Global Stability for Holomorphic Foliations on Kaehler Manifolds

J. V. Pereira

Instituto de Matemática Pura e Aplicada, IMPA, Estrada Dona Castorina, 110 Jardim Botânico, 22460-320 - Rio de Janeiro, RJ, Brasil.
E-mail: jvp@impa.br

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We prove the following theorem for Holomorphic Foliations on compact complex kaehler manifolds: if there exist a compact leaf with finite holonomy, then every leaf is compact with finite holonomy.

Key Words: Holomorphic Foliations, holonomy, compact leaf.

1. INTRODUCTION

The question of global stability is recurrent in the theory of foliations. The work of Ehresmann and Reeb establishes the so called global stability theorem, which says that if $\mathcal{F}$ is a transversely orientable codimension one foliation in a compact connected manifold $M$ that has a compact leaf $L$ with finite fundamental group, then every leaf of $L$ is compact with finite holonomy group [3]. Counterexamples for codimension greater than one are known since the birth of the theorem. Here we want to abolish the hypothesis on the codimension for a special kind of foliation, namely holomorphic foliations on complex Kaehler manifolds. In other words we are going to prove the following:

**Theorem 1.** Let $\mathcal{F}$ be a holomorphic foliation of codimension $q$ on a compact complex Kaehler manifold. If $\mathcal{F}$ has a compact leaf with finite holonomy group then every leaf of $\mathcal{F}$ is compact with finite holonomy group.

Another kind of stability problem was posed by Reeb and Haefliger. The question was the stability of compact foliations, that is, if a foliation has all leaves compact is the leaf space Hausdorff? Positive answers to
Corollary 2. Suppose $M$ is a complex Kaehler manifold, not necessarily compact. If $F$ is a compact foliation, i.e., every leaf is compact, then every leaf has finite holonomy group. Consequently, there is an upper bound on the volume of the leaves, and the leaf space is Hausdorff.

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2. THE LEAF VOLUME FUNCTION

Let $F$ be a holomorphic foliation of a complex Kaehler manifold $(M, \omega)$. As in [1] we define

$$\Omega = \{ p \in M | \text{the leaf } L_p \text{ through } p \text{ is compact with finite holonomy} \}$$

By the local stability theorem of Reeb, see for example [3], $\Omega$ is an open set of $M$. Set, for every $p \in \Omega$, $n(p) \in \mathbb{N}$ to be the cardinality of the holonomy group of $L_p$. If $d$ is the dimension of the leaves then we define volume function of $F$:

$$T : \Omega \rightarrow \mathbb{R}^+, \quad T(p) = n(p) \int_{L_p} \omega^d$$

Lemma 3. $T$ is a continuous locally constant function in $\Omega$.

Proof: The continuity is obvious. We have to prove that $T$ is locally constant. To do this we have just to observe that it is constant in the residual subset of $\Omega$, formed by the union of leaves without holonomy, see [6] p. 96. By the Reeb local stability theorem there is a saturated neighborhood for each leaf in this set where all leaves are homologous. Then using the closedness of $\omega^d$ and Stokes Theorem we prove the lemma.

Remark 4. In fact, the proof of this lemma is essentially contained in [8].
3. A LEMMA ABOUT DIFF($\mathbb{C}^N, 0$)

In 1905 Burnside [2] proved that if $G$ is a subgroup of $GL(n, F)$, where $F$ is a field of characteristic zero, with exponent $e$, then $G$ is finite with cardinality($G$) $\leq e^n$. Recall that a group has exponent $e$ if every element $g$ belonging to the group is such that $g^e = 1$. From the generalization of this result by Herzog-Praeger [7] we obtain:

**Lemma 5.** If $G$ is a subgroup of $Diff(\mathbb{C}^N, 0)$ with exponent $e$ then $G$ is finite with cardinality($G$) $\leq e^n$.

*Proof:* If for each element of $G$ we consider its derivative we obtain a subgroup of $GL(n, \mathbb{C})$ with exponent $e$. Thus we only have to prove that the normal subgroup $G_0$ of $G$, formed by its elements tangent to the identity is the trivial group.

Let $g \in G_0$, then $g^e = Id$. Defining $H(x) = \sum_{i=1}^{e} Dg(0)^{-i}g^i(x)$, we see that:

$$H \circ g(x) = Dg(0)Dg(0)^{-1}\sum_{i=1}^{e} Dg(0)^{-i}g^{i+1}(x) = Dg(0)H(x)$$

Hence $g$ is conjugated to its linear part, and therefore $g$ must be the identity.

4. PROOF OF THE RESULTS

Let $F$ be as in the theorem. Consider the connected component of $\Omega$ containing the leaf $L$ that is compact and with finite holonomy, and call it $\Omega_L$. By lemma 3, the volume function $T$ is constant in $\Omega_L$, therefore if $p \in \partial \Omega_L$ we have that the leaf through $p$ is approximated by leaves in $\Omega_L$. Observing that the leaves in $\Omega_L$ have the volume bounded by the value of $T|_{\Omega_L}$, one can conclude that $L_p$ has bounded volume. Since for compact manifolds a leaf has bounded volume if, and only if, it is a compact leaf, we conclude that $L_p$ is also compact.

Take any transversal $\Sigma$ of $L_p$. Hence $\Sigma \cap \Omega_L$ will be an open set such that every leaf of $\Omega_L$ cuts it in at most $m$ points, otherwise the volume function wouldn’t be bounded in $\Omega_L$. Thus for every holonomy germ $h$ of $L_p$, $(h^m)|_{\Sigma \cap \Omega_L} = Id$. Analytic continuation implies that $h^m = Id$. This is sufficient to prove that the holonomy group of $L_p$ has finite exponent. Finally, using Lemma 5, we see that $\partial \Omega_L = \emptyset$, and prove the theorem.

The Corollary follows observing that the set of leaves without holonomy is residual and that we don’t need the compactness of the manifold to assure that a limit leaf is compact. Then the holonomy group of each leaf is finite and by the results of Epstein [5] we get the consequence.
Remark 6. The same proof works in a more general context. We have just to suppose that our foliation is transversely quasi-analytic and that there is a closed form which is positive on the \((n - q)\)-planes of the distribution associated to the foliation.

REFERENCES
3. C. Camacho and A. Lins Neto, Geometric theory of foliations, Birkhauser, 1985