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Liouvillian Integration of the Lotka-Volterra System

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The Lotka-Volterra system of autonomous differential equations consists in three homogeneous polynomial equations of degree 2 in three variables. This system, or the corresponding vector field $LV(A, B, C)$, depends on three non-zero (complex) parameters and may be written as $LV(A, B, C) = V_x\partial_x + V_y\partial_y + V_z\partial_z$ where

$$V_x = x(Cy + z),\ V_y = y(Az + x),\ V_z = z(Bx + y).$$

Similar systems of equations have been studied by Volterra in his mathematical approach of the competition of species and this is the reason why this name has been given to such systems.

In fact, $LV(A, B, C)$ can be chosen as a normal form for most of factored quadratic systems; the study of its first integrals of degree 0 is thus of great mathematical interest.

Given a homogeneous vector field, there is a foliation whose leaves are homogeneous surfaces in the three-dimensional space (or curves in the corresponding projective plane), such that the trajectories of the vector field are completely contained in a leaf. A first integral of degree 0 is then a function on the set of all leaves of the previous foliation.

In the present paper, we give all values of the triple (A, B, C) of non-zero parameters for which $LV(A, B, C)$ has a homogeneous liouvillian first integral of degree 0. We also discuss the corresponding problem of liouvillian integration for quadratic factored vector fields that cannot be put in Lotka-Volterra normal form.

Our proof essentially relies on combinatorics and elementary algebraic geometry, especially in proving that some conditions are necessary.

Key Words: Integrability, Darboux polynomials, Lotka-Volterra system.

1. INTRODUCTION

The search of first integrals is a classical tool in the classification of all trajectories of a dynamical system. Let us simply recall the role of energy in Hamiltonian systems.

We are interested in systems consisting of three ordinary autonomous differential equations in three variables: $\dot{x} = V_x,\ \dot{y} = V_y,\ \dot{z} = V_z$.

A *first integral* of such a system of equations (or of the corresponding vector field V) is a non-constant function f that satisfies the partial derivative equation

$$V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} + V_z \frac{\partial f}{\partial z} = 0.$$

That means that f is constant along all trajectories of the one-parameter local semi-group generated by the vector field V.

The *local* existence of first integrals in a neighborhood of a regular point is a consequence of some classical theorems of differential calculus.

The interesting point for us is the search of *global* solutions; this problem has an algebraic nature when the coordinate functions V_x, V_y and V_z are polynomials in the space variables x, y and z.

Related to this problem is the study of singular integrable differential forms, which is of great interest [2, 3, 4, 5].

A key point is the specification of the class in which we look for first integrals. In the algebraic case, we follow the classical way of "Integration of differential equations in finite terms" [29, 30, 31]. In this frame, it seems reasonable to consider the class of all liouvillian elements over the differential field $\mathbb{C}(x, y, z)$ of all rational fractions in three variables on the constant field $\mathbb{C}$ of complex numbers. We follow the definition of liouvillian elements given by Singer [33].

Despite some specific methods [34], the search of liouvillian first integrals of polynomial vector fields relies mainly on the study of particular solutions whose use dates back to a memoir of Darboux [7] and we are now used to calling them *Darboux polynomials* of these vector fields [14]. With this vocabulary, polynomial first integrals are Darboux polynomials with the *eigenvalue* or *cofactor* 0.

So far as we know, Poincaré [26, 27, 28] was the first to notice the difficulty of a decision procedure for the existence of Darboux polynomials; he wrote: *Il en résulte qu'on ne peut trouver une limite supérieure du degré de l'intégrale générale algébrique, à moins qu'on ne trouve un moyen quelconque d'exprimer, dans les inégalités, que cette intégrale est irréductible.*

No procedure is known up to now; Jouanolou gives a theorem about this subject but his result is not effective [9].

The Lotka-Volterra vector field $LV(A,B,C)$ [8], that we study here, can be considered as a normal form of a factored vector field of degree 2 in three variables, at least in what concerns the search of first integrals of degree 0, i. e. first integrals of $LV(A,B,C)$ that are also first integrals of the vector field $E = x\partial_x + y\partial_y + z\partial_z$, that we call the *Euler field* for evident reasons.

In the case of factored vector fields, some *necessary* conditions can be given for the existence of a liouvillian first integral of degree 0 and a categorical result can be obtained: non-integrability is generic [13]. Many *sufficient* conditions can also be given to ensure this existence: it suffices to exhibit a fourth Darboux polynomial, i. e. a Darboux polynomial which is not divisible by x, y or z.

A systematic search of such Darboux polynomials of a given degree can be carried out with the help of a computer algebra system [1]. Looking carefully to the lists of parameters produced by such a search shows that the situation remains very intricate.

We turned therefore our interest to the determination of all values of the parameters for which $LV(A,B,C)$ has a first integral of degree 0, which is not only liouvillian, but is a rational fraction. This means that the space of leaves of the foliation generated by the field together with the Euler field, can be well described as an algebraic variety.

In other words, we can take into consideration the 1-form $\omega_0 = i_V i_E \Omega$, where Ω is the 3-form $dx\,dy\,dz$. This 1-form is projective, i. e. $i_E\omega_0 = 0$, and satisfies the Pfaff condition $\omega_0 \wedge d\omega_0 = 0$, which allows the search of an integrating factor ϕ, according to a theorem of Frobenius, and ϕ has to be a rational fraction such that $\phi\omega_0$ has a rational primitive.

This problem, that can be called *rational integration*, has been studied by Poincaré [26, 27, 28] and Painlevé [20, 21, 22, 23, 24, 25] for instance under the name *algebraic integration*. In the case of the Lotka-Volterra system, it turned out to be amenable and its solution was the subject of a previous work [17]. As in the paper [15] in which we dealt with the search of polynomial first integrals of $LV(A,B,C)$, a combinatorial approach of some systems of linear equations has been a key tool to deduce necessary conditions on the parameters.

The purpose of the present paper in to give necessary and sufficient conditions on the parameters $[A,B,C]$ in order to get a liouvillian first integral of $LV(A,B,C)$. Sufficient conditions have been the result of a careful search of Darboux polynomials up to the degree 12. During this search, we discovered an exceptional family of triples of parameters, $(A = 2, B = \frac{2l+1}{2l-1}, C = \frac{1}{2})$, where l is an integer, for which there exists a strict irreducible Darboux polynomial of degree $2l$ and no strict Darboux polynomials of lower degree. In a certain sense, this family of parameters correspond to a degeneracy of the problem: in this case, there are six Darboux points instead of seven.

Except this family of parameters, when there exists a fourth Darboux polynomial for $LV(A, B, C)$, then there exists a fourth Darboux polynomial with a degree at most 6.

According to a previous paper of ours [14], we can expect some very exceptional situations in which a liouvillian first integral of degree 0 can be built without a fourth Darboux polynomial. We shall exhibit these cases by rather elementary methods.

The main task will then be to find necessary conditions on the parameters to have a strict Darboux polynomial of $LV(A, B, C)$. Our analysis will begin with a combinatorial result leading to some arithmetic conditions on the parameters and producing a finite number of sub-cases for a further analysis.

One of these sub-cases consists in all triples of positive integers greater than 1. For this sub-case, graph-theoretical arguments can be used to conclude that the only possibility is $A = B = C = 2$. We defer this strange proof to another paper [16], if these ideas turn out to be interesting enough. Indeed, we have been unable to use a variation of these elementary but intricate arguments in the other sub-cases.

On the contrary, we had to study the factorization of a strict Darboux polynomial in branches (irreducible two-variable power series) at every Darboux point. According to these arguments, an *irreducible* strict Darboux polynomial has a bounded degree in most sub-cases (in fact all of them except the exceptional family corresponding to a degeneracy). With such a bound, we receive the results from our systematic search of Darboux polynomials up to degree 12.

The present paper is organized as follows:

1. This introduction,
2. General tools,
3. Various reductions,
4. The classification result,
5. The main specific analysis,
6. A more precise study of branches of Darboux curves at Darboux points,
7. Completing the proof that there is nothing else,
8. Dealing with non-normal situations,
9. Final remarks on Lotka-Volterra system,
10. Some more algebraic geometry.

We tried to make the paper as self-contained as possible. Nevertheless, the reader is invited to refer to [10, 12] for details about Levelt's method and an application to Jouanolou derivation in the many-variable (i. e. with more than three variables) case.

2. GENERAL TOOLS

2.1. Some vocabulary of differential algebra

Given two fields $k \subset K$, a k-derivation of the extension field K is a k-linear map δ from K in itself that satisfies Leibniz rule for the derivation of a product. Endowed with this mapping δ, K is called a *differential field*. The kernel of δ is then a subfield of K. It is known as the *field of constants* of the derivation and it contains the base field k.

When K is the field $k(x, y, z)$ of rational fractions in three unknowns, the usual partial derivatives ∂_x, ∂_y and ∂_z are derivations and they commute with one another. Their common field of constants is exactly k.

In what follows, k is some finite extension of $\mathbb{Q}$ by some parameters and can be thought of as a subfield of the field $\mathbb{C}$ of complex numbers. In particular, the characteristic of all fields will be 0.

A polynomial vector field $V = V_x\partial_x + V_y\partial_y + V_z\partial_z$, where V_x, V_y and V_z are homogeneous polynomials of the same degree in $k[x, y, z]$, defines a k-derivation δ_V of K. When L is an extension of $k(x, y, z)$ in which ∂_x, ∂_y and ∂_z have been extended as commuting k-derivations of L, δ_V is thus extended from K to L.

In particular, we will consider the special vector field $E = x\partial_x + y\partial_y + z\partial_z$ and call it the Euler field. Indeed, an element f of $k(x, y, z)$ is homogeneous of degree m if and only if $\delta_E(f) = mf$, according to Euler relation. In a differential extension field L, this identity may be taken as a *definition of homogeneity*.

We will also freely use the ideas of differential calculus such as exterior derivatives or n-forms in the frame of differential algebra.

A 1-form is an element $\omega = \omega_x\,dx + \omega_y\,dy + \omega_z\,dz$ in the three-dimensional vector space K^3 expressed in the "canonical" base $[dx, dy, dz]$. The exterior derivative of an element f of K is the 1-form $df = \partial_x(f)\,dx + \partial_y(f)\,dy + \partial_z(f)\,dz$.

Given a polynomial vector field V, a *homogeneous first integral of degree 0* is an element f of some differential extension field L of K that satisfies $\delta_V(f) = \delta_E(f) = 0$ (f is then a constant for these two derivations) without being a "constant"; we find it more convenient to keep the word *constant* for the elements c such that $dc = 0$, i. e. $\partial_x(c) = \partial_y(c) = \partial_z(c) = 0$.

2.2. Darboux polynomials

Consider a vector field $V = V_x\partial_x + V_y\partial_y + V_z\partial_z$, where V_x, V_y and V_z are homogeneous polynomials in the space variables x, y and z and have the same degree m .

A non-zero polynomial f is said to be a *Darboux polynomial* of V if there exists some polynomial Λ such that

$$V_x \frac{\partial f}{\partial x} + V_y \frac{\partial f}{\partial y} + V_z \frac{\partial f}{\partial z} = \Lambda f,$$

and Λ is then the corresponding *eigenvalue* or *cofactor*.

As V is homogeneous, the consideration of the homogeneous component of highest degree of the above identity shows that Λ has to be a homogeneous polynomial of degree $m-1$. Moreover, the homogeneous component f^+ of highest degree of a Darboux polynomial f is a Darboux polynomial. It is therefore sufficient to study homogeneous Darboux polynomials.

Suppose now that some homogeneous Darboux polynomial f, for a given homogeneous polynomial vector field V and an cofactor Λ, is the product $f = gh$ of two relatively prime homogeneous polynomials. As a polynomial ring is a unique factorization domain, Gauss lemma shows that the factors g and h have to be Darboux polynomials for V with some cofactors Λ_1 and Λ_2 such that $\Lambda = \Lambda_1 + \Lambda_2$.

Recall that we suppose the characteristic of the base field k to be 0. Then, if some positive power f^n of an irreducible homogeneous polynomial f is a Darboux polynomial for V, f itself is a Darboux polynomial for V.

Thus, the determination of all Darboux polynomials of a given polynomial vector field V amounts to finding all irreducible Darboux polynomials for V.

According to Euler identity, homogeneous polynomials are Darboux polynomials for the Euler field; with respect to E, the cofactor of a homogeneous polynomial is its degree.

2.3. Darboux points

As we have previously seen, a homogeneous first integral f of degree 0 of some homogeneous polynomial vector field V belongs to the common kernel of δ_V and δ_E:

$$\begin{aligned} &V_x \tfrac{\partial f}{\partial x} + V_y \tfrac{\partial f}{\partial y} + V_z \tfrac{\partial f}{\partial z} = 0, \\ &x \tfrac{\partial f}{\partial x} + y \tfrac{\partial f}{\partial y} + z \tfrac{\partial f}{\partial z} = 0. \end{aligned}$$

Its derivative df is then proportional to the 1-form

$$\omega_0 = (yV_z - zV_y)dx + (zV_x - xV_z)dy + (xV_y - yV_x)dz.$$

According to [14], finding a liouvillian first integral f of degree 0 amounts to finding a liouvillian integrating factor for the Pfaff form ω_0. If s is the common degree of the coordinates of V, this integrating factor is a homogeneous first integral of degree $-(s+2)$ of the vector field $\overline{V}$, where

$\overline{V}$ leads to the same ω_0 and has a zero divergence:

$$\overline{V} = V - \frac{\mathrm{div}(V)}{s+2}E.$$

In this framework, the lines in $\mathbb{C}^3$ (or points of the projective plane $\mathcal{P}_2(\mathbb{C})$) where ω_0 vanishes (i. e. where V and E are collinear) are of special interest. We call these singular points of ω_0 the *Darboux points* of V.

When the degree of V is 2 and ω_0 is irreducible (*its three coordinates are relatively prime*), there are generically seven Darboux points according to a result of Darboux; in the special case of a quadratic factored V, these points will be described more precisely.

2.4. Singular points of irreducible Darboux curves

Let us begin with a general result on homogeneous differential ideals.

Proposition 1. *Consider a homogeneous polynomial derivation $d = V_1\partial_1 + \cdots + V_n\partial_n$ of the n-variable polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$, where $V_1, \cdots, V_n$ are homogeneous polynomials in $x_1, \ldots, x_n$ of the same degree s. Let $\mathcal{I}$ be a homogeneous d-invariant ideal of $\mathbb{K}[x_1, \ldots, x_n]$. Denote by $S(\mathcal{I})$ the "singular" ideal of $\mathcal{I}$, i. e. the ideal generated by the partial derivatives of the elements of $\mathcal{I}$. Then, $S(\mathcal{I})$ is also homogeneous and d-invariant.*

Proof: The homogeneity of $S(\mathcal{I})$ is a standard result. To prove that $S(\mathcal{I})$ is d-invariant, it suffices to show that the images $d(g_i)$ of all partial derivatives $g_i = \partial_i(g)$ of all homogeneous g of $\mathcal{I}$ also belong to $S(\mathcal{I})$:

$$d(g_i) \ = \ \sum_j V_j\partial_j(g_i) \ = \ \sum_j V_j\partial_i(g_j).$$

As $\mathcal{I}$ is d-invariant, the sum $d(g) = \sum_j V_j g_j$ belongs to $\mathcal{I}$ and every partial derivative $\partial_i(d(g))$ then belongs to $S(\mathcal{I})$. Such a $\partial_i(d(g))$ may be written as:

$$\partial_i(d(g)) \ = \ \sum_j \partial_i(V_j)g_j + \sum_j V_j\partial_i(g_j),$$

whence

$$d(g_i) \ = \ \sum_j V_j\partial_i(g_j) \ = \ \partial_i(d(g)) - \sum_j \partial_i(V_j)g_j,$$

and $d(g_i)$ belongs to $S(\mathcal{I})$ as the difference of two elements of $S(\mathcal{I})$. ∎

Corollary 2. *Let f be an irreducible homogeneous Darboux polynomial of the homogeneous three-variable polynomial vector field $V = V_x\partial_x + V_y\partial_y +$*

$V_z\partial_z$. Then, the singular points of f in $\mathcal{P}_2(\mathbb{C})$ are some of the Darboux points of V.

Proof: The ideal generated by f is homogeneous and V-invariant. Its singular ideal, generated by the three partial derivatives f_x, f_y, f_z of f, is then homogeneous and V-invariant, according to Proposition 1. This singular ideal has a finite number of zeroes (the singular points of the curve) in the projective plane. The ideal corresponding to each such point is then V-invariant. To conclude, it suffices to notice that, if the homogeneous ideal of all polynomials vanishing at some point of the projective plane is V-invariant, then this point is a Darboux point of V. ∎

It is now natural to study Darboux polynomials around Darboux points. This is the purpose of Levelt's method. We will restrict ourselves to the special case of factored quadratic vector fields, which is the only one that we are interested in in the present paper; we thus defer the details to the next section. It is nevertheless useful to say some words about this method in the general case.

2.5. Levelt's method

These ideas are useful in the general n-variable homogeneous case. We are looking for a homogeneous irreducible non-trivial Darboux polynomial F of some homogeneous vector field V of degree s in n variables and we look for necessary conditions for its existence. The idea is to study the vector field at a Darboux point in order to find such a condition, which relates the degree m of F, its cofactor Λ and the components of V.

The cofactor of F as a Darboux polynomial of V is a homogeneous polynomial Λ of degree $s-1$ while its cofactor for the n-variable Euler field is its degree m:

$$d_V(F) = \sum_{i=1}^{n} V_i \frac{\partial F}{\partial x_i} = \Lambda F, \tag{1}$$

$$\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} = mF. \tag{2}$$

Let Z be a Darboux point of V; without lost of generality, we can suppose that the last coordinate z_n of $z = (z_1, \ldots, z_n)$ is equal to 1.

Then, adding the product of Equation (1) by x_n and the product of Equation (2) by $(-V_n)$, we get an equation in which the partial derivative of F with respect to the last variable x_n no longer appears:

$$\sum_{i=1}^{n-1}(x_nV_i - x_iV_n)\frac{\partial F}{\partial x_i} = (x_n\Lambda - mV_n)F. \tag{3}$$

According to Euler formula (2), Equations (1) and (3) are in fact equivalent for homogeneous polynomials F of degree m.

By the definition of a Darboux point, all differences $V_i(z_1,\dots,1) - z_iV_n(z_1,\dots,1)$ vanish so that $[\Lambda(z_1,\dots,1) - mV_n(z_1,\dots,1)]F(z_1,\dots,1) = 0$. Let us stress the fact that we cannot a priori exclude the possibility that $F(z_1,\dots,1) \neq 0$.

Choose now the local affine coordinates $y_1,\dots,y_{n-1}$ defined by $x_1 = z_1 + y_1,\dots,x_{n-1} = z_{n-1} + y_{n-1}$. This change of coordinates sends the Darboux point Z to the origin of our new coordinate system.

In what follows, we will adopt the following convention: if some homogeneous polynomial in n variables $z_1,\dots,z_n$ is denoted by a capital letter, we denote by the corresponding small letter the non-homogeneous polynomial in $n-1$ variables $y_1,\dots,y_{n-1}$, that we get from the homogeneous polynomial in n variables. For instance, we define f by

$$f(y_1,\dots,y_{n-1}) = F(z_1 + y_1,\dots,z_{n-1} + y_{n-1}, 1). \tag{4}$$

In this local system of coordinates, Equation (3) becomes

$$\sum_{i=1}^{n-1}(v_i - (z_i + y_i)v_n)\frac{\partial f}{\partial y_i} = (\lambda - mv_n)f. \tag{5}$$

The study of this equation will be called the *local analysis* of V at Z. Looking simultaneously at many or all such equations at various Darboux points and at their relationships will be called a *global analysis* of the vector field.

The polynomials involved in Equation (5) are in general non-homogeneous in $n-1$ variables and they can be decomposed into their homogeneous components:

$$\phi = \sum_{i=0}^{\deg(\phi)} \phi_{(i)},$$

where the polynomial $\phi_{(i)}$ is homogeneous of degree i; in particular, $\phi_{(0)}$ is the constant term of ϕ.

Let $\mu_Z(F)$ be the lowest integer such that $f_{(i)} \neq 0$, i. e. the multiplicity of F at Z.

When $\lambda \neq mv_n$, the minimal degree on the right-hand side of Equation (5) is $\mu_Z(F)$ while it seems to be $\mu_Z(F) - 1$ on the left-hand side. The contradiction is only apparent since all constant terms $(v_i - (z_i + y_i)v_n)_{(0)}$ are all 0. Indeed, Z is a Darboux point of V.

Comparing now the terms of minimal degree $\mu_Z(F)$ of both sides of Equation (5) yields

$$\sum_{i=1}^{n-1} (v_i - (z_i + y_i)v_n)_{(1)} \frac{\partial h}{\partial y_i} = (\lambda - mv_n)_{(0)} h, \tag{6}$$

where h is the non-trivial homogeneous component $f_{(\mu_Z(F))}$ of lowest degree of f.

In Equation (6), partial derivatives of h are multiplied by linear homogeneous polynomials and h by a constant.

Then, the homogeneous polynomial h is a non-trivial eigenvector of a *linear derivation* (or linear differential operator) d_L from $\mathbb{C}[t_1, \ldots, t_\nu]$ to itself defined by

$$d_L(h) = \sum_{i=1}^{\nu} l_i \frac{\partial h}{\partial t_i} = \chi h, \tag{7}$$

where the coefficients $l_i(t_1, \ldots, t_\nu) = \sum_{j=1}^{\nu} l_{ij} t_j$ are linear forms in the variables $t_1, \ldots, t_\nu$ and where $L = (l_{ij})_{1 \leq i,j \leq \nu}$ is the $\nu \times \nu$ corresponding matrix.

Of course, in our case, $t_i = y_i, 1 \leq i \leq n-1 = \nu$, χ is the constant term $(\lambda - mv_n)_{(0)}$ while the l_i are the linear components $(v_i - (z_i + y_i)v_n)_{(1)}$.

When the matrix L is diagonalizable, the following lemma is easy to be proved. We gave in [12] two different proofs of it in the general case.

Lemma 3. *Let h be a non-trivial homogeneous polynomial eigenvector of derivation d_L defined in Equation (7) where χ is the corresponding eigenvalue. Denote by $\rho_1, \ldots, \rho_\nu$. the ν eigenvalues of L.*

Then, there exist ν non-negative integers $i_1, \ldots, i_\nu$ such that

$$\begin{aligned} \sum_{j=1}^{\nu} \rho_j i_j &= \chi, \\ \sum_{j=1}^{\nu} i_j &= deg(h). \end{aligned} \tag{8}$$

The existence of these nonnegative integers is a key argument to find necessary conditions on the parameters of the vector field to have a Darboux polynomial with prescribed degree and cofactor.

This is the main step of a method that we are used to calling *Levelt's method*. As we know now, Levelt was the referee of Jouanolou's book [9] and he gave this argument as a shortcut for some proof in the last chapter.

3. VARIOUS REDUCTIONS

3.1. Quadratic factored vector fields

In the present subsection, we explain to what extend the Lotka-Volterra vector field $LV(A,B,C), ABC \neq 0$ may be considered as a normal form of a quadratic factored vector field.

A quadratic vector field $V = V_x\partial_x + V_y\partial_y + V_z\partial_z$, where V_x, V_y and V_z are homogeneous quadratic polynomials in x, y and z, is said to be *factored* if x divides V_x, y divides V_y and z divides V_z. So far as we are interested in integration, a linear change of coordinates preserves qualitative properties of the vector field. A factored quadratic vector field is then a vector field with three linearly independent first degree Darboux polynomials.

Taking these polynomials as coordinates leads to the factored form:

$$V_x = x\phi_x, V_y = y\phi_y, V_z = z\phi_z,$$

where ϕ_x, ϕ_y, ϕ_z are homogeneous first degree polynomials.

As we look for homogeneous first integrals of degree 0, V may be freely translated by some multiple of the Euler field E. By removing "diagonal elements", we get a first normalization of the factored vector fields that we study:

$$V_x = x(a_{12}\,y + a_{13}\,z),\ V_y = y(a_{21}\,x + a_{23}\,z),\ V_z = z(a_{31}\,x + a_{32}\,y).$$

The 1-form $\omega_0 = (yV_z - zV_y)dx + (zV_x - xV_z)dy + (xV_y - yV_x)dz$ is then written as

$$\begin{array}{rcl} \omega_0 & = & yz((a_{31} - a_{21})\,x + a_{32}\,y - a_{23}\,z)dx, \\ & + & zx(-a_{31}\,x + (a_{12} - a_{32})\,y + a_{13}\,z)dy, \\ & + & xy(a_{21}\,x - a_{12}\,y + (a_{23} - a_{13})\,z)dz. \end{array}$$

Solving the system $\{\omega_0 = 0\}$ of three homogeneous polynomial equations, we find seven Darboux points that can be expressed with homogeneous coordinates as:

$$\begin{array}{l} M_1 = [1,0,0], M_2 = [0,1,0], M_3 = [0,0,1], \\ M_4 = [0,a_{23},a_{32}], M_5 = [a_{13},0,a_{31}], M_6 = [a_{12},a_{21},0], \\ M_7 = [a_{13}a_{32}+a_{12}a_{23}-a_{23}a_{32}, \\ \qquad a_{21}a_{13}+a_{23}a_{31}-a_{31}a_{13}, \\ \qquad a_{32}a_{21}+a_{31}a_{12}-a_{12}a_{21}]. \end{array}$$

If ω_0 is not irreducible, we can simplify ω_0 by the greatest common divisor of all its coordinates. Then, by an exterior derivation, we obtain a 2-form, which is isomorphic to a vector field (there are 3 variables here); this vector field is no longer quadratic but linear (or even constant or zero) [11]. This is another (simpler) problem so we decide to restrict ourselves to an irreducible ω_0.

If $a_{i,j} = a_{j,i} = 0$ for some $i \neq j$, then ω_0 is reducible (x, y or z is a common factor of the coordinates of ω_0).

If $a_{i,j} = a_{j,k} = a_{i,k} = 0$ for some permutation of $\{1,2,3\}$, the other parameters being non-zero, then a suitable linear change of variables put the vector field in the following form:

$$V = xy\partial_y + (x+y)z\partial_z.$$

Except this situation, we need either $a_{12}a_{23}a_{31} \neq 0$ or $a_{21}a_{32}a_{13} \neq 0$ in order to have an irreducible ω_0. Then with a diagonal change of variables $x' = ax, y' = by, z' = cz$ (with a previous odd permutation of the variables in the first case), the vector field can be put in *Lotka-Volterra form*:

$$V = x(Cy+z)\partial_x + y(Az+x)\partial_y + z(Bx+y)\partial_z, \quad A, B, C \in k.$$

A quadratic factored vector field with an irreducible ω_0 is thus linearly equivalent to one of the following:

(i) A Lotka-Volterra field $LV(A,B,C)$ with $ABC \neq 0$,
(ii) A Lotka-Volterra field $LV(A,B,0)$ with $AB \neq 0$,
(iii) A Lotka-Volterra field $LV(A,0,0)$ with $A \neq 0$,
(iv) The Lotka-Volterra field $LV(0,0,0) = xz\partial_x + yx\partial_y + zx\partial_z$,
(v) The field $xy\partial_y + (x+y)z\partial_z$,

In the Lotka-Volterra cases, the Darboux points are

$$\begin{aligned}&M_1 = [1,0,0], M_2 = [0,1,0], M_3 = [0,0,1],\\ &M_4 = [0,A,1], M_5 = [1,0,B], M_6 = [C,1,0],\\ &M_7 = [1+CA-A, 1+AB-B, 1+BC-C].\end{aligned}$$

In the last case they are

$$\begin{aligned}&M_1 = [1,0,0], M_2 = [0,1,0], M_3 = [0,0,1],\\ &M_4 = [0,0,1] = M_3, M_5 = [0,0,1] = M_3, M_6 = [0,1,0] = M_2,\\ &M_7 = [0,0,1] = M_3.\end{aligned}$$

Looking at these points, we see that all cases but $LV(A,B,C), ABC \neq 0$ are degenerate: there are less than seven Darboux points.

Moreover, $LV(A,B,C), ABC \neq 0$ can degenerate with respect to Darboux points. In the special case where $1-A+AC=0$ (resp. $1-B+AB=0$, $1-C+BC=0$), M_7 is equal to M_4 (resp. M_5, M_6); we will nevertheless consider this situation as normal. As a special case of this special case, if two among the numbers $1-A+AC, 1-B+AB, 1-C+BC$ are zero, then the third is also 0 and "$M_7=[0,0,0]$" disappears as a point of the projective plane; ω_0 is then reducible.

3.2. Linear formal symmetries of the problem

Some linear changes of variables preserve the factored form of the given vector field in Lotka-Volterra normal form. Adding thereafter a multiple of the Euler field and conjugating by a diagonal linear change of variables put the image in Lotka-Volterra normal form for another triple (A', B', C') of parameters.

Thus, such linear changes of variables provide transformations of the set of parameters. We will refer to them as *natural transformations*.

Permutations of the coordinates lead to natural transformations: applying a circular permutation of x, y, z yields a circular permutation of the parameters, in which (A,B,C) becomes (B,C,A) or (C,A,B). On the other hand, exchanging two coordinates changes (A,B,C) for $(1/B, 1/A, 1/C)$, $(1/C, 1/B, 1/A)$ or $(1/A, 1/C, 1/B)$. Of course, if one of the parameters vanishes, the only natural transformations are the circular permutations.

When (A', B', C') is the image of (A,B,C) by a natural transformation, the vector field $LV(A,B,C)$ has a liouvillian (resp. rational) first integral of degree 0 if and only if $LV(A', B', C')$ has a liouvillian (resp. rational) first integral of degree 0.

So, it is "natural" to describe classification results up to natural transformations.

3.3. Strict Darboux polynomials at Darboux points

Proposition 4. *Let f be a strict (homogeneous) Darboux polynomial of degree m and cofactor Λ. Let (u,v) be a system of local affine coordinates around some Darboux point M of $LV(A,B,C)$, i. e. $u(M)=v(M)=0$. Denote now by $g(u,v)$ the non-zero non-homogeneous irreducible two-variable polynomial expressing f in the new local coordinates.*

Then g satisfies the following differential equation:

$$(U_{(1)}+U_{(2)})g_u + (V_{(1)}+V_{(2)})g_v = (W_{(0)}+W_{(1)})g, \tag{9}$$

in which $W_{(0)}$ is a constant, $U_{(1)}, V_{(1)}, W_{(1)}$ are homogeneous two-variable polynomials of degree 1 and $U_{(2)}, V_{(2)}$ are homogeneous two-variable polynomials of degree 2 (in u,v).

Proof: As the vector field is factored, the Darboux property of f and the fact that f is homogeneous can be written as

$$\begin{aligned} x\phi_x\tfrac{\partial f}{\partial x} + y\phi_y\tfrac{\partial f}{\partial y} + z\phi_z\tfrac{\partial f}{\partial z} = \Lambda f, \\ x\tfrac{\partial f}{\partial x} + y\tfrac{\partial f}{\partial y} + z\tfrac{\partial f}{\partial z} = mf, \end{aligned}$$

where ϕ_y, ϕ_y, ϕ_z are homogeneous first degree polynomials in x, y, z.

One of the coordinates of M is not zero. To fix matters, suppose that $M_z \neq 0$ and choose affine local coordinates u, v such that $M_u = M_v = 0$, for instance $u = x - M_x, v = y - M_y$.

In the difference between the first equation and the product of the second by ϕ_z, the partial derivative with respect to z disappears. Moreover, as M is a Darboux point, coefficients near f_u and f_v have no homogeneous component of degree 0. ∎

Consider now the linear part $U_{(1)}\partial_u + V_{(1)}\partial_v$ of the previous differential operator. The corresponding matrix can generally be put in diagonal form with different eigenvalues ρ and σ; in fact this is the case for all seven Darboux points of $LV(A, B, C)$.

In this general situation, a linear change of variables can be performed on u and v in such a way that Equation (9) takes the special form

$$(\rho u + U_{(2)})g_u + (\sigma v + V_{(2)})g_v = (\tau + W_{(1)})g, \tag{10}$$

where we denote by τ the constant $W_{(0)}$.

Proposition 5. *Let f be a strict irreducible Darboux polynomial of degree $m > 1$ and cofactor Λ of $LV(A, B, C)$. Let M be a Darboux point and let ρ, σ, τ be the values appearing in Equation (10) at M. Then the following equation has at least one solution*

$$\rho i + \sigma j = \tau, i \in \mathbb{N}, j \in \mathbb{N}, \tag{11}$$

Sketch of the proof: This is a special case of Levelt's method, that we described above. Consider indeed a non-zero solution g of Equation (10) in the polynomial ring $\mathbb{C}[u, v]$ or even in the power series ring $\mathbb{C}[[u, v]]$. This g can be written as the sum of its homogeneous components: $g = \sum_{k=\mu}^{\infty} g_k$, with $g_\mu \neq 0$. Restricted to the degree μ, Equation (10) becomes $\rho\partial g_\mu/\partial u + \sigma\partial g_\mu/\partial v = \tau g_\mu$. To satisfy this last equation g_μ has to be a monomial $u^i v^j$ with $\rho i + \sigma j = \tau$. ∎

Proposition 6. *Let f be a strict irreducible Darboux polynomial of degree $m > 1$ and cofactor Λ of $LV(A, B, C)$. Let M be a Darboux point and let ρ, σ, τ be the values appearing in Equation (10) at M. Suppose moreover that the local coordinate u is a Darboux polynomial, i. e. that $U_{(2)}$ is* *divisible by u in Equation (10). Then τ/ρ is a nonnegative integer and the coefficient $g_{\tau/\rho,0}$ is not 0.*

Sketch of the proof: The polynomial g can be developed as $g = \sum_{d=0} g_d$, where g_d is a homogeneous two-variable polynomial of degree d. Consider the lowest degree d_0 for which $g_d \neq 0$. Then there exists two nonnegative integers i_0 and j_0 such that $\rho i_0 + \sigma j_0 = \tau,\ i_0 + j_0 = d_0, g_{i_0,j_0} \neq 0$.

Unless $\rho = \sigma$, there is a unique solution to the previous two equations. In this case, consider the solution (i_0, j_0) with the smallest possible j_0.

If $j_0 \neq 0$ and $U_{(2)}$ is divisible by u , $g_{d,0}, d > d_0$ depends linearly on $g_{d-1,0}$ and an easy induction in Equation (10) shows that all $g_{d,0}, d > d_0$ vanish as the first ones $g_{d,0}, d \leq d_0$ do. But v is supposed not to be a factor of g, a contradiction. ∎

3.4. Darboux polynomials and supply functions

In [14], we proved that a homogeneous polynomial three-variable vector field W whose divergence is 0 has a liouvillian first integral of degree 0 if and only if there exists a 1-form $\omega = \omega_x dx + \omega_y dy + \omega_z dz$ such that:

(a) $\omega_x, \omega_y, \omega_z$ are homogeneous rational fractions in $\mathbb{C}(x, y, z)$ of degree -1,

(b) ω is a closed form: $\partial\omega_x/\partial y = \partial\omega_y/\partial x$, $\partial\omega_y/\partial z = \partial\omega_z/\partial y$, $\partial\omega_z/\partial x = \partial\omega_x/\partial z$,

(c) ω is orthogonal to W: $W_x\omega_x + W_y\omega_y + W_z\omega_z = 0$.

In this case, the least common multiple of the denominators of the components $\omega_x, \omega_y, \omega_z$ is a Darboux polynomial of W.

Finding a strict Darboux polynomial f (with cofactor Λ) is the simplest way to build such an ω for $LV(A, B, C)$. Indeed, in the non-zero 1-form $\omega = \alpha dx/x + \beta dy/y + \gamma dz/z + df/f$, the coefficients α, β, γ can be chosen in such a way that the scalar product $W_x\omega_x + W_y\omega_y + W_z\omega_z$ is zero. Of course, W denote the sum $V + \phi E$ with a zero divergence.

The following result gives another possibility to build ω.

Proposition 7. *Let $\omega = \omega_x dx + \omega_y dy + \omega_z dz$ be a homogeneous closed 1-form of degree -1 such that the least common multiple of the denominators of $\omega_x, \omega_y, \omega_z$ is $x^\alpha y^\beta z^\gamma$, $\alpha, \beta, \gamma \in \mathbb{N}$. Then ω can be decomposed as $\omega = adx/x + bdy/y + cdz/z + d(\Phi)$, where a, b, c are constants and where Φ is a homogeneous rational fraction of degree 0, whose denominator is a product of powers of x, y, z.*

Sketch of the proof: First, ω_x can be decomposed as

$$\sum_{i=-\alpha}^{i=\beta+\gamma} \omega_{x,i}(y, z) x^i,$$

where $\omega_{x,i}(y,z)$ is a homogeneous rational fraction of total degree $-1-i$ in y and z. The same can be done for the other two coefficients ω_y and ω_z.

The coefficient $\omega_{x,-1}(y,z)$ before x^{-1} is a homogeneous fraction of degree 0. Using the "crossed derivatives" relation $\partial\omega_x/\partial y = \partial\omega_y/\partial x$, the partial derivative with respect to y of this coefficient is easily shown to be 0, and the same is true for its derivative with respect to z. Thus, $\omega_{x,-1}(y,z)$ is a constant R_x that we call the *residue of* ω *with respect to* x.

In the same way, we define R_y and R_z. The difference $\tilde{\omega} = \omega - R_x dx/x - R_y dy/y - R_z dz/z$ is a closed form of degree -1 without residues; $\tilde{\omega}$ is then easily shown to be the exterior derivative of a homogeneous function of degree 0. ∎

Thus, even if there is no strict Darboux polynomial of $LV(A,B,C)$, there is nevertheless a last chance to build ω and thus to perform the liouvillian integration: find a *supply function*, i. e. a homogeneous rational function of degree 0, with a denominator $x^\alpha y^\beta z^\gamma$, such that $V_x\partial\Phi/\partial x + V_y\partial\Phi/\partial y + V_z\partial\Phi/\partial z = \Lambda$, where Λ is homogeneous polynomial of degree 1. Later on, we will see that this new possibility is in fact very exceptional.

3.5. Darboux polynomials of $LV(A,B,C)$, $ABC \neq 0$

By the very definition of a factored vector field, like $LV(A,B,C)$, the space variables x, y and z are Darboux polynomials.

Every homogeneous non-zero polynomial f can be written as the product

$$f = x^\alpha y^\beta z^\gamma g$$

where g is not divisible by x, y or z.

If f is a Darboux polynomial, so is g. Such polynomials as g will play an important role in our combinatorial analysis. We will call them *strict* Darboux polynomials.

Let g be a strict Darboux polynomial of degree m for $LV(A,B,C)$:

$$x(Cy+z)\partial_x g + y(Az+x)\partial_y g + z(Bx+y)\partial_z g = (\lambda x + \mu y + \nu z)g$$

As g is supposed not to be divisible by x, y or z, we can consider the three homogeneous non-zero two-variable polynomials of degree m obtained by setting $x=0$, $y=0$ and $z=0$ in g and call them P, Q and R respectively.

From the previous relation involving g, we deduce some partial differential equations concerning these two-variable polynomials:

$$\begin{array}{rcl} (\mu y + \nu z)P &=& yz(A\partial_y P + \partial_z P), \\ (\nu z + \lambda x)Q &=& zx(B\partial_z Q + \partial_x Q), \\ (\lambda x + \mu y)R &=& xy(C\partial_x R + \partial_y R). \end{array} \tag{12}$$

As $ABC \neq 0$, it is not very difficult to prove that there exists six non-negative integers β_1, γ_1, α_2, γ_2, α_3 and β_3 such that

$$\begin{array}{rcl} P & \equiv & y^{\beta_1} z^{\gamma_1} (y - Az)^{m-\beta_1-\gamma_1}, \\ Q & \equiv & z^{\gamma_2} x^{\alpha_2} (z - Bx)^{m-\gamma_2-\alpha_2}, \\ R & \equiv & x^{\alpha_3} y^{\beta_3} (x - Cy)^{m-\alpha_3-\beta_3}. \end{array} \tag{13}$$

The equivalence sign $\equiv$ means that the two sides are non-zero multiples of one another by some constants.

Moreover, these numbers satisfy the following equations and inequalities:

$$\begin{array}{lclcl} \lambda & = & \beta_3 & = & \gamma_2 B, \\ \mu & = & \gamma_1 & = & \alpha_3 C, \\ \nu & = & \alpha_2 & = & \beta_1 A, \\ \beta_1 & + & \gamma_1 & \leq & m, \\ \alpha_2 & + & \gamma_2 & \leq & m, \\ \alpha_3 & + & \beta_3 & \leq & m. \end{array} \tag{14}$$

In particular, the cofactor corresponding to a strict Darboux polynomial of $LV(A, B, C)$ is a linear form $\Lambda = \lambda x + \mu y + \nu z$ where λ, μ and ν are nonnegative integers.

Moreover, if A (resp. B, C) is not a positive rational number, then $\alpha_2 = \beta_1 = 0$ (resp. $\beta_3 = \gamma_2 = 0$, $\gamma_1 = \alpha_3 = 0$), which means that f does not vanish at M_3 (resp. M_1, M_2).

With the help of these notations, we can give more precise results at the seven Darboux points of $LV(A, B, C)$ than Equation (11) and thus improve Proposition 5.

PROPOSITION 8. *Let f be a strict irreducible Darboux polynomial of degree m and cofactor Λ of $LV(A, B, C)$. In M_1, M_2, M_3, Equation (11) takes the following form:*

$$\begin{array}{lclclcl} i_1 & + & B\ j_1 & = & \lambda & = & \beta_3, \\ i_2 & + & C\ j_2 & = & \mu & = & \gamma_1, \\ i_3 & + & A\ j_3 & = & \nu & = & \alpha_2. \end{array} \tag{15}$$

In this case, both local coordinates are Darboux polynomials, and the previous equations have solutions with $i = 0$ and solutions with $j = 0$.

In M_4, M_5, M_6, Equation (11) takes the following form:

$$\begin{array}{lclcl} i_4 & + & (1 - C - 1/A)\ j_4 & = & m - \gamma_1 - \beta_1, \\ i_5 & + & (1 - A - 1/B)\ j_5 & = & m - \alpha_2 - \gamma_2, \\ i_6 & + & (1 - B - 1/C)\ j_6 & = & m - \beta_3 - \alpha_3. \end{array} \tag{16}$$

In this case, the first local coordinates is a Darboux polynomial, and the previous equations have solutions with $j = 0$.

In M_7, the local coordinates can be chosen in such a way that Equation (11) becomes

$$\rho_7 i_7 + \sigma_7 j_7 = \tau_7 = m - \beta_3 \frac{AC - A + 1}{ABC + 1} - \gamma_1 \frac{AB - B + 1}{ABC + 1} - \alpha_2 \frac{BC - C + 1}{ABC + 1}, \tag{17}$$

where ρ_7 and σ_7 are defined by their sum and their product:

$$\rho_7 + \sigma_7 = 1, \quad \rho_7 \sigma_7 = \frac{(AC - A + 1)(AB - B + 1)(BC - C + 1)}{(ABC + 1)^2}.$$

In this case, ρ_7 and σ_7 are the two roots of a second order equation whose discriminant is $\Delta_7 = 1 - 4P_7$. In the sequel, we will simply refer to this number as Δ.

Sketch of the proof: These are elementary computations. ∎

3.6. Darboux polynomials in degenerate cases

In this subsection, we derive necessary conditions to have a strict Darboux polynomial in the four degenerate cases of factored quadratic vector fields described at the end of Subsection 3.1.

Proposition 9. *The quadratic factored vector field $xy\partial_y + (x+y)z\partial_z$ has no strict Darboux polynomial.*

Proof: With the notations of Subsection 3.5 for the two-variable homogeneous "trace" polynomials of a supposed strict Darboux polynomial f, Equation (12) becomes here

$$\begin{array}{rcl} (\mu y + \nu z)P & = & yz\partial_z P, \\ (\nu z + \lambda x)Q & = & zx\partial_z Q, \\ (\lambda x + \mu y)R & = & xy\partial_y R. \end{array}$$

The result (13) can then be written as:

$$P \equiv y^{\beta_1} z^{\gamma_1}, \quad Q \equiv z^{\gamma_2} x^{\alpha_2}, \quad R \equiv x^{\alpha_3} y^{\beta_3}.$$

Since y^{β_1+1} divides $(\mu y + \nu z)P$, $\nu = 0$; since x^{α_3+1} divides $(\lambda x + \mu y)R$, $\mu = 0$. Then $\partial_z P = 0$ and $P \equiv y^m, Q \equiv z^\lambda x^{m-\lambda}, R \equiv x^{m-\lambda} y^\lambda$. P and R have then to be coherent with respect to the coefficient of y^m: $\lambda = m$. On the other hand, P and Q have to be coherent with respect to the coefficient of z^m: $\lambda < m$, a contradiction. ∎

Proposition 10. *$LV(0,0,0)$ has no strict Darboux polynomial.*

Proof: With the notations of Subsection 3.5 for the two-variable homogeneous "trace" polynomials of a supposed strict Darboux polynomial f, Equation (12) becomes here

$$\begin{array}{rcl}(\mu y+\nu z)P & = & yz\partial_z P,\\ (\nu z+\lambda x)Q & = & zx\partial_x Q,\\ (\lambda x+\mu y)R & = & xy\partial_y R.\end{array}$$

The result (13) can then be written as:

$$P\equiv y^{\beta_1}z^{\gamma_1},\quad Q\equiv z^{\gamma_2}x^{\alpha_2},\quad R\equiv x^{\alpha_3}y^{\beta_3}.$$

Since y^{β_1+1} divides $(\mu y+\nu z)P$, $\nu=0$; in a similar way $\lambda=\mu=0$. Then $\partial_z P=0, P\equiv y^m$ and also $\partial_x Q=0, Q\equiv z^m, \partial_y R=0, R\equiv x^m$. This is not coherent with respect to the coefficients of x^m, y^m and z^m. ∎

PROPOSITION 11. *$LV(A,0,0), A\neq 0$, has no strict Darboux polynomial.*

Proof: With the notations of Subsection 3.5 for the two-variable homogeneous "trace" polynomials of a supposed strict Darboux polynomial f, Equation (12) becomes here

$$\begin{array}{rcl}(\mu y+\nu z)P & = & yz(A\partial_y P+\partial_z P),\\ (\nu z+\lambda x)Q & = & zx\partial_x Q,\\ (\lambda x+\mu y)R & = & xy\partial_y R.\end{array}$$

The result (13) can then be written as:

$$\begin{array}{rcl}P & \equiv & y^{\beta_1}z^{\gamma_1}(y-Az)^{m-\beta_1-\gamma_1},\\ Q & \equiv & z^{\gamma_2}x^{\alpha_2},\\ R & \equiv & x^{\alpha_3}y^{\beta_3}.\end{array}$$

Since z^{γ_2+1} divides $(\nu z+\lambda x)Q$, $\lambda=0$; similarly $\mu=0$. Then $\partial_y R=0, R\equiv x^m$.

To ensure coherence of the coefficients of x^m, $Q\equiv x^m, \nu=m$, which leads to $mP=y(A\partial_y P+\partial_z P)$ and the coefficient of y^m in P is not 0, a contradiction with $R\equiv x^m$. ∎

We defer the study of $LV(A,B,0), A,B\neq 0$, to the end of the paper. Indeed, this degenerate case is in fact the less degenerate and we need the whole strength of the tools of Section 7, adapted to the special situation $C=0$, to deal with this family of vector fields.

4. THE CLASSIFICATION RESULT FOR $LV(A, B, C)$

THEOREM 12. *If the triple $[A, B, C]$ of (generally non-zero) parameters falls, up to natural symmetries, in one of the cases of the following list, then there exists a strict Darboux polynomial f for $LV(A, B, C)$. When they belong to $\mathbb{N}^\star$, we give the values of p, q, r. Moreover, if $LV(A, B, C)$ has a strict Darboux polynomial of degree at most 12, then the triple $[A, B, C]$ can be found in the following list.*

In the first two cases, A, B, C are related by a unique condition and f has degree 1.

1. $ABC + 1 = 0$,
$f = x - Cy + ACz$ *is a first integral.*

2. $B = 1$ *(resp.* $A = 1$, $C = 1$*),*
$f = y - Az$. *Remark that* $C = 0$ *is possible.*

In the next two cases, A, B, C are related by two conditions and f has degree 2.

3. $p = -A - 1/B = 1, q = -B - 1/C = 1$, $ABC = 1$ *and* $r = -C - 1/A = 1$, *with the change of variables* $x = -\frac{X}{A}, y = \frac{Y}{A+1}, z = \frac{Z}{A(A+1)}$,
$f = X^2 - 2\,XY - 2\,XZ + Y^2 - 2\,YZ + Z^2$ *is a first integral.*

4. $A = 2, q = -B - 1/C = 1$ *(or natural transform),*
$f = (x - Cy)^2 - 2C^2yz$. *Remark that* $[B = 0, C = -1]$ *is possible.*

There is a finite number of isolated triples of complex numbers leading to a polynomial first integral of $LV(A, B, C)$ with a degree 3, 4 or 6. We will denote as usual by i the square root of -1 and by $j = \frac{-1+i\sqrt{3}}{2}$ a primitive third root of 1. Let us remark that these cases have been found in [8] using the so-called Painlevé analysis.

5. $[A, B, C] = [(j-1)/3, j-1, j]$, $[p, q, r] = [1, 2, 2]$,
$f = 9\,x^3 - 27\,jx^2y + 9\,jx^2z + 18\,x^2z - 27\,xy^2 - 27\,jxy^2 + 9\,jxyz + 18\,xyz + 9\,xz^2 + 9\,jxz^2 - 9\,y^3 - 9\,y^2z + 9\,jy^2z + 9\,jyz^2 + z^3 + 2\,jz^3$.

6. $[A, B, C] = [(i-2)/5, (i-3)/2, i-1]$, $[p, q, r] = [1, 2, 3]$,
$f = 625\,x^4 - 2500\,ix^3y + 2500\,x^3y + 500\,ix^3z + 1500\,x^3z - 7500\,ix^2y^2 + 5000\,x^2yz + 900\,ix^2z^2 + 1200\,x^2z^2 - 5000\,ixy^3 - 5000\,xy^3 - 3000ixy^2z + 1000xy^2z + 1800xyz^2 + 2600\,ixyz^2 + 520\,ixz^3 + 360\,xz^3 - 2500\,y^4 - 4000\,y^3z + 2000\,iy^3z + 2400\,iy^2z^2 - 1800\,y^2z^2 - 160\,yz^3 + 880\,iyz^3 + 96\,iz^4 + 28\,z^4$.

7. $[A, B, C] = [(j-2)/7, (j-4)/3, j-1]$, $[p, q, r] = [1, 2, 4]$,
$f = 117649x^6 + 705894x^5y - 705894jx^5y + 504210x^5z + 100842jx^5z - 5294205jx^4y^2 + 3327786x^4yz - 605052jx^4yz + 864360x^4z^2 + 324135jx^4z^2 - 14117880jx^3y^3 - 7058940x^3y^3 + 5142942x^3y^2z - 7865676jx^3y^2z + 5099724 \times x^3yz^2 + 1382976jx^3yz^2 + 761460x^3z^3 + 411600jx^3z^3 - 15882615x^2y^4 -$

$15882615jx^2y^4-6353046x^2y^3z-19059138jx^2y^3z+8297856x^2y^2z^2-64827\times jx^2y^2z^2+3408048x^2yz^3+2185596jx^2yz^3+363825x^2z^4+257985jx^2z^4-12706092xy^5-6353046jxy^5-16336404xy^4z-10890936jxy^4z+259308xy^3z^2+1685502jxy^3z^2+3537702jxy^2z^3+3685878xy^2z^3+1055754jxyz^4+1000188xyz^4+89208xz^5+79758jxz^5-3176523y^6-5445468y^5z+2722734jy^5z-2917215y^4z^2+4862025jy^4z^2-185220y^3z^3+3333960jy^3z^3+1091475jy^2z^4+317520y^2z^4+168966jyz^5+98658yz^5+8721z^6+9720jz^6$.

There are two isolated triples of rational numbers leading to a rational first integral of degree 0 for $LV(A,B,C)$ and the smallest possible degree is then 3 or 4.

8. $A=-7/3, B=3, C=-4/7,\ p=2, r=1,$
$f = -259308\,x^3z - 185220\,x^2yz + 259308\,x^2z^2 + 567\,xy^3 - 13230\,xy^2z - 71001\,xyz^2 - 86436\,xz^3 + 324\,y^4 + 3024\,y^3z + 10584\,y^2z^2 + 16464\,yz^3 + 9604\,z^4.$

9. $A=-3/2, B=2, C=-4/3,\ p=1, r=2,$
$f = 108\,x^2z + 6\,xy^2 + 180\,xyz - 108\,xz^2 + 8\,y^3 + 36\,y^2z + 54\,yz^2 + 27\,z^3.$

There are some isolated triples of rational numbers for which the degree of f is 3 or 4 (with $L_3=0$ in the notations of Lemma 14).

10. $[A,B,C]=[2,4,-1/6],$
$f = 216\,x^3 + 108\,x^2y - 54\,x^2z + 18\,xy^2 - 36\,xyz + y^3 - 4\,y^2z + 4\,yz^2.$

11. $[A,B,C]=[2,-8/7,1/3],$
$f = 216\,x^3 + 189\,x^2z + 882\,xyz - 343\,y^2z + 686\,yz^2.$

12. $[A,B,C]=[6,1/2,-2/3],$
$f = 9\,x^2y + 12\,xy^2 - 144\,xyz + 432\,xz^2 + 4\,y^3 - 72\,y^2z + 432\,yz^2 - 864\,z^3.$

13. $[A,B,C]=[-6,1/2,1/2],$
$f = 3\,x^2y + 24\,xyz + 144\,xz^2 - 8\,y^2z - 96\,yz^2 - 288\,z^3.$

14. $[A,B,C]=[3,1/5,-5/6],$
$f = 1296\,x^4 + 4320\,x^3y - 6480\,x^3z + 5400\,x^2y^2 - 18900\,x^2yz + 3000\,xy^3 - 18000\,xy^2z + 27000\,xyz^2 + 625\,y^4 - 5625\,y^3z + 16875\,y^2z^2 - 16875\,yz^3.$

15. $[A,B,C]=[2,-13/7,1/3],$
$f = 648\,x^4 - 216\,x^3y - 252\,x^2yz + 1176\,xy^2z - 343\,y^3z + 686\,y^2z^2.$

There are some isolated triples of rational numbers for which the degree of f is 3, 4 or 6 (with $L_1\neq 0, L_2\neq 0, L_3\neq 0$ in the notations of Lemma 14).

16. $[A,B,C]=[2,2,2],$
$f = x^2z + xy^2 - 3\,xyz + yz^2.$

17. $[A,B,C]=[2,3,-3/2],\ r=2,$
$f = 8\,x^2z + 16\,xyz - y^3 + 4\,y^2z - 4\,yz^2.$

18. $[A,B,C]=[2,2,-5/2],\ r=3,$
$f = 8\,x^2z^2 - 4\,xy^2z + 24\,xyz^2 + y^4 - 6\,y^3z + 12\,y^2z^2 - 8\,yz^3.$

19. $[A,B,C] = [-4/3, 3, -5/4]$, $p=1, r=2$,
$f = 576\,x^2z + 864\,xyz - 384\,xz^2 + 27\,y^3 + 108\,y^2z + 144\,yz^2 + 64\,z^3$.

20. $[A,B,C] = [-9/4, 4, -5/9]$, $p=2, r=1$,
$f = 419904\,x^3z + 279936\,x^2yz - 314928\,x^2z^2 + 15552\,xy^2z + 69984\,xyz^2 + 78732\,xz^3 - 256\,y^4 - 2304\,y^3z - 7776\,y^2z^2 - 11664\,yz^3 - 6561\,z^4$.

21. $[A,B,C] = [-3/2, 2, -7/3]$, $p=1, r=3$,
$f = 324\,x^2z^2 + 72\,xy^2z + 864\,xyz^2 - 324\,xz^3 + 16\,y^4 + 96\,y^3z + 216\,y^2z^2 + 216\,yz^3 + 81\,z^4$.

22. $[A,B,C] = [-5/2, 2, -8/5]$, $p=2, r=2$,
$f = 125000\,x^3z^3 - 5000\,x^2y^2z^2 + 225000\,x^2yz^3 - 187500\,x^2z^4 - 1600\,xy^4z - 6000\,xy^3z^2 + 15000\,xy^2z^3 + 87500\,xyz^4 + 93750\,xz^5 - 64\,y^6 - 960\,y^5z - 6000\,y^4z^2 - 20000\,y^3z^3 - 37500\,y^2z^4 - 37500\,yz^5 - 15625\,z^6$.

23. $[A,B,C] = [-10/3, 3, -7/10]$, $p=3, r=1$,
$f = 81000000x^4z^2 + 64800000x^3yz^2 - 108000000x^3z^3 - 243000x^2y^3z + 3240000x^2y^2z^2 + 29700000x^2yz^3 + 54000000x^2z^4 - 97200xy^4z - 1296000\,xy^3z^2 - 6480000xy^2z^3 - 14400000xyz^4 - 12000000xz^5 + 729y^6 + 14580y^5z + 121500y^4z^2 + 540000y^3z^3 + 1350000y^2z^4 + 1800000yz^5 + 1000000z^6$.

There is a "sporadic family" where the degree of f is unbounded. This case is in fact degenerate with respect to Darboux points.

24. $[A,B,C] = [A_l = -\frac{2l+1}{2l-1}, 1/2, 2], l \in \mathbb{N}^\star$.
It is not easy to write f in "closed form".

Proof: We performed a systematic search of irreducible strict Darboux polynomials of $LV(A,B,C)$ up to the degree 12 and then classified the results; all of them (except the last one) can be checked by (machine) computations.

In the last case, the existence of f can be proven by studying the linear overdetermined system of Subsection 5.1 in order to prove that its kernel is not trivial in this case. We did it in Subsection 5.2 as an example of these elementary linear algebra techniques. ∎

Here are the only known possibilities. Of course, it remains to be *proven* that there are no other cases. The next three sections are devoted to this proof.

Remark 13. As a description of the Darboux polynomial in the last case 24 of the list, we simply write that f is not easy to describe in closed form. As the reader can also see, the results in the other cases are not easy too and their expression is sometimes long. In fact, there seems to be a deep structure of these polynomials, which is hidden by their writing in the given system of coordinates. Each of them could be written as a short sum of generalized monomials, i. e. products of homogeneous first degree polynomials. A systematic study of this phenomenon remains to be done.

5. THE MAIN SPECIFIC ANALYSIS

We will now restrict ourselves to $LV(A,B,C)$, with $ABC \neq 0$; many results will also hold for triples $[A,B,C]$ where one (and only one) of the parameters is 0. We look for all values of the triple $[A,B,C]$ for which a strict Darboux polynomial enables to perform the liouvillian integration.

In the last section, we will study the exceptional situations in which this liouvillian integration is achieved with a supply function.

5.1. Darboux polynomials and linear algebra

Consider some strict Darboux polynomial f for $LV(A,B,C), ABC \neq 0$. The degree of f is m and its cofactor is $\Lambda = \lambda x + \mu y + \nu z$. The relation $V_x f_x + V_y f_y + V_z f_z - \Lambda f = 0$ means that some homogeneous polynomial of degree $m+1$ is 0. This identity then appears as a homogeneous linear system of equations; there is one equation for every monomial in the polynomial of degree $m+1$ and the unknowns are the coefficients of the three-variable homogeneous polynomial f of degree m.

The unknowns are then parameterized by the triples (i,j,k) of nonnegative integers summing to m and there are $n = \binom{m+2}{2}$ of them. The equations are parameterized by the triples (i,j,k) of nonnegative integers summing to $m+1$ and there are $N = \binom{m+3}{2}$ of them.

The corresponding matrix M is very sparse; there are at most three non-zero coefficients per row or per column and they are given by affine forms of the indices:

$$\begin{array}{rcl} M_{(i+1,j,k),(i,j,k)} & = & j + Bk - \lambda, \\ M_{(i,j+1,k),(i,j,k)} & = & Ci + k - \mu, \\ M_{(i,j,k+1),(i,j,k)} & = & i + Aj - \nu. \end{array} \tag{18}$$

As this over-determined system is supposed to have a nontrivial solution, determinants of maximal order are equal to 0 but most of them are not easy to compute.

On the contrary, the extra assumption that f is a strict Darboux polynomial can be used to deduce that some smaller determinants are 0. Moreover, these determinants will be amenable to combinatorial computations.

Consider now the linear square subsystem whose unknowns have their index i equal to 0 or 1 and whose equations have their index i equal also to 0 or 1 (except $(0,0,m+1)$ and $(0,m+1,0)$). As f is a (non-zero) strict Darboux polynomial, this square subsystem is not Cramer and its determinant, that we call D_1, is equal to 0.

It turns out that combinatorial computations yield to a factorization of determinant D_1. Then saying that $D_1 = 0$ becomes a useful necessary condition on parameters (A,B,C) of the vector field and parameters λ, μ, ν, m of the polynomial.

The same is true for analogous determinants D_2 and D_3 defined in a natural way by a "circular permutation". We give this important factorization in the following lemma.

LEMMA 14. *If $ABC \neq 0$, then, up to factors that cannot vanish, determinants D_1, D_2 and D_3 can be written as the following products, in which m stands for the degree of the supposed strict Darboux polynomial and where $\alpha_2 = \nu, \alpha_3, \beta_1, \beta_3 = \lambda, \gamma_1 = \mu, \gamma_2$ are the previously introduced nonnegative integers:*

$$\begin{aligned}
D_1 &\equiv [1/A - \beta_1]^{\overline{\beta_1}}\,[C - \gamma_1]^{\overline{\gamma_1}}\,[1/A + C + 1]^{\overline{m-\beta_1-\gamma_1-1}}\,L_1,\\
D_2 &\equiv [1/B - \gamma_2]^{\overline{\gamma_2}}\,[A - \alpha_2]^{\overline{\alpha_2}}\,[1/B + A + 1]^{\overline{m-\gamma_2-\alpha_2-1}}\,L_2,\\
D_3 &\equiv [1/C - \alpha_3]^{\overline{\alpha_3}}\,[B - \beta_3]^{\overline{\beta_3}}\,[1/C + B + 1]^{\overline{m-\alpha_3-\beta_3-1}}\,L_3,
\end{aligned}$$

where

$$\begin{aligned}
L_1 &= \{m(1+ABC) - \lambda(1+AC) - \mu(1-B) + \nu C(1-B)\},\\
L_2 &= \{m(1+ABC) - \mu(1+BA) - \nu(1-C) + \lambda A(1-C)\},\\
L_3 &= \{m(1+ABC) - \nu(1+CB) - \lambda(1-A) + \mu B(1-A)\}.
\end{aligned}$$

The notation $[t]^{\overline{k}}$ for the rising factorial power will be explained in the proof. If the exponent is negative in it, the corresponding factor is not significant and may be canceled.

Proof: With a suitable choice in the order of equations and unknowns, the square matrix $\mathcal{M}$ of our linear sub-system, whose order is $2m+1$, has the following form

$$\mathcal{M} = \begin{pmatrix} \mathcal{O} & \mathcal{A} \\ \mathcal{C} & \mathcal{B} \end{pmatrix},$$

where $\mathcal{O}$ is the zero square matrix of order m, $\mathcal{A}$ is a rectangular matrix with m rows and $m+1$ columns, $\mathcal{C}$ is a rectangular matrix with $m+1$ rows and m columns and $\mathcal{B}$ is a diagonal square matrix of order $m+1$.

These last three matrices look like what follows and they have a lot of zeroes.

$$\mathcal{A} = \begin{pmatrix}
m-\mu & A-\nu & 0 & 0 & \cdots & \cdots & \cdots \\
0 & m-1-\mu & 2A-\nu & 0 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & 0 & 2-\mu & (m-1)A-\nu & 0 \\
\cdots & \cdots & \cdots & 0 & 0 & 1-\mu & mA-\nu
\end{pmatrix}.$$

$$\mathcal{C} = \begin{pmatrix} 1-\nu & 0 & \cdots & \cdots & \cdots \\ \mathcal{C}_{2,1} & 1+A-\nu & \cdots & \cdots & \cdots \\ 0 & \mathcal{C}_{3,2} & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & 1+(m-2)A-\nu & 0 \\ \cdots & \cdots & \cdots & \mathcal{C}_{m-1,m-2} & 1+(m-1)A-\nu \\ \cdots & \cdots & \cdots & 0 & \mathcal{C}_{m,m-1} \end{pmatrix},$$

where $\mathcal{C}_{i+1,i} = C + m - 1 - i - \mu, 0 \leq i \leq m-1$.

$$\mathcal{B} = \begin{pmatrix} Bm-\lambda & 0 & \cdots & \cdots & 0 \\ 0 & 1+B(m-1)-\lambda & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & (m-1)+B-\lambda & 0 \\ 0 & \cdots & \cdots & 0 & m-\lambda \end{pmatrix}.$$

The development of the determinant D_1 of $\mathcal{M}$ has only $m+1$ non-zero terms, each of them being of course a product of $2m+1$ factors. As all corresponding permutations have the same signature, this leads to the following expression

$$D_1 = \sum_{i=0}^{m}(i+(m-i)B-\lambda)\prod_{j=0}^{i-1}(m-j-\mu)(1+jA-\nu)\prod_{j=i+1}^{m}(jA-\nu)(C+m-j-\mu).$$

The reason is the following: in every non-zero term of D_1, some of the diagonal elements of $\mathcal{B}$ appear as factors. Trying to avoid diagonal elements of $\mathcal{B}$ yields null terms of D_m. In a similar way, if we take two diagonal elements of $\mathcal{B}$, the corresponding cofactor is a $2m-1$ determinant; a permutation puts it in a triangular form with a zero element on the diagonal.

The determinant D_1 is thus the sum of products of all diagonal elements of $\mathcal{B}$ by their cofactors; these cofactors are triangular determinants of order $2m$ and thus easy to compute.

Using the notation $[t]^{\overline{k}}$ for the *rising factorial power* $\prod_{i=0}^{k-1}(t+i)$ and expressing λ, μ and ν in terms of other parameters lead to

$$D_1 = \sum_{i=0}^{m}(i+(m-i)B-\beta_3)\ \prod_{j=0}^{i-1}(m-j-\gamma_1)(1+jA-\beta_1 A)\times \prod_{j=i+1}^{m}(jA-\beta_1 A)(C+m-j-\alpha_3 C).$$

$$D_1 = A^m\sum_{i=0}^{m}(i+(m-i)B-\beta_3)\ \prod_{j=0}^{i-1}(m-j-\gamma_1)(1/A+j-\beta_1)\times \prod_{j=i+1}^{m}(j-\beta_1)(C+m-j-\alpha_3 C).$$

$$D_1 = A^m \textstyle\sum_{i=0}^{m}(i+(m-i)B-\beta_3) \quad \begin{array}{l}[m-i+1-\gamma_1]^{\overline{i}}[1/A-\beta_1]^{\overline{i}}\times \\ [i+1-\beta_1]^{\overline{m-i}}[C-\gamma_1]^{\overline{m-i}}.\end{array}$$

The terms of the sum between β_1 and $m-\gamma_1$ are the only non-zero ones since $[i+1-\beta_1]^{\overline{m-i}}=0$ if $i<\beta_1$ and $[m-i+1-\gamma_1]^{\overline{i}}=0$ if $i>m-\gamma_1$, whence the expression

$$D_1 = A^m \textstyle\sum_{i=\beta_1}^{m-\gamma_1}(i+(m-i)B-\beta_3) \quad \begin{array}{l}[m-i+1-\gamma_1]^{\overline{i}}[1/A-\beta_1]^{\overline{i}}\times \\ [i+1-\beta_1]^{\overline{m-i}}[C-\gamma_1]^{\overline{m-i}}.\end{array}$$

When $m=\beta_1+\gamma_1$, there is only one term in this sum, and some non-zero factors may be ignored to get

$$\begin{array}{l} D_1 \equiv A^m(\beta_1+\gamma_1 B-\beta_3)[1]^{\overline{\beta_1}}[1/A-\beta_1]^{\overline{\beta_1}}[1]^{\overline{\gamma_1}}[C-\gamma_1]^{\overline{\gamma_1}}, \\ D_1 \equiv (\beta_1+\gamma_1 B-\beta_3)[1/A-\beta_1]^{\overline{\beta_1}}[C-\gamma_1]^{\overline{\gamma_1}}. \end{array}$$

When $n=m-\beta_1-\gamma_1-1\geq 0$, there are many terms in the sum, and the first factor, $(i+(m-i)B-\beta_3)$ can be written as a linear combination $\phi(i-\beta_1)+\psi(m-\gamma_1-i)$, where ϕ and ψ are constants with respect to the index i.

These constants are determined by the linear system

$$\begin{array}{rcrcl} \phi & - & \psi & = & 1-B \\ -\beta_1\phi & + & (m-\gamma_1)\psi & = & mB-\lambda \end{array}$$

which can be solved in

$$\begin{array}{rcl} (m-\beta_1-\gamma_1)\phi & = & mB-\beta_3+(m-\gamma_1)(1-B) \\ (m-\beta_1-\gamma_1)\psi & = & mB-\beta_3+\beta_1(1-B) \end{array}$$

Determinant D_1 is then equal to $A^m\frac{(m-\beta_1)!(m-\gamma_1)!}{n!}(\phi D_{1,1}+\psi D_{1,2})$ where

$$D_{1,1} = [1/A-\beta_1]^{\overline{\beta_1+1}}[C-\gamma_1]^{\overline{\gamma_1}} \sum_{i=\beta_1+1}^{m-\gamma_1} \begin{array}{l}\dfrac{n!}{(i-\beta_1-1)!(m-\gamma_1-i)!}\times \\ [1/A+1]^{\overline{i-\beta_1-1}}[C]^{\overline{m-\gamma_1-i}},\end{array}$$

$$D_{1,2} = [1/A-\beta_1]^{\overline{\beta_1}}[C-\gamma_1]^{\overline{\gamma_1+1}} \sum_{i=\beta_1}^{m-\gamma_1-1} \begin{array}{l}\dfrac{n!}{(i-\beta_1)!(m-\gamma_1-i-1)!}\times \\ [1/A]^{\overline{i-\beta_1}}[C+1]^{\overline{m-\gamma_1-i+1}}.\end{array}$$

Discarding some factors, that cannot vanish, we get

$$D_1 \equiv [\frac{1}{A} - \beta_1]^{\overline{\beta_1}}[C - \gamma_1]^{\overline{\gamma_1}}\{T_0 + T_1\}$$

where

$$T_0 = [\frac{1}{A} + C]^{\overline{m-\beta_1-\gamma_1}}(mB - \beta_3 + \beta_1(1 - B))$$

and

$$T_1 = (m - \beta_1 - \gamma_1)(\frac{1-B}{A})[\frac{1}{A} + C + 1]^{\overline{m-\beta_1-\gamma_1-1}}$$

Finally, D_1 may be written as announced, which includes the special case $m = \beta_1 + \gamma_1$, the rising factorial power $[1/A + C + 1]^{\overline{-1}}$ corresponding in this case to some non-zero factor.

The analogous expressions of D_2 and D_3 are obtained in the same way. ∎

The previous lemma can be adapted to the case where one of the parameters, say C, is 0.

Lemma 15. *If $C = 0, AB \neq 0$, then $\mu = \gamma_1 = \alpha_3 = 0$, $\lambda = \beta_3 = \gamma_2 B = m$, $D_1 = 0$ as $L_1 = 0$, and, up to factors that cannot vanish, determinants D_2 and D_3 factor as follows:*

$$\begin{aligned} D_2 &\equiv [1/B - \gamma_2]^{\overline{\gamma_2}}\,[A - \alpha_2]^{\overline{\alpha_2}}\,[1/B + A + 1]^{\overline{m-\gamma_2-\alpha_2-1}}\,L_2 \\ D_3 &\equiv [B - m]^{\overline{m}}\,L_3, \end{aligned}$$

where $L_2 = m - \nu + mA$ and $L_3 = -\nu + mA$.

Proof: Elementary computations give the first equalities and the factorization of D_2 and D_3 follows the proof of Lemma 14. ∎

5.2. A sporadic family

In this subsection, we are interested in the following family of Lotka-Volterra systems:

$$SLV_l = LV(2, B_l = -\frac{2l+1}{2l-1}, 1/2), \quad l \in \mathbb{N}^\star.$$

This family provides a negative answer [18] to the conjecture that there exists a uniform bound M_2 such that, if some homogeneous three-variable vector field of degree 2 has a particular solution (an irreducible Darboux polynomial in our words) of degree at least M_2, then the vector field is rationally integrable (has a homogeneous rational first integral of degree 0). Indeed:

(a) SLV_l has no homogeneous rational first integral of degree 0.
(b) SLV_l has an irreducible Darboux polynomial f_l of degree $m = 2l$:

$$x(y/2+z)\frac{\partial f_l}{\partial_x}+y(2z+x)\frac{\partial f_l}{\partial_y}+z(B_l x+y)\frac{\partial f_l}{\partial_z}=((l-1)y+2lz)f_l. \tag{19}$$

In the present paper, we are concerned in the second point: proving that SLV_l has an irreducible Darboux polynomial f_l of degree $m = 2l$. This can be done by elementary techniques of linear algebra. This proof is necessary for our complete classification and we are not able to give a simple closed form for f_l.

Proposition 16. *SLV_l has an irreducible Darboux polynomial f_l of degree $m = 2l$ and cofactor $(l-1)y + 2lz$.*

Proof: First, we prove that there exists a non-zero homogeneous polynomial f of degree $2l$ that satisfies Equation (19) and afterwards, we check that f is irreducible.

To be precise, Equation (19) can be written as:

$$\begin{aligned} x(y/2+z)\partial f/\partial_x + y(2z+dx)\partial f/\partial_y + z(y - \tfrac{2l+1}{2l-1}x)\partial f/\partial_z \\ -((l-1)y+2lz)\,f \;=\; 0. \end{aligned} \tag{20}$$

In the present special situation, the linear system described in Subsection 5.1 can be partially put in "triangular form", which leads to a partial solution of it: all $f_{i,j,k}$, where $k > j$ and all $f_{i,j,k}$, where $j > k+2$ are 0. Moreover, $M_{(i,j+1,k),(i,j,k)} = Ci + k - \mu = 0$ when $j = k+2$ and $M_{(i,j,k+1),(i,j,k)} = i + Aj - \nu = 0$ when $j = k$.

We are thus left with a linear sub-system: the remaining unknowns correspond to the triples (i,j,k), $i+j+k = 2l$ where $k \le j \le k+2$ and the remaining equations to the triples (i,j,k), $i+j+k = 2l+1$ where $k \le j \le k+2$.

There are as many equations as unknowns in this subsystem: $3l+1$. To prove that this square linear sub-system has a one-dimensional kernel, do as follows:

(a) choose an arbitrary non-zero value for $f_{2l,0,0}$,
(b) determine the other coefficients in a unique way by a suitable induction, using all equations but the last one indexed by $(0, l, l+1)$,
(c) the given value of B_l is exactly the good one to solve this last equation.

In other words, the corresponding determinant has many factors and it has in fact only two non-zero terms; it is easy to compute and the value of B_l is exactly the condition for this determinant to vanish.

It remains to be proven that f is irreducible. First remark that f is a *strict Darboux polynomial* of a Lotka-Volterra vector field, which means that f is divisible neither by x, nor by y nor by z.

Now, if f were the product $f_1 f_2$ of two non-constant factors, f_1 and f_2 would also be homogeneous strict Darboux polynomials of the same vector field. According to relations (14), the degree of f_1 would be $2l_1$ and its cofactor $(l_1 - \epsilon_1)y + 2l_1 z$ whereas the degree of f_2 would be $2l_2$ and its cofactor $(l_2 - \epsilon_2)y + 2l_2 z$; l_1 and l_2 are positive integers. Moreover, one of the ϵ_i would take the value 0 and the other the value 1. Solving the linear system corresponding to f_1 or f_2 would produce (arguments are similar to those used before) another value of B, a contradiction. ∎

6. BRANCHES AT DARBOUX POINTS

The number of irreducible branches of an irreducible strict Darboux polynomial of degree m at a Darboux point can be computed when the Darboux operator in the local power series ring has special properties; moreover, this situation occurs sufficiently often in the Lotka-Volterra system to be used in our further analysis.

In the following three propositions, a Darboux element of $\mathbb{C}[[u,v]]$ with a given cofactor is completely known from its trace on the "exposed line" $\{(i,j) \in \mathbb{N}^2, \rho i + \sigma j = \tau\}$.

Proposition 17. *Suppose that the equation $\rho i + \sigma j = 0$ has a unique solution in $\mathbb{N}^2$, i. e. that it is only possible if $i = j = 0$.*

Let R, S be two two-variable power series of valuation 2 and let T be a two-variable power series of valuation 1 (without constant term). Then the differential equation

$$(\rho u + R) f_u + (\sigma v + S) f_v = T f$$

has a unique invertible solution in $\mathbb{C}[[u,v]]$ with a constant term 1.

Proof: Write $f = \sum_{i,j} f_{i,j} u^i v^j$, with $f_{0,0} = 1$ and consider the differential equation as an infinite linear system where the coefficients $f_{i,j}$ are the unknowns. This system is triangular: the $f_{i,j}$, where $(i,j) \neq (0,0)$, can be computed from coefficients with lower indices. ∎

Proposition 18. *Suppose that the equation $\rho i + \sigma j = 0$ has a unique solution in $\mathbb{N}^2$, i. e. that it is only possible if $i = j = 0$. Let R, S be two two-variable power series of valuation 2 and let T be a two-variable power series of valuation 1 (without constant term). Suppose that the equation*

$$(\rho u + R) f_u + (\sigma v + S) f_v = (\tau + T) f \tag{21}$$

has some non-zero solution in $\mathbb{C}[[u,v]]$. Then the equation $\rho i + \sigma j = \tau$ has a finite number of solutions in $\mathbb{N}^2$. Their set will be referred to as the exposed line. *Moreover, two solutions of Equation (21) are equal as soon as they agree on the exposed line.*

Proof: Write again $f = \sum_{i,j} f_{i,j} u^i v^j$ and consider the differential equation as an infinite linear system where the coefficients $f_{i,j}$ are the unknowns.

If the equation $\rho i + \sigma j = \tau$ had no solution, this system would be triangular Cramer, a contradiction with $f \neq 0$.

If it had infinitely many solutions, then ρ and σ would have a negative rational ratio, a contradiction with the first hypothesis.

So, the "exposed line" exists, either ρ and σ are rationally independent or they have a positive rational ratio. In the case of a positive rational ratio, we can fix matters in Equation (21) by a multiplication to have $\rho \in \mathbb{N}^\star, \sigma \in \mathbb{N}^\star, \tau \in \mathbb{N}$ and ρ and σ relatively prime.

In the irrational case, the exposed line has only one element (i_0, j_0).

The last assertion is easily proven by induction on the total degree $i+j$: indeed, the infinite linear system is triangular and every coefficient $f_{i,j}$ for which $\rho i + \sigma j \neq \tau$ can be computed in a unique way from coefficients with indices of lower total degree. Thus, in the case of a positive rational ratio,

(i) the coefficients $f_{i,j}$ on the exposed line are free,
(ii) other coefficients ($\rho i + \sigma j \neq \tau$) are determined from the free ones (some of them are 0).

while in the case of an irrational ratio ρ/σ,

(i) all coefficients with a total degree $i + j \leq i_0 + j_0$ except f_{i_0,j_0} are 0,
(ii) f_{i_0,j_0} is the only free unknown,
(iii) other coefficients are determined from f_{i_0,j_0}.

∎

The following easy corollary will be useful to describe the factorization of the solutions of Equation (21) in irreducible elements in $\mathbb{C}[[u,v]$.

Corollary 19. *Suppose that the equation $\rho i + \sigma j = 0$ has a unique solution in $\mathbb{N}^2$, i. e. that it is only possible if $i = j = 0$. Let R, S be two two-variable power series of valuation 2.*

In $\mathbb{C}[[u,v]]$, the differential equation

$$(\rho u + R) f_u + (\sigma v + S) f_v = \rho f$$

has a unique solution such that $f_{1,0} = 1$, $U = u + \sum_{i+j>1} U_{i;j} u^i v^j$, while the differential equation

$$(\rho u + R) f_u + (\sigma v + S) f_v = \sigma f$$

has a unique solution such that $f_{0,1} = 1$, $V = v + \sum_{i+j>1} V_{i,j} u^i v^j$.

The previous results can be used to compute the number of branches of an irreducible Darboux curve passing through some Darboux point of a quadratic homogeneous three-variable vector field.

PROPOSITION 20. *Under the hypotheses of Proposition 18, denote by* $i_{\min}$ *(resp.* $j_{\min}$*) the minimal value of the first (resp. second) coordinate of a pair in the exposed line for which* $f_{i,j} \neq 0$. *Then* f *can be written in* $\mathbb{C}[[u, v]]$ *as the product*

$$f \;=\; K U^{i_{\min}} V^{j_{\min}} \prod_k F_k,$$

where K *is an invertible power series, where* U *and* V *are the special solutions defined in Corollary 19 and where there are as many elements in the product as the number* $\sharp(\mathcal{E})$ *of solutions of* $\{\rho i + \sigma j = \tau, i \geq i_{\min}, j \geq j_{\min}\}$ *minus one.*

When the ratio ρ/σ *is not rational, there are no* F_k. *When this ratio is a positive rational number, we choose* ρ *and* σ *relatively prime positive integers and the* F_k *are some irreducible solutions of*

$$(\rho u + R) f_u + (\sigma v + S) f_v = \rho\sigma f. \tag{22}$$

Sketch of the proof: First consider the case of an irrational ratio. There exits a $g \;=\; K U^{i_0} V^{j_0}$ such that

(i) $(\rho u + R) g_u + (\sigma v + S) g_v = (\tau + T) g$,

(ii) $g_{i_0,j_0} = f_{i_0,j_0}$.

According to Proposition 18, $f = g$.

Now consider the case of a rational ratio.

We call the polynomial $\gamma(f) \;=\; \sum_{\rho i+\sigma j=\tau} f_{i,j} u^i v^i$, i. e. the restriction of f to the exposed line, the *germ* of f. This germ is a homogeneous polynomial of degree $\sharp(\mathcal{E}) - 1$ in u^σ and v^ρ that can be factored as $\gamma(f) = \prod(\lambda_k u^\sigma + \mu_k v^\rho)$, where the λ_k and μ_k are constants. Every such factor gives an irreducible $F_k \;=\; \lambda_k U^\sigma + \mu_k V^\rho$ and the conclusion follows from Proposition 18. ∎

PROPOSITION 21. *Let* M *be a Darboux point of some homogeneous three-variable vector field. Let* f *be a homogeneous irreducible Darboux polynomial of degree* m *and cofactor* Λ *of this vector field. The local Equation (10) can be written with the notations of the present subsection as Equation (21). Suppose that the equation* $\rho i + \sigma j = 0$ *has a unique solution in*

$\mathbb{N}^2$ *so that the exposed line $\mathcal{E}$ is not empty; in this case, there is no multiple factor in the decomposition of g in $\mathbb{C}[[u,v]]$ given by Proposition 20; hence, the minimum index $i_{\min}$ (resp. $j_{\min}$) is equal either to 0 or to 1.*

Proof: In g had a multiple factor ϕ, then g, g_u and g_v would have this ϕ as a common non-invertible factor in $\mathbb{C}[[u,v]]$. The ideal generated by them would then be included in the principal ideal generated by ϕ; the quotient ring could not be a finite-dimensional vector space over $\mathbb{C}$, a contradiction with the fact that f is an irreducible polynomial. ∎

Remark 22. If u (resp. v) is a Darboux polynomial, then u and U (resp. v and V) are associates in the unique factorization domain $\mathbb{C}[[u,v]]$ and so $i_{\min} = 0$ (resp. $j_{\min}) = 0$.

The next three corollaries are applications of the previous propositions to the points $M_1, M_2, M_3, M_4, M_5, M_6$ of $LV(A,B,C)$ when they are nodes (eigenvalues are rationally dependent with a positive ratio) and also to M_7 when ρ_7 and σ_7 are rationally independent (Δ is not the square of a rational number).

Corollary 23. *Parameters being as usual (Equations and Inequalities (14)), f is an irreducible Darboux polynomial of degree m of $LV(A,B,C)$. If $A \in \mathbb{N}^{\star\star}$ (resp. $B \in \mathbb{N}^{\star\star}$, $C \in \mathbb{N}^{\star\star}$), then the projective curve corresponding to f has β_1 irreducible branches at the point M_3 (resp. γ_2 branches at M_1, α_3 branches at M_2).*

Sketch of the proof: Equation (15) in Proposition 8 becomes $i_3 + Aj_3 = \nu = \alpha_2$. Hypotheses of Proposition 17 are fulfilled and $i_{\min} = j_{\min} = 0$ according to Proposition 6, whence the result. ∎

Corollary 24. *Parameters being as usual (Equations and Inequalities (14)), f is an irreducible Darboux polynomial of degree m of $LV(A,B,C)$. If $p = -A + 1/B \in \mathbb{N}^\star$ (resp. $q \in \mathbb{N}^\star$, $r \in \mathbb{N}^\star$), then the projective curve corresponding to f has $i_5 + j_5$ irreducible branches at the point M_5, where the non-negative integers i_5 and j_5 are the remainder and the quotient of the Euclidean division of $m - \alpha_2 - \gamma_2$ by $1 + p$ (resp. $i_6 + j_6$ branches at M_6, $i_4 + j_4$ branches at M_4 with corresponding definitions of the integers). Moreover $i_5 = 0$ or $i_5 = 1$ (resp. $i_6^2 = i_6$, $i_4^2 = i_4$).*

Sketch of the proof: Equation (16) in Proposition 8 becomes $i_5 + (p+1)j_5 = m - \alpha_2 - \gamma_2$. Hypotheses of Proposition 17 are fulfilled and $j_{\min} = 0$ according to Proposition 6 and $i_{\min} = 0, 1$ according to Proposition 21, whence the result. ∎

Corollary 25. *Parameters being as usual (Equations and Inequalities (14)), f is an irreducible Darboux polynomial of degree m of $LV(A,B,C)$. If Δ is not the square of a rational number, i. e. if the eigenvalues ρ_7 and*

σ_7 are rationally independent, then either f does not pass through M_7 or M_7 is an ordinary double point of f. Thus,

$$m - \frac{(AC - A + 1)\lambda + (AB - B + 1)\mu + (BC - C + 1)\nu}{ABC + 1} = \epsilon, \quad \epsilon = 0, 1.$$

Sketch of the proof: In this case $i_7 = j_7 = \epsilon$ according to Proposition 21 and $\tau_7 = \epsilon$ in Equation (17), whence the result from Proposition 8. ∎

The next proposition deals with Darboux polynomials at some saddle Darboux points.

PROPOSITION 26. *Let R, S be two two-variable power series in $\mathbb{C}[[u, v]$ of valuation 2 and let T be a two-variable power series of valuation 1 (without constant term). Let p and q be relatively prime nonnegative integers ($q > 0$, p may be 0). If the differential equation*

$$(u + R)g_u + (-\frac{p}{q}\,v + S)g_v = (I + T)g, I \in \mathbb{N}^{\star\star}$$

has a solution $g = u^I + \sum_{d>I} g_d$, where each g_d is a homogeneous two-variable polynomial of degree d, then there exists a $\phi \in \mathbb{C}[[u, v]]$ such that

(i) $\phi = u + \sum_{d>1} \phi_d$,
(ii) $\phi^I = f$,
(iii) $(u + R)\phi_u + (-\frac{p}{q}\,v + S)\phi_v = (1 + T/I)\phi$.

Sketch of the proof: By induction on $n \in \mathbb{N}^\star$, there exists a unique homogeneous two-variable polynomial ϕ_n of degree n such that the partial sum $\Psi_n = u + \sum_{d=2}^{d=n} \phi_d$ satisfies the approximate identities ("val" stands for the valuation of a power series):

$$\begin{aligned} &\mathrm{val}\left((\Psi_n)^I - f\right) \geq n + I, \\ &\mathrm{val}\left((u + R)\,\partial_u(\Psi_n) + (-\frac{p}{q}\,v + S)\,\partial_v(\Psi_n) - (1 + \frac{T}{I})(\Psi_n)\right) \geq n + 1. \end{aligned}$$

The assertion is true for $n = 1$ as $\phi_1 = u$ is the good choice.

Now suppose that the assertion has been proven for all $n' < n$, where $n > 1$ and prove it for n. Call the step n ordinary if $n - 1$ is not a multiple of $p + q$; otherwise the step n is critical.

The second property holds as soon as ϕ_n satisfies an equation

$$u\,\partial_u\,(\phi_n) - \frac{p}{q}\,v\,\partial_v\,(\phi_n) = \Phi_n,$$

where Φ_n depends on the previous ϕ_d.

In the case of an ordinary step, as $i - pj/q \neq 0$ for all $(i,j), i+j = n$, this equation has a unique solution which also satisfies the first property.

Indeed, the difference $\Delta_n = \big((\Psi_n)^I - f\big)$ satisfies

$$\begin{aligned} &\mathrm{val}(\Delta_n) \geq n + I - 1, \\ &\mathrm{val}\left((u+R)\,\partial_u(\Delta_n) + (-\frac{p}{q}\,v + S)\,\partial_v(\Delta_n) - (I+T)(\Delta_n)\right) \geq n + I - 1, \end{aligned}$$

whence $\mathrm{val}(\Delta_n) \geq n + I$.

In the case of a critical step, there is a pair $(i,j), i+j = n, i - pj/q = 0$. The existence of f_{n+I-1} shows that the coefficient $(\Phi_n)_{i,j}$ is 0, so that the differential equation in ϕ_n has solutions. The coefficient $(\phi_n)_{i,j}$ remains free. It is possible to fix it in a unique way to satisfy the first property. ∎

Proposition 27. *A* saddle *point of a vector field is a Darboux point at which eigenvalues have a non positive rational ratio, i. e. where $\rho = 1, \sigma = -p/q, q \in \mathbb{N}^\star, p \in \mathbb{N}, (p,q) = 1$. If the local equation at a saddle point M can be written as*

$$(u+R)g_u + (-p/qv + S)g_v = (K+T)g, K' \in \mathbb{N}^\star,$$

then the local solution in $\mathbb{C}[[u,v]]$ corresponding to a strict Darboux polynomial f vanishing at M has the following form:

$$g = u^I v^J + \sum_{d > I+J} g_d,$$

where the g_d is a homogeneous two-variable polynomial of degree d. If f is irreducible and if v is a Darboux coordinate (v divides S), then $J = 0$ and $I = 1$, which means that M is a simple point of f and that $K = 1$.

Sketch of the proof: It it easy to show that v divides g as soon as v is Darboux and $J \geq 1$. This is to be excluded in such a way that the previous Proposition 26 gives $g = \phi^I$. As f is an irreducible polynomial, $I = 1$ according to the argument in the proof of Proposition 21. ∎

7. COMPLETING THE PROOF: THERE IS NOTHING ELSE

In this section, we look for *necessary* conditions on A, B, C for the existence of a strict Darboux polynomial for $LV(A,B,C)$. In other words, this section is devoted to the proof of the following main theorem.

Theorem 28. *If $LV(A,B,C)$ has a strict Darboux polynomial f, then the triple $[A,B,C]$ of parameters falls, up to natural symmetries, in one of the cases of Theorem 12.*

As in Theorem 12, p stands for $-A-\dfrac{1}{B}$, q for $-B-\dfrac{1}{C}$ and r for $-C-\dfrac{1}{A}$.

7.1. Strict Darboux polynomials of degree 1

LEMMA 29. *If $LV(A,B,C)$ has a strict Darboux polynomial of degree 1 than either $ABC+1=0$ or one of the parameters A,B,C is equal to 1.*

Proof: Applying the analysis of Subsection 3.5, it is not difficult to see that either $\lambda=\gamma_2=1$ (or some circular permutation of it), in which case $B=1$, or $\lambda=\mu=\nu=0$; then P is a non-zero multiple of $y-Az$, Q a non-zero multiple of $z-Bx$ and R a non-zero multiple of $x-Cy$, which implies $ABC+1=0$. ∎

On the other hand, we gave the sought Darboux polynomial corresponding to these two situations (cases **1** and **2** in the list of Theorem 12) in such a way that the previous necessary conditions are also sufficient.

So, we may now look for strict Darboux polynomials of some degree $m\geq 2$ and conditions on A,B,C for them to exist. In our following considerations, we will thus suppose the extra "open" condition

$$BC+1\neq 0,\ A\neq 1,\ B\neq 1,\ C\neq 1, \tag{23}$$

which means that there is no strict Darboux polynomial of degree 1 for $LV(A,B,C)$.

7.2. Strict Darboux polynomials of degree $m\geq 2$

According to Lemma 14, determinants D_1, D_2 and D_3 have to vanish in order to allow the existence of some strict Darboux polynomial f of degree $m\geq 2$.

The condition $D_1=0$ is a disjunction: either $1/A$ is a positive integer in the range $[1,\beta 1]$, or C is a positive integer in the range $[1,\gamma_1]$, or $r=-1/A-C$ is positive integer in the range $[1,m-\beta 1-\gamma 1-1]$, or

$$L_1=\{m(1+ABC)-\lambda(1+AC)-\mu(1-B)+\nu C(1-B)\}=0.$$

Conditions $D_2=0$ and $D_3=0$ are translated in a similar way and two new parameters $p=-A-1/B$ and $q=-B-1/C$ are convenient to express the conditions we get.

Many tracks may now be followed: there are four possibilities corresponding to $D_1=0$, four also for $D_2=0$ and $D_3=0$. So, sixty-four branches open in our case analysis! In order to reduce their number, we have to take into account the possible natural transformations of the parameters. We summarize these simplifications in the following proposition.

PROPOSITION 30. *If $LV(A,B,C)$, with the conditions (23) on the parameters, has a strict Darboux polynomial f of degree $m\geq 2$, then (up to* *usual natural symmetries) one of the following conditions has to be fulfilled (with our usual notations):*

I. $L_1 = L_2 = L_3 = 0$,
II. $L_1 \neq 0, L_2 = L_3 = 0, 1/A \in \mathbb{N}^{\star\star}, \nu \in \mathbb{N}^{\star}$,
III. $L_1 \neq 0, L_2 = L_3 = 0, r = -C - 1/A \in \mathbb{N}^{\star}, m - \beta_1 - \gamma_1 - 1 \in \mathbb{N}^{\star}$,
IV. $L_1 \neq 0, L_2 \neq 0, L_3 = 0, A \in \mathbb{N}^{\star\star}, C \in \mathbb{N}^{\star\star}, \mu \in \mathbb{N}^{\star}, \nu \in \mathbb{N}^{\star}$,
V. $L_1 \neq 0, L_2 \neq 0, L_3 = 0, A \in \mathbb{N}^{\star\star}, r = -C - 1/A \in \mathbb{N}^{\star}, \nu \in \mathbb{N}^{\star}, m - \beta_1 - \gamma_1 - 1 \in \mathbb{N}^{\star}$,
VI. $L_1 \neq 0, L_2 \neq 0, L_3 = 0, C \in \mathbb{N}^{\star\star}, p = -A - 1/B \in \mathbb{N}^{\star}, \mu \in \mathbb{N}^{\star}, m - \alpha_2 - \gamma_2 - 1 \in \mathbb{N}^{\star}$,
VII. $L_1 \neq 0, L_2 \neq 0, L_3 = 0, C \in \mathbb{N}^{\star\star}, 1/B \in \mathbb{N}^{\star\star}, \mu \in \mathbb{N}^{\star}, \lambda \in \mathbb{N}^{\star}$,
VIII. $L_1 \neq 0, L_2 \neq 0, L_3 = 0, p = -A - 1/B \in \mathbb{N}^{\star}, r = -C - 1/A \in \mathbb{N}^{\star}$, $m - \alpha_2 - \gamma_2 - 1 \in \mathbb{N}^{\star}, m - \beta_1 - \gamma_1 - 1 \in \mathbb{N}^{\star}$.

In these other cases $L_1 \neq 0, L_2 \neq 0, L_2 \neq 0$.

IX. $p \in \mathbb{N}^{\star}, q \in \mathbb{N}^{\star}, r \in \mathbb{N}^{\star}$,
X. $p \in \mathbb{N}^{\star}, q \in \mathbb{N}^{\star}, C \in \mathbb{N}^{\star\star}$,
XI. $p \in \mathbb{N}^{\star}, B \in \mathbb{N}^{\star\star}, C \in \mathbb{N}^{\star\star}$,
XII. $A \in \mathbb{N}^{\star\star}, B \in \mathbb{N}^{\star\star}, C \in \mathbb{N}^{\star\star}$.

Proof: Building this list consists in gathering some of the 64 previous subcases when they are images of one another under natural transformations, and canceling those leading to a Darboux polynomial of degree 1; let us remark that a condition like $L_3 = 0$ is invariant under the transformation $A \to 1/A, B \to 1/C, C \to 1/B$. ∎

Now, we take into consideration each of the previous twelve cases. It turns out that the first one is empty.

Proposition 31. [case I of Proposition 30] *$LV(A, B, C)$, where the parameters satisfy the extra condition (23), has no strict Darboux polynomial of degree $m \geq 2$ such that $L_1 = L_2 = L_3 = 0$.*

Proof: The linear system $L_1 = L_2 = L_3 = 0$ of equations in $\{\lambda, \mu, \nu\}$ would be solved (as the determinant is not 0) in a unique way to yield

$$\lambda = m\frac{1+B}{2},\ \mu = m\frac{1+C}{2},\ \nu = m\frac{1+A}{2}.$$

Hence, the parameters would be rational numbers such that $A \geq -1, B \geq -1, C \geq -1$.

According to the equations and inequalities of Subsection 3.5, if $B < 0$ then $\lambda = 0$ which implies in turn $B = -1$. The same is true for A and C: either they are positive or they are equal to -1.

Up to evident symmetries, four sub-cases are to be considered to achieve the proof:

(1) $A = B = C = -1$, whence $ABC + 1 = 0$, which has been excluded.
(2) $A = B = -1$ and $C > 0$, whence

$$\lambda = \gamma_2 = \nu = \beta_1 = 0, \mu = m\frac{1+C}{2} \leq m, \alpha_3 = \frac{\mu}{C} = m\frac{1+C}{2C} \leq m,$$

which would imply $C \leq 1, 1 \leq C$, i. e. $C = 1$, an excluded possibility.
(3) $A = -1$ and $B > 0, C > 0$, whence

$$\begin{array}{l} \nu = \beta_1 = 0, \mu = m\frac{1+C}{2} \leq m, \gamma_2 = \frac{\lambda}{B} = m\frac{1+B}{2B} \leq m, \\ \alpha_3 + \beta_3 = \frac{\mu}{C} + \lambda = m\left(\frac{1+C}{2C} + \frac{1+B}{2}\right) \leq m, \end{array}$$

which would imply $C \leq 1, 1 \leq B, \frac{1}{2C} + \frac{B}{2} \leq 0$, which is impossible.
(4) $A > 0, B > 0, C > 0$; with the same kind of remarks, we receive the contradictory results $A + 1/B \leq 0, B + 1/C \leq 0, C + 1/A \leq 0$.

∎

Dealing with other elements of the list in Proposition 30 will be more difficult. A general idea will be useful to deduce new necessary conditions when one of the L_i is zero. We call this tool "the transfer principle" and study it in a next subsection. We will first describe strict Darboux polynomials of degree 2.

7.3. Strict Darboux polynomials of degree 2

In Theorem 12, two families of triples $[A, B, C]$ lead to strict Darboux polynomials of degree 2. It is therefore convenient to check that these families are the only ones to yield them. Moreover, this will help us to simplify the sequel by excluding these triples in the same manner as we have excluded those with the open condition (23).

Lemma 32. *If $LV(A, B, C)$ has a strict Darboux polynomial of degree 2 then, up to a natural transformation, either $p = q = 1$ (and then $r = 1$ and $ABC = 1$) or $A = 2, q = 1$.*

Proof: Either $\lambda = \mu = \nu = 0$ and f is a polynomial first integral of degree 2, which is only possible if $p = q = r = 1$ or some of them, for instance ν is equal to 1 or 2; as we exclude $A = 1$, $\beta_1 \neq \nu$. Then we can choose $A = 2, \nu = 2, \beta_1 = 1$ whence $q = 1$. ∎

On the other hand, we gave the sought Darboux polynomial corresponding to these two situations (cases **3** and **4** in the list of Theorem 12) in such a way that the previous necessary conditions are also sufficient.

So, we may now look for strict Darboux polynomials of some degree $m \geq 3$ and conditions on A, B, C for them to exist. We will also suppose the triples do not produce a strict Darboux polynomial of degree 2, i. e. that they satisfy the new "open" conditions:

$$[p,q,r] \neq [0,0,0], [A,q] \neq [2,1], [1/2,1], [B,r] \neq [2,1], [1/2,1], [C,p] \neq [2,1], [1/2,1]. \tag{24}$$

7.4. The transfer principle

PROPOSITION 33. *Let f be a strict Darboux polynomial of $LV(A,B,C)$ with a degree m and an cofactor $\Lambda = \lambda x + \mu y + \nu z$ such that $L_3 = 0$. Then*

$$D_3' = [2/C - \alpha_3]^{\overline{\alpha_3}} [2B - \beta_3]^{\overline{\beta_3}} [2/C + 2B + 1]^{\overline{m - \alpha_3 - \beta_3 - 2}} = 0. \tag{25}$$

Sketch of the proof: In the general case of $LV(A,B,C)$, f can be decomposed as $f = \sum_k z^k R_{m-k}(x,y)$, where $R_{m-k}(x,y)$ is a homogeneous polynomial of degree $m-k$ is x, y. The assumption that f is a Darboux polynomial of cofactor Λ can be written as a finite set of equations, for all $k \in [0,m]$:

$$\begin{aligned}[(Cyx\partial_x + xy\partial_y + k(Bx+y)) - {} & (\lambda x + \mu y)](R_{m-k}) = \\ & [-x\partial_x - Ay\partial_y + \nu](R_{m-k+1}).\end{aligned}$$

The first of them, for $k = 0$, provides the special form of $R_m = R$, that we described in Equation (13) in Subsection 3.5.

The second one allows us to compute R_{m-1} from R_m. When $L_3 = 0$, it can be shown that R_{m-1} is similar to the R_m:

$$R_{m-1} \equiv x^{\alpha_3} y^{\beta_3} (x - Cy)^{m - \alpha_3 - \beta_3 - 1}.$$

Thus, we can change the values of the coefficients $M_{(i+1,j,k),(i,j,k)}$ and $M_{(i,j+1,k),(i,j,k)}$ in the linear system (18) for all triples such that $k = 1$ to translate directly the combinatorial nature of R_{m-1}: this is our *transfer principle*.

In this modified system, we consider the square subsystem of all unknowns for which $k = 1$ or $k = 2$ together with the suitable equations. The corresponding determinant can be computed in a similar way as D_3 in Subsection 3.5. ∎

COROLLARY 34. *With the usual notations, when $L_3 = 0$, the parameters have to fulfill one of the following conditions:*

(i) $2/C \in \mathbb{N}^\star$ *and* $\alpha_3 \geq 2/C$,
(ii) $2B \in \mathbb{N}^\star$ *and* $\beta_3 \geq 2B$,
(iii) $-(2/C + 2B + 1) \in \mathbb{N}$ *and* $0 \leq -(2/C + 2B + 1) \leq m - \alpha_3 - \beta_3 - 3$

With the transfer principle, it is possible to solve the second and third cases, where $L_2 = L_3 = 0$, of Proposition 30 and to make more precise the cases where $L_3 = 0$.

Proposition 35. [case II and III of Proposition 30] *$LV(A, B, C)$ has no strict Darboux polynomial of degree $m \geq 2$ such that $L_1 \neq 0, L_2 = L_3 = 0$ except if $[A, B, C] = [-2, 2, -1/2]$ or $[A, B, C] = [-2, 1/2, -1/2]$, in which case condition (23) is not fulfilled; this is a sub-case of case* **4** *of Theorem 12.*

Sketch of the proof: From the assumptions, it is easy to deduce that A, B, C are rational. The three integers $p_1 = m - \beta_1 - C\alpha_3, p_2 = m - \gamma_2 - A\beta_1, p_3 = m - \alpha_3 - B\gamma_2$ are nonnegative. Solving the linear system $\{L_2 = 0, L_3 = 0\}$ in $\{\alpha_3, \beta_1\}$ gives a unique solution. Substituting this solution in p_1, p_2, p_3 shows that γ_2 could be 0. In this case, $\alpha_3 = 1$ and $\beta_1 = 1/A$ ($\alpha_2 = 1$) and $L_2 = L_3 = (m - 1)(1 + ABC)$. This would imply $m = 1$ or $ABC = -1$, which is excluded. Thus, γ_2 is in fact a strictly positive integer, whence $B > 0$. As $p_1 \geq 0$ and $p_3 \geq 0$, $A = -1/Ba, 0 < a \leq 1$ and $C = -c/B, 0 < c \leq 1$. In particular, A and C are strictly negative in such a way that $\alpha_3 = \beta_1 = 0$. Two values of γ_2 are then deduced from $L_2 = L_3 = 0$ in this case. Equating them gives $AC = 1$ and $-1 + 2Ba = n \in \mathbb{N}$ is the only way to fulfill Lemma 14. Elementary computations show that the transfer principle cannot be achieved unless $2/B$ or $2B$ is a positive integer. Using a natural transformation, we can restrict ourselves to the case $2B = e \in \mathbb{N}^\star$. Number theoretic simple arguments only give $[A = -2, B = 2, C = -1/2]$, which is the announced result. ∎

Proposition 36. [cases IV to VIII of Proposition 30 revisited] *If the parameters satisfy the extra conditions (23,24), and if $LV(A, B, C)$ has a strict Darboux polynomial of degree $m \geq 3$ with $L_1 \neq 0, L_2 \neq 0, L_3 = 0$, then (up to usual natural symmetries) one of the following conditions has to be fulfilled:*

(i) (from case IV*)* $A \in \mathbb{N}^{\star\star}, C = 2, \mu \in \mathbb{N}^\star, \nu \in \mathbb{N}^\star$,
(ii) (from case V*)* $A \in \mathbb{N}^{\star\star}, r = -C - 1/A = 1, \nu \in \mathbb{N}^\star, 2q = -2(B + 1/C) = 1$,
(iii) (from case VI*)* $C = 2, p = -A - 1/B \in \mathbb{N}^{\star\star}, \mu \in \mathbb{N}^\star$,
(iv) (from case VI*)* $C \in \mathbb{N}^{\star\star}, p = -A - 1/B \in \mathbb{N}^\star, \mu \in \mathbb{N}^\star$,
$2q = -2(B + 1/C) = 1$,
(v) (from case VII*)* $C = 2, 1/B \in \mathbb{N}^{\star\star}, \mu \in \mathbb{N}^\star, \lambda \in \mathbb{N}^\star$,
(vi) (from case VIII*)* $p = -A - 1/B = 1, r = -C - 1/A = 2, C = 2/3$,
or f is a polynomial first integral,

Sketch of the proof: We simply divide each case according to the three possibilities $2B \in \mathbb{N}^\star$, $2/C \in \mathbb{N}^\star$, $2q = -2(B+1/C) \in \mathbb{N}^\star$ coming from Proposition 33 applied to $L_3 = 0$ and then cancel sub-cases that are contradictory with the two open conditions (23,24). ∎

7.5. Counting branches and bounding the degree when $\boldsymbol{L_1 L_2 L_3 \neq 0}$

In order to achieve our study of most of the situations, we will deduce stronger necessary conditions on the parameters in order to have an *irreducible* strict Darboux polynomial; a useful tool is then the analysis of an irreducible Darboux polynomial around the seven Darboux points of $LV(A,B,C)$.

In Section 6, we were able to compute the number of branches of a Darboux curve through some Darboux points in several situations.

It turns out that this argument is a decisive tool to produce an upper bound on the degree m of an irreducible strict Darboux polynomial in almost all cases. The conclusion is then a computation.

We will start with the four cases in which $L_1L_2L_3 \neq 0$ (cases IX to XII of Proposition 30). In each case, we distinguish two possibilities: either Δ is the square of a rational number, which imply strong necessary conditions on the parameters, or the number of branches at M_7 is known (0 or 2) and Corollary 25 leads to an upper bound on the degree m.

PROPOSITION 37. [case IX of Proposition 30] *Suppose that the parameters satisfy the extra conditions (23,24), and that $LV(A,B,C)$ has a strict Darboux polynomial f of degree $m \geq 3$ with $p,q,r \in \mathbb{N}^\star$. Then, up to a natural transformation, the triple $[p,q,r]$ is equal to $[1,2,2],[1,2,3],[1,2,4]$; in these cases, f is a polynomial first integral of $LV(A,B,C)$ (cases* **5** *to* **7** *of Theorem 12).*

Proof: Except if $p=q=r=1$, in which case $LV(A,B,C)$ has a Darboux polynomial of degree 2, a second degree equation for A can be deduced from the definitions of p,q,r; its discriminant cannot be the square of a rational number and A is not rational. The same is true for B and C in such a way that $\lambda=\mu=\nu=0$ according to relations (14), which means that f is a polynomial first integral of $LV(A,B,C)$. We could conclude with the help of [15]. But it is also possible to notice that Δ has to be the square of a rational number: indeed, if it were not, Corollary 25 would give $m=0$ or $m=1$, which is impossible. Δ can be expressed from p,q,r as

$$\Delta \;=\; -\frac{3pqr+7pq+7pr+7qr+16p+16q+16r+36}{pqr+pq+pr+qr-4},$$

provided that the denominator is not 0. In the cases where this denominator is 0, either $ABC+1=0$ or $p=q=r=1$ and $LV(A,B,C)$ has a strict Darboux polynomial of degree 1 or 2. Otherwise, an elementary discussion using sign arguments shows that the previous fraction is the square of a rational number if and only if the triple $[p, q, r]$ is equal to $[1, 2, 2], [1, 2, 3], [1, 2, 4]$ or a permutation of them. ∎

Proposition 38. [case X of Proposition 30] *If the parameters satisfy the extra conditions (23,24), and if $LV(A, B, C)$ has a strict Darboux polynomial f of degree $m \geq 3$ with $p, q \in \mathbb{N}^\star, C \in \mathbb{N}^{\star\star}$, then, up to a natural transformation, the triple $[A, B, C]$ is equal to $[-4/7, -7/3, 3]$, $[-4/3, -3/2, 2]$ (in cases **8** and **9** of Theorem 12) or (in cases **19** to **23** of Theorem 12) to $[-5/4, -4/3, 3]$, $[-5/9, -9/4, 4]$, $[-7/3, -3/2, 2]$, $[-8/5, -5/2, 2]$, $[-7/10, -10/3, 3]$.*

Sketch of the proof: First suppose that Δ is the square of a rational number. Then, an elementary discussion using sign arguments [17] shows that the only possible triples are $[C, p, q] = [3, 1, 2]$ and $[C, p, q] = [2, 2, 1]$, which correspond to the two first given results.

When Δ is not the square of a rational number, Corollary 25 gives the following equality

$$m - \frac{(p+1)(qC+1)\alpha_3}{1 + pqC + p - C} = \epsilon, \quad \epsilon = 0, 1,$$

while Corollary 24 gives the two following ones

$$\begin{aligned} m &= (p+1)j_5 + i_5 & i_5 = 0, 1, \\ m &= (q+1)j_6 + i_6 + \alpha_3 & i_6 = 0, 1. \end{aligned}$$

The three previous linear equations in the unknowns α_3, j_5, j_6 have a unique solution.

Now, substituting this solution in the inequality $C\alpha_3 \leq m$ gives an inequality involving C, p, q and m.

This inequality then gives an upper bound for m as a function of C, p and q except if $[C = 2, p = 1]$ or $[p = 1, q = 1]$; but in these cases, $LV(A, B, C)$ has a Darboux polynomial of degree 2.

Excluding also the case $[p = 1, q = 2, C = 3]$, which corresponds to the existence of a rational first integral, we receive the upper bound 10 for the degree m; as we have explored all possibilities up to degree 12, we are sure that there is nothing else but the last five given triples. ∎

Proposition 39. [case XI of Proposition 30] *Suppose that the parameters satisfy the extra conditions (23,24), and that $LV(A, B, C)$ has a strict Darboux polynomial f of degree $m \geq 3$ with $p \in \mathbb{N}^\star, B, C \in \mathbb{N}^{\star\star}$. Then, up to a natural transformation, either $[A, B, C] = [-3/2, 2, 3]$ or $[A, B, C] = [-5/2, 2, 2]$ (cases **17** and **18** of Theorem 12).*

Sketch of the proof: An elementary discussion using sign arguments [17] shows that Δ is the square of a rational number only if $[C = 2, p = 1]$, which gives an f with a degree 2.

Thus Δ is not the square of a rational number and Corollary 25 gives

$$m - \frac{C\alpha_3 B(p+1)}{CpB + C - 1} - \gamma_2 \left(1 - \frac{B(p+1)}{CpB + C - 1}\right) = \epsilon, \quad \epsilon = 0, 1,$$

The parameters $m, \beta_1, \gamma_2, \alpha_3$ satisfy the three inequalities from (14) which become:

$$\begin{array}{lcl} m - C\alpha_3 & \geq & 0, \\ m - \gamma_2 & \geq & 0, \\ m - \alpha_3 - B\gamma_2 & \geq & 0. \end{array}$$

The second inequality may be dropped: the third is stronger.

According to Corollary 24, $m - \gamma_2 = i_5 + j_5(p+1)$, where $i_5 = 0$ or $i_5 = 1$.

Afterwards, we deduce an inequality leading to an upper bound for j_5, which turns out to be 0 or 1. As $j_5 = 0$ is excluded, some $[C, p, B]$ are to be canceled. Then, tedious computations (with the help of a computer algebra system) give two possibilities: $[C = 2, p = 2, B = 2]$ and $[p = 1, C = 3, B = 2]$. The first one correspond to $[A, B, C] = [-5/2, 2, 2]$ (case **18** of Theorem 12) and the second one to $[A, B, C] = [-3/2, 2, 3]$ (case **17** of Theorem 12). ∎

Proposition 40. [case XII of Proposition 30] *Suppose that the parameters satisfy the extra conditions (23,24), and that $LV(A, B, C)$ has a strict Darboux polynomial f of degree $m \geq 3$ with $A, B, C \in \mathbb{N}^{\star\star}$. Then, up to a natural transformation, the triple $[A, B, C]$ is equal to $[2, 2, 2]$ (case* **16** *of Theorem 12).*

Proof: In the present situation, the parameters $m, \beta_1, \gamma_2, \alpha_3$ satisfy the three inequalities from (14):

$$\begin{array}{lclcl} m - \beta_1 - \gamma_1 & = & m - \beta_1 - C\alpha_3 & \geq & 0, \\ m - \alpha_2 - \gamma_2 & = & m - \gamma_2 - A\beta_1 & \geq & 0, \\ m - \alpha_3 - \beta_3 & = & m - \alpha_3 - B\gamma_2 & \geq & 0. \end{array}$$

Moreover, Δ is easily shown to be negative here and thus cannot be the square of a rational number.

According to Corollary 25,

$$m - \frac{(AC - A + 1)\lambda + (AB - B + 1)\mu + (BC - C + 1)\nu}{ABC + 1} = \epsilon, \quad \epsilon = 0, 1.$$

A linear combination

$$C_1(m-\beta_1-C\alpha_3)+C_2(m-\gamma_2-A\beta_1)+C_3(m-\alpha_3-B\gamma_2),$$
$$-m+\frac{(AC-A+1)\lambda+(AB-B+1)\mu+(BC-C+1)\nu}{ABC+1}+\epsilon,$$

in which $\beta_1, \gamma_2, \alpha_3$ are no longer involved, can be built with positive coefficients C_1, C_2, C_3.

In this combination, the coefficient of m is negative in such a way that we get an upper bound M for m, which depends on $A, B, C \in \mathbb{N}^{\star\star}$:

$$M=\frac{\epsilon(1+ABC)^2}{(AC-A+1)(BA-B+1)(CB-B+1)}$$

Then, $3-M$ is shown to be nonnegative by an easy-for-a-machine computation: substituting $A=a+2, B=b+2, C=c+2$, we receive a numerator which is a polynomial in the nonnegative a, b, c whose coefficients are all nonnegative.

Thus, the only possibility is $m=3$, with a solution (case **16** of Theorem 12). ∎

7.6. Concluding when $L_3 = 0$

Using some arguments similar to the previous ones and also some very specific ones, we now finish the classification by concluding in all cases of Proposition 36 where $L_3 = 0$.

PROPOSITION 41. [case i of Proposition 36] *If f is an irreducible Darboux curve of degree $m > 2$ of $LV(A, B, C)$, with $L_3 = 0, A \in \mathbb{N}^{\star\star}, C = 2$ and the conditions (23,24), then, up to a natural transformation, the triple $[A, B, C]$ is equal to $[4, -1/6, 2]$ or $[2, -1/6, 2]$ (cases* **10** *and* **13** *of Theorem 12).*

Sketch of the proof: To deal with this special case, it is convenient to perform a linear change of coordinates: $X = x, T = x - Cy, Z = z$. Indeed, a well chosen 5×5 determinant has to vanish, which gives a new necessary condition. In the new coordinate system, involved unknowns are $(\alpha_3, m-1-\alpha_3, 1)$, $(\alpha_3, m-2-\alpha_3, 2)$, $(\alpha_3-1, m-1-\alpha_3, 2)$, $(\alpha_3, m-3-\alpha_3, 3)$ and $(\alpha_3-1, m-2-\alpha_3, 3)$; involved equations are parameterized by $(\alpha_3+1, m-2-\alpha_3, 2)$, $(\alpha_3, m-1-\alpha_3, 2)$, $(\alpha_3, m-2-\alpha_3, 3)$, $(\alpha_3-1, m-1-\alpha_3, 3)$ and $(\alpha_3-1, m-\alpha_3, 3)$.

It turns out that the only possibilities for A are 2 and 4. ∎

PROPOSITION 42. [case ii of Proposition 36] *If f is an irreducible Darboux curve of degree $m > 2$ of $LV(A, B, C)$, with $L_3 = 0, A \in \mathbb{N}^{\star\star}, C = -1 - 1/A, B = -1/2 - 1/C = (A-1)/2(A+1)$ and the conditions (23,24), then, up to a natural transformation, the triple $[A, B, C]$ is equal to $[2, 1/6, -3/2]$ or $[5, 1/3, -6/5]$ (cases* **12** *and* **14** *of Theorem 12).*

Sketch of the proof: In this situation, $\rho_7 i_7 + \sigma_7 j_7 < 0$, which is impossible if Δ is not the square of a rational number. But Δ is the square of a rational number if and only if $25A^2 - 6A + 81$ is the square of an integer, which is only possible when $A \in \mathbb{N}^{\star\star}$ is either 2 or 5. ∎

PROPOSITION 43. [case iii of Proposition 36] *If f is an irreducible Darboux curve of degree $m > 2$ of $LV(A, B, C)$, with $L_3 = 0, p \in \mathbb{N}^{\star\star}, C = 2$ and the conditions (23,24), then, up to a natural transformation, the triple $[A, B, C]$ is equal to $[4, -1/6, 2]$ $(p = 2)$ or $[2, -1/6, 2]$ $(p = 4)$(cases* **10** *and* **13** *of Theorem 12).*

Sketch of the proof: Computations are similar to those of Proposition 41. ∎

PROPOSITION 44. [case iv of Proposition 36] *If f is an irreducible Darboux curve of degree $m > 2$ of $LV(A, B, C)$, with $L_3 = 0, C \in \mathbb{N}^{\star\star}, p \in \mathbb{N}^{\star}, 2q \in \mathbb{N}^{\star}$ and the conditions (23,24), then, up to a natural transformation, the triple $[A, B, C]$ is equal to $[1/5, -5/6, 3]$ or $[1/2, -2/3, 6]$, $(p = 1, q = 1/2$ in both cases* **12** *and* **14** *of Theorem 12).*

Sketch of the proof: It is not very difficult to prove that $p = 1$ and $q = 1/2$. Then, Δ has to be the square of a rational number; setting $C = 2 + c$, this is equivalent to saying that $25c^2 + 44c + 100$, with $c \in \mathbb{N}$, is the square of a positive integer. ∎

PROPOSITION 45. [case v of Proposition 36] *If f is an irreducible Darboux curve of degree $m > 2$ of $LV(A, B, C)$, with $L_3 = 0, 1/B = K \in \mathbb{N}^{\star\star}, C = 2$ and the conditions (23,24), then, up to a natural transformation, the triple $[A, B, C]$ is equal to $[-8/7, 1/3, 2]$ or $[-13/7, 1/3, 2]$ or belongs to the sporadic family $[A, B, C] = [A_l = -\frac{2l+1}{2l-1}, 1/2, 2], l \in \mathbb{N}^{\star}$ (cases* **11**, **15** *and* **24** *of Theorem 12).*

Sketch of the proof: First, some linear combination of the usual inequalities and the equation $L_3 = 0$ in which α_3 and γ_2 are no longer involved is nonnegative when $A > 0$ which gives a contradiction.

Thus $A < 0$, whence $\beta_1 = \alpha_2 = \nu = 0$.

If $K > 2$ then Proposition 16 gives $i_6 + (1 - B - 1/C) j_6 = m - \beta_3 - \alpha_3$ and this equation has to be fulfilled with a pair (i, j) such that $i_6 = 0, 1, i_6 + j_6 < m$. On the other hand, denoting $m - \gamma_2$ by δ_2 and $m - \gamma_1$ by δ_1, we find that $m - \beta_3 - \alpha_3$ is bounded from below by $m(K - 2)/2K + \delta_1/2 + \delta_2/K$. The only possibilities (simple computations) are $(\delta_1 = 0, \delta_2 = 1, K = 3, 4)$, $(\delta_1 = 0, \delta_2 = 2, m = 2\alpha 3 = 2 + K\beta_3)$, $(\delta_1 = 1, \delta_2, 0, K = 3, 4)$.

The case $\delta_2 = 2$ can be excluded: τ_7 cannot be 0 or 1, and so Δ has to be the square of a rational number, which is impossible for $m > 2$. The same argument excludes all possibilities for $K = 3, 4$ except cases **11** and **15** of Theorem 12.

Now, in the case $K = 2$, Equation (16) at $M_6(M_7)$ becomes $i_6 = m - \beta_3 - \alpha_3$ and must be satisfied with $j_6 = 0$ as the local coordinate v is a Darboux power series and Proposition 26 can be applied: $m = \alpha_3 + \beta_3 + 1$. This is case **24** of Theorem 12. ∎

PROPOSITION 46. [case vi of Proposition 36] *There is no irreducible Darboux curve of degree* $m > 2$ *of* $LV(A, B, C)$*, with* $L_3 = 0, p = 1, r = 2, C = 2/3$ *and* $L_3 = 0$.

Proof: In this case, we would have $[A, B, C] = [-3/8, -8/5, 2/3]$. Hence $\lambda = \nu = 0$ and, as $L_3 = 0$, $\mu = 7m/11$. As $\Delta = 313/49$ is not the square of a rational number, $i_7 = j_7 = -5m/11 < 0$, a contradiction. ∎

We have now to deal with $LV(A, B, 0), A, B \neq 0$.

PROPOSITION 47. *The vector field* $LV(A, B, 0), A, B \neq 0$*, has no strict Darboux polynomial except in the following cases:* $[A \neq 0, B = 1, C = 0]$, $[A = -1, B = 2, C = 0]$.

Sketch of the proof: The given two possibilities are limits of cases **2** and **4** of Theorem 12 and it is easy to check that $y - Az$ is a strict Darboux polynomial of $LV(A, 1, 0)$ while $(x+y)^2 - 2yz$ is a strict Darboux polynomial of $LV(-1, 2, 0)$.

We leave the details of the proof that these are the only cases to the reader. Here are the headlines.

From Lemma 15 we know that $\mu = \gamma_1 = \alpha_3 = 0$ and $\lambda = \beta_3 = B\gamma_2 = m$ and $L_1 = 0$.

If there existed a strict irreducible Darboux polynomial with a degree greater than 2, then the transfer principle would lead to a contradiction. ∎

8. DEALING WITH NON-NORMAL SITUATIONS

In this section we adapt the previous remarks and constructions to find all cases where a factored quadratic vector field, which cannot be put in Lotka-Volterra normal form, has a strict Darboux polynomial or an supply function.

As far as strict Darboux polynomials are involved in the liouvillian integration of a factored quadratic vector field, the search is finished. Indeed, on the one hand, we have a complete description in the Lotka-Volterra case, even if the vector field degenerates; on the other hand, we proved in Proposition 9 that the only quadratic factored vector field that cannot be put in Lotka-Volterra normal form has no strict Darboux polynomial.

Thus, it remains to describe all cases of quadratic factored vector fields (in Lotka-Volterra normal form and degenerate ones) for which the liouvillian integration can be performed with a supply function instead of a strict Darboux polynomial.

Proposition 48. *The homogeneous rational fraction y/x is a supply function for the quadratic factored vector field $xy\partial_y + (x+y)z\partial_z$; $\log(z/y) - y/x$ is then a homogeneous liouvillian first integral of degree 0 of the vector field.*

Sketch of the proof: A simple verification. ∎

Proposition 49. *If $LV(A,B,C)$ (A,B,C may vanish) has no strict Darboux polynomial, and if there exits a supply function, then, up to a natural transformation (circular permutations only!), $[A,B,C] = [-1, 1/2, 0]$.*

Sketch of the proof: Let $\dfrac{N}{x^\alpha y^\beta z^\gamma}$ be a supply function for $LV(A,B,C)$ that we supposed written in reduced form.

By definition, the following identity holds for some cofactor $\Lambda = \lambda x + \mu y + \nu z$:

$$\begin{aligned}&x(Cy+z)N_x + y(Az+x)N_y + z(Bx+y)N_z\\ &\quad -N((\beta + B\gamma)x + (\gamma + C\alpha)y + (\alpha + A\beta)z) \;=\; \Lambda x^\alpha y^\beta z^\gamma.\end{aligned}$$

First, consider that α, β, γ are not 0. Two-variable marginal polynomials can be derived from N as in Equation (13). It turns out that A, B, C would be positive rational numbers such that $1/A + C \leq 1, 1/B + A \leq 1, 1/C + B \leq 1$, which is impossible.

Of course, one of the α, β, γ has to be strictly positive. If there is only one, say γ, then it is easy to derive from the consideration of the two-variable marginal polynomials that $\Lambda = 0$, i. e. that N is a Darboux polynomial of degree γ and cofactor $\gamma(Bx + y)$; this is excluded.

Now, the only possibility is $\alpha \neq 0, \beta \neq 0, \gamma = 0$ up to a natural symmetry. Considering once more marginal polynomials of N we get the following facts: $C\alpha \in \mathbb{N}$, $A \neq 0$, $\pm\alpha/A \in \mathbb{N}^\star$, $C + 1/A \leq 1$, $\pm\beta A \in \mathbb{N}^\star$, $B \neq 0$,$\beta/B \in \mathbb{N}^\star$, $A + 1/B \leq 1$, $\lambda = 0$, $\alpha + \beta A = 0$.

If $\beta > 1$, as y does not divide N, a determinant, analogous to D_2 of Lemma 14, vanishes. To compute it, replace λ by β, μ by $C\alpha$, ν by 0. This is only possible if $\beta/B = \alpha + \beta$ or when the counterpart of L_2 is 0, which gives $\alpha C = \beta/B$ and $ABC + 1 = 0$.

Two possibilities are now to be considered: either $C = 0$ or $C > 0$.

First suppose $C > 0$.

In this case $\mu = 0$. If $\alpha > 1$, as x does not divide N, a determinant, analogous to D_1 of Lemma 14, vanishes. To compute it, replace λ by β, μ by $C\alpha$, ν by 0. This is only possible if $\alpha C = \alpha + \beta$ or when the counterpart of L_1 is 0, which gives $\alpha C = \beta/B$ and $ABC + 1 = 0$.

As $\alpha C = \beta/B$ means $ABC + 1 = 0$, which is excluded, either $\alpha = 1$ or $\beta = 1$. If $\alpha = 1, \beta \geq 2$, then $\beta/B = 1 + \beta$, which implies $C = 1$ ($N_{0,\beta,1} \neq 0$). This is also to be excluded. In the same way $\alpha > 1, \beta = 1$ leads to an excluded situation. Thus, $\alpha = \beta = 1$. This, too, is impossible: it would imply $\nu = 0$.

We are now left with $C = 0$.

Then $\mu \neq 0$ and $\alpha = 1$, otherwise, x would divide N. Either $\beta > 1$ and then $\beta/B = 1 + \beta$ or $\beta = 1$. In any case, the only non-zero coefficients of N are the $N_{0,j,1+\beta-j}, 0 \leq j \leq 1 + \beta$. If $\beta > 1$, the coefficient $N_{1,\beta-2,2}$ can be computed from $N_{0,\beta-1,2}$: $N_{1,\beta-2,2} \neq 0$; it can also be computed by induction from $N_{\beta-1,0,2}$: $N_{1,\beta-2,2} = 0$. ∎

Remark 50. This very exceptional case appeared as an example in a previous paper of mine [14] (with printing errors for which I would like to apologize).

9. FINAL REMARKS ON LOTKA-VOLTERRA SYSTEM

Remark 51. In every "completely normal" case, where $ABC \neq 0$ and where there are exactly seven Darboux points, if there exists a strict Darboux polynomial, then there exists one whose degree is at most 6.

Remark 52. Whereas it is only a countable union of closed sets in the Zariski topology, the set of parameters $[A, B, C]$ for which the Lotka-Volterra system has a strict Darboux polynomial turns out to be closed in $\mathbb{C}^3$ for the ordinary topology.

Remark 53. All isolated Darboux curves are rational (they have a genus 0). We call isolated Darboux curve the fourth Darboux curve when there is no fifth one i. e. when $LV(A, B, C)$ is not rationally integrable. When the vector field is rationally integrable, the simplest rational first integral of degree 0 has a numerator N and a denominator D of the same degree δ. Isolated Darboux curves are the factors of $\alpha N + \beta D$ when it has factors. Irreducible $\alpha N + \beta D$ are level curves of the first integral and they are not "isolated"; their genus also seems to be 0.

10. SOME MORE ALGEBRAIC GEOMETRY

In order to find necessary conditions on the parameters that allow the existence of a strict Darboux polynomial, it has been important to get rid of the degree of such a sought polynomial.

Due to the simplification of computations coming from the factored form of the vector field, we have been able to get such a bound in many cases by looking at nodes for instance.

Similar bounds can be obtained thanks to a formula due to Darboux [7], another one due to Milnor [19], a result of Serre's [32] together with a careful analysis of the local properties of an irreducible strict Darboux polynomial around its singular points.

We give now some development of these ideas in the case of the Lotka-Volterra system. Although these tools are not necessary for our classification result, which is complete, they will certainly be useful in a more general context.

As we have seen in Subsection 2.4, all of them are some Darboux points of the vector field. But it turns out that a Darboux curve has to be highly singular as the next subsection shows.

The first idea consists in the following remark: if f is an irreducible Darboux polynomial of $LV(A,B,C)$ with an cofactor Λ and a degree m, then the two corresponding identities can be combined to give

$$W_x \frac{\partial f}{\partial x} + W_y \frac{\partial f}{\partial y} + W_z \frac{\partial f}{\partial z} = 0, \tag{26}$$

where W is the vector field $W = V - \frac{\Lambda}{m} E$, the *vector cofactor* of f.

Such an orthogonality relation can be used to get un upper bound on the total number of zeroes in $\mathcal{P}_2(\mathbb{C})$ of the homogeneous ideal of $\mathbb{C}[x,y,z]$ generated by the three partial derivatives f_x, f_y, f_z of f.

10.1. A relation of Darboux and its generalization

The following formula of enumerative geometry has been first given by Darboux [7] and then clearly established by Jouanolou [9]; an alternate proof with a useful generalization can be found in [6].

Proposition 54. *Let $F_1, F_2, F_3, G_1, G_3, G_3$ be six homogeneous polynomials in three variables with coefficients in an algebraically closed field $\mathbb{K}$ such that:*

(a) the "vectors" $V = [F_1, F_2, F_3]$ and $W = [G_1, G_3, G_3]$ are irreducible: F_1, F_2, F_3 (resp. G_1, G_3, G_3) are relatively prime,
(b) the orthogonality *condition $F_1G_1 + F_2G_2 + F_3G_3 = 0$ holds, (with the equality between degrees: $a_1 + b_1 = a_2 + b_2 = a_3 + b_3 = r$),*
(c) the homogeneous ideal $(F_1, F_2, F_3, G_1, G_3, G_3)$ has no zero in $\mathcal{P}_2(\mathbb{K})$.

Then, the ideals generated by triples V and W have only finitely many zeroes in the projective plane and

$$\sharp\mathcal{Z}(V) + \sharp\mathcal{Z}(W) = \frac{a_1a_2a_3 + b_1b_2b_3}{r},$$

where $\sharp\mathcal{Z}(.)$ is the total number (when finite) of zeroes of a homogeneous ideal in the projective plane, provided that they are counted according to their multiplicity.

Sketch of the proof: The original proof of Darboux is known to be erroneous since Jouanolou noticed it. In particular, Darboux does not pay attention to the last hypothesis (no common zeroes) and a counterexample is easy to find.

Jouanolou gave a correct proof of the result, but his proof is far from being elementary. In [6], we propose a simpler proof with a useful generalization, that we do not really need in the present work. ∎

Proposition 55. *Let f be an irreducible Darboux polynomial of degree $m > 1$ and cofactor Λ of $LV(A, B, C)$. The only Darboux point which can be a common zero of df and W is M_7.*

When M_7 is not a common zero of df and W, the sum $\Sigma\tau_i$ of Tjurina numbers of f at all Darboux points plus the total number of zeroes of the homogeneous ideal (W_x, W_y, W_z) in the projective plane is equal to $[(m-1)^3 + 1^3]/m = m^2 - 3m + 3$

Sketch of the proof: According to Proposition 8, if $f(M_1) = 0$ then i_1 or j_1 is positive and so is λ; $W_x(M1) = -\lambda/m$ is thus strictly negative. So, M_1 cannot be a common zero of df and W, and the same is true for M_2 and M_3.

Similarly, if $f(M_4) = 0$, then i_4 or j_4 is positive and so is $m - \gamma_1 - \beta_1$; $W_y(M_4) = V_y(M_4) + W_x(M_4)$ and M_4 cannot be a common zero of W_x and W_y since $V_y(M_4) = y(M_4)(Az(M_4) + x(M_4)) = A^2 \neq 0$. The same is true for M_5 and M_6.

Thus, only M_7 can be a common zero of df and W; later on, we will see that this possibility is very seldom.

The Tjurina number τ_i of f at M_i is exactly the multiplicity of M_i as a common zero of f_x, f_y, f_z. The announced result is then an easy application of the Darboux formula given in Proposition 54. ∎

10.2. Bounding the total number of branches at Darboux points

Proposition 55 leads to an upper bound on the total number of irreducible branches of an irreducible Darboux curve through all Darboux points. We will use results of Milnor [19] and Serre [32] to derive the following conclusion:

Proposition 56. *The total number of irreducible branches of an irreducible strict Darboux polynomial of degree m at all Darboux points is at most $m + 2$ except in the exceptional case where df and W vanish at M_7.*

Sketch of the proof: When df and W have no common zero in the projective plane, the Darboux relation is

$$\sharp\mathcal{Z}(df) + \sharp\mathcal{Z}(W) = m^2 - 4m + 7,$$

where $\sharp\mathcal{Z}(df)$ is the sum of all Tjurina numbers τ_M.

At every singular point M of the curve, as $W(M) \neq 0$, the Tjurina number τ_M and its Milnor number μ_M agree.

According to Milnor [19], at every singular point, and even at an ordinary point, $2\delta_M = \mu_M + r_M - 1$, where δ_M is a number introduced by Serre to give a measurement of the fact that the local ring is not integrally closed and where r_M is the number of irreducible branches of the curve at M.

According to Serre [32],

$$2\sum \delta_M + 2g \ = \ m^2 - 3m + 2,$$

where g is the genus of the curve, a nonnegative integer.

Every Darboux point is either a simple zero of W or a zero of f.

Thus, $\sum \mu_M = m^2 - 4m + d$, where d is the number of Darboux points where $f = 0$.

An easy difference gives

$$\sum r_M = m^2 - 3m + 2 - 2g - m^2 + 4m = m + 2 - 2g.$$

This is the announced result. ∎

Remark 57. The last formula can also be useful in the other direction to compute the genus of a Darboux curve when the number of its branches through all Darboux points is known.

Proposition 58. *Suppose that f is an irreducible Darboux curve of degree $m \geq 3$ of $LV(A, B, C)$ and that there is no strict Darboux polynomial of degree $m \leq 2$ for these values of the parameters. then, f and W do not simultaneously vanish at M_7; thus the total number of branches of f through all Darboux points is at most $m + 2$.*

Proof: Suppose $f(M_7) = W_x(M_7) = W_y(M_7) = W_z(M_7) = 0$. If $df(M_7) \neq 0$, then M_7 is an ordinary point of the curve and the Levelt equation has to be solved with $(i, j) = (1, 0)$ or $(i, j) = (0, 1)$, which implies $\rho\sigma = 0$, i. e. M_7 is equal to M_4, M_5 or M_6, a degenerate situation that we have previously studied.

So, $df(M_7)$ is also 0 under our hypothesis, which implies that the product $\rho\sigma$ is a negative rational number.

Simple computations lead to $p = 1, C = 2$ or $p = q = 1$, in which case we have a Darboux polynomial of degree 2. ∎

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