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Some Remarks About the Integration of Polynomial Planar Vector Fields

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In this paper, we present some tools, especially but not only from enumerative and algebraic geometry, that are involved in the search of first integrals of polynomial planar vector fields. As an illustration, we described how these tools have been useful in our recent characterization of all cases of projective liouvillian integration of the homogeneous three-dimensional Lotka-Volterra system.

Key Words: Integrability, Darboux polynomials, ordinary differential equations.

1. INTRODUCTION

The aim of this paper is first to make precise the framework in which we look for first integrals of planar vector fields.

Many definitions, that are well-known for us, are given, general methods are described and the results of our work on a special class of examples, the Lotka-Volterra system, are given in details.

Last, we describe a new tool which is useful in the study of the integration of some polynomial planar vector fields, those with at least a Darboux line. Usual affine polynomial planar vector fields have this property, which is then not so seldom.

The idea is to decompose a candidate Darboux polynomial in its two-variable homogeneous components; the component of highest degree can be expressed without knowing its exact degree and it is possible to start from it to derive necessary conditions for the existence of a Darboux polynomial.

2. AFFINE AND PROJECTIVE POINTS OF VIEW

2.1. The affine model

In the first naive affine point of view, a *polynomial planar vector field* V is a $\mathbb{C}$-derivation $V = P\partial_x + Q\partial_y$ of the ring $\mathbb{C}[x, y]$, where $P, Q \in \mathbb{C}[x, y]$.

In this context, a *first integral of* V is a non-constant function F such that

$$PF_x + QF_y = 0.$$

In order to make things algebraically clear, we demand that the "function" F belongs to some differential field extension $\mathbb{K}$ of $\mathbb{C}(x, y)$ for the two commuting derivations ∂_x and ∂_y.

Moreover, saying that F is not a constant means that its two-component exterior derivative $[\partial_x F, \partial_y F] = [F_x, F_y]$ is not equal to $[0, 0]$.

2.2. The projective model

In this point of view, a *polynomial planar vector field* V is a particular $\mathbb{C}$-derivation of the ring $\mathbb{C}[x, y, z]$

$$V = P\partial_x + Q\partial_y + R\partial_z,$$

where P, Q, R are *homogeneous* polynomials with the *same degree* m.
The triple $[P, Q, R]$ thus defines a map from the projective plane to itself with some singularities.

A first integral of V is a non-constant "function" F such that

$$PF_x + QF_y + RF_z = 0.$$

F belongs here to some differential field extension $\mathbb{K}$ of $\mathbb{C}(x, y, z)$ for the three commuting derivations ∂_x, ∂_y and ∂_z. That F is not a constant is given by

$$[\partial_x F, \partial_y F, \partial_z F] = [F_x, F_y, F_z] \neq [0, 0, 0].$$

We will say that F is a true *first integral* if it is a function on the projective plane. This means that F does not depend on the chosen representative $[x, y, z]$ of a point in the projective plane, but only on this point; in other words, F is *homogeneous of degree 0*, which can be written with the help of the Euler field $E = x\partial_x + y\partial_y + z\partial_z$:

$$xF_x + yF_y + zF_z = 0.$$

The exterior derivative $[F_x, F_y, F_z]$ of a true first integral is collinear to the 1-form $\omega = (Qz - Ry)dx + (Rx - Pz)dy + (Py - Qx)dz$.

It turns out that finding a true first integral amounts to finding an integrating factor for ω. In turn, an integrating factor for ω is a homogeneous (not true) first integral of the curl of ω.

The curl of ω is the sum of the given vector field together with a suitable multiple of E, in order to get a vector field with a zero divergence.

3. CHANGING THE POINT OF VIEW

3.1. From the affine model to the projective one

This is very easy: make P and Q homogeneous with the same degree m in the three variables x, y, z: $P \to z^m P(x/z, y/z)$, $Q \to z^m Q(x/z, y/z)$, then set $R = 0$. Then, if F is a first integral in the affine sense, $F(x/z, y/z)$ is a true first integral in the projective sense.

3.2. From the projective model to the affine one

Let $V = P\partial_x + Q\partial_y + R\partial_z$. Consider its transform $V_0 = zV - RE$.

Up to multiplication by z, V_0 leads to the same 1-form ω and moreover has a zero third coordinate. Let $z = 1$ in it to get the sought affine planar polynomial vector field.

Evaluating in $z = 1$ a true first integral of the projective model gives a first integral of the affine model.

3.3. A slight difference, the degree

Though without common factor for its polynomial coordinates, the affine model may have a degree higher by 1. For example, this happens for the Jouanolou vector field

$$y^2\partial_x + z^2\partial_y + x^2\partial_z,$$

for which $zV - RE = [y^2 - x^3, 1 - x^2 y, 0]$. This vector field and its generalizations have been studied in [**?**, **?**, **?**].

Later on, we will discuss this difference. In fact, the affine model has a special property, which will turn out to be very useful in the computational treatment of its integrability:

There exists a Darboux polynomial of degree 1, the extra coordinate z.

4. DARBOUX POLYNOMIALS AND DARBOUX POINTS

Now, we adopt the projective point of view in which V is a polynomial vector field of degree m. To study the integrability of such a vector field, an old method of Darboux consists in finding sufficiently many "particular algebraic solutions" of the system of differential equations.

Due to the great interest of this concept, we have got the habit to call these solutions *Darboux polynomials*. Indeed, these polynomials are not only meaningful as equations of plane projective algebraic curves; they are also useful on the whole projective plane.

Given a polynomial planar vector field $V = P\partial_x + Q\partial_y + R\partial_z$, where P, Q, R are homogeneous three-variable polynomials with the same degree m, a homogeneous polynomial F in $\mathbb{C}[x, y, z]$ is said to be a Darboux polynomial of V with *cofactor* (or eigenvalue) Λ if

$$PF_x + QF_y + RF_z = \Lambda F.$$

The cofactor Λ has to be a homogeneous polynomial of degree $m - 1$.

4.1. The method of Darboux

Many elementary remarks can be done about Darboux polynomials.

First, the homogeneous components of a non-homogeneous Darboux polynomial are Darboux polynomials, and that was a reason to define only homogeneous Darboux polynomials.

Second, the product of two Darboux polynomials is Darboux and its cofactor is the sum of cofactors; conversely, factors of Darboux polynomials are Darboux (characteristic 0 is important here as in many places in differential algebra).

Thus, if there are sufficiently many irreducible pairwise relatively prime homogeneous Darboux polynomials, a well-suited product of (maybe non-integer) powers of them is a Darboux object with cofactor 0.

This "function" is a first integral but not a true one: it is homogeneous, but there is no reason for its degree to be 0.

In the case this degree is not 0, an integrating factor of ω can be deduced, which ensures the liouvillian integration of the vector field.

When there is a non-trivial linear combination with *integer* coefficients of cofactors of Darboux polynomials, the previous object is a homogeneous rational first integral; this rational first integral is not a true one except if the same linear combination of the degrees moreover yields 0.

These ideas date back to Darboux.

4.2. Vector cofactors

If the degree of the Darboux polynomial F is n, the new vector field $V_F = V - \frac{\Lambda}{n}E$ is the *vector cofactor* of F, which means the following relation between the vector field V_F and the closed 1-form $[F_x, F_y, F_z]$

$$(V_F)_x F_x + (V_F)_y F_y + (V_F)_z F_z = 0.$$

This orthogonality relation leads to an interesting formula of enumerative geometry concerning the intersection indices of the homogeneous ideals generated by the coordinates of V_F on the one hand and by the three partial derivatives of F on the other hand (the Tjurina ideal of F) [**?**, **?**, **?**].

4.3. Darboux points

Irreducible homogeneous Darboux polynomials of a polynomial homogeneous vector field generate minimal homogeneous prime ideals in $\mathbb{C}[x, y, z]$ that are invariant under the corresponding derivation.

Similarly, a maximal homogeneous ideal corresponds to a point in the projective plane. The ideal is the one of all polynomials vanishing at the point; if $[x_0, y_0, z_0]$ is a representative of the point, the ideal is generated by the three polynomials $zy_0 - z_0y, xz_0 - x_0z, yx_0 - y_0x$.

It is not very difficult to check that such a maximal homogeneous ideal is invariant if and only if the vector field and the Euler field are collinear at the corresponding point.

The points where this happens are thus of special interest. They are the singular points of ω or the *Darboux points* of the vector field. There is a finite number of them in the projective plane.

The first-order study of the behavior of Darboux polynomials around Darboux points is known as the Levelt (or Lagutinskii-Levelt) procedure.

This is a very powerful tool to deal with the integrability of a given vector field. Though powerful, this first-order study has to be completed in many applications; for instance, in our recent treatment of the liouvillian integration [**?**] of the Lotka-Volterra system, we factored irreducible Darboux polynomials into branches (two-variable formal power series) in order to find more efficient necessary conditions of integrability.

5. WHAT KINDS OF FIRST INTEGRALS

Up to now, we did not precise the class of functions in which we look for a first integral of a given polynomial planar vector field, either in the affine case or in the projective case.

The first problem, that dates back at least to Poincaré and Painlevé [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**], consists in finding a rational first integral in the affine case or a homogeneous rational first integral of degree 0 in the projective case. We will call this problem the *rational integration* of the vector field.

The only general remark we have to do on this problem deals with linear algebra: the rational integration of a polynomial planar projective vector field holds if and only if, for some degree n and cofactor Λ, the Darboux operator $f \to Pf_x + Qf_y + Rf_z - \Lambda f$ has a kernel whose dimension is ≥ 2. We used this remark in [**?**] to find necessary conditions (rationality) on the parameters for the rational integration of the Lotka-Volterra system.

Let us remark that Poincaré proposed necessary conditions for the rational integration by looking at the singular points of the 1-form in the projective plane (the Darboux points of the vector field); he did not really succeed in giving completely efficient conditions [**?**, **?**, **?**].

The other interesting problem of integration of polynomial planar vector fields can be called *liouvillian integration*. In this case, we want to decide if there exists a liouvillian first integral (in the affine case) or a homogeneous liouvillian first integral of degree 0 (in the projective case).

A liouvillian first integral f is an element of some liouvillian extension $\mathbb{K}$ of $\mathbb{C}(x, y, z)$. Liouvillian extensions have been defined by Singer in the case of several commuting derivations (the partial derivatives here) [**?**]. As the three partial derivatives are defined on $\mathbb{K}$, the initial vector field V as well as the Euler field E are defined on $\mathbb{K}$. A homogeneous first integral of degree 0 is then a constant for both fields, without being a constant for the three partial derivatives.

Typically, this problem can be solved by a two-step process. First find sufficiently many Darboux polynomials to build an integrating factor for the 1-form ω, and then integrate it. If the integrating factor is liouvillian, so is the first integral [**?**, **?**].

6. THE LOTKA-VOLTERRA SYSTEM

We started the study of the integration, i. e. the search of first integrals, of the Lotka-Volterra system (LV for short) several years ago [**?**]; because the most difficult, this system was the most interesting example that we found to illustrate a method of Strelcyn and Wojciechowski, the *compatibility analysis* [**?**]. This Lotka-Volterra system can be written as

$$LV(A, B, C) = V_x\partial_x + V_y\partial_y + V_z\partial_z,$$

where

$$V_x = x(Cy + z),\ V_y = y(Az + x),\ V_z = z(Bx + y),\ ABC \neq 0.$$

Many people were and are still interested in this system and in related ones [**?**].

6.1. Is LV a so particular system ?

LV has three Darboux lines, the coordinates x, y, z. Conversely, if a quadratic vector field has three Darboux lines without common point, the equations of these lines can be chosen as coordinates to get a field in *factored form*. Thereafter, except in some very exceptional cases, a diagonal linear change of variable and the addition of a well-chosen multiple of the Euler field puts it in the LV form.

Thus, with respect to integrability questions, LV is a class of normal forms of some quadratic systems.

We say that a Darboux polynomial for LV is *strict* if it is not divisible by any of the coordinates x, y, z. The cofactors of strict Darboux polynomials of degree n have the form $\lambda x + \mu y + \nu z$ where (λ, μ, ν) is a triple of nonnegative integers such that $\lambda, \mu, \nu \leq n$.
Remark that this is a special aspect of the "finiteness of cofactors".

Now, if LV has *one* extra (strict) Darboux polynomial besides x, y, z, the liouvillian integration holds.

Darboux points (the seven zeroes of ω in the projective plane) are easy to compute and the local study of Darboux polynomials is a possible computation.

A finite group of automorphisms of the whole family $LV(A, B, C)$, where $(A, B, C) \in \mathbb{C}^3$, allows us to reduce a painful case analysis.

The local study around the three Darboux lines (this point of view is dual to Levelt's method) is amenable to combinatorial efforts [**?**].

In the last section of the present paper, we present a deeper treatment of this aspect, which turns out to be useful even with only one Darboux line.

7. INTEGRATION OF THE LOTKA-VOLTERRA SYSTEM

We found necessary and sufficient conditions on the triple (A, B, C) of parameters to ensure the rational integration [**?**] of the Lotka-Volterra system or its liouvillian integration [**?**].

It is not very easy to describe shortly the arguments that were involved in these classifications. Nevertheless, it is possible to give some insight about them.

To be fair, we have to say that these works have first been experimental. Using a computer algebra system, we look for strict Darboux polynomials of degree $1, 2, 3, 4, \ldots, 12$. Indeed, we know that there is a finite number of possible cofactors (and we know them) of strict Darboux polynomials of a given degree and we can try.

Thereafter, we performed a classification of the results taking into account the finite group of symmetries (circular permutations in particular) of the family.

The hard job was to find necessary conditions to be sure that the list was complete. A standard idea, dating back to Poincaré, consists in looking for an upper bound on the the degree of irreducible Darboux polynomials. This idea is a dream; indeed, we discovered the sporadic family, with unbounded degree [**?**].

Thus, our analysis was different. We used Levelt's method around the seven Darboux points, but this was not sufficient. According to the nature of every Darboux point (saddle, node), we were able to split Darboux polynomials into branches and we found some arithmetic conditions on the involved parameters.

Another important tool, that we have previously noticed, was the study around the three Darboux lines of coordinates; a very interesting combinatorial treatment was possible, that led to a rather intricate case analysis, in which the original parameters A, B, C or some functions of them, denoted by p, q, r, take integer values.

8. WITH ONE DARBOUX LINE (AT INFINITY)...

In this last section, we describe roughly how the idea of expanding Darboux polynomials around Darboux lines, instead of around Darboux points, that we used successfully in our work on the integration of the LV system, can be powerful in a much more general context.

Indeed, if we start from an affine model, the projective one receives a Darboux polynomial, the extra coordinate z; thus, having a Darboux line is indeed a common fact, if we have in mind the affine model as a starting point.

To be clear, we will restrict ourselves to the case of quadratic planar vector fields and first give a normal form for them.

Afterwards, we will look for necessary conditions on the "free" parameters in order to allow the existence of a Darboux polynomial. Therefore, we will start by the homogeneous two-variable homogeneous component of highest degree of this supposed Darboux polynomial F. Thereafter, we have to look for homogeneous two-variable polynomial solutions of some linear differential equations.

8.1. A suitable normal form

Thanks to a linear change of variables and to the addition of a multiple of E, it is possible to choose coordinates in which a quadratic vector field with a Darboux line can be written with a, b, c, d, e as free parameters:

$$V = [x\,(ax + by + y) + cxz + ez^2]\partial_x + [y\,(by + ax + x) + dyz + ez^2]\partial_y.$$

Let F be a (homogeneous three-variable) Darboux polynomial for V with cofactor $\lambda x + \mu y + \nu z$.

Up to a constant, F "starts" by $y^{\alpha_0} x^{\alpha_1} (x-y)^{\alpha_2}$ as its (x, y)-homogeneous component of highest degree, where $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{N}$.

The degree of F is $\alpha_0 + \alpha_1 + \alpha_2$ and λ, μ are given by

$$\lambda = \alpha_0 + a(\alpha_0 + \alpha_1 + \alpha_2), \mu = b(\alpha_0 + \alpha_1 + \alpha_2) + \alpha_1.$$

8.2. Effective computations

With the previous normal form, it is possible to solve equations providing successive (x, y)-components of decreasing degree starting from the highest degree. F can be written as

$$F = y^{\alpha_0} x^{\alpha_1} (x-y)^{\alpha_2} \left(1 + \sum_k \frac{z^k F_k}{D^k}\right),$$

where F_k is homogeneous of degree $2k$ and $D = xy(x-y)$ (the zeroes of D correspond to the three Darboux points of the vector field on the line at infinity).

The fact that F is a Darboux polynomial can be written as a sequence of differential equations for the F_k:

$$\begin{aligned}
&[P_2\partial_x + Q_2\partial_y - kL](F_k) = \\
&\quad [-D(P_1\partial_x + Q_1\partial_y) + (k-1)M + N)](F_{k-1}) + \\
&\quad D[-D(P_0\partial_x + P_1\partial_y) + (k-2)S + T)](F_{k-2}),
\end{aligned}$$

where $F_0 = 1$, $F_{-1} = 0$ and L, M, N, S, T are polynomials computed from the vector field and from the integer parameters α_i.

The differential equation for F_1 gives the value of ν. This was our combinatorial analysis for the LV system and this first step was almost sufficient. The differential equation for F_2 gives the value of e. The differential equation for F_3 then gives strong constraints on the 4 remaining parameters a, b, c, d.

With the help of a computer algebra system, we can at least find interesting examples in many situations.

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