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On the Period Function of Centers in Planar Polynomial Hamiltonian Systems of Degree Four

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In this paper we study non-degenerate centers of planar polynomial Hamiltonian systems. We prove that if the differential system has degree four then the period function of the center tends to infinity as we approach to the boundary of its period annulus. The proof takes advantage of the geometric properties of the period annulus in the Poincaré disc and it requires the study of the so called cubic-like Hamiltonian systems, namely the differential systems associated to a Hamiltonian function of the form $H(x,y) = A(x) + B(x)y + C(x)y^2 + D(x)y^3$. Concerning the centers of this family of differential systems, we obtain an analytic expression of its period function. From our point of view this expression constitutes the first step in order to find the isochronicity conditions in the family.

Key Words: Isochronicity, period function, hamiltonian systems

1. INTRODUCTION

In this paper we study the behaviour of the period function of the centers of planar polynomial Hamiltonian systems, i.e., differential systems of the

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form

$$\begin{cases} \dot{x} = -H_y(x,y), \\ \dot{y} = H_x(x,y), \end{cases} \tag{1}$$

where H is a real polynomial in x and y. The period function of a center gives the period of each periodic orbit inside its period annulus. Questions relating to the behaviour of the period function have been extensively studied by a number of authors. Let us mention for instance the problems of isochronicity (see [?, ?, ?]), monotonicity (see [?, ?, ?]) or the bifurcation of critical periods (see [?, ?, ?]). In this paper we prove that if the differential system (??) has degree four then the period function of the center tends to infinity as we approach to the boundary of its period annulus.

The results that we present here are a continuation of the ones that we obtain in [?] and this continuation is twofold. This is so because in that paper we develop a machinery to study the cubic-like Hamiltonian systems, namely the ones given by $H(x,y) = A(x) + B(x)y + C(x)y^2 + D(x)y^3$, and we use it to show that every center of a polynomial Hamiltonian system of degree four is non-isochronous. In the present paper we first improve this machinery and then we apply it to prove that, in fact, the period function of these centers tends to infinity (see Theorem **??**).

In order to prove Theorem **??** we first study the shape of the period annulus of the center in the Poincaré disc and this enables us to decrease the number of parameters in the system. This reduction lead us to four possible cases depending in some parameter values. In two of these cases the theorem already follows from results in [?]. Concerning the other two cases, one falls into the cubic-like family and the other one into the quadratic-like family, namely the Hamiltonian systems associated to $H(x,y) = A(x) + B(x)y + C(x)y^2$, which was previously studied in [?]. In that paper it is given the isochronicity condition for the centers of the quadratic-like family, and to this end it was first necessary to dispose of an expression of its period function. An interesting problem for further research is to obtain the isochronicity condition for the centers in the cubic-like family and, following the approach in [?], in the present paper we provide the expression of its period function (see Corollary **??**).

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2. DEFINITIONS AND SETTING OF THE PROBLEM

Definition 1. We say that a critical point of a planar system is a *center* if there is a punctured neighbourhood of the point that consists entirely of periodic orbits surrounding that point. The largest punctured neighbourhood with this property is called the *period annulus* of the center and it will be denoted by $\mathcal{P}$. The function which associates to any periodic orbit γ in $\mathcal{P}$ its period is called the *period function* of the center. The center is said to be *non-degenerate* if the linearized vector field at the point has two non-zero eigenvalues.

Our goal in this paper is to prove that the period function of any non-degenerate center of a polynomial Hamiltonian system of degree four tends to infinity as we approach to the boundary of its period annulus. A polynomial Hamiltonian system of degree n is a differential system of the form

$$\begin{cases} \dot{x} = -H_y(x,y), \\ \dot{y} = H_x(x,y), \end{cases} \tag{2}$$

where $H(x,y)$ is a polynomial of degree $n+1$. Let us fix that $H(0,0)=0$ and denote the homogeneous part of degree i of $H(x,y)$ by $H_i(x,y)$. Notice that, by definition, if (**??**) is a system of degree n then $H_{n+1} \not\equiv 0$. We can assume that the non-degenerate center that we study is located at the origin and that

$$H(x,y) = \frac{x^2+y^2}{2} + H_3(x,y) + \ldots + H_{n+1}(x,y). \tag{3}$$

It is clear that this can be done without loss of generality taking an appropriate coordinate system and scaling the time by a constant amount. In this case (see [**?**]) it follows that $H(x,y) > 0$ for any point $(x,y) \in \mathcal{P}$, and therefore $H(\mathcal{P}) = (0, h_0)$ for some $h_0 \in \mathbb{R}^+ \cup \{+\infty\}$. Note also that the solutions of (**??**) are contained in the level curves $\{H(x,y) = h,\ h \in \mathbb{R}\}$. When the center is nonglobal (i.e., $\mathcal{P} \neq \mathbb{R}^2$) this implies that h_0 is finite and that $\partial\mathcal{P}$, the boundary of its period annulus, is contained in the level curve $\{H(x,y) = h_0\}$. We will use this notation all over the paper.

On the other hand, one can show (see [**?**]) that the set of periodic orbits in the period annulus can be parametrized by the energy h. Thus, for each $h \in (0, h_0)$ we will denote the periodic orbit in $\mathcal{P}$ of energy level h by γ_h. This allows us to consider the period function over $(0,h_0)$ instead of the original period function which is defined over the set of periodic orbits contained in the period annulus. Therefore in the sequel we will talk about the period function $h \longmapsto T(h)$ which gives the period of the periodic orbit with energy $h \in (0, h_0)$. With this notation we can now state the main result of the paper.

THEOREM 2. *Let $h \longmapsto T(h)$ be the period function of a non-degenerate center of a polynomial Hamiltonian system of degree four. Then $T(h) \longrightarrow +\infty$ as $h \nearrow h_0$.*

It is worth pointing out that, in some sense, it is not necessary to assume that the center is non-degenerate in order that its period function tends to infinity. Indeed, Theorem 2.2 in [**?**] shows that if the center is degenerate then its period function tends to infinity as the periodic orbits tend to the center. In our setting this means that $T(h) \longrightarrow +\infty$ as $h \searrow 0$.

There is another result in connection with Theorem **??** that should be referred. It is proved in [**?**] that the period function of the center at the origin of a Hamiltonian system given by

$$H(x,y) = \frac{x^2+y^2}{2} + H_{2n+1}(x,y)$$

tends to infinity as $h \nearrow h_0$. In particular, this shows that the conclusion of Theorem **??** is also true for polynomial Hamiltonian systems of second degree. In view of this one may wonder if the result is true for any polynomial Hamiltonian system of even degree. In Section **??** we will show by means of an example that this is not so.

For the remainder of this section we shall show that it suffices to prove Theorem **??** for a particular type of center. Simultaneously we shall introduce the definitions and notation that will be used henceforth.

Notice first that when n is even then any center of the Hamiltonian system given by (**??**) is nonglobal. This is so because if $H(x,y)$ is a polynomial of odd degree then it is not possible that $H(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Consequently, since we deal with a Hamiltonian system of degree four, the period annulus of the center at the origin cannot be the whole plane, and note then that $h_0 < +\infty$. We can also assume that $\mathcal{P}$ is unbounded because otherwise there would be some finite critical point in $\partial\mathcal{P}$ and it is well known that then $T(h) \longrightarrow +\infty$ as $h \nearrow h_0$. Summarizing, we have shown that in order to prove Theorem **??** we can assume without loss of generality that $\mathcal{P}$ is nonglobal and unbounded.

At this point it is convenient to recall that when we deal with a polynomial system then the Poincaré compactification provides an analytic extension of the vector field at infinity. It is then possible to study the behaviour of the flow at infinity, which corresponds to the equator of the Poincaré sphere. This fact will be a crucial item in our work because in order to study the boundary of the period annulus we shall turn account of a property of the Hamiltonian systems at infinity. To state this property precisely we need an additional definition.

Definition 3. Let q be an infinite critical point of a planar polynomial Hamiltonian system in the Poincaré compactification and let $\mathcal{H}$ be a hyperbolic sector associated to q. We say that $\mathcal{H}$ *does not contain straight lines* if for any finite straight line ℓ (in the Poincaré compactification) which passes through q there exists a neighbourhood V of q such that $\ell \cap V$ is not contained in the interior of $\mathcal{H}$.

We are now ready to state the above-named property. Thus, in [?] is proved the following.

LEMMA 4. *Let q be an infinite critical point of a polynomial Hamiltonian system with a hyperbolic sector $\mathcal{H}$. Then either $\mathcal{H}$ does not contain straight lines or its two separatrices are contained in the equator of the Poincaré sphere. In the first case both separatrices are finite and belong to the same level curve of the Hamiltonian.*

Now, according to Lemma **??**, the fact that $\mathcal{P}$ is unbounded and non-global implies the existence of an infinite critical point with a hyperbolic sector that has both separatrices lying in the finite part and inside $\partial\mathcal{P}$. Let us say that, in the Poincaré disc, this infinite critical point is given by the direction $\theta^\star \in [0, 2\pi)$. In this case Proposition 4.1 in [?] shows that if the Hamiltonian system associated to (**??**) has such an infinite critical point then

$$g_{n+1}(\theta^\star) = g'_{n+1}(\theta^\star) = g_n(\theta^\star) = 0,$$

where $g_i(\theta) = H_i(\cos\theta, \sin\theta)$ for $i = 2, 3, \ldots, n+1$. Notice that by means of a rotation of axis we can suppose without loss of generality that $\theta^\star = \pi/2$. This is so because after such a change of coordinates the quadratic part of the Hamiltonian remains unchanged. In view of this it is natural to introduce the following definition.

DEFINITION 5. In the sequel we shall say that a period annulus $\mathcal{P}$ is *admissible* when $\mathcal{P}$ is unbounded, nonglobal and its boundary, in the Poincaré disc, contains the infinite critical point given by the direction $\theta = \pi/2$. We shall denote by $\mathcal{H}^\star$ the hyperbolic sector of this infinite critical point that has both separatrices inside $\partial\mathcal{P}$.

In the preceding discussion we only used that we deal with a polynomial Hamiltonian system of even degree. We should now focus our attention in the case we shall study, the polynomial Hamiltonian systems of degree four. In this case we have shown that there is no loss of generality in assuming that

$$H_5(x, y) = x^2(a_0y^3 + a_1y^2x + a_2yx^2 + a_3x^3)$$

and

$$H_4(x, y) = x(b_0y^3 + b_1y^2x + b_2yx^2 + b_3x^3).$$

Consequently, setting $H_3(x, y) = c_0y^3 + c_1y^2x + c_2yx^2 + c_3x^3$, the Hamiltonian function (**??**) can be rewritten as

$$H(x, y) = A(x) + B(x)y + C(x)y^2 + D(x)y^3, \tag{4}$$

where

$$
\begin{aligned}
A(x) &= x^2(1/2 + c_3 x + b_3 x^2 + a_3 x^3), \\
B(x) &= x^2(c_2 + b_2 x + a_2 x^2), \\
C(x) &= 1/2 + c_1 x + b_1 x^2 + a_1 x^3, \\
D(x) &= c_0 + b_0 x + a_0 x^2.
\end{aligned} \tag{5}
$$

In fact we have done better than this, let us gather it in short:

Remark 6. In order to prove Theorem **??** it suffices to see that any center at the origin of the Hamiltonian system given by (**??**) and having an admissible period annulus satisfies $T(h) \longrightarrow +\infty$ as $h \nearrow h_0$. Here by "any center" we mean for any possible choice of parameters in (**??**), and note that $a_0 = a_1 = a_2 = a_3 = 0$ is not allowed because then $H_5 \equiv 0$.

With this remark in mind we divide the proof of Theorem **??** in the following four possible situations:

I	$a_0 \neq 0$,
II	$a_0 = 0$ and $b_0 \neq 0$,
III	$a_0 = b_0 = 0$ and $c_0 \neq 0$,
IV	$a_0 = b_0 = c_0 = 0$.

In fact, Theorem **??** under the parameter values I and II follows respectively from Theorem 4.1 and Proposition 5.1 in [**?**]. More precisely, when $a_0 \neq 0$ we proved that if $\mathcal{P}$ is admissible then $T(h) \longrightarrow +\infty$ as $h \nearrow h_0$, and in case that $a_0 = 0$ and $b_0 \neq 0$ we showed that if $\mathcal{P}$ is admissible then $H_5 \equiv 0$. Consequently, to prove Theorem **??** we only need to consider the cases III and IV.

The paper is organized as follows. Section **??** is devoted to prove Theorem **??**, which gives an expression of the period function of the centers of the cubic-like Hamiltonian systems, namely the ones given by a Hamiltonian function as in (**??**). This is done in a more general setting than the one we shall need to prove Theorem **??** because we shall only assume that A, B, C and D are analytic functions on $\mathbb{R}$. In Section **??** we study the case III, in which by means of Theorem **??** we show that if $\mathcal{P}$ is admissible then $T(h) \longrightarrow +\infty$ as $h \nearrow h_0$. The case IV is studied in Section **??**. Note that in this case we have $D \equiv 0$, and so the Hamiltonian function (**??**) gives a quadratic-like Hamiltonian system with $\deg(C) \leq 3$. We shall prove (see Proposition **??**) that $T(h) \longrightarrow +\infty$ as $h \nearrow h_0$ for any non-global center of a quadratic-like Hamiltonian system with $\deg(C) \leq 3$. Finally, in Section **??** we show by means of an example that the conclusion of Theorem **??** is not true for polynomial Hamiltonian systems of degree six.

3. CUBIC-LIKE HAMILTONIAN SYSTEMS

In this section we assume that A, B, C and D are analytic functions on $\mathbb{R}$ such that the Hamiltonian system given by

$$H(x,y) = A(x) + B(x)y + C(x)y^2 + D(x)y^3$$

has a non-degenerate center at the origin. Since the associated differential system is

$$\begin{cases} \dot{x} = -B(x) - 2C(x)y - 3D(x)y^2, \\ \dot{y} = A'(x) + B'(x)y + C'(x)y^2 + D'(x)y^3, \end{cases} \tag{6}$$

this corresponds to require that

$$A'(0) = B(0) = 0 \text{ and } 2C(0)A''(0) - B'(0)^2 > 0. \tag{7}$$

Notice that this inequality implies that $C(0) \neq 0$. In order that the period annulus of the center satisfies $H(\mathcal{P}) = (0, h_0)$ with $h_0 > 0$ we shall also assume that

$$A(0) = 0 \text{ and } C(0) > 0. \tag{8}$$

It is clear that this can be done without loss of generality. This is a more general situation than the one we shall need to prove Theorem **??** but we find it interesting in itself. The main result that we obtain in this section is Theorem **??**, which provides an expression to compute $T(h)$ for any $h \in (0, h_0)$. From now on we define $(x_{\mathrm{L}}, x_{\mathrm{R}})$ to be the projection of $\mathcal{P}$ to the x-axis, that is

$$(x_{\mathrm{L}}, x_{\mathrm{R}}) = \{x \in \mathbb{R} : \text{ there exists } y \in \mathbb{R} \text{ such that } (x,y) \in \mathcal{P}\}.$$

Consequently $x_{\mathrm{L}} < 0 < x_{\mathrm{R}}$, and note that x_{L} or x_{R} may not be finite.

Remark 7. The key point in the proofs is that, for each fixed x, $H(x,y)$ is a polynomial of third degree with respect to y. Since we are interested in the behaviour of $T(h)$ when $h \nearrow h_0$ (i.e., for periodic orbits near $\partial\mathcal{P}$), this is the reason why we shall always assume that $D(x) \neq 0$ for all $x \in (x_{\mathrm{L}}, x_{\mathrm{R}})$. It will be clear however that this assumption is not necessary if one is interested only in $T(h)$ when $h \gtrsim 0$ (i.e., for periodic orbits near the center at the origin). In this case, for the results that we obtain, it suffices that $D(0) \neq 0$ holds.

To prove Theorem **??** we need the following three lemmas from [**?**] that we include here for the sake of completeness.

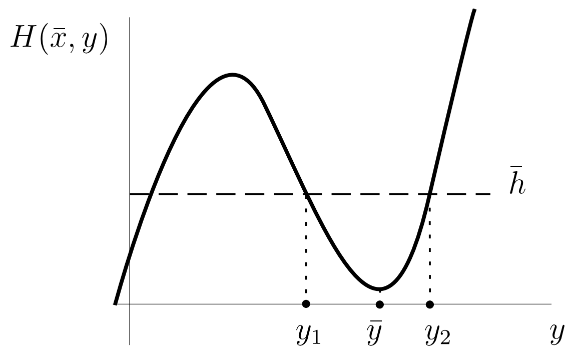

FIG. 1. The graph of $H(\bar{x}, y)$ in case that $D(\bar{x}) > 0$.

LEMMA 8. *If $D(x) \neq 0$ for all $x \in (x_L, x_R)$ then $C(x)^2 - 3D(x)B(x) > 0$ and*

$$\left(x, \frac{-C(x) + \sqrt{C(x)^2 - 3D(x)B(x)}}{3D(x)} \right) \in \mathcal{P} \text{ for any } x \in (x_L, x_R).$$

Proof. Assume for instance that $D(x) > 0$ for all $x \in (x_{\mathrm{L}}, x_{\mathrm{R}})$ and consider any $\bar{x}$ in this interval. Then there exists $\bar{h} \in (0, h_0)$ such that the periodic orbit $\gamma_{\bar{h}}$ has two intersection points with the straight line $x = \bar{x}$, say $(\bar{x}, y_1)$ and $(\bar{x}, y_2)$ with $y_1 < y_2$. Notice that the discriminant of

$$H_y(\bar{x}, y) = B(\bar{x}) + 2C(\bar{x})y + 3D(\bar{x})y^2$$

with respect to y is precisely $4\big(C(\bar{x})^2 - 3D(\bar{x})B(\bar{x})\big)$. So, it is clear that, if $C(\bar{x})^2 - 3D(\bar{x})B(\bar{x}) \leq 0$ then on account of $D(\bar{x}) > 0$ it follows that $\dot{x} \leq 0$ on $x = \bar{x}$. This clearly contradicts that $\gamma_{\bar{h}}$ intersects twice with $x = \bar{x}$, and accordingly $C(\bar{x})^2 - 3D(\bar{x})B(\bar{x}) > 0$.

Note now that $(\bar{x}, y) \in \mathcal{P}$ for all $y \in (y_1, y_2)$ because otherwise there exist at least four different values of y such that $H(\bar{x}, y) = \bar{h}$. In addition one can show that y_1 and y_2 are located as in Figure **??**. This is a consequence of the last observation and the fact that, as h increases, the two intersection points of γ_h with $x = \bar{x}$ must go away from each other. The minimum of the function $y \longmapsto H(\bar{x}, y)$ is given precisely by

$$\bar{y} = \frac{-C(\bar{x}) + \sqrt{C(\bar{x})^2 - 3D(\bar{x})B(\bar{x})}}{3D(\bar{x})},$$

and therefore in view of Figure **??** we conclude that $(\bar{x}, \bar{y}) \in \mathcal{P}$. ▮

At this point, given $x \in (x_{\mathrm{L}}, x_{\mathrm{R}})$ and $h \in (0, h_0)$, we need to precise how many real roots has the equation $H(x,y) = h$. If $D(x) \neq 0$ then

$$A(x) - h + B(x)y + C(x)y^2 + D(x)y^3 = 0$$

is a third degree equation with respect to y and its discriminant (see [**?**]) is given by

$$\Delta(x,h) = \frac{1}{108D(x)^4}\Big(18\big(A(x)-h\big)B(x)C(x)D(x) - 4D(x)B(x)^3 - 4C(x)^3\big(A(x)-h\big) + C(x)^2B(x)^2 - 27D(x)^2\big(A(x)-h\big)^2\Big).$$

It is well known that the signum of Δ provides us the information that we need. Thus, if $\Delta < 0$ then one root is real and the other two are complex, if $\Delta > 0$ then the three roots are real and different, and finally if $\Delta = 0$ then the three roots are real and at least two of them are equal.

Notice in addition that the numerator of $\Delta(x,h)$ is a second degree polynomial with respect to h, the coefficient of h^2 is $-27D(x)^2$, and on the other hand one can check that its discriminant is given by $\big(C(x)^2 - 3D(x)B(x)\big)^3$. Thus, for each fixed $x \in (x_{\mathrm{L}}, x_{\mathrm{R}})$, Lemma **??** shows that the equation $\Delta(x,h) = 0$ has two real solutions, say $h = F(x)$ and $h = G(x)$. From now on we fix that $F(x)$ and $G(x)$ are given respectively by

$$\frac{2C(x)^3 + 27A(x)D(x)^2 - 9D(x)C(x)B(x) + 2\sqrt{\big(C(x)^2 - 3D(x)B(x)\big)^3}}{27D(x)^2}$$

and

$$\frac{2C(x)^3 + 27A(x)D(x)^2 - 9D(x)C(x)B(x) - 2\sqrt{\big(C(x)^2 - 3D(x)B(x)\big)^3}}{27D(x)^2},$$

and consequently this allows us to write

$$\Delta(x,h) = \frac{1}{4D(x)^2}\big(F(x)-h\big)\big(h-G(x)\big) \text{ for all } x \in (x_{\mathrm{L}}, x_{\mathrm{R}}). \quad (9)$$

These functions will play an important role in order to compute the period of each periodic orbit in $\mathcal{P}$.

Remark 9. Let us mention that in fact we could assume without loss of generality that $B \equiv 0$ and $D \equiv \pm 1$. Indeed, one can verify that canonical change of coordinates

$$u = \psi(x) := \int_0^x \frac{ds}{|D(s)|^{1/3}}\,,$$
$$v = |D(x)|^{1/3}\left(y + \frac{C(x) - \sqrt{C(x)^2 - 3D(x)B(x)}}{3D(x)}\right),$$

brings system (**??**) to the Hamiltonian system given by

$$\widetilde{H}(u,v) = \widetilde{A}(u) + \widetilde{C}(u)v^2 \pm v^3,$$

where the coefficient ± 1 of v^3 depends on the signum of D in $(x_{\mathrm{L}}, x_{\mathrm{R}})$ and

$$\widetilde{A}(u) = G\big(\psi^{-1}(u)\big) \text{ and } \widetilde{C}(u) = 3\left(\frac{F\big(\psi^{-1}(u)\big) - G\big(\psi^{-1}(u)\big)}{4}\right)^{1/3}.$$

Note that, on account of Lemma **??**, if $D(x) \neq 0$ for all $x \in (x_{\mathrm{L}}, x_{\mathrm{R}})$ then this coordinate transformation is well defined in the whole period annulus.

Lemma 10. *If $D(x) \neq 0$ for all $x \in (x_L, x_R)$ then*

(a)$G(x) = \beta x^2 + O(x^3)$ with $\beta > 0$ and $G'(x) \neq 0$ for all $x \in (x_L, x_R) \setminus \{0\}$.

(b)$F(x) \geq h_0$ for all $x \in (x_L, x_R)$.

Proof. Consider any $\bar{x} \in (x_{\mathrm{L}}, x_{\mathrm{R}}) \setminus \{0\}$ and note that if we denote

$$\bar{y} = \frac{-C(\bar{x}) + \sqrt{C(\bar{x})^2 - 3D(\bar{x})B(\bar{x})}}{3D(\bar{x})}$$

then $H_y(\bar{x}, \bar{y}) = 0$. Thus, taking into account that $(\bar{x}, \bar{y}) \in \mathcal{P}$ by Lemma **??**, it follows that $H_x(\bar{x}, \bar{y}) \neq 0$ (otherwise $\mathcal{P}$ would contain a critical point different than the origin). On the other hand a computation shows that

$$G'(x) = H_x\left(x, \frac{-C(x) + \sqrt{C(x)^2 - 3D(x)B(x)}}{3D(x)}\right),$$

and so we can assert that $G'(\bar{x}) \neq 0$. This proves (a) because one can verify that $G(x) = \beta x^2 + O(x^3)$ with

$$\beta = \frac{2C(0)A''(0) - B'(0)^2}{4C(0)}$$

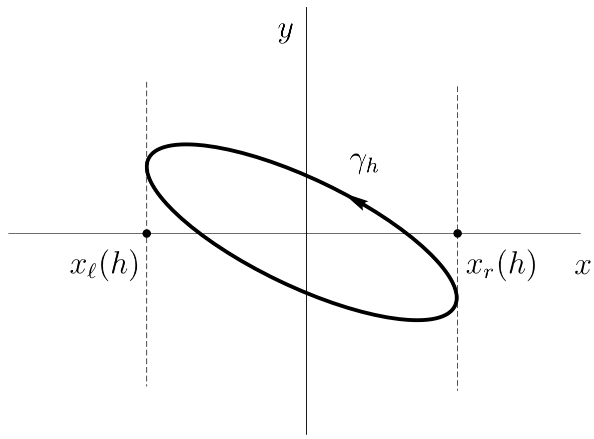

FIG. 2. Interpretation of $x_\ell(h)$ and $x_r(h)$ in terms γ_h.

and the assumptions (**??**) and (**??**) imply that $\beta > 0$.

Part (b) will be proved by contradiction. So assume that we have $F(\widetilde{x}) < h_0$ for some $\widetilde{x} \in (x_{\rm L}, x_{\rm R})$. Then there exists $\varepsilon > 0$ so that $F(\widetilde{x}) < h$ for all $h \in (h_0 - \varepsilon, h_0)$. Notice also that $G(\widetilde{x}) < h$ for all $h \in (h_0 - \varepsilon, h_0)$ because from Lemma **??** it follows that $F(\widetilde{x}) > G(\widetilde{x})$. Hence

$$\Delta(\widetilde{x}, h) = \frac{1}{4D(\widetilde{x})^2}\big(F(\widetilde{x}) - h\big)\big(h - G(\widetilde{x})\big) < 0,$$

and therefore $H(\widetilde{x}, y) = h$ has only one real solution for all $h \in (h_0 - \varepsilon, h_0)$. This obviously contradicts that $\widetilde{x} \in (x_{\rm L}, x_{\rm R})$, and so (b) follows. ▮

For each $h \in (0, h_0)$, let $[x_\ell(h), x_r(h)]$ be the projection of the periodic orbit γ_h to the x-axis, that is

$$[x_\ell(h), x_r(h)] = \{x \in \mathbb{R} : \text{ there exists } y \in \mathbb{R} \text{ such that } (x, y) \in \gamma_h\}.$$

We have thus a geometrical definition of the endpoints $x_\ell(h)$ and $x_r(h)$ in terms γ_h (see Figure **??**). We shall next obtain a result that provides an analytic way to compute them. To this end we define

$$g(x) = \operatorname{sgn}(x)\sqrt{G(x)} = x\sqrt{\frac{G(x)}{x^2}}\,,$$

which by (a) in Lemma **??** is an analytic function on $(x_{\rm L}, x_{\rm R})$ satisfying $g'(x) > 0$.

LEMMA 11. *If $D(x) \neq 0$ for all $x \in (x_L, x_R)$ then*

(a)For each $h \in (0, h_0)$, $\Delta(\,\cdot\,, h)$ is strictly posit ive in the interior of $[x_\ell(h), x_r(h)]$ and zero at the endpoints. In addition

$$x_\ell(h) = g^{-1}\left(-\sqrt{h}\right) \text{ and } x_r(h) = g^{-1}\left(\sqrt{h}\right).$$

(b)$G(x) \longrightarrow h_0$ as $x \searrow x_L$ or $x \nearrow x_R$.

Proof. Fix $h \in (0, h_0)$ and note that

$$x_{\mathrm{L}} < x_\ell(h) < 0 < x_r(h) < x_{\mathrm{R}}. \tag{10}$$

In view of Figure **??** it is clear that $H\big(x_\ell(h), y\big) = h$ has a double root for some value of y and therefore $\Delta\big(x_\ell(h), h\big) = 0$. Now, since

$$\Delta(x, h) = \frac{1}{4D(x)^2}\big(F(x) - h\big)\big(h - G(x)\big), \tag{11}$$

we conclude that $G\big(x_\ell(h)\big) = h$ because, taking (**??**) into account, (b) in Lemma **??** shows that $F\big(x_\ell(h)\big) \geq h_0 > h$. Thus $x_\ell(h) = g^{-1}\big(-\sqrt{h}\big)$. The fact that $x_r(h) = g^{-1}\big(\sqrt{h}\big)$ follows exactly the same way. Consider next any $\bar{x} \in \big(x_\ell(h), x_r(h)\big)$ and notice that, according to (**??**), then $\bar{x} \in (x_{\mathrm{L}}, x_{\mathrm{R}})$. In this case by applying Lemma **??** we can assert that $F(\bar{x}) \geq h_0 > h$ and, due to

$$G\big(x_\ell(h)\big) = G\big(x_r(h)\big) = h, \tag{12}$$

that $G(\bar{x}) < h$. Therefore from (**??**) it turns out that $\Delta(\bar{x}, h) > 0$, and this proves (a). Finally (b) follows readily from (**??**) using that $x_\ell(h) \searrow x_{\mathrm{L}}$ and $x_r(h) \nearrow x_{\mathrm{R}}$ as $h \nearrow h_0$. ∎

Remark 12. Part (a) of Lemma **??** and expression (**??**) show that, for each $h \in (0, h_0)$,

$$\big(F(x) - h\big)\big(h - G(x)\big) > 0$$

holds for all $x \in \big(x_\ell(h), x_r(h)\big)$. We shall turn account of this fact later.

We are now almost in position to prove the main result of this section. Before we have to describe precisely the three roots of the equation $H(x, y) = h$ (see [**?**] for instance). Thus, setting

$$\Psi(x, h) = \frac{9B(x)C(x)D(x) - 27(A(x) - h)D(x)^2 - 2C(x)^3}{54D(x)^3},$$

we define

$$R(x,h) = \big(\Psi(x,h) - i\,\Delta(x,h)^{1/2}\big)^{1/3}$$
$$S(x,h) = \big(\Psi(x,h) + i\,\Delta(x,h)^{1/2}\big)^{1/3}.$$

Then the three roots are given by

$$y_1(x,h) = R(x,h) + S(x,h) - \frac{1}{3}\frac{C}{D}\,(x),$$
$$y_2(x,h) = -\frac{1}{2}\big(R(x,h) + S(x,h)\big) - \frac{1}{3}\frac{C}{D}\,(x) + \frac{i\sqrt{3}}{2}\,\big(S(x,h) - R(x,h)\big)$$

and

$$y_3(x,h) = -\frac{1}{2}\big(R(x,h) + S(x,h)\big) - \frac{1}{3}\frac{C}{D}\,(x) + \frac{i\sqrt{3}}{2}\,\big(R(x,h) - S(x,h)\big).$$

In the sequel when we refer to some root we shall use the above notation. From now on, given $z \in \mathbb{C}$, $Re(z)$ and $Im(z)$ will denote respectively the real and the imaginary part of z. In addition, its argument and modulus will be denoted respectively by $\arg(z)$ and $|z|$. We shall also use that we can write Ψ in terms of F and G as we did with Δ in (**??**). Indeed, a computation shows that

$$\Psi(x,h) = \frac{2h - \big(F(x) + G(x)\big)}{4D(x)}. \tag{13}$$

The straight lines $x = x_\ell(h)$ and $x = x_r(h)$ split-up the periodic orbit γ_h into two components (see Figure **??**). The following result determines them.

Proposition 13. *Assume that $D(x) > 0$ for all $x \in (x_L, x_R)$. Then, for each $h \in (0,h_0)$, the upper component of γ_h is given by the the graph of $y_1(\,\cdot\,,h)\colon \big(x_\ell(h), x_r(h)\big) \longrightarrow \mathbb{R}$ and the lower one is given by the the graph of $y_3(\,\cdot\,,h)\colon \big(x_\ell(h), x_r(h)\big) \longrightarrow \mathbb{R}$.*

Proof. Fix $h \in (0,h_0)$ and notice that from (a) in Lemma **??** it follows that $\Delta(x,h) > 0$ for all $x \in \big(x_\ell(h), x_r(h)\big)$. Consequently, for each x in this interval, the three roots of

$$A(x) + B(x)y + C(x)y^2 + D(x)y^3 = h \tag{14}$$

are different and real. The two roots that are contained in γ_h (see Figure **??**) are the ones that become equal at the endpoints of $\big[x_\ell(h), x_r(h)\big]$.

In these endpoints, according to (a) in Lemma **??**, the function G takes the value h. Consequently, in order to decide which two branches of (**??**) give γ_h we must compute $y_1(x,h)$, $y_2(x,h)$ and $y_3(x,h)$ at $h = G(x)$. To do this we shall first evaluate $R(x,h)$ and $S(x,h)$ at $h = G(x)$. Note that, since $h \in (0,h_0)$ and $D(x) > 0$ for all $x \in (x_{\mathrm{L}}, x_{\mathrm{R}})$, from (**??**) it follows that

$$\Psi\big(x, G(x)\big) = \frac{h - F(x)}{4D(x)} < \frac{h_0 - F(x)}{4D(x)}\,.$$

Thus, taking into account that $F(x) \geq h_0$ by (b) in Lemma **??**, this shows that $\Psi\big(x,G(x)\big) < 0$. On account of this, and using that $\Delta\big(x,G(x)\big) = 0$, one can verify that

$$R\big(x,G(x)\big) = \frac{1 - i\sqrt{3}}{2}\,\big|\Psi\big(x,G(x)\big)\big|^{1/3}$$

and

$$S\big(x,G(x)\big) = \frac{1 + i\sqrt{3}}{2}\,\big|\Psi\big(x,G(x)\big)\big|^{1/3}.$$

Now it is easy to check that

$$y_1\big(x,G(x)\big) = \big|\Psi\big(x,G(x)\big)\big|^{1/3} - \frac{1}{3}\frac{C}{D}(x),$$
$$y_2\big(x,G(x)\big) = -2\,\big|\Psi\big(x,G(x)\big)\big|^{1/3} - \frac{1}{3}\frac{C}{D}(x),$$

and

$$y_3\big(x,G(x)\big) = \big|\Psi\big(x,G(x)\big)\big|^{1/3} - \frac{1}{3}\frac{C}{D}(x).$$

We thus get $y_1\big(x,G(x)\big) = y_3\big(x,G(x)\big) \neq y_2\big(x,G(x)\big)$. We conclude therefore that the solutions of the equation (**??**) which give the periodic orbit γ_h are $y_1\big(\,\cdot\,,h\big)$ and $y_3\big(\,\cdot\,,h\big)$. Finally, notice that

$$y_1(x,h) - y_3(x,h) = 3\,Re\big(S(x,h)\big) - \sqrt{3}\,Im\big(S(x,h)\big) > 0,$$

which is due to the fact that $3\cos(x/3) - \sqrt{3}\sin(x/3) > 0$ for all $x \in (0,\pi)$. So we deduce that $y_1(x,h) > y_3(x,h)$, and this completes the proof of the result. ∎

In case that $D(x) < 0$ for all $x \in (x_{\mathrm{L}}, x_{\mathrm{R}})$, a similar reasoning shows that the upper and lower components of γ_h are given by $y_3(\,\cdot\,,h)$ and $y_2(\,\cdot\,,h)$

respectively. We point out that Lemma 3.5 in [**?**] corresponds only to this case. However, as we will see in the proof of the next result, it is only necessary to study one of the two cases ($D > 0$ or $D < 0$) because the coordinate transformation $(x, y) \longmapsto (-x, -y)$ brings one to the other. Now, the main result of this section is the following.

THEOREM 14. *If $D(x) \neq 0$ for all $x \in (x_{\rm L}, x_{\rm R})$ then, for each $h \in (0, h_0)$, the period of the periodic orbit γ_h is given by*

$$T(h) = 2\int_{g^{-1}(-\sqrt{h})}^{g^{-1}(\sqrt{h})} \frac{\cos\left(\frac{1}{3}\arg\big(Z(x,h)\big) - \frac{\pi}{3}\right)}{\sqrt{3\big(F(x)-h\big)\big(h-G(x)\big)}} \left(\frac{F(x)-G(x)}{4|D(x)|}\right)^{1/3} dx,$$

where

$$Z(x,h) = h - \frac{F(x)+G(x)}{2} + i\sqrt{\big(F(x)-h\big)\big(h-G(x)\big)}\,.$$

Proof. We begin by proving the result for the case in which $D(x) > 0$ for all $x \in (x_{\rm L}, x_{\rm R})$. After, using that it is true for this case, we will prove the result in case that $D(x) < 0$ for all $x \in (x_{\rm L}, x_{\rm R})$.

So assume that $D(x) > 0$ for all $x \in (x_{\rm L}, x_{\rm R})$. In this situation the period of the periodic orbit γ_h is given by

$$T(h) = \int_{g^{-1}(-\sqrt{h}\,)}^{g^{-1}(\sqrt{h}\,)} \left(\frac{1}{H_y\big(x, y_1(x,h)\big)} - \frac{1}{H_y\big(x, y_3(x,h)\big)}\right) dx. \qquad (15)$$

This follows from Proposition **??** using the equality $\dot{x} = -H_y(x, y)$ to compute the time and taking into account that, according to (a) in Lemma **??**,

$$x_\ell(h) = g^{-1}(-\sqrt{h}\,) \ \text{ and } \ x_r(h) = g^{-1}(\sqrt{h}\,).$$

Our next goal is to compute $H_y\big(x, y_1(x,h)\big)$ and $H_y\big(x, y_3(x,h)\big)$. To this end notice first that

$$H(x,y) - h = D(x)\big(y - y_1(x,h)\big)\big(y - y_2(x,h)\big)\big(y - y_3(x,h)\big).$$

Hence

$$\begin{aligned} H_y(x,y) = D(x)\big\{&\big(y - y_2(x,h)\big)\big(y - y_3(x,h)\big) \\ &+ \big(y - y_1(x,h)\big)\big(y - y_3(x,h)\big) + \big(y - y_1(x,h)\big)\big(y - y_2(x,h)\big)\big\}, \end{aligned}$$

and so we obtain

$$H_y\big(x, y_1\big) = D(x)\big(y_1 - y_2\big)\big(y_1 - y_3\big) \ \text{ and } \ H_y\big(x, y_3\big) = D(x)\big(y_3 - y_1\big)\big(y_3 - y_2\big).$$

Now one can verify that

$$\frac{1}{H_y\big(x,y_1\big)}-\frac{1}{H_y\big(x,y_3\big)}=\frac{1}{D(x)\big(y_1-y_3\big)}\left\{\frac{1}{y_1-y_2}+\frac{1}{y_3-y_2}\right\}.$$

Then, using the fact that $(y_1-y_3)(y_1-y_2)(y_3-y_2)=\sqrt{108\,\Delta}$ (see [**?**]), this equality yields

$$\frac{1}{H_y\big(x,y_1\big)}-\frac{1}{H_y\big(x,y_3\big)}=\frac{y_1+y_3-2y_2}{D(x)\sqrt{108\,\Delta}}. \tag{16}$$

The task is now to compute $y_1+y_3-2y_2$. Thus, since $\Delta(x,h)>0$ for all $x\in\big(x_\ell(h),x_r(h)\big)$ by (a) in Lemma **??**, we obtain

$$y_1(x,h)+y_3(x,h)-2y_2(x,h)=3\,Re\big(S(x,h)\big)+3\sqrt{3}\,Im\big(S(x,h)\big). \tag{17}$$

On the other hand notice that

$$\begin{aligned}S(x,h)=&\frac{1}{\big(2D(x)\big)^{1/3}}\left\{h-\frac{F(x)+G(x)}{2}+i\sqrt{\big(F(x)-h\big)\big(h-G(x)\big)}\right\}^{1/3}\\=&\frac{Z(x,h)^{1/3}}{\big(2D(x)\big)^{1/3}}.\end{aligned}$$

Here we use (**??**) and (**??**), which give respectively the expression of Δ and Ψ in terms of F and G, and we take into account that $D(x)>0$ for all $x\in(x_{\text{L}},x_{\text{R}})$. Therefore, since one can verify that $|Z(x,h)|=\big(F(x)-G(x)\big)/2$, it turns out that

$$S(x,h)=\left(\frac{F(x)-G(x)}{4D(x)}\right)^{1/3}\left\{\cos\left(\frac{1}{3}\arg(Z)\right)+i\sin\left(\frac{1}{3}\arg(Z)\right)\right\}.$$

Consequently, we can rewrite (**??**) as

$$\begin{aligned}y_1+y_3-2y_2&=3\left(\frac{F(x)-G(x)}{4D(x)}\right)^{1/3}\left\{\cos\left(\frac{\arg(Z)}{3}\right)+\sqrt{3}\sin\left(\frac{\arg(Z)}{3}\right)\right\}\\&=6\left(\frac{F(x)-G(x)}{4D(x)}\right)^{1/3}\cos\left(\frac{\arg(Z)-\pi}{3}\right).\end{aligned}$$

Taking into account this equality we return to the expression in (**??**). Thus, using that $D(x)>0$ for all $x\in(x_{\text{L}},x_{\text{R}})$, from (**??**) we get

$$D(x)\sqrt{108\,\Delta}=3\sqrt{3\big(F(x)-h\big)\big(h-G(x)\big)}$$

and hence

$$\frac{1}{H_y\big(x,y_1\big)}-\frac{1}{H_y\big(x,y_3\big)}=\frac{2\cos\big(\frac{1}{3}\arg(Z\,)-\frac{\pi}{3}\big)}{\sqrt{3\big(F(x)-h\big)\big(h-G(x)\big)}}\left(\frac{F(x)-G(x)}{4D(x)}\right)^{1/3}$$

Thus, on account of the expression of $T(h)$ given in (**??**), this proves the validity of the theorem for the case $D(x) >$ for all $x \in (x_{\rm L}, x_{\rm R})$.

Assume now that $D(x) < 0$ for all $x \in (x_{\rm L}, x_{\rm R})$. In order to prove the result in this situation we will use that we already prove it for the case $D > 0$. To this end we perform the canonical change of coordinates $\{u = -x, v = -y\}$ to the initial Hamiltonian system, and we obtain a new one which is given by the Hamiltonian function

$$\widetilde{H}(u,v)=\widetilde{A}(u)+\widetilde{B}(u)v+\widetilde{C}(u)v^2+\widetilde{D}(u)v^3, \tag{18}$$

where $\widetilde{A}(u) = A(-u)$, $\widetilde{B}(u) = -B(-u)$, $\widetilde{C}(u) = C(-u)$ and $\widetilde{D}(u) = -D(-u)$. It is obvious that the phase portrait of the new differential system can be obtained with a rotation of π radians of the initial one. Consequently, it is clear that the projection of its period annulus is $(-x_{\rm R}, -x_{\rm L})$ and that, for the values of u inside this interval, $\widetilde{D}(u) > 0$ holds. Therefore we can use the theorem to compute $\widetilde{T}(h)$, the period of the periodic orbit inside the energy level h of the new Hamiltonian system given by (**??**). Note also that since the change of coordinates is area-preserving we have $\widetilde{T}(h) = T(h)$. Moreover, following the obvious notation, one can check that

$$\widetilde{G}(u)=G(-u)\ \text{ and } \widetilde{F}(u)=F(-u),$$

and so $\widetilde{g}(u) = -g(u)$ and $\widetilde{Z}(u,h) = Z(-u,h)$. Taking this into account it follows that

$$\begin{aligned}\widetilde{T}(h) &= 2\int_{\widetilde{g}^{\,-1}(-\sqrt{h})}^{\widetilde{g}^{\,-1}(\sqrt{h})}\frac{\cos\big(\frac{1}{3}\arg\big(\widetilde{Z}(u,h)\big)-\frac{\pi}{3}\big)}{\sqrt{3\big(\widetilde{F}(u)-h\big)\big(h-\widetilde{G}(u)\big)}}\left(\frac{\widetilde{F}(u)-\widetilde{G}(u)}{4\widetilde{D}(u)}\right)^{1/3}du\\ &= 2\int_{g^{\,-1}(\sqrt{h})}^{g^{\,-1}(-\sqrt{h})}\frac{\cos\big(\frac{1}{3}\arg\big(Z(-u,h)\big)-\frac{\pi}{3}\big)}{\sqrt{3\big(F(-u)-h\big)\big(h-G(-u)\big)}}\left(\frac{F(-u)-G(-u)}{-4D(-u)}\right)^{1/3}du.\end{aligned}$$

Finally, the change of variable $u = -x$ yields

$$\widetilde{T}(h)=2\int_{g^{\,-1}(-\sqrt{h})}^{g^{\,-1}(\sqrt{h})}\frac{\cos\big(\frac{1}{3}\arg\big(Z(x,h)\big)-\frac{\pi}{3}\big)}{\sqrt{3\big(F(x)-h\big)\big(h-G(x)\big)}}\left(\frac{F(x)-G(x)}{-4D(x)}\right)^{1/3}dx,$$

and this proves the result for the case $D<0$ because $|D(x)|=-D(x)$ and $\widetilde{T}(h)=T(h)$. ∎

Remark 15. It is easy to show that the real part of $Z(x,h)$ is always negative near the endpoints of $\big(x_\ell(h),x_r(h)\big)$. However, for an arbitrary $h\in(0,h_0)$, it may occur that the real part of $Z(x,h)$ becomes positive at some value of x in the middle of this interval. This is the reason why we do not use the function $\arctan(x)$ to compute the argument of $Z(x,h)$. This is not the case when $h\approx 0$. Indeed, for these values of h we have

$$h-\frac{F(x)+G(x)}{2}<0 \text{ for all } \big(x_\ell(h),x_r(h)\big)$$

since $G(0)=0$ and $F(0)>0$. Thus, for periodic orbits γ_h near the center we can use that

$$\arg\big(Z(x,h)\big)=\arctan\left(\frac{2\sqrt{\big(F(x)-h\big)\big(h-G(x)\big)}}{2h-\big(F(x)+G(x)\big)}\right)+\pi$$

to compute $T(h)$.

Following Remark **??**, we point out that the assumption $D(x)\neq 0$ for all $x\in(x_{\mathrm{L}},x_{\mathrm{R}})$ in Theorem **??** is not essential in order to compute $T(h)$ for periodic orbits γ_h near the center at the origin (i.e., with $h\gtrsim 0$). Thus, for these periodic orbits, the result is still true if we only assume that $D(0)\neq 0$ holds. In view of Remark **??** this leads us to the following corollary.

COROLLARY 16. *If $D(0)\neq 0$ then, for each $h\gtrsim 0$, the period of the periodic orbit γ_h is given by*

$$T(h)=\int_{g^{-1}(-\sqrt{h})}^{g^{-1}(\sqrt{h})}\frac{2}{\sqrt{3\big(F(x)-h\big)\big(h-G(x)\big)}}\left(\frac{F(x)-G(x)}{4|D(x)|}\right)^{1/3}$$
$$\cos\left(\frac{1}{3}\arctan\left(\frac{2\sqrt{\big(F(x)-h\big)\big(h-G(x)\big)}}{2h-\big(F(x)+G(x)\big)}\right)\right)dx.$$

An interesting question for further research concerning the centers of the cubic-like Hamiltonian systems is to obtain the isochronicity condition. From our point of view, Corollary **??** can play an important role in this direction because it provides an expression of the function $h\longmapsto T(h)$ which depends explicitly on A, B, C and D.

4. CASE III: $\boldsymbol{A}_0 = \boldsymbol{B}_0 = 0$ AND $\boldsymbol{C}_0 \neq 0$

The remainder of the paper is devoted to prove Theorem **??**. So recall that in Section **??** (see Remark **??**) we reduce its proof to study the centers of the cubic-like Hamiltonian systems given by (**??**) and having an admissible period annulus. Note in particular that $D(x) = a_0x^2 + b_0x + c_0$. This section is devoted to the case $a_0 = b_0 = 0$ and $c_0 \neq 0$. Thus $D \equiv c_0$, and we can therefore apply Theorem **??** to compute $T(h)$. It will be shown the following proposition.

PROPOSITION 17. *Assume that $a_0 = b_0 = 0$ and $c_0 \neq 0$. If $\mathcal{P}$ is admissible then $T(h) \longrightarrow +\infty$ as $h \nearrow h_0$.*

Proof. To show this we shall first use the Poincaré compactification of the vector field (see Section 7 in [**?**] for details). One can verify that the characteristic polynomial of the critical point at the origin of the local chart U_2 is given by $W(u, v) = -3c_o v^3$. Hence the two separatrices of the hyperbolic sector $\mathcal{H}^\star$ must be tangent to $v = 0$. In the Poincaré disc this means that they are tangent to infinity and, accordingly, either $x_{\text{L}} = -\infty$ or $x_{\text{R}} = +\infty$. On the other hand, Proposition 6.3 in [**?**] shows that $a_1 = 0$ is a necessary condition in order that $\mathcal{P}$ is admissible. Consequently, since $C(x) = 1/2 + c_1x + b_1x^2 + a_1x^3$, we have $\deg(C) \leq 2$.

Note that we can apply Theorem **??** because $D(x) \neq 0$ for all $x \in \mathbb{R}$. Thus, for each $h \in (0, h_0)$, the period of the periodic orbit γ_h is given by

$$T(h) = 2\int_{g^{-1}(-\sqrt{h})}^{g^{-1}(\sqrt{h})} \frac{\cos\big(\frac{1}{3}\arg\big(Z(x,h)\big) - \frac{\pi}{3}\big)}{\sqrt{3\big(F(x)-h\big)\big(h-G(x)\big)}} \left(\frac{F(x)-G(x)}{4|D(x)|}\right)^{1/3} dx,$$

where

$$Z(x,h) = h - \frac{F(x)+G(x)}{2} + i\sqrt{\big(F(x)-h\big)\big(h-G(x)\big)}.$$

Remark **??** shows that $\arg\big(Z(x,h)\big) \in (0,\pi)$ if $x \in \big(g^{-1}(-\sqrt{h}), g^{-1}(\sqrt{h})\big)$ and so, taking into account that $2\cos\big((x-\pi)/3\big) \geq 1$ for all $x \in (0,\pi)$, we can assert that

$$T(h) \geq \int_{g^{-1}(-\sqrt{h})}^{g^{-1}(\sqrt{h})} \frac{1}{\sqrt{3\big(F(x)-h\big)\big(h-G(x)\big)}} \left(\frac{F(x)-G(x)}{4|D(x)|}\right)^{1/3} dx.$$

Note that, according to Lemma **??**, $g^{-1}(-\sqrt{h}) \longrightarrow x_{\text{L}}$ and $g^{-1}(\sqrt{h}) \longrightarrow x_{\text{R}}$ as $h \nearrow h_0$. Therefore, applying Fatou's Lemma it follows that

$$\varliminf_{h \nearrow h_0} T(h) \geq \int_{x_{\text{L}}}^{x_{\text{R}}} \Phi(x)\, dx, \tag{19}$$

where

$$\Phi(x) = \frac{1}{\sqrt{3\big(F(x)-h_0\big)\big(h_0-G(x)\big)}}\left(\frac{F(x)-G(x)}{4|D(x)|}\right)^{1/3}.$$

Let us assume for instance that $x_{\rm R} = +\infty$. Then, on account of (**??**), to prove the result it suffices to show that $\Phi(x)$ has order ≥ -1 at infinity. To estimate this order notice first that, from Lemma **??**, we have $G(x) = h_0 + O(x^{-\alpha})$ with $\alpha > 0$ as $x \longrightarrow +\infty$. On the other hand one can verify that

$$\left(\frac{F(x)-G(x)}{4|D(x)|}\right)^{2/3} = \frac{C(x)^2 - 3D(x)B(x)}{9D(x)^2}\ ,.$$

In our situation $\deg(C) \leq 2$, $\deg(B) \leq 4$ and $D(x) \equiv c_0$. Consequently $F(x) - G(x) = P(x)^{3/2}$ where $P(x)$ is a polynomial of degree $n \leq 4$, and so we can write

$$\begin{aligned}\Phi(x) =& \frac{1}{(4|c_0|)^{1/3}}\frac{\big(F(x)-G(x)\big)^{1/3}}{\sqrt{3\big(F(x)-h_0\big)\big(h_0-G(x)\big)}}\\ =& \frac{1}{(4|c_0|)^{1/3}}\frac{P(x)^{1/2}}{\sqrt{3\big(P(x)^{3/2}+G(x)-h_0\big)\big(h_0-G(x)\big)}}.\end{aligned}$$

This shows that the order of $\Phi(x)$ at infinity is equal to

$$\frac{1}{2}\,n - \frac{3}{4}\,n + \frac{1}{2}\,\alpha = \frac{1}{2}\,\alpha - \frac{1}{4}\,n \geq -1$$

and completes the proof of the result. ▮

5. CASE IV: $A_0 = B_0 = C_0 = 0$

In this section we prove Theorem **??** under the parameter values $a_0 = b_0 = c_0 = 0$. This corresponds to the case IV, and so it will complete the proof of Theorem **??** (recall that the result in the case I and II is proved in [**?**]). Note that, since $D(x) = a_0x^2 + b_0x + c_0$, in this case we have $D \equiv 0$. Consequently, according to (**??**), the Hamiltonian function is

$$H(x,y) = A(x) + B(x)y + C(x)y^2 \tag{20}$$

where $\deg(A) \leq 5$, $\deg(B) \leq 4$ and $\deg(C) \leq 3$. We have to show that if $\mathcal{P}$ is admissible then $T(h) \longrightarrow +\infty$ as $h \nearrow h_0$. Notice that, by definition, a center with an admissible period annulus is, in particular, nonglobal. Hence, it is clear that the result will follow once we prove the following more general proposition.

PROPOSITION 18. *Assume that the Hamiltonian system given by* (**??**), *where* A, B *and* C *are polynomials, has a non-degenerate center at the origin with a nonglobal period annulus. If* $\deg(C) \leq 3$ *then* $T(h) \longrightarrow +\infty$ *as* $h \nearrow h_0$.

Proof. For each $h \in (0, h_0)$, Lemma 3.3 in [**?**] shows that the period of the periodic orbit γ_h is given by

$$T(h) = 2\int_{x_\ell(h)}^{x_r(h)} \frac{dx}{\sqrt{4h\,C(x) - G(x)}}\,,$$

where $[x_\ell(h), x_r(h)]$ is the projection of γ_h on the x-axis and $G := 4AC - B^2$. Note also that

$$x_\ell(h) \searrow x_{\mathrm{L}} \ \text{ and } x_r(h) \nearrow x_{\mathrm{R}} \ \text{ as } h \nearrow h_0,$$

where $(x_{\mathrm{L}}, x_{\mathrm{R}})$ is the projection of the period annulus on the x-axis. Consequently, by applying Fatou's Lemma we can assert that

$$\lim_{h \nearrow h_0} T(h) \geq 2\int_{x_{\mathrm{L}}}^{x_{\mathrm{R}}} \frac{dx}{\sqrt{4h_0 C(x) - G(x)}}\,. \tag{21}$$

On the other hand, from (a) of Lemma 3.3 in [**?**] it follows that

$$\frac{1}{4}\frac{G(x)}{C(x)} \longrightarrow h_0 \ \text{ as } x \searrow x_{\mathrm{L}} \text{ or } x \nearrow x_{\mathrm{R}}. \tag{22}$$

In order to prove the result we must only consider the case in which $(x_{\mathrm{L}}, x_{\mathrm{R}})$ is unbounded, otherwise Corollary 3.7 in [**?**] shows that $T(h) \longrightarrow +\infty$ as $h \nearrow h_0$. So we can assume that either $x_{\mathrm{L}} = -\infty$ or $x_{\mathrm{R}} = +\infty$, and notice then that from (**??**) it follows

$$\deg(4h_0C - G) < \deg(C).$$

Here we use that, since the period annulus is nonglobal, we have $h_0 < +\infty$. Hence the assumption on $\deg(C)$ implies that $\deg(4h_0C - G) \leq 2$. Now, taking into account (**??**) and the fact that $(x_{\mathrm{L}}, x_{\mathrm{R}})$ is unbounded, this clearly forces that $T(h) \longrightarrow +\infty$ as $h \nearrow h_0$. ∎

6. A COUNTEREXAMPLE FOR HAMILTONIAN SYSTEMS OF DEGREE SIX

As we mention in Section **??**, from a result in [**?**] it follows that the period function of any non-degenerate center of a polynomial Hamiltonian system of second degree tends to infinity as we approach to the boundary of its period annulus. Since Theorem **??** shows the same result for Hamiltonian systems of fourth degree, one may wonder if the result is true for any Hamiltonian system of even degree. We conclude the paper showing that this is not so. To this end, taking

$$A(x) = \frac{1}{3}\, x^2\,(3x+7),$$
$$B(x) = \frac{1}{3}\, x\,(x+2)(3x^2+4x+2),$$

and

$$C(x) = \frac{1}{12}\,(x^2+x+1)(x+2)(3x^2+4x+2),$$

we consider the quadratic-like Hamiltonian function $H(x,y) = A(x) + B(x)y + C(x)y^2$. We have thus a Hamiltonian system of degree six which one can verify that has a center at the origin. In order to study its period annulus notice first that $C(x) > 0$ for all $x > -2$. Hence, for these values of x, the energy level $H(x,y) = h$ is given by the graphs of the functions

$$x \longmapsto \frac{-B(x) \pm \sqrt{4h\,C(x) - G(x)}}{2C(x)},$$

where

$$G := 4AC - B^2 = \frac{1}{3}\, x^2 (x+2)(3x^2+4x+2).$$

For each fixed h, these functions are well defined for values of x such that $4h\,C(x) - G(x) \geq 0$ and this condition is equivalent to

$$\frac{G(x)}{4C(x)} = \frac{x^2}{x^2+x+1} \leq h.$$

Thus, on account of Figure **??**, we can assert that the period annulus of the center at the origin is

$$\mathcal{P} = \left\{(x,y) \in \mathbb{R}^2 \,:\, 0 < H(x,y) < 1\right\}.$$

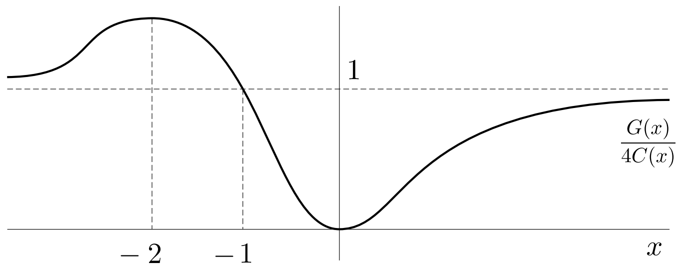

FIG. 3. The graph of $\frac{G}{4C}$.

With our notation this means that $h_0 = 1$. In addition, one can also verify that, apart from the center at the origin, the unique critical point is located at $(-2, 4/3)$. Note that this critical point is not in $\partial\mathcal{P}$ because $H(-2, 4/3) = 4/3 > 1$.

We turn now to study the function $h \longmapsto T(h)$. For each $h \in (0,1)$, Lemma 3.3 in [**?**] shows that the period of the periodic orbit γ_h is given by

$$T(h) = \int_{g^{-1}(-\sqrt{h})}^{g^{-1}(\sqrt{h})} \frac{2dx}{\sqrt{4hC(x) - G(x)}} \text{ where } g(x) = \frac{\operatorname{sgn}(x)}{2}\sqrt{\frac{G(x)}{C(x)}}.$$

Thus, the change of coordinates $x = g(u)$ in the expression of $T(h^2)$ yields

$$T(h^2) = \int_{g^{-1}(-h)}^{g^{-1}(h)} \frac{2dx}{\sqrt{4h^2C(x) - G(x)}} = \int_{-h}^{h} \frac{1}{F\big(g^{-1}(u)\big)} \frac{du}{\sqrt{h^2 - u^2}},$$

where

$$F(x) = \sqrt{C(x)}g'(x) = \frac{\operatorname{sgn}(x)}{4} \frac{C(x)}{\sqrt{G(x)}} \left(\frac{G}{C}\right)'(x).$$

It is easily seen that F is a nonvanishing analytic function on $(-2, +\infty)$ with order $1/2$ at $+\infty$. Consequently there exists some $m > 0$ such that $F(x) > m$ for all $x \in (-1, +\infty)$. Since we have that $g^{-1}\left([-h, h]\right) \subset (-1, +\infty)$ for $0 < h < 1$ (see Figure **??**), this shows that

$$T(h^2) \leq \frac{1}{m} \int_{-h}^{h} \frac{du}{\sqrt{h^2 - u^2}} = \frac{\pi}{m} \text{ for all } h \in (0,1).$$

We can assert therefore that the period function of the center at the origin does not tend to infinity as we approach to $\partial\mathcal{P}$.

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