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Codimension-two Singularities of Reversible Vector Fields in 3D

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This paper is concerned with the dynamics near an equilibrium point of reversible systems. For a large class of reversible vector fields on the three dimensional space we present all the topological types and their respective normal forms of the codimension-two symmetric singularities. Such classification comes from useful new results, also proved here, on dynamical systems defined on manifolds with boundary.

Key Words: Reversibility, Singularity, Bifurcation, Normal Forms

1. INTRODUCTION

This paper investigates generic two-parameter families of reversible vector fields on a three dimensional space. We continue the results contained in [5] where the codimension-one bifurcations have been analyzed.

The orbits of the vector fields X studied here are anti-symmetric with respect to the reflection $\varphi(x, y, z) = (x, y, -z)$ in such a way that the critical points contained in the plane $Fix(\varphi) = S = \{z = 0\}$ are the symmetric singularities of X. In our approach we will present all the different types of symmetric singularities of such vector fields of codimension-two, their nor-

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mal forms and their respective unfolding. So a complete list of topological models for all possible two-parameter families of reversible vector fields is given.

For general reference on time-reversible systems see [4] and [8]. Let X be a (germ of) C^r vector field on $R^3, 0$ with $r > 3$.

We say that X is $(\varphi-)$*reversible* if

$$D\varphi(p).X(p) = -X(\varphi(p)).$$

In this way we derive that the vector field is expressed by the following form:

$$x' = zf^1(x, y, z^2), y' = zf^2(x, y, z^2), z' = g(x, y, z^2).$$

where f^1, f^2 and g are C^∞ real functions on $R^3, 0$.

Because the structure of time reversibility does not impose significant restriction on the dynamics of the system which is far away from $Fix(\varphi)$, uniformly, we concentrate our attention to these equilibria p in $Fix(\varphi)$.

Let Ω be the space of the germs of C^r reversible vector fields at 0 on R^3 with the C^r- topology.

The concept of structural stability in Ω comes from the following definition:

"Two vector fields X_1 and X_2 in Ω are said to be $C^0 -$ equivalent if there is a phase space homeomorphism that casts the phase trajectories of X_1 to the phase trajectories of X_2".

We fix the following notations:

ν_0 is the set of elements in Ω which are structurally stable (codimension-zero singularities).

$\Omega_1 = \Omega/\nu_0$ and ν_1 is the set of elements in Ω_1 which are structurally stable relative to Ω_1 (codimension-one singularities).

$\Omega_2 = \Omega_1/\nu_1$ and ν_2 is the set of elements in Ω_2 which are structurally stable relative to Ω_2 (codimension-two singularities).

We know from [5] that υ_0 (resp. υ_1) is open and dense in Ω (resp. Ω_1). Moreover, υ_1 is a codimension-one manifold of Ω_1.

Roughly speaking, our approach consists to make a special change of coordinates around 0 and then address the analysis to the study of general vector fields defined on a 3-manifold (in our case $K = \{z \geq 0\}$) with boundary (in our case $\{z = 0\}$). We should mention that the knowledge of the phase portrait of X in K determines the phase portrait of X in $R^3, 0$. So the paper is centered on the study of vector fields near the boundary of a 3-manifold.

The remainder of the paper is organized as follows. In Section 2, we state our main result. Section 3 is devoted to the study of codimension-two singularities of vector fields defined in 3-dimensional manifolds with non-empty boundary; such study is essential for the proof of Theorem A which is given in Section 5.

Observe that results contained in Section 3 can be applied in PDE problems as stated by Arnold in [3].

2. STATEMENT OF MAIN RESULT

Theorem A:

1. ν_2 is open and dense in Ω_2 ;
2. ν_2 is a codimension-two submanifold of Ω;
3. Let p be a codimension-two singularity and fix a coordinate system around p such that $x(p) = y(p) = z(p) = 0$.

Then, in the space of two-parameter families of vector fields in Ω, an everywhere dense set is formed by generic families such that their normal forms are:

(0.1) $X_{\alpha,\beta}(x, y, z) = (0, 0, 1)$
(0.2) $X_{\alpha,\beta}(x, y, z) = (z, 0, \pm\frac{x}{2})$
(0.3) $X_{\alpha,\beta}(x, y, z) = (z, 0, \frac{x^2+y}{2})$
(1.1) $X_{\alpha,\beta}(x, y, z) = (z, 0, \frac{-3x^2+y^2+\alpha}{2})$
(1.2) $X_{\alpha,\beta}(x, y, z) = (z, 0, \frac{-3x^2-y^2+\alpha}{2})$
(1.3) $X_{\alpha,\beta}(x, y, z) = (z, 0, \frac{4\delta x^3+y+\alpha x}{2})$, with $\delta = \pm 1$
(1.4) $X_{\alpha,\beta}(x, y, z) = (axz, byz, \frac{ax+by+cz^2+\alpha}{2})$, with $(a, b, c) = \delta(3, 2, 1)$, $\delta = \pm 1$
(1.5) $X_{\alpha,\beta}(x, y, z) = (axz, byz, \frac{ax+by+cz^2+\alpha}{2})$, with $(a, b, c) = \delta(1, 3, 2)$, $\delta = \pm 1$
(1.6) $X_{\alpha,\beta}(x, y, z) = (axz, byz, \frac{ax+by+cz^2+\alpha}{2})$, with $(a, b, c) = \delta(1, 2, 3)$, $\delta = \pm 1$
(1.7) $X_{\alpha,\beta}(x, y, z) = (xz, 2yz, \frac{x+2y-cz^2+\alpha}{2})$
(1.8) $X_{\alpha,\beta}(x, y, z) = ((-x + y)z, (-x - y)z, \frac{-3x-y+z^2+\alpha}{2})$
(2.1) $X_{\alpha,\beta}(x, y, z) = (z(\varepsilon(x + y)^2 + \alpha, z(x - y), \frac{1}{2}(3x + y - 2z^2 + \varepsilon(x + y)^2 + \beta))$, with $\varepsilon = \pm 1$
(2.2) $X_{\alpha,\beta}(x, y, z) = (z(-y + x(\alpha - x^2 - y^2)), z(x + y(\alpha - x^2 - y^2)),$
$-x - 3y + 2z^2 + (x + y)(\alpha - x^2 - y^2) + \beta))$
(2.3) $X_{\alpha,\beta}(x, y, z) = (axz, byz, \frac{1}{2}(y + cz^2 + \varepsilon(c - 2a)x^2 + \alpha x + \beta))$, with $(a, b, c) = \delta(5, 3, 1)$, if $\varepsilon < 0$, and $(a, b, c) = \delta(1, 3, 5)$, if $\varepsilon > 0$ e $\delta = \pm 1$;
(2.4) $X_{\alpha,\beta}(x, y, z) = (axz, byz, \frac{1}{2}(y + cz^2 + \varepsilon(c - 2a)x^2 + \alpha u + \beta))$, with $(a, b, c) = \delta(3, 1, 5)$ and $\delta = \pm 1$;

(2.5) $X_{\alpha,\beta}(x,y,z) = (axz, byz, \frac{1}{2}(y + cz^2 + \varepsilon(c-2a)x^2 + \alpha u + \beta))$, with $(a,b,c) = \delta(1,3,5)$, if $\varepsilon < 0$, $(a,b,c) = \delta(5,3,1)$, if $\varepsilon > 0$ and $\delta = \pm 1$;

(2.6) $X_{\alpha,\beta}(x,y,z) = (axz, byz, \frac{1}{2}(y + cz^2 + \varepsilon(c-2a)x^2 + \alpha u + \beta))$, with $(a,b,c) = \delta(-3,-1,1)$, if $\varepsilon < 0$, $(a,b,c) = \delta(-1,-3,1)$, if $\varepsilon > 0$ and $\delta = \pm 1$;

(2.7) $X_{\alpha,\beta}(x,y,z) = (axz, byz, \frac{1}{2}(y + cz^2 + \varepsilon(c-2a)x^2 + \alpha u + \beta))$, with $(a,b,c) = \delta(-1,-3,1)$, if $\varepsilon < 0$, $(a,b,c) = \delta(-3,-1,1)$, if $\varepsilon > 0$ and $\delta = \pm 1$;

(2.8) $X_{\alpha,\beta}(x,y,z) = (axz, (a^2 - 2ab - b^2)yz + 2abz^3 - 2abx^2z - 2ab\alpha xz, \frac{1}{2}(-2by + (a+b)z^2 - (a+b+2c)x^2 - \alpha(a+b+c)x + \beta))$, with $(a,b,c) \in \{\delta(1,2,3), \delta(1,2,-3)\}$ and $\delta = \pm 1$;

(2.9) $X_{\alpha,\beta}(x,y,z) = (axz + \alpha yz, xz + ayz, \frac{1}{2}(x + y + cz^2 + \beta))$, with $(a,c) \in \{\delta(1,2), \delta(1,-2)\}$ and $\delta = \pm 1$;

(2.10) $X_{\alpha,\beta}(x,y,z) = (axz + byz + \alpha xz, -bxz + ayz, \frac{1}{2}(x + y + az^2 + \varepsilon_1 x^2 + \varepsilon_2 y^2 + (\varepsilon_1 - \varepsilon_2)xy + \beta))$, with $(a,b) \in \{\delta(1,2), \delta(1,-2)\}$, $\delta = \pm 1$ and $\varepsilon_1, \varepsilon_2 \in \{1,-1\}$;

(2.11) $X_{\alpha,\beta}(x,y,z) = (axz + byz, -bxz + ayz, \frac{1}{2}(cz^2 + x^2 + y^2 + \alpha x + \beta))$, with $(a,b,c) \in \{\delta(1,2,3), \delta(1,2,-3)\}$ and $\delta = \pm 1$;

(2.12) $X_{\alpha,\beta}(x,y,z) = (axz + byz, -bxz + ayz, \frac{1}{2}(cz^2 + x^2 - y^2 + \alpha x + \beta))$, with $(a,b,c) \in \{\delta(1,2,3), \delta(1,2,-3)\}$ and $\delta = \pm 1$;

(2.13) $X_{\alpha,\beta}(x,y,z) = (z, 0, \frac{1}{2}(3x^2 + y^3 + \alpha y + \beta))$;

(2.14) $X_{\alpha,\beta}(x,y,z) = (z, 0, \frac{1}{2}(4x^3 + 2xy - y^2 + \alpha y + \beta))$;

(2.15) $X_{\alpha,\beta}(x,y,z) = (z, 0, \frac{1}{2}(5x^4 + 3\varepsilon x^2 y + y + \alpha x^2 + \beta x))$ with $\varepsilon = \pm 1$;

We remark that the above mentioned forms $0.j$ and $1.k$ with $j = 1,2,3$ and $k = 1,2,...8$, concern the codimension-zero and codimension-one singularities respectively and they have been studied in [5].

3. VECTOR FIELDS ON MANIFOLDS WITH BOUNDARY

Let (u,v,w) be any coordinate system on $\mathbb{R}^3, 0$.

Consider $M = \{(u,v,w) \in \mathbb{R}^3, 0; w \geq 0\}$ and the set $\mathfrak{X}^r$ of all germs of C^r vector fields on $\mathbb{R}^3, 0$ $(r > 3)$ endowed with the C^r topology. We denote the boundary of M by S.

Rather than considering vector fields on $M, 0$ we deal, as usual, with those elements in $\mathfrak{X}^r$. The transition between those objects is made via the concept of S-structural stability in $\mathfrak{X}^r$, which is reached from the following definition.

Definition 1. Two vector fields F_1 and F_2 in $\mathfrak{X}^r$ are said to be $S - C^0-$ equivalent if there is a S-preserving homeomorphism of $R^3, 0$ that casts the phase trajectories of F_1 to the phase trajectories of F_2.

Definition 2. A point $p \in S$ is a S -singularity (or simply singularity) of $F \in \mathfrak{X}^r$ if $F(p)$ is tangent to S at p.

Let $h : \mathbb{R}^3, 0 \to \mathbb{R}$ be given by $h(u, v, w) = w$. In these coordinates the $S-$ singular set of F in $\mathfrak{X}^r$ is determined by the equations:

$$h = 0 \quad \text{and} \quad Fh = 0.$$

Denote by Σ_0 the set of all elements in $\mathfrak{X}^r$ which are structurally stable at 0. In what follows we establish some notations and review some basic facts and concepts. We refer the reader to see [10].

Theorem B (Sotomayor and Teixeira): *The following statements hold:*

(i) A vector field $F \in \mathfrak{X}^r$ is in Σ_0 if and only if:

(a) $F(p) \neq 0$ for every p in S;

(b) for every local implicit definition, h, of S at p, one of the following conditions is satisfied:

(b.1) $Fh(p) \neq 0$ (regular case);

(b.2) $Fh(p) = 0$ and $F^2h(p) \neq 0$ (fold case);

(b.3) $Fh(p) = F^2h(p) = 0$, $F^3h(p) \neq 0$ and the set of vectors $\{Dh(p), DFh(p), DF^2h(p)\}$ are linearly independent (cusp case);

(ii) Fixing $h(u, v, w) = w$, the generic normal forms are:

1. *$F(u, v, w) = (0, 0, 1)$ (regular case);*
2. *$F(u, v, w) = (\delta, 0, u)$, with $\delta = \pm 1$ (Fold singularity);*
3. *$F(u, v, w) = (1, 0, u^2 + v)$ (Cusp singularity);*

(iii) Σ_0 is open and dense in Γ^r.

The points of S where (b.2) is satisfied are called *fold singularities*; they form a smooth curve σ_X in S, along which X has quadratic contact with S. This tangency can be internal (when $F^2h(p) > 0$) or external (when $F^2h(p) < 0$).

The points where (b.3) is satisfied are called *cusp singularities*.

In our context, the codimension-one S -singularities of F are firstly classified by (i) $F(p) \neq 0$ and (ii) p is a critical point of F . The second case may occur generically for one parameter family of vector fields.

In the course of our arguments, although omitted in the text, we used known techniques (see [10] and [6]) to find those local homeomorphisms which are the C^0- equivalences between two vector fields defined around the boundary of a region in $\mathbb{R}^3$. For C^0- equivalence between two families of such vector fields we do not require continuity with respect to the parameters.

Let F be a C^∞ vector field on $I\!R^3$ parameterized by (x, y, z) such that $X(x_0, y_0, z_0) \neq (0, 0, 0)$. A flow box construction K_F at (x_0, y_0, z_0) consists of a smooth change of coordinates (i.e. a germ of a $C^\infty-$ diffeomorphism) $K_F(x, y, z) = (u, v, w)$ in a neighborhood of (x_0, y_0, z_0), such that $K_F(x_0, y_0, z_0) = (0, 0, 0)$ and in the new coordinates $X(u, v, w) = (1, 0, 0)$. It is well known that if G is a small perturbation of F, then the map $G \longrightarrow K_G$ is smooth.

Firstly we consider the sets $\boldsymbol{\mathcal{X}}_1^r = \boldsymbol{\mathcal{X}}^r \backslash \Sigma_0$ (the bifurcation set of $\boldsymbol{\mathcal{X}}^r$). Let Σ_1 be the subset of $\boldsymbol{\mathcal{X}}_1^r$ constituted by the elements which are structurally stable relative to $\boldsymbol{\mathcal{X}}_1^r$.

The first step to be considered is to find preliminary normal forms for the boundary codimension-one singularities. This is done in Lemma 5 below.

Given $F = (f_1, f_2, g)$ in $\boldsymbol{\mathcal{X}}^r$ such that $g(0) = 0$, we have the following possibilities:

$$\text{(i) } (f_1(0), f_2(0)) \neq (0, 0) \quad \text{or} \quad \text{(ii) } (f_1(0), f_2(0)) = (0, 0).$$

In the case (i) the required normal forms for F, can be derived from Vishik's Normal Form Lemma (in [13]) . Case (ii) means that 0 is a critical point of F. In what follows we analyze the main result contained in [10] concerning the classification of the codimension-one $S-$ singularities. We begin by defining the subsets $\Sigma_1(a)$ and $\Sigma_2(b)$ of $\boldsymbol{\mathcal{X}}_1^r$.

We denote by $Hessian(f)$ the Hessian Matrix of the function f, and $Hess(f)$ the determinant of $Hessian(f)$.

Let h be any local implicit definition of S at 0.

Definition 3. We say that F is in $\Sigma_1(a)$ if the following conditions hold:

(i) 0 is a hyperbolic critical point of F;

(ii) the eigenvalues of $DF(0)$ are pairwise distinct and the corresponding eigenspaces are transversal to S at 0;

(iii) each pair of non complex conjugate eigenvalues of $DF(0)$ have distinct real parts.

Definition 4. We say that F is in $\Sigma_1(b)$ if $F(0) \neq 0$, $Fh(0) = 0$, $F^2h(0) = 0$ and one of the following conditions hold:

(1) $F^3h(0) \neq 0$, $rank\{Dh(0), DFh(0), DF^2h(0)\} = 2$ and the function $Fh_{|S}$ has at 0 a non degenerate (Morse) critical point;
(2) $F^3h(0) = 0$, $F^4h(0) \neq 0$ and 0 is a regular point of $Fh_{|S}$.

Next results are in [10].

Theorem C: *The following statements hold:*

1. $\Sigma_1 = \Sigma_1(a) \cup \Sigma_2(b)$;
2. Σ_1 *is a codimension-one submanifold of* $\mathfrak{X}^r$;
3. Σ_1 *is open and dense in* $\mathfrak{X}^r_1$, *in the topology induced from* $\mathfrak{X}^r$;
4. For a residual set of smooth curves $\gamma : I\!R \to \mathfrak{X}^r$, γ *meets* Σ_1 *transversally and* $\gamma^{-1}(\mathfrak{X}^r_2) = \emptyset$, $\mathfrak{X}^r_2 = \mathfrak{X}^r_1 \backslash \Sigma_1$.

Throughout this section we fix the function $h(u, v, w) = w$

Lemma 5. *(Classification Lemma)*

1.The elements of $\Sigma_1(a)$ *are classified in the following way:*

(a_{11}) *Nodal S-Singularity:* $F(0) = 0$, *the eigenvalues of* $DF(0)$, λ_1, λ_2, *and* λ_3, *are real, distinct,* $\lambda_1\lambda_j > 0$, $j = 2, 3$ *and the eigenspaces are transverse to* S *at* 0;

(a_{12}) *Saddle S-Singularity:* $F(0) = 0$, *the eigenvalues of* $DF(0)$, λ_1, λ_2, *and* λ_3, *are real, distinct,* $\lambda_1\lambda_j < 0$, $j = 2$ *or* 3 *and the eigenspaces are transverse to* S *at* 0;

(a_{13}) *Focus S-Singularity:* 0 *is a hyperbolic critical point of* F, *the eigenvalues of* $DF(0)$ *are* $\lambda_{12} = a \pm ib$, $\lambda_3 = c$, *with* a, b, c *distinct from zero and* $c \neq a$, *and the eigenspaces are transverse to* S *at* 0.

2.The elements of $\Sigma_1(b)$ *are classified by means of the following normal forms:*

(b_{11}) *Lips S-Singularity: defined in Definition 4, item* (1), *with* $Hess(Fh_{/S}(0)) > 0$;

(b_{12}) *Bec to Bec S-Singularity: defined in Definition 4, item* (1), *with* $Hess(Fh_{/S}(0)) < 0$;

(b_{13}) *Dove's Tail S-Singularity: defined in Definition 4, item* (2).

Now we characterize the set $\sum_2$.

Definition 6. We denote by $\Sigma_2(a)$ the set of the vector fields $F \in \Sigma_2$, such that $F(0) = 0$ and one of the following conditions is satisfied:

(b_{21}) *(Saddle-Node)* 0 is a Saddle-Node singularity of F (one eigenvalue is 0) and the eigenspaces of $DF(0)$ are transverse to S at 0;

(b_{22}) *(Hopf)* 0 is a Hopf singularity (a pair of pure imaginary eigenvalues) of codimension-one of F and the eigenspaces of $DF(0)$ are transverse to S at 0;

(b_{23}) *(2-Node)* The eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in I\!R$ of $DF(0)$ satisfy $\lambda_1 = \lambda_2 \neq \lambda_3$, $\text{rank}(DF(0) - \lambda_1 Id) = 2$. Moreover, the eigenspaces $V_{\lambda_1}, V_{\lambda_2}, V_{\lambda_3}$ are transverse to S at 0;

(b_{24}) *(2-Hyperbolic)* The eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in I\!R$ of $DF(0)$ are distinct and there is one dimensional invariant manifold at 0 with quadratic tangency with S at 0. Moreover, the other eigenspaces are transverse to S at 0;

(b_{25}) *(2-Focus)* The eigenvalues of $Df(0)$ are $\lambda_1 \in I\!R$ and $\lambda, \bar{\lambda} \in \boldsymbol{C}$ with Re $(\lambda) = \lambda_1$. Moreover, the eigenspaces $V_{\lambda_1}, V_{\lambda_2}, V_{\lambda_3}$ are transverse to S at 0 and $rank(Hessian(Fh_{/S})$ is not zero;

(b_{26}) *(Tangent Focus)* The eigenvalues of $Df(0)$ are $\lambda_1 \in I\!R$ and $\lambda, \bar{\lambda} \in \boldsymbol{C}$ with Re $(\lambda) \neq \lambda_1$. Moreover, the eigenspaces V_λ has a quadratic tangency with S at 0 and the eigenspace V_{λ_1} is transverse to S at 0.

Definition 7. We denote by $\Sigma_2(b)$ the set of the vector fields $F \in \Sigma_2$, which has a unique point $p \in S$ such that $F(p) \neq 0$, $Fh(p) = 0$, $F^2h(p) = 0$ and one of the following conditions is satisfied:

(a_{21}) *(Goose)* $F^3h(p) \neq 0$, $\text{rank}\{dh(p), d(Fh)(p), d(F^2h)(p)\} = 2$ and the function $Fh_{/S}$ is equivalent, in p, to a simple germ of codimension-one;

(a_{22}) *(Gull)* $F^3h(p) = 0$, $F^4h(p) \neq 0$ and p is a non degenerate critical point of $Fh_{/S}$.

(a_{23}) *(Butterfly)* $F^3h(p) = 0$, $F^4h(p) = 0$, $F^5h(p) \neq 0$ and p is a regular point of $Fh_{/S}$.

Theorem D The following statements hold:

(i) $\Sigma_2 = \Sigma_2(a) \cup \Sigma_2(b)$ is open and dense in $\mathfrak{X}_2^r$;

(ii) F is structurally stable relative to $\mathfrak{X}_2^r$ if and only if $F \in \Sigma_2$;

(iii) Σ_2 is a codimension-two submanifold of $\mathfrak{X}^r$;

(iv) the normal forms of any 2-parameter family $F_{\alpha\beta}$ transverse to Σ_2 at F_{00} are:

(2.1) (Saddle-Node) $F_{\alpha\beta} = (\varepsilon(u+v)^2 + \alpha, u - v, u + v + \lambda_3 w + \beta)$, where $\lambda_3 \notin \{0, -1\}, \varepsilon = \pm 1$;

(2.2) (Hopf) $F_{\alpha\beta} = (-v + u(\alpha - u^2 - v^2), u + v(\alpha - u^2 - v^2), ku + v + \lambda_3 w + \beta)$, where $\lambda_3 \notin \{0, 1\}, k \neq 1$;

(2.3) (2-node) $F_{\alpha\beta} = (\lambda_1 + \alpha v, u + \lambda_1 v, u + v + \lambda_3 w + \beta)$, where $\lambda_3 \neq 0$ and $\lambda_3 \neq \lambda_1$;

(2.4) (2-Hyperbolic a) $F_{\alpha\beta} = (\lambda_1 u, \lambda_2 v, \lambda_3 w + v + \varepsilon_1(\lambda_3 - 2\lambda_1)u^2 + \alpha u + \beta)$, where $\lambda_3 \notin \{0, 2\lambda_1, 2\lambda_2\}$ and $\lambda_2 \neq 2\lambda_1$;

(2.5) (2-Hyperbolic b) $F_{\alpha\beta} = (\lambda_1 u, (a^2 - 2ab - b^2)v + 2abw - 2abu^2 - 2ab\alpha u, -2bv + (a+b)w - (a+b+2\lambda)u^2 - \alpha(a+b+\lambda)u + \beta)$, where a, b, λ are real nonzero numbers and pairwise distinct;

(2.6) (2-Focus) $F_{\alpha\beta} = (au+bv+\alpha u, -bu+av, u+v+aw+\varepsilon_1 u^2+\varepsilon_2 v^2-(\varepsilon_1-\varepsilon_2)uv+\beta)$, where $\varepsilon_1, \varepsilon_2 \in \{+1,-1\}$, a, b are nonzero real numbers and pairwise distinct, and $\varepsilon_1, \varepsilon_2 \in \{+1,-1\}$;

(2.7) (Tangent focus) $F_{\alpha\beta} = (au+bv, -bu+av, \lambda_3+u^2+\varepsilon v^2+\alpha u+\beta)$, where a, λ_3 are real nonzero numbers and pairwise distinct, and $\varepsilon = \pm 1$;

(2.8) (Goose) $F_{\alpha\beta} = (1, 0, -3u^2 - v^3 + \alpha v + \beta)$;

(2.9) (Gull) $F_{\alpha\beta} = (1, 0, 4u^3 + 2uv + v^2 + \alpha v + \beta)$;

(2.10) (Buttterfly) $F_{\alpha\beta} = (1, 0, 5u^4 + 3\varepsilon u^2 v + v + \alpha u^2 + \beta u)$, where $\varepsilon = \pm 1$.

4. PROOF OF THEOREM D

Our analysis will have two different approachs: in the first one we work with the vector fields having critical points in the boundary. The later concerns with those vector fields which present tangency points.

4.1. Critical points in the boundary

Next result will be useful in the sequel and its proof is in [9].

LEMMA 8. *Consider neighborhoods V of $p \in M$ and B of $F \in \mathfrak{X}^r(M)$, where p is a simple critical point of F. Then there are, a neighborhood $B_1 \subset B$ of F and a $C^{r-1}-$ function $\eta : B_1 \to V$, such that $G(p) = 0 \Leftrightarrow p = \eta(G)$.*

Denote $\eta(G) = p_G$.

LEMMA 9. *Let $p \in S$ be a Saddle-Node singularity of $F \in \mathfrak{X}^r$. Then, there exist neighborhoods B of F in $\mathfrak{X}^r$, N of p in M and a $C^{r-1}-$ function $\xi : B \to I\!R^2$, such that:*

(1) $\xi(G) = 0 \Leftrightarrow G$ has a unique critical point $p_G \in S \cap N$, which is a Saddle-Node singularity;
(2) $d\xi_F$ is surjective.

Proof. We only provide an outline of the proof.

Fix a coordinate system $u = (u_1, u_2, u_3)$ around p such that:

$$u_1(p) = u_2(p) = u_3(p) = 0 \text{ and } \frac{\partial}{\partial u_i}(p) \in T_i, \quad i = 1, 2, 3,$$

where T_i is the eigenspace associated to the eigenvalue λ_i of $(DF)_p$. From Takens's normal form, we can write the vector field F in the form:

$$F = (F^1(u), F^2(u), F^3(u)) = (u_1^2, \lambda_2 u_2, \lambda_3 u_3) + h.o.t.,$$

where $u=(u_1,u_2,u_3)$ and λ_2,λ_3 are nonzero and distinct real numbers.

So S is expressed by the equation:

$$h(u_1,u_2,u_3)=au_1+bu_2+cu_3+\sum_{i+j+k>1}a_{i,j,k}u_1^iu_2^ju_3^k=0,$$

where a,b,c are non zero numbers.

We have that: $F^1_{u_1}(0)=0$; $F^1_{u_1u_1}(0)\neq 0$; $F^2_{u_2}(0)\neq 0$; $F^3_{u_3}(0)\neq 0$.

Let B_0 and N_0 be small neighborhoods of F and 0, in $\mathfrak{X}^r$ and $I\!R^3,0$ respectively. Define the C^r- application $\zeta_1: B_0\times N_0\to I\!R^2$ by $\zeta_1(G;u)=(G^2(u),G^3(u))$ where $G=(G^1,G^2,G^3)\in B_0$.

We have that:

$$\zeta_1(F;0,0,0)=(0,0) \text{ and } \frac{\partial\zeta_1}{\partial u_2\partial u_3}(F;0,0,0)=\begin{pmatrix}\lambda_2 & 0\\ 0 & \lambda_3\end{pmatrix}.$$

From the Implicit Function Theorem, there exist neighborhoods $B_1\times I_1$ of $(F,0)$ and $I_2\times I_3$ of $(0,0)$ and a unique pair of C^r- applications $\iota_1,\iota_2: B_1\times I_1\to I_2\times I_3$ such that:

$$\zeta_1(F;0,0,0)=(0,0) \text{ and } \zeta_1(G;u_1,u_2,u_3)=0$$

with $(G,u_1)\in B_1\times I_1$. Moreover, $(u_2,u_3)\in I_2\times I_3$ only if $u_2=\iota_1(G,u_1)$ and $u_3=\iota_2(G,u_1)$.

We define a C^r- application: $\zeta_2: B_1\times I_1\to I\!R$ by

$$\zeta_2(G,u_1)=G^1(u_1,\iota_1(G,u_1),\iota_2(G,u_1))$$

It satisfies: $\zeta_2(F,0)=0$, $\frac{\partial\zeta_2}{\partial u_1}(F,0)=0$ and $\frac{\partial^2\zeta_2}{\partial u_1^2}(F,0)=2$.

So there exist, a neighborhood B of F, $B\subset B_1$, I of $0\in I\!R$ and a unique $C^{r-1}-$ function $\zeta_3: B\to I$ such that: $\zeta_1(F)=0$ and $\frac{\partial\zeta_2}{\partial u_1}(G,u_1)=0$ with $G\in B$. Moreover, $u_1\in I$ only if $u_1=\zeta_3(G)$.

Consider now $\xi_1: B\to I\!R$ and $\eta: B\to I\!R^3$ by where

$$\xi_1(G)=\zeta_2(Y,\zeta_3(G))=G^1(\zeta_3(Y),\iota_1(G,\zeta_3(G)),\iota_2(G,\zeta_3(G)) \text{ and}$$

$$\eta(G)=(\zeta_3(Y),\iota_1(G,\zeta_3(G)),\iota_2(G,\zeta_3(G)),\ \eta(G)=p_G.$$

Hence, $G\in B$ has a critical point $(u_1,u_2,u_3)\in N=I_1\times I_2\times I_3$ if and only if $u_2=\iota_1(G,u_1)$, $u_3=\iota_2(G,u_1)$ and $\zeta_2(G,u_1)=0$.

Moreover, $\xi_1(G)$ is a minimum of $\zeta_2(G,u_1)$, $u_1\in I_1$ and this minimum is reached at p_G.

So G has no critical points in N provided that $\xi_1(G) > 0$; G has a unique critical point in N provided that $\xi_1(G) = 0$; it is of Saddle-Node singularity. The mapping G has two critical points provided that $\xi_1(G) < 0$.

Let Π be the projection of N on T^3 parallel to S. Define a C^{r-1}, $\xi_2 : B \to \mathbb{R}$ by $\xi_2 = \Pi(p_G(G))$ and $\xi : B \to \mathbb{R}^2$ by $\xi(G) = (\xi_1(G), \xi_2(G))$. We have that $\xi(F) = (0,0)$. Let us observe that $\xi(G) = (0,0)$. This means that G has a critical point p_G in $N \cap S$. The reciprocal assertion is immediate.

Considering the curves $\alpha, \beta : \mathbb{R} \to B$ defined by $\alpha(s) = F + s(1,0,0)$ and $\beta(s) = F + s(0,0,\lambda_3)$, we are able to prove that $(d\xi)_F$ is surjective. ▌

Remark 10. Following Lemma 9, observe that G has no critical points in N provided that $\xi_1(G) > 0$; G has a unique critical point in N provided that $\xi_1(G) = 0$. Moreover, if $\xi_2(G) = 0$, then $p_G \in S$; otherwise, $p_G \notin S$. If $\xi_1(G) < 0$, then G has two critical points in N (a *saddle* and a *node*).

LEMMA 11. *Let $p \in S$ be a Hopf singularity of $F \in \mathfrak{X}^r$. Then, there exist neighborhoods B of F in $\mathfrak{X}^r$, N of p in $\mathbb{R}^3, 0$, and a $C^{r-1}-$ function $\xi : B \to \mathbb{R}^2$ such that:*

(1) $\xi(G) = 0 \Leftrightarrow G \in B$ has a unique critical point $p_G \in S \cap N$, which is a Hopf singularity;
(2) $d(\xi)_F$ is surjective.

Proof. We only provide an outline of the proof.

Fix a coordinate system $u = (u_1, u_2, u_3)$ around p such that:

$$u_1(p) = u_2(p) = u_3(p) = 0 \text{ and } \frac{\partial}{\partial u_i}(p) \in T_i, \quad i = 1,2,3,$$

where T_i is the eigenspace associated to the eigenvalue λ_i of $(DF)_p$. Let F_α be a generic one-parameter family of vector fields representing a Hopf bifurcation in $\mathfrak{X}^r$, such that $F_0 = F$. That means:

(h1) the eigenvalues of $(DF)_0$ are $\lambda_{12} = \pm i$ and $\lambda_3 \neq 0$;
(h2) $(DF_\alpha)_p$ has eigenvalues: $\lambda_{12}(\alpha) = \alpha \pm i$; $\lambda_3 \neq 0$;
(h3) $\frac{\partial}{\partial\alpha}(\mathrm{Re}(\lambda_{12})(\alpha))\big|_{\alpha=0} \neq 0$.

In these coordinates, we can write the vector field F_α in the form:

$$F_\alpha(u) = (-u_2 + u_1\Lambda + h.o.t., u_1 + u_2\Lambda + h.o.t., \lambda_3 u_3 + h.o.t.),$$

where $u = (u_1, u_2, u_3)$ and $\Lambda = (\alpha - u_1^2 - u_2^2)$.

Consider $B_0(F)$ and $N_0(0)$ as before and let S be expressed by:

$$h(u_1, u_2, u_3) = au_1 + bu_2 + cu_3 + \sum_{i+j+k>1} a_{i,j,k} u_1^i u_2^j u_3^k.$$

From Lemma 8, we have that the correspondence $G \in B_0 \to p_G \in N_0$ is C^{r-1} with $p_F = 0$ and $G(p_G) = 0$.

The characteristic polynomial associated to $A_G = (DG)_{p_G}$ is:

$$P_\lambda(A_G) = \lambda^3 - \sigma(A_G)\lambda^2 + \sigma(A_G^*)\lambda - \det(A_G).$$

We know that $\det(A_F) \neq 0$. Let us consider $R_1 \subset I\!R$, a small neighborhood of λ_3.

The application: $\eta : B_1 \times R_1 \to I\!R$ given by $\eta(G,\lambda) = P_\lambda(A_G)$ satisfies $\eta(F,\lambda_3) = 0$, $\frac{\partial\eta}{\partial\lambda}(F,\lambda_3) \neq 0$, and $\frac{\partial\eta}{\partial\lambda}(G,\lambda) = 3\lambda^2 - 2\sigma(A_G)\lambda + \sigma(A_G^*)$.

Thus, we get neighborhoods $B \subset \mathfrak{X}^r$ of F, N_1 of λ_3, and a function $\Lambda_3 : B \to N_1$ such that $\eta(G,\lambda) = 0 \Leftrightarrow \lambda = \Lambda_3(G)$.

Define $\xi_1 : B \to I\!R$ by $\xi_1(G) = \Lambda_3(G) - \sigma((A_G)_{p_g})$. Hence, G has eigenvalues $\Lambda_3(G) \in I\!R$ and $\lambda_{12} = \pm ei$ provided that $\xi_1(G) = 0$. This means that, p_G is a critical point of Hopf type of G.

Let Π be the projection of N on T^3 parallel to S and define $\xi_2 : B \to I\!R$ given by $\xi_2 = \Pi(p_G(G))$. We take $\xi : B \to I\!R^2$ given by $\xi(G) = (\xi_1(G), \xi_2(G))$. If $\xi_1(G) = 0$ then p_G is a Hopf singularity of G. Let G_μ be any generic one-parameter family of vector fields in $\mathfrak{X}^r$ (generic Hopf bifurcation) with $G_0 = F$. We know that any complex eigenvalues λ_μ of (DG_μ) satisfies $\frac{\partial\lambda_\mu}{\partial\mu} \neq 0$.

As before it can be shown that $(D\xi)_F$ is surjective. ∎

Remark 12. Following Lemma 11, observe that G has a unique critical point p_G and $(DG)_{p_G}$ has eigenvalues $\lambda_{12} = \zeta_1 \pm \zeta_2 i$ and λ_3, where ζ_1, λ_3 are non zero numbers, provided that $\xi_1(G) \leq 0$ (if $\xi_1(G) = 0$, then $\zeta_1 = 0$). Moreover, the critical point is a focus. In case $\xi_1(G) > 0$, the critical point is a focus, but a periodic orbit γ_G emerges. We recall that G has a unique critical point p_G in S, provided that $\xi_2(G) = 0$. If $\xi_2(G) > 0$, then G has no critical points in M.

LEMMA 13. *Let $p \in S$ be a Degenerate Node singularity of $F \in \mathfrak{X}^r$. Then, there exist neighborhoods B of F in $\mathfrak{X}^r$, N of p in $I\!R^3, 0$, and a $C^{r-1}-$ function $\xi : B \to I\!R^2$ such that:*

(1) $\xi(G) = 0 \Leftrightarrow G \in B$ has a unique critical point, which is a Degenerate Node singularity, $p_G \in N$;

(2) $d(\xi)_F$ is surjective.

Proof. Consider a coordinate system $u = (u_1, u_2, u_3)$ in $I\!R^3, 0$, as Lemma 9, and F in the form:

$$F(u_1, u_2, u_3) = (\lambda_1 u_1 + h.o.t, u_1 + \lambda_1 u_2 + h.o.t, \lambda_3 u_3 + h.o.t),$$

and S expressed by $h(u_1,u_2,u_3) = au_1+bu_2+cu_3+\sum_{i+j+k>1} a_{i,j,k}u_1^i u_2^j u_3^k$. Consider the application $\xi = (\xi_1, \xi_2) : B \to I\!R^2$ by

$$\xi_1(G) = (\sigma(G) - \lambda_3(G))^2 - 4\frac{\det(G)}{\lambda_3(G)} \text{ and } \xi_2(G) = \Pi(p_G(G)),$$

where Π is the projection of N on T^3 parallel to S.

Now we finish this proof as in Lemma 11. ▌

Remark 14. Following Lemma 13, we have that the vector field $G \in B$ has a critical point p_G (Degenerate Node singularity) in $N \cap S$, provided that $\xi_1(G) = 0$; If $\xi_1(G) > 0$, then G has a hyperbolic critical point which is a focus singularity; G has a unique critical point p_G in S, provided that $\xi_2(G) = 0$; If $\xi_2(G) > 0$, then G has no critical points.

LEMMA 15. *Let $p \in S$ be a Hyperbolic Tangent singularity of $F \in \mathfrak{X}^r$. Then there exist: a neighborhood B of F in $\mathfrak{X}^r$, a neighborhood N of p in M and a $C^{r-1}-$ function $\xi : B \to I\!R^2$ such that:*

(1) $\xi(G) = 0 \Leftrightarrow G \in B$ has a unique critical point, that is a Hyperbolic Tangent singularity, $p_G \in N$;

(2) $d\xi_F$ is surjective.

Proof. Consider a coordinate system $u = (u_1, u_2, u_3)$ in $I\!R^3, 0$ as in Lemma 9, and F in the form:

$$F(u) = (\lambda_1 u_1 + h.o.t., \lambda_2 u_2 + h.o.t., \lambda_3 u_3 + u_2 + \varepsilon_1(\lambda_3 - 2\lambda_1)u_1^2 + h.o.t.),$$

where $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$. The vector field has an invariant manifold, γ, tangent to S, parameterized by $m(s) = (s, 0, -\varepsilon_1 s^2)$ and S is expressed by the equation

$$au_1 + bu_2 + cu_3 + \sum_{i+j+k>1} a_{i,j,k}u_1^i u_2^j u_3^k = 0.$$

Consider the application $\xi = (\xi_1, \xi_2) : B \to I\!R^2$ by

$$\xi_1(G) = G^3_{u_1}(p_G) \text{ and } \xi_2(G) = \Pi(p_G(G)),$$

where Π is the projection of N on T^3 parallel to S.

The proof now follows immediately. ▌

Remark 16. Following Lemma 15, we observe that: If $\xi(G) \neq 0$, then G has a hyperbolic critical point in such a way that the invariant submanifold is transverse to S. G has a unique critical point p_G in S, provided that $\xi_2(G) = 0$. If $\xi_2(G) > 0$, then G has no critical points.

LEMMA 17. *Let $p \in S$ be a Degenerate Focus singularity of $F \in \mathfrak{X}^r$. Then there exist: a neighborhood B of F in $\mathfrak{X}^r$, a neighborhood N of p in M, and a $C^{r-1}-$ function $\xi : B \to I\!R^2$ such that:*

(1) $\xi(G) = 0 \Leftrightarrow G \in B$ has a unique critical point, that is a Degenerate Focus singularity, $p_G \in N$;
(2) $d\xi_F$ is surjective.

Proof. We prove this lemma in the same way as Lemma 11. We observe that F has the form:

$$F(u) = (k_1u_1+k_2u_2+h.o.t., -k_2u_1+k_1u_2+h.o.t., -k_2u+k_1v+k_1w+h.o.t.),$$

where $u = (u_1, u_2, u_3)$, and k_1, k_2, λ_3 are nonzero real numbers, with $\lambda_3 \neq k_1$; $\lambda_3 \neq 2k_1$.

We have that: $\Delta_F = (\sigma(F) - k_1)^2 - \frac{4\det(F)}{k_1} < 0$.

Consider the functions $\xi = (\xi_1, \xi_2) : B \to I\!R$ given by

$$\xi_1(G) = \sigma(G) - 3\lambda_3^G \text{ and } \xi_2(G) = \Pi(p_G(G)),$$

where $\sigma(G)$ is the trace of $(DG)_{p_G}$ and Π being the projection of N on T^3 parallel to S.

The proof follows as above. ∎

Remark 18. Following Lemma 17, we observe that $G \in B$ has a critical point p_G in $N \cap S$, which is a Degenerate Focus singularity provided that $\xi(G) = 0$. If $\xi(G) \neq 0$, then G has a hyperbolic critical point that is a focus and $\mathrm{Re}((\lambda_{12}^G) \neq \lambda_3^G$. G has a unique critical point p_G in S, provided that $\xi_2(G) = 0$. In the case that $\xi_2(G) > 0$, then G has no critical points in M.

LEMMA 19. *Let $p \in S$ be a Tangent Focus singularity of $F \in \mathfrak{X}^r$. Then there exist: a neighborhood B of F in $\mathfrak{X}^r$, a neighborhood N of p in M, and a $C^{r-1}-$ function $\xi : B \to I\!R^2$ such that:*

(1) $\xi(G) = 0 \Leftrightarrow G \in B$ has a unique critical point, that is a Tangent Focus singularity, $p_G \in N$;
(2) $d\xi_F$ is surjective.

Proof. First of all take

$$F(u) = (F_1(u), F_2(u), F_3(u));$$

where $u = (u_1, u_2, u_3)$,

$$\begin{aligned} F_1(u) &= au_1 + bu_2 + h.o.t.; \\ F_2(u) &= -bu_1 + au_2 + h.o.t.; \\ F_3(u) &= \lambda_3 u_3 + \varepsilon_1(2a-\lambda_3)u_1^2 + \varepsilon_2(2a-\lambda_3)u_2^2 + 2b(\varepsilon_1-\varepsilon_2) + h.o.t., \end{aligned}$$

a, b, λ_3 are nonzero real numbers, $\lambda_3 \neq the;\ \lambda_3 \neq 2a$ and $\varepsilon_1 = \pm 1;\ \varepsilon_2 = \pm 1$. Now we proceed as in Lemma 11. ▮

Remark 20. Following Lemma 19, we observe that $G \in B$ has a critical point p_G in $N \cap S$, which is a Tangent Focus singularity, and G has a 2-dimensional invariant manifold tangent to S, provided that $\xi(G) = 0$. If $\xi(G) \neq 0$, then G has a hyperbolic critical point at which the stable or unstable invariant manifolds are transverse to S.

So G has a unique critical point p_G in S provided that $\xi_2(G) = 0$; if $\xi_2(G) < 0$, then G has a unique critical point p_G out S. In the case $\xi_2(G) > 0$, G has no critical points in M.

4.2. Vector fields tangent to the boundary

In this section, we prove that $\Sigma_2(b)$ is a codimension-two submanifold of $\mathfrak{X}^r$.

LEMMA 21. *Let 0 be a $S-$Singularity of Goose type of $F \in \mathfrak{X}^r$. Then there exist neighborhoods B of F in $\mathfrak{X}^r$, V of 0 in $\mathbb{R}^3$ and a $C^\infty-$mapping, $\xi : B \to \mathbb{R}^2$ such that:*

(1) Any $G \in B$, has a unique $S-$ singularity in V of Goose type, provided that $\xi(G) = 0$;
(2) $D\xi(F)$ is surjective.

Proof. In the coordinates given in Section 1 one has

$$F(u, v, w) = (f^1(u, v, w), f^2(u, v, w), g(u, v, w)),$$

where $h(u, v, w) = w$, $F(0) \neq 0$, and

$$\begin{aligned} f^1(u,v,w) &= \textstyle\sum_{i,j,k=0}^{\infty} a_{ijk} u^i v^j w^k, \\ f^2(u,v,w) &= \textstyle\sum_{i,j,k=0}^{\infty} b_{ijk} u^i v^j w^k, \\ g(u,v,w) &= \textstyle\sum_{i,j,k=0}^{\infty} c_{ijk} u^i v^j w^k. \end{aligned}$$

Also observe that $Fh(u, v, w) = g(u, v, w)$ and from our hyphotheses we get that:

$$c_{000} = b_{000} = c_{100} = 0, ; a_{000} \neq 0, \text{ and } c_{200}.c_{030} \neq 0,$$

and in this case we have that $Fh_{/S}$ is (germ-)equivalent to the germ $\pm\zeta^3 + \varepsilon_1.\iota^2; \varepsilon_1 = \pm 1$.

So we may write $g(u,v,0) = c_{200}u^2 + c_{030}v^3 + h.o.t.$.

If $G = (G^1, G^2, G^3)$ is a small perturbation of $F \in \mathfrak{X}^r$, we still have by continuity that $G^1(u,v,w) \neq 0$, $G^2_{uu}(u,v,w) \neq 0$, and $G^3_{vvv}(u,v,w) \neq 0$ for (u,v,w) in a small neighborhood of 0 in $\mathbb{R}^3$. Observe now that $g(u,v,0)$ unfolds generically as $g_{\alpha,\beta}(u,v,0) = c_{200}u^2 + c_{030}v^3 + \alpha v + \beta + h.o.t.$(for details, see [2]).

We want to prove that any G in B is equivalent to $G_{\alpha,\beta}$ where the singular set of $G_{\alpha,\beta}$ is given by the equation:

$$c^1_{200}u^3 + c^1_{030}uv^3 + \alpha uv + \beta u + h.o.t. = 0,$$

for some α, β.

As a matter of fact we follow the ideas and techniques contained in [6]; that is, the tools in singularity theory is strongly used to get an equivalence between the $S-singular$ sets of the vector fields G and $G_{\alpha,\beta}$. After then we extend some how this equivalence to a full neighborhood of the singularity in M.

For each G in a small neighborhood(say B) of F we may take a flow box construction K_G and new coordinates (r_1, r_2, r_3) such that $G(r_1, r_2, r_3) = (1,0,0)$ and $S = \{(r_1, r_2, r_3) : r_3 = m_G(r_1, r_2)\}$, where m_G is a $C^\infty-$function in such way that $m_F(r_1, r_2) = b^1u^3 + b^2uv^3$. Notice that the correspondences $G \to K_G$ and $G \to m_G$ are continuous with $K_G \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ and $m_G \in C^\infty(\mathbb{R}^2, \mathbb{R})$.

Fix G for instance. We look for the points (r_1, r_2, r_3) in S such that G is tangent to S; this means that we look for those points $(r_1, r_2, m_G(r_1, r_2))$, that are solutions of $\frac{dm_G(r_1,r_2)}{dr_1} = 0$. So we define a C^r- correspondence $\xi : B, F \to \mathbb{R}^2, 0$ by $\xi(G) = (\frac{\partial G^3}{\partial v}(p), G^3(p))$. So we have that if $\xi(G) = (\alpha, \beta)$, then G is equivalent to $G_{\alpha,\beta} = (1,0,0)$. In the system of coordinates given for K_G, the manifold S is expressed by $m_{\alpha,\beta} = b_1u^3 + b_2uv^3 + \alpha uv + \beta u$.

Now the conclusion of the proof is immediate.

∎

The proofs of next results are similar to that of Lemma 21 and they will be omitted.

LEMMA 22. *Let 0 be a $S-$ singularity of Gull type of $F \in \mathfrak{X}^r$. Then there exist neighborhoods B of F in $\mathfrak{X}^r$, V of 0 in $\mathbb{R}^3$ and a $C^\infty-$ function, $\xi : B \to \mathbb{R}^2$ such that:*

(1) If $\xi(G) = 0$, then there exists a unique $S-$singularity of G in V of Gull type;

(2) $D\xi(F)$ is surjective.

Lemma 23. *Let* 0 *be a* $S-$ *singularity of Butterfly type of* $F \in \mathfrak{X}^r$*. Then, there exist neighborhoods* B *of* F *in* $\mathfrak{X}^r$*,* V *of* 0 *in* $\mathbb{R}^3$ *and a* C^∞ *function,* $\xi : B \to \mathbb{R}^2$ *such that:*

(1) If $\xi(G) = 0$*, then there exists a unique* $S-$*singularity of* G *in* V *of Butterfly type;*

(2) $D\xi(F)$ *is surjective.*

4.3. Characterization of tangency sets

Let $p \in M$ be a critical point of the vector field $F \in \mathfrak{X}^r$. In the sequel we may use the following notations: $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, or $\lambda_{12} = a \pm ib,\ \lambda_3 \in \mathbb{R}$, for the eigenvalues of $(DF)_p$.

Let (u_1, u_2, u_3) be a coordinate system in a neighborhood of p such that:

$$u_i(p) = 0 \text{ and } \frac{\partial}{\partial u_i}(p) \in T_i, \quad i = 1, 2, 3,$$

where T_i is the eigenspace associated to the eigenvalue λ_i of $(DF)_p$.

Let B_0 and N_0 be, neighborhoods of F and p, in $\mathfrak{X}^r$ and $\mathbb{R}^3, 0$, respectively. Let us denote by L_F the tangency set between F and S. That is: $L_F = \{q \in M : h(q) = Fh(q) = 0\}$, where h is any implicit definition of S such that $h(p) = 0$.

Lemma 24. *Let* $p \in S$ *be a critical point of* $F \in \mathfrak{X}^r$*. (a) If we have two eigenspaces transverse to* S *and* $Re(\lambda_i) \neq Re(\lambda_j)$*, for some* $i \neq j,\ i, j \in \{1, 2, 3\}$ *then* L_F *is a regular curve; (b) If two eigenspaces are tangent to* S *and the other is transverse to* S *and* $(Re(\lambda_1) - 2\lambda_3) \neq 0,\ i \in \{1, 2, 3\}$ *then* L_F *is a conic curve.*

Proof. Let S be expressed by the equation: $h(u_1, u_2, u_3) = au_1 + bu_2 + cu_3 + \sum_{i+j+k>1} a_{i,j,k} u_1^i u_2^j u_3^k = 0$.

We derive that $b^2 + c^2 \neq 0$, provided that the eigenspaces T_2, T_3 are transverse to S.

Call F by F_R(resp. F_M) where all the eigenvalues are real (resp. the eigenvalues are $\lambda_{12} = k_1 \pm ik_2,\ k_2 \neq 0;\ \ \lambda_3 \in \mathbb{R}$).

We may write:

$$F_R(u) = \lambda_1 u_1 + \tilde{F}^1(u), \lambda_2 u_2 + \tilde{F}^2(u), \lambda_3 u_3 + \tilde{F}^3(u)) \text{ and}$$

$$F_M(u) = (k_1 u_1 + k_2 u_2 + \tilde{F}^1(u), -k_2 u_1 + k_1 u_2 + \tilde{F}^2(u), \lambda_3 u_3 + \tilde{F}^3(u)),$$

where $u = (u_1, u_2, u_3)$.

We have that,

$$(F_R h)(u) = (a\lambda_1)u_1 + (b\lambda_2)u_2 + (c\lambda_3)u_3 + h.o.t., \tag{1}$$

$$(F_M h)(u) = (ak_1 - bk_2)u_1 + (ak_2 + bk_1)u_2 + (c\lambda_3)u_3 + h.o.t. \tag{2}$$

We define $\zeta_R : N_0 \to I\!R^2$ by $\zeta_R(u) = (h(u), Fh(u))$ where $u = (u_1, u_2, u_3)$, and Fh, is either $F_R h$ or $F_M h$.

We have that $\zeta_R(0) = 0$ and

$$(D\zeta_R)_0 = \begin{pmatrix} a & b & c \\ (Fh)_{u_1}(0) & (Fh)_{u_2}(0) & (Fh)_{u_3}(0) \end{pmatrix},$$

where Fh, is either $F_R h$ or $F_M h$.

From Equations 1 and 2, we derive the vector fields:

$$\nabla(F_R h)(0) = (a\lambda_1, b\lambda_2, c\lambda_3) \text{ and}$$

$$\nabla(F_M h)(0) = (ak_1 - bk_2, ak_2 + bk_1, c\lambda_3).$$

By a direct calculation we get that set $\{(\nabla h)(0), (\nabla(Fh)_R)(0)\}$ is linearly independent and $(d\zeta_R)_0$ is surjective. So $L_F = \zeta_R^{-1}(0)$ is a regular curve in S and so the assertion (a) is reached. We prove item b.

Assume that the tangent eigenspaces to S are T_1, T_2 and h has the form:

$$h_M(u_1, u_2, u_3) = au_1^2 + bu_2^2 + cu_3 + h.o.t.,$$

with $a = \pm 1$, $b = \pm 1$ and $c \neq 0$.
Hence,

$$(F_R h_M)(u) = (2a\lambda_1)u_1^2 + (2b\lambda_2)u_2^2 + (c\lambda_3)u_3 + h.o.t.,$$

$$(F_M h_M)(u) = (2ak_1)u_1^2 + (2bk_1)u_2^2 + (2ak_2 - 2bk_2)u_1u_2 + (c\lambda_3)u_3 + h.o.t..$$

As before Fh_M means either $F_R h_M$ or $F_M h_M$. Let us observe that $(\nabla Fh_m)(0) = 0$, and so 0 is a critical point of $Fh_{/S}$.

Moreover,

$$(F_R h_M)_{/S}(u) = (2\lambda_1 - \lambda_3)au_1^2 + (2\lambda_2 - \lambda_3)bu_2^2 + h.o.t.,$$

$$(F_M h_M)_{/S}(u) = (2k_1 - \lambda_3)au_1^2 + (2k_1 - \lambda_3)bu_2^2 + (2ak_2 - 2bk_2)u_1u_2 + h.o.t..$$

We define the application $\zeta_M : N_0 \to I\!R^2$ by

$$\zeta_M(u) = ((\widehat{Fh})_{u_1}(u), (\widehat{Fh})_{u_2}(u)),$$

where $\widehat{Fh}$, is either $(F_R h_M)_{/S}$ or $(F_M h_M)_{/S}$.

We have that,

$$\zeta_M(0) = 0 \text{ and } (D\zeta_M)_0 = \begin{pmatrix} (\nabla(\widehat{Fh})_{u_1})(0) \\ (\nabla(\widehat{Fh})_{u_2})(0) \end{pmatrix},$$

where $\widehat{Fh}$, is either $(F_R h_M)_{/S}$ or $(F_M h_M)_{/S}$.

So,

$$(\nabla((F_R h_M)_{/S})_{u_1}(0) = ((2\lambda_1 - \lambda_3)a, 0),$$

$$(\nabla((F_R h_M)_{/S})_{u_2}(0) = (0, (2\lambda_2 - \lambda_3)b),$$

$$(\nabla((F_M h_M)_{/S})_{u_1}(0) = (2a(2k_1 - \lambda_3), 2k_2(a-b)),$$

$$(\nabla((F_M h_M)_{/S})_{u_2}(0) = (2k_2(a-b), 2b(2k_1 - \lambda_3)),$$

As,

$$\det((D\zeta_M)_0) = (2\lambda_1 - \lambda_3)(2\lambda_2 - \lambda_3) \neq 0$$

or,

$$\det((D\zeta_M)_0) = 4ab(2k_1 - \lambda_3)^2 - 4k_2^2(a-b)^2 \neq 0.$$

Hence, $(D\zeta_M)_0$ is surjective and therefore 0 is a non degenerate critical point of $Fh_{/S}$ and L_F is a conic curve. Moreover, L_F is a point or an ellipse provided that $(2\text{Re}(\lambda_1) - \lambda_3)(2\text{Re}(\lambda_2) - \lambda_3)ab > 0$. Otherwise, in the case that the last expression is negative L_F is a pair of straight lines or a hyperbola. ▮

Let us take a coordinate system $u = (u_1, u_2, u_3)$ with $u(p) = 0$, such that: $h(u_1, u_2, u_3) = u_3$.

So L_F is expressed by: $L_F = \{q \in I\!R^3 : q = (u_1, u_2, 0),\ F^3(q) = 0\}$. Denote $\alpha_i = \frac{\partial F^3}{\partial u_i}(p),\ i = 1, 2$. We have that:

Corollary 25. *Let $p \in S$ be a critical point of $F \in \mathfrak{X}^r$.*

(1) If $\alpha_1^2 + \alpha_2^2 \neq 0$, then L_F is a regular curve in S;
(2) if $\alpha_1^2 + \alpha_2^2 = 0$, and $Hess(F^3_{/S}) \neq 0$, then L_F is a conic curve in S.

LEMMA 26. *Consider as above B_0 and N_0. Let $G \in B_0$.*

(1) if we have two eigenspaces transverse to S and $Re(\lambda_i) \neq Re(\lambda_j)$, for some $i \neq j,\ i, j \in \{1, 2, 3\}$, then G is tranverse to S at p_G;

(2) if the eigenspaces is such that, at least two of them are tangent to S, and $(Re(\lambda_1) - 2\lambda_3) \neq 0,\ i \in \{1, 2, 3\}$, then, either L_G is a conic curve in S, or L_G is empty.

Proof. We define applications $\zeta^R : B_0 \times N_0 \to I\!R^2$ and $\zeta^M : B_0 \times N_0 \to I\!R^3$, by $\zeta^R(G, u) = \zeta_R(u)$ and $\zeta^M(G, u) = \zeta_M(u)$ where $u = (u_1, u_2, u_3)$.

The first item is reached from Lemma 24. Moreover, Lemma 24 guarantees the existence of a unique point $\tilde{p}_G \in N_0$ such that is

$$(Gh_{/S})_{u_1}(\tilde{p}_G) = 0 \text{ and } (Gh_{/S})_{u_2}(\tilde{p}_G) = 0,$$

for each $G \in B_0$. Moreover, $Hess((Gh_{/S})(\tilde{p}_G)) \neq 0,\ (Gh_{/S})(\tilde{p}_G) = 0$ and $(Gh_{/S})(\tilde{p}_G) \neq 0$.

If $(Gh_{/S})(\tilde{p}_G) = 0$, then $\tilde{p}_G$ is a non degenerated critical point of $(Gh_{/S})$ and $Hess((Gh_{/S}(\tilde{p}_G)))$ has the same sign as $Hess((Fh_{/S}(0)))$.

So if $Hess(Gh_{/S}(\tilde{p}_G)) < 0$, then L_G, either is a *a pair of straight lines*, or a *hyperbola*. Otherwise, L_G, is either a *point* or a *ellipse*.

If $(Gh_{/S})(\tilde{p}_G) \neq 0$, then at $\tilde{p}_G$ we have that G is transverse to S. ∎

LEMMA 27. *Let $p \in S$ be a singularity of $F \in \mathfrak{X}^r$ and $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of $(DF)_p$. Call by $T_i,\ i = 1, 2, 3$ the eigenspace associated to $\lambda_i,\ i = 1, 2, 3$. Assume one of the following possibilities:*

(1) $\lambda_1, \lambda_2, \lambda_3 \in I\!R$, and $\lambda_i \neq \lambda_j\ \ \forall i \neq j$;

(2) $\lambda_1 = \lambda_2 \neq \lambda_3 \in I\!R$, and $rank((DF)_p - \lambda_1 Id) = 2$;

(3) $\lambda_{12} = a \pm ib,\ b \neq 0,\ \lambda_3 \in I\!R$. Then, the set $\{(\nabla h)_p, (\nabla Fh)_p, (\nabla F^2 h)_p\}$ is linearly independent.

Proof. Let (u_1, u_2, u_3) be a coordinate system in a neighborhood of p such that $u_i(p) = 0$, and $\frac{\partial}{\partial u_i}(p) \in T_i,\ \ i = 1, 2, 3$.

Let $h(u_1, u_2, u_3) = au_1 + bu_2 + cu_3 + \sum_{i+j+k>1} a_{i,j,k} u_1^i u_2^j u_3^k$, where a, b, c, are nonzero real numbers.

Case (1) Write F in the following way:

$$F(u) = (\lambda_1 u_1 + F^1(u), \lambda_2 u_2 + F^2(u), \lambda_3 u_3 + F^3(u)),$$

with $u = (u_1, u_2, u_3)$ and $F^i(0) = 0,\ i = 1, 2, 3$.

Thus, $D_r = \det((\nabla h)_0, (\nabla Fh)_0, (\nabla F^2 h)_0) = 0$ if and only if $\lambda_i = \lambda_j$, for some $i \neq j,\ i, j = 1, 2, 3$.

Case(2) We write F as:

$$F(u_1,u_2,u_3) = (\lambda_1 u_1 + h.o.t., u_1 + \lambda_1 u_2 + h.o.t., \lambda_3 u_3 + h.o.t.).$$

Thus, $D_r^* = \det((\nabla h)_0, (\nabla F h)_0, (\nabla F^2 h)_0) = 0$ if and only if $\lambda_1 = \lambda_3$.
Case (3) In this case F can be expressed by:

$$F(u_1,u_2,u_3) = \alpha u_1 + \beta u_2 + h.o.t, -\beta u_1 + \alpha u_2 + h.o.t, \lambda_3 u_3 + h.o.t).$$

So, we get that

$$D_c = \det((\nabla h)_0, (\nabla F h)_0, (\nabla F^2 h)_0) \neq 0,$$

and the proof is finished. ▌

Lemma 28. *Let $p \in S$ be a critical point of the vector field F such that all associated eigenspaces T_i, $i = 1,2,3$ are transverse to S at p. Then there are neighborhoods B of F in $\mathfrak{X}^r$ and N of p in M, and a unique $C^{r-1}-$ application $\sigma : B \to I\!R^3$, $\sigma(G) = \tilde{p}_G$, such that:*

(1) $\tilde{p}_G$ is a unique point in N, satisfying $h(\tilde{p}_G) = Gh(\tilde{p}_G) = G^2h(\tilde{p}_G) = 0$;

(2) $\tilde{p}_G$ is either a critical point of G(when $G^3h(\tilde{p}_G) = 0$) or a cusp singularity of G;

(3) If $q \in L_G$, with $q \neq \tilde{p}_G$, then q is a Fold singularity of G.

Proof. Let (u_1,u_2,u_3) be a coordinate system in a neighborhood of p such that: $u_i(p) = 0$, $\frac{\partial}{\partial u_i}(p) \in T_i$, $i = 1,2,3$ where T_i is the eigenspace associated to the eigenvalue λ_i of $(DF)_p$, and $h(u_1,u_2,u_3) = au_1 + bu_2 + cu_3 + \sum_{i+j+k>1} a_{i,j,k} u_1^i u_2^j u_3^k$, where a, b, c nonzero real numbers.

Let us consider neighborhoods $B_0 \subset \mathfrak{X}^r$ of $F \in \Sigma_2(a)$, and N_0 of p in M. We define the application: $\zeta : B_0 \times N_0 \to I\!R^3$ by $\zeta(G,p) = (h(p), Gh(p), G^2h(p))$.

We have that $\zeta(F,0) = 0$ and $\frac{\partial \zeta}{\partial p}(F,0) = \begin{pmatrix} (\nabla h)_0 \\ (\nabla F h)_0 \\ (\nabla F^2 h)_0 \end{pmatrix}$ and from Lemma 27 we have that $\det\left(\dfrac{\partial \zeta}{\partial p}(F,0)\right) \neq 0$.

So there exist neighborhood $B \subset B_0$ of F and $N \subset N_0$ of p and a unique function $\sigma : B \to I\!R^3$, $\sigma(G) = \tilde{p}_G$, such that for $G \in B$, $\zeta(G,p) = 0$ only if $p = \sigma(G)$. This proves the first item.

Assume now that $G^3h(\tilde{p}_G(G)) = 0$. We have that:

$$(Gh)(\tilde{p}_G) + (G^2h)(\tilde{p}_G) + (G^3h)(\tilde{p}_G) = 0 \text{ and}$$

$$G(\tilde{p}_G)(\nabla h)_{\tilde{p}_G} + G(\tilde{p}_G)(\nabla Gh)_{\tilde{p}_G} + G(\tilde{p}_G)(\nabla G^2 h)_{\tilde{p}_G} = 0.$$

From Lemma 27 we have that the set $\{(\nabla h)_{\tilde{p}_G}, (\nabla Gh)_{\tilde{p}_G}, (\nabla G^2 h)_{\tilde{p}_G}\}$ is linearly independent. Thus, $G(\tilde{p}_G) = 0$.

Now, let $q \in L_G$, with $q \neq \tilde{p}_G$. We have that $H(q) = 0$; $Gh(q) = 0$.

If $G^2h(q) = 0$, then$q = \tilde{p}_G$. Otherwise, we have that q is a singularity of G of Fold type. ▮

LEMMA 29. *Let $p \in S$ be a Tangent Focus singularity of $F \in \mathfrak{X}^r$ and $Hess(Fh_{/S}) > 0$. If G is close of F, in $\mathfrak{X}^r$ then L_G, either has two singularities of Cusp type, or has a singularity of Dove's Tail type, or all singularities are of Fold type.*

Proof. By Lemma 26, we know that for a vector field G in a neighborhood of F, L_G is a conic curve. In this case, or L_G is reduced to a point, or it is a *ellipse* or it is still empty(in the last case G is transverse to S) .

Assume that $h(u_1, u_2, u_3) = ku_3 + h.o.t.$ In this case, the eigenvalues of $(DF)_p$ are $a+bi, a-bi, \lambda_3 \neq the$ and the respective eigenspace is transverse to S. From Lemma 15, and [14], a $C^\infty-$ normal form of G is $G(u) = (G^1(u), G^2(u), G^3(u))$ where:

$$\begin{aligned} G^1(u) &= au_1 - bu_2 + \iota_1 u_1\sigma - \iota_2 u_2\sigma + \iota_3 u_1\sigma^2, \\ G^2(u) &= bu_1 + au_2 + \iota_2 u_1\sigma + \iota_1 u_2\sigma + \iota_3 u_2\sigma^2, \\ G^3(u) &= ku_1 + u_2 + \lambda_3 u_3 + u_1^2 + u_2^2 + \iota_4 u_3\sigma, \end{aligned}$$

$u = (u_1, u_2, u_3)$, $k \neq 1$, $\iota_1, \iota_2, \iota_3, \iota_4$ are constant and $\sigma = (u_1^2 + u_2^2)^m u_3^n$, with m, n are the smallest non negative integers such that $m + n > 0$ and $2ma + n\lambda_3 = 0$. The genericity condition is given by $2m\iota_1 + n\iota_4 \neq 0$.

For simplicity in the analysis of the diagram of bifurcation, we consider $h(u_1, u_2, u_3) = u_3$ and the C^0- universal unfolding of G, given by:

$$G(u) = F_{\alpha,\beta}(u) = (au_1 + bu_2, -bu_1 + au_2, \lambda_3 u_3 + \epsilon_1 u_1^2 + \epsilon_2 u_2^2 + \alpha u_1 + \beta),$$

where $u = (u_1, u_2, u_3)$, $\epsilon_1\epsilon_2 > 0$ and $a \neq 0, b \neq 0$, and $a \neq \lambda_3$.

Let us calculate $Gh_{/S}$,$G^2h_{/S}$. For simplicity, consider $a = b = -1$ and $\varepsilon_1 = \varepsilon_2 = 1$.

$$Gh_{/S}(u_1, u_2) = u_1^2 + u_2^2 + \alpha u + \beta, \text{ and} \tag{3}$$

$$G^2h_{/S}(u_1, u_2) = -2u_1^2 - 2u_2^2 - a(u_1 + u_2). \tag{4}$$

Along L_G, we have:

$$u_2 = u_1 + \frac{2\beta}{\alpha};\ \alpha \neq 0 \text{ and} \tag{5}$$

$$G^2 h_{/L_G(u_1)=2u_1^2+\frac{\alpha^2+4\beta}{\alpha}u_1} + \frac{(\alpha^2+4\beta)\beta}{\alpha^2}. \tag{6}$$

If $\alpha = 0$, we have that $G^2 h_{/L_G} = 2\beta = 0 \Leftrightarrow \beta = 0$(that is to say $G = F$.)

We have that $\Delta = 0 \Leftrightarrow \alpha^2 = 4\beta$ or $\alpha^2 = -4\beta$, where $\Delta = \frac{(\alpha^2+4\beta)(\alpha^2-4\beta)}{\alpha^2}$.
So the equation $G^2 h_{/L_G} = 0$ has either two simple roots, or a double root, or no roots. Let q be any such root.

If $\Delta = 0$ then we have that q is a singularity of G of Dove's Tail type; if $\Delta > 0$, then q is a singularity of G of Cusp type. In this case we have two Cusps singularities. Finally, if $G^2 h_{/L_G}(q) \neq 0, \forall q$ ($\Delta < 0$), then L_G, only has folds singularities of G. ▮

The proof of next Lemma will be omitted.

LEMMA 30. *Let $p \in S$ be a Tangent Focus singularity of F such that $Hess(Fh_{/S})(p) < 0$. If G is close to F, then L_G, or has two singularities of the Cusp type, or a singularity Dove's Tail type, or all singularities are of Fold type.*

4.4. Structural Stability

In this section we study the relative structural stability in $\mathfrak{X}_2^r = \mathfrak{X}^r - (\Sigma_0 \cup \Sigma_1)$.

We divide the study in two parts. First, we deal with $\Sigma_2(a)$. In this case, we used the geometric road to find the homeomorphism (equivalence) between the vector fields. The other case, $\Sigma_2(b)$, is treated via theory of singularities of applications.

4.4.1. Hyperbolic Critical Point

In what follows we outline the construction of the equivalences that assures the structural stability in Σ_2. We will use the geometric road , and we will use several results and techniques contained in [9], [10] and [11].

Let $F \in \mathfrak{X}^r$ with $F(p) = 0$. In the proceeding sections we have characterized Σ_2 by means of disjoint union of several subsets. Observe that associated to each subset always there is a germ of mapping $\xi : \Sigma_2 \to \mathbb{R}^2$. In this section this notation is assumed.

Let V and B be small neighborhoods of p and F respectively. For each $G \in B$ we distinguish the subsets in V :

(1) the boundary of V;

(2) The submanifold S, the curve, L_G, and $I_G = \cup_{q \in L_G}(\varphi_t(q)), I_G \cap S$, where $\varphi_t(q)$ is the flow of $G \in B$ with $\varphi_0(q) = q \in V$;

(3) The singularity p_G and $\tilde{p}_G$ are distinguished, as well as their trajectories;

(4) For each singularity listed in Definition 6, we consider:

(a) The two-dimensional invariant manifold W_{12} of G tangent to the linear space generated by T_1 and T_2;

(b) The unidimensional invariant manifold W_3 of G tangent to T_3.

(5) We also include all the intersections between any distinguished sets listed above.

Associated to each G close to F in $\mathfrak{X}^r$ we may (and do) define an useful stratification, E_0, E_1, E_2, and $E_3 = V$ with $\dim(E_i) = i$. They are essential in our analysis. We will use the notation contained in [11]. In fact the strata are distinguished sets for any required equivalence between the systems in question.

If $p_G = \tilde{p}_G$ then, from Lemmas 26 and 28, $E_0 \subset S$. If $p_G \neq \tilde{p}_G$, then we have that $p_G \in E_0(G)$ and $p_G \notin S$.

We have to consider the following cases:

(1) If p_G is not a saddle, we have that G is transverse to ∂V;

(2) If p_G is a saddle, we distinguish the one-dimensional submanifold of ∂V, formed by the external tangencies between G and ∂V. This submanifold is included in the list of distinguished set and it is far away from L_G;

(3) When p_G is a tangent focus, we distinguish the intersection between W_{12} and ∂V.

Proposition 31. *Let $F \in \sigma_2$. There exist neighborhoods V of p in M and B of F in $\mathfrak{X}^r$ such that:*

(1) (F,p) is $C^\infty-$ equivalent to the $(G,p_G) \in (B \times V)$ if and only if $\xi(G) = 0$;

(2) if $G_1, G_2 \in B$ and $\xi_2(G_1).\xi_2(G_2) > 0$ then $(G_1)_{/V}$ is C^0- equivalent to $(G_2)_{/V}$, where $\xi = (\xi_1, \xi_2)$ is the submersion defined in Section 4.

Proof. We only provide an outline of the proof.

Let $p \in S$ be a critical point of F of the type $\Sigma_2(a)$ and we prove the sufficiency.

Innitially, we observe that due to the definition of critical point of type $\Sigma_2(a)$, this behavior persists for small perturbations of F in $\mathfrak{X}_2^r$ and the homeomorphism is firstly defined on ∂V, and then extended to a full neighborhood as a stratificated application. From Lemma 3.2, in [10], we get that any trajectory passing throught $q \in I_F \cap \partial V$ meets L_F just once.

Except the Tangent Focus case, the homeomorphism is built in the same way as in Theorem A of [11] applied for W_{12}. Observe that W_{12} separates V in two distinct components, and depending on the sign of λ_3 (positive or negative), we consider the homeomorphism built just as in [10] (Proposition 3.6, pg 182), in the cases Node ($\lambda_3 < 0$) or Saddle-Node ($\lambda_3 > 0$). The homeomorphism still preserves the elements: $C = W_{12} \cap \partial V$, $q_3 = W_3 \cap \partial V$ and $U = I_F \cap \partial V$. In the case $\lambda_3 > 0$, U is formed by two curves U_1 and U_2, and we still have a curve T, formed by the external tangency of F with ∂V. See Figure 1.

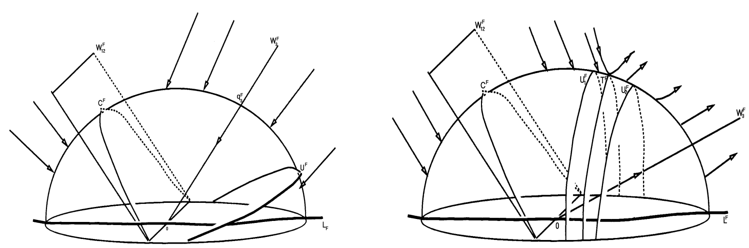

FIG. 1. Cases: Saddle-Node, Hopf, 2-Node, 2-Hyperbolic and 2-focus

In the Tangent Focus case, we separate the study in two subcases (see Lemmas 19, 29 and 30):

$$(i)\ Hess(Fh_{/S}) > 0 \text{ and } (ii)\ Hess(Fh_{/S}) < 0.$$

In the first case, we should observe:

(1) L_F is a point, and it coincides with p_F;
(2) $W_{12} \cap \partial V$ is a closed curve C;
(3) F is transverse to the ∂V;
(4) $q_3 = W_3 \cap \partial V$.

We impose that C, q_3 must be preserved by the homeomorphism. See Figure 2.

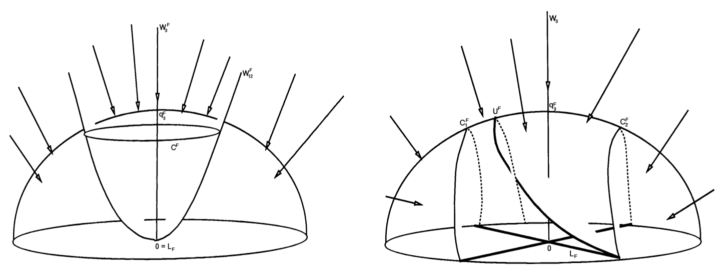

FIG. 2. Case: Tangent Focus

When $Hess(Fh_{/S}) < 0$, we observe that:

(1) L_F is a pair of straight lines at p_F;
(2) $W_{12} \cap \partial V$ is a pair of arcs C_1, C_2;
(3) F is transverse to the ∂V;
(4) $q_3 = W_3 \cap \partial V$.

Lemmas 29 and 30, guarantee the persistence of the elements above mentioned after perturbations. As before we define the homeomorphism on ∂V taking in account the distinguished elements, and then we extend to V (applying for example, the technique of arc length).

We now prove the reciprocal of item (1) of Proposition 31.

Let $G_1, G_2 \in B$ be, such that $\xi_2(G_1)\xi_2(G_2) > 0$. We will proceed the construction of the equivalence between G_1 and G_2 as in the part 1.

Let us notice that for any $G \in B$ with $\xi(G) \neq 0$, we still distinguish the following set: $\tilde{p}_G$, $\mu_G = L_G \cap W$, where W is a 2-dimensional invariant manifold of G or ∂V, as well as the trajectories of G passing through it. Let us consider W_{12} be an attractor and:

(1) If W_3 is an attractor, then $U = I_G \cap \partial V$ is a closed interval in ∂V with extreme point on L_G, transverse to the strong two-dimensional invariant manifold of F ;

(2) If W_3 is repeller, then $U = I_G \cap \partial V$ is composed two semi-intervals $U_1 = [a_1, b_1), U_3 = [a_3, b_3)$ and a closed interval $U_2 = [a_2, b_2]$ where b_1, b_3 are in the one-dimensional invariant manifold of G and a_1, a_2, a_3, b_2, are in the trajectories of G passing through $L_G \cap \partial V$. We should mention, G

come in(or comes out) of V in U_1 and U_3 and come out (or come in) of V in U_2. In addition there trajectories of G passing through $\tilde{p}_G$ and μ_G meet ∂V inside U_2.

We classify the trajectories of G in the following way:

(1) the trajectories of G which enter or leave V in finite time, and stay away from L_G;

(2) the trajectories which has a unique external tangency point with ∂V;

(3) the trajectories which enter or leave V passing through points different from μ_G;

(4) the trajectories passing through U_1, U_2, U_3;

(5) This class is composed by P_G.

Now we conclude the proof as above (proof of sufficiency). ▮

4.4.2. Tangencial Singularities

Consider coordinates $\xi = (\xi_1, \xi_2, \xi_3)$, in a neighborhood of $p \in S$ such that $F = \frac{\partial}{\partial \xi_1}$. Let $\xi_3 = \eta(\xi_1, \xi_2)$ be a solution of $h(\xi_1, \xi_2, \xi_3) = 0$ with $\eta(0,0) = 0$, where $S = h^{-1}(0)$. We fix $N = \{\xi_1 = 0\}$ being a transverse section to F in p.

Define now the application, $\sigma_F : S, p \to N, p$ by:

$$\sigma_F(\xi_1, \xi_2, \eta(\xi_1, \xi_2)) = (0, \xi_2, \eta(\xi_1, \xi_2))$$

This mapping σ_F is of the same diferenciabillity class as F and it is called projection of S along the orbits of F on N (details in [9]).

In this section, we will use the theory of singularities of applications. The following results are fundamental:

THEOREM 32 (V. I. Arnol'd, [1]). *In the space of the smooth compact hipersurfaces in* $\mathbb{R}^3$, *there is an open and dense set formed by surfaces whose projections of any observation point and in any direction it is locally equivalent to the projection of one of the 10 surfaces, of the following list, in* $(0,0,0)$, *along the axis* x *:*

1-Regular: $z = x$;

2-fold: $z = x^2$;

3-Cusp: $z = x^3 + xy$;

4-Lips: $z = x^3 + xy^2$;

5-Bec to Bec: $z = x^3 - xy^2$;
6-Dove's Tail: $z = x^4 + xy$;
7-Goose: $z = x^3 + xy^3$;
8-Gull: $z = x^4 + x^2y + xy^2$;
9-Butterfly: $z = x^5 + x^3y \pm xy$.

These projections are pairwise inequivalent. The singularities of the projections of the list are not removable by a small perturbations of the surface.

Singularities of types 2 and 3 are used in the classification of codimension-zero singularities whereas types $4, 5$ and 6 are used in the classification of codimension-one singularities (see [5]). Rieger in [7], proved that the types $7, 8$ and 9 are of codimension-two in this space. We use these types in the classification of the codimension-two singularities(em $\mathfrak{X}^r$).

Next result is given in [7] (Lemmas 1.3.2 and 1.3.3.)

THEOREM 33 (J. H. Rieger, [7]). *The normal forms:*

Goose: $z = x^3 + xy^3 + \alpha xy + \beta x$;
Gull: $z = x^4 + x^2y + xy^2 + \alpha xy + \beta x$;
Butterfly: $z = x^5 + x^3y \pm xy + \alpha x^3 + \beta x^2$,

are the universal unfolding of $z = x^3 + xy^3$, $z = x^4 + x^2y + xy^2$, $z = x^5 + x^3y \pm xy$, *respectively.*

Next two results come directly from Theorem 32 and of Theorem 33.

PROPOSITION 34. *The point* $p \in S$ *is a singularity of* F *of type* $\Sigma_2(b)$ *if and only if* σ_F *is* C^r- *equivalent (in the usual sense of theory of singularities of applications) to one of the following applications:*

Goose: $\sigma_{21} = (x^3 + xy^3, y)$;
Gull: $\sigma_{22} = (x^4 + x^2y + xy^2, y)$;
Butterfly: $\sigma_{23} = (x^5 + x^3y \pm xy, y)$

PROPOSITION 35. *The normal forms of the unfolding of a singularity of the* F *in* $\Sigma_2(b)$, *are:*

Goose: $F(x, y, z) = (1, 0, 0)$ *and* $h(x, y, z) = z - x^3 + xy^3 + \alpha xy + \beta x$;
Gull: $F(x, y, z) = (1, 0, 0)$ *and* $h(x, y, z) = z - x^4 + x^2y + xy^2 + \alpha xy + \beta x$;
Butterfly: $F(x, y, z) = (1, 0, 0)$ *and* $h(x, y, z) = z - x^5 + x^3y \pm xy + \alpha x^3 + \beta x^2$.

The proof of the following result will be omitted.

LEMMA 36. *The normal forms of the unfolding of a singularity of the vector field F in $\Sigma_2(b)$, are given by:*

Goose: $F_{\alpha,\beta}(u,v,w) = (1,0,-3u^2 - v^3 + \alpha v + \beta)$;
Gull: $F_{\alpha,\beta}(u,v,w) = (1,0,4u^3 + 2uv + v^2 + \alpha v + \beta)$;
Butterfly: $F_{\alpha,\beta}(u,v,w) = (1,0,5u^4 + 3\varepsilon u^2 v + v + \alpha u^2 + \beta u)$.

Let us observe that the stability in $\Sigma_2(b)$, proceeds directly from Theorem 32.

5. CODIMENSION-TWO SINGULARITIES OF REVERSIBLE VECTOR FIELDS

Denote $\Omega_2 = \Omega \setminus \nu_1$.

We define the subset ν_2 of Ω_2 in the following way: "$X \in \nu_2$ if and only if the origin is a codimension-two singularity of $F = F(X) \in \Sigma_2$."

It proceeds from the definition of ν_2 and Theorem D that:

PROPOSITION 37.

(i) ν_2 *is open and dense in* Ω_2;

(ii) any element X *in* ν_2 *is structurally stable relative to* Ω_2;

(iii) ν_2 *is a codimension-two submanifold of* Ω;

(iv) any mapping $\eta : I\!R^2 \to \Omega$ *transverse to* ν_2 *at* X *in* ν_2 *is (pointwise)* C^0- *equivalent to*

$$X_{\alpha\beta} = (zf^1_{\alpha\beta}(x,y,z^2), zf^2_{\alpha\beta}(x,y,z^2), g_{\alpha\beta}(x,y,z^2))$$

where $F_{\alpha\beta}(u,v,w) = (f^1_{\alpha\beta}(u,v,w), f^2_{\alpha\beta}(u,v,w), g_{\alpha\beta}(u,v,w))$ *is one of the normal forms listed in the Theorem D.*

Remark 38. We say that two families of vector fields Z_λ and Z_μ in χ^r with $\lambda \in R^m, 0$ and $\mu \in R^k$ are C^0- equivalent if there is a reparametrization $\mu = \mu(\lambda)$ such that Z_λ is C^0- equivalent to $Z_{\mu(\lambda)}$ for every λ. Here, we are not requiring continuity with respect to λ.

The proof of Theorem A is a direct consequence of the last proposition.

REFERENCES

1. V. I. ARNOLD, *Indices of Singular Points of 1-forms on a Manifold with Boundary, Convolution of Invariants of Reflection Groups, and Singular Projection of Smooth Surfaces*, Russian Math. Surveys, **34**:2, (1979), 1–42.

2. V. I. Arnold, *Normal Forms for Functions near Degenerate Critical Points, the Weil Groups of A_k, D_k, E_k, and Lagrangian Singularities*, Functional Anal. Appl., **6**, (1972), 254–272.

3. V. I. Arnold, *On Local Problems of Analysis* , Vestn. Mos. UN-TA, Matematika, Mekhan.,**25**:2, (1970), 52–56.

4. J. S. W. Lamb, *Reversing Symmetries in Dynamical Systems*, Thesis, University of Amsterdam, (1994).

5. J. C. R. Medrado and M. A. Teixeira, *Symmetric Singularities of Reversible vector Fields in dimension Three*, in "Physica D" ,**112**, (1998), 122–131

6. W. Melo and J. Palis, *Geometric theory of dynamical systems, an introduction*, Springer Verlag, NY, (1982).

7. J. H. Rieger, *Versal Topological Stratification and the bifurcation geometry of Map Germs of the Plane*, Math. Proc. Camb. Phil. Soc., **107**, (1990), 127–147.

8. M. B. Sevryuk, *Reversible Systems*, Edited by A.Dold and B. Eckmann, Springer-Verlag, **1211**, (1986).

9. J. Sotomayor, *Generic One-Parameter of vector Fields on Two-Dimensional Manifolds*, I.H.E.S., **43**, (1974), 5–46.

10. J. Sotomayor and M. A. Teixeira, *Vector fields near the boundary of for 3-manifold*, in "Lecture Notes in Mathematics" **1331**, Springer-Verlag, (1988), 169–195.

11. M. A. Teixeira, *Singularities of reversible vector fields*, in "Physica D",**1482**, (1996), 1–18

12. M. A. Teixeira, *Local Reversibility and applications*, in "Real and Complex Singularities, Ed. Bruce J. W. and Tari, F.",**412**, Research Notes in Mathmatics, (2000), Chapman&Hall/CRC, 251–265.

13. S. M. Vishik, *Vector Fields near the Boundary of the Manifold*, Vestnik Moskovskogo Universiteta Matematika, **27**, (1972), 21–28.

14. J. Yang, *Polynomial normal forms for vector fields on $\mathbb{R}^3$*, Duke Math. J., **106**, (2001), 1–18.