variables

$$x = y + \epsilon \mathcal{H}(t) \tag{6}$$

transforms (**??**) into

$$\ddot{y} + g(y + \epsilon \mathcal{H}(t)) = \frac{\epsilon}{T}. \tag{7}$$

Now let us assume that p is a function in $C^\omega(\mathbb{R}/T\mathbf{Z})$ with

$$\int_0^T p(t)dt = 1.$$

Then we can find $\mathcal{P} \in C^\omega(\mathbb{R}/T\mathbf{Z})$ such that

$$\ddot{\mathcal{P}}(t) = -\frac{1}{T} + p(t).$$

The change

$$x = z + \epsilon \mathcal{P}(t)$$

will transfom (**??**) into

$$\ddot{z} + g(z + \epsilon \mathcal{P}(t)) = \frac{\epsilon}{T}. \tag{8}$$

To prove theorem **??** it will be sufficient to find $\mathcal{P}$ in $C^\omega(\mathbb{R}/T\mathbf{Z})$ and such that all the solutions of (**??**) satisfy

$$|z(t)| + |\dot{z}(t)| \to \infty \quad \text{as } t \to +\infty. \tag{9}$$

This is so because then we can undo the second change of variables and find an equation of the type (**??**) with the solutions going to infinity.

Proof of theorem ??. Let Π be the Poincaré map associated to (**??**). More precisely,

$$\Pi : \mathbb{R}^2 \to \mathbb{R}^2, \ \ (\xi, \eta) \mapsto (\xi_1, \eta_1), \ \ \xi_1 = z(T), \eta_1 = \dot{z}(T),$$

where $z(t)$ is the solution of (**??**) with $z(0) = \xi$, $\dot{z}(0) = \eta$. To prove (**??**) it will be sufficient to show that, for every orbit $(\xi_n, \eta_n) = \Pi^n(\xi, \eta)$,

$$|\xi_n| + |\eta_n| \to \infty \quad \text{as } n \to +\infty.$$

This is so because g is Lipschitz-continuous.

To fix the ideas let us assume that ϵ has a sign, say $\epsilon > 0$. We shall prove that, for an arbitrary point (ξ, η), one has

$$\eta_1 \geq \eta + \frac{\epsilon}{2}. \tag{10}$$

From here $\eta_n \geq \eta + n\epsilon/2$ and the conclusion follows.

To prove (**??**) we compare Π with the Poincaré map associated to (**??**). This map is just a translation along the vertical direction, namely

$$T_\epsilon(\xi, \eta) = (\xi, \eta + \epsilon).$$

This formula can be easily justified from the change given by (**??**) and the discussions in the previous section. Notice that, for this change,

$$\dot{x}(nT\pm) = \dot{y}(nT) + \epsilon\dot{\mathcal{H}}(nT\pm).$$

Given a solution $z(t)$ of (**??**) we can employ (**??**) to deduce that

$$|\ddot{z}(t) + g(z(t) + \epsilon\mathcal{H}(t)) - \frac{\epsilon}{T}| \leq \epsilon L||\mathcal{H} - \mathcal{P}||$$

where $||.||$ stands for the uniform norm. This means that we can interpret $z(t)$ as an approximate solution of (**??**) for $||\mathcal{H} - \mathcal{P}||$ small. Let $y(t)$ be the solution of (**??**) with the same initial conditions as $z(t)$ at $t = 0$. An application of the fundamental inequality for approximate solutions (see for instance [**?**], page 8) leads to

$$|y(t) - z(t)| + |\dot{y}(t) - \dot{z}(t)| \leq (e^{L^*|t|} - 1)\epsilon||\mathcal{H} - \mathcal{P}||$$

where $L^* = \max\{L, 1\}$. For $t = T$ we obtain

$$|\eta_1 - \eta - \epsilon| \leq (e^{L^*T} - 1)\epsilon||\mathcal{H} - \mathcal{P}||.$$

If $\mathcal{P}$ is close enough to $\mathcal{H}$, namely

$$(e^{L^*T} - 1)||\mathcal{H} - \mathcal{P}|| < \frac{1}{2},$$

then (**??**) holds and the proof is finished.

4. ADDITIONAL REMARKS.

Isochronous centers and the Lipschitz condition

As we mentioned already there are many isochronous centers satisfying (**??**). We follow [**?**] to make an analytic construction. Let $S : \mathbb{R} \to \mathbb{R}$ be an analytic and odd function satisfying

$$C_0 := \sup_{X \in \mathbb{R}} |S(X)| < 1, \quad C_1 := \sup_{X \in \mathbb{R}} |XS'(X)| < \infty$$

(example: $S(X) = \alpha \arctan X$, $|\alpha| < \frac{2}{\pi}$).

Consider the initial value problem

$$\frac{dX}{dx} = \frac{\omega}{1 + S(X)}, \quad X(0) = 0,$$

where $\omega = \frac{2\pi}{T}$. The solution $X(x)$ is defined in $(-\infty, +\infty)$ and it is known that the function $g(x) = X'(x)X(x)$ produces an isochronous center. We notice that $g(\pm\infty) = \pm\infty$ because $X'(x)$ remains between $\omega/(1 + C_0)$ and $\omega/(1 - C_0)$. To check the condition (**??**) we must find a bound of

$$g'(x) = X'(x)^2 + X(x)X''(x).$$

Since $|X'| \le \omega/(1 - C_0)$ we can concentrate on the second term,

$$|XX''| = \frac{\omega^2}{(1 + S(X))^3} |XS'(X)| \le \frac{\omega^2}{(1 - C_0)^3} C_1.$$

To construct examples of isochronous centers which are not Lipschitz continuous one can repeat the previous construction with $C_0 < 1$ and $C_1 = \infty$. This is the case for $S(X) = \frac{1}{2} \sin X$.

Local isochronous centers

Let us now consider the case of a local isochronous center. This means that g is defined on some interval $I = (a, b)$, $-\infty \le a < 0 < b \le +\infty$, satisfies

$$g(0) = 0, \quad xg(x) > 0, \ x \in I - \{0\}$$

and there is a neighborhood $V \subset \mathbb{R}^2$ of the origin such that every orbit of (**??**) lying on V has period T. The ideas of section 2 still work and lead to the following conclusion:

*Let $x(t)$ be a solution of (**??**) lying in V; that is, for each $n \in \mathbf{Z}$,*

$$(x(nT), \dot{x}(nT\pm)) \in V \ \text{ and } \ (x(t), \dot{x}(t)) \in V, \quad \forall t \in (nT, (n+1)T).$$

Then $\frac{1}{2}\dot{x}(t)^2 + G(x(t)) \to +\infty$ as $|t| \to \infty$, $t \notin T\mathbf{Z}$. (Here G is a primitive of g).

This result applies to the equation

$$\ddot{x} + x + 1 - \frac{1}{(1+x)^3} = \epsilon \delta_\sharp(t).$$

In fact the solutions of the autonomous equation ($\epsilon = 0$) can be explicitly computed and they all have period π (see [?]). We can take $a = -1$, $b = +\infty$, $V = \{(x, \dot{x}) \in \mathbb{R}^2 / \, x > -1\}$.

The asymmetric oscillator

Consider the piecewise linear equation

$$\ddot{x} + ax^+ - bx^- = 0$$

where a, $b > 0$, $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. A direct computation shows that that this equation is isochronous with minimal period $\frac{\pi}{\sqrt{a}} + \frac{\pi}{\sqrt{b}}$. The periodic perturbation

$$\ddot{x} + ax^+ - bx^- = f(t) \qquad (11)$$

has been studied by many authors. We refer to [**?**, **?**] for the origins and to the survey papers [**?**, **?**] for more recent results.

Notice that theorem **??** applies in this case and gives new examples of non-existence of periodic solutions of (**??**).

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