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Bifurcations of Simple Umbilical Points defined by Vector Fields Normal to a Surface Immersed in $\mathbb{R}^4$

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The ν-principal configuration of an immersed surface M in $\mathbb{R}^4$ is the set formed by the umbilical points and the lines of principal curvatures with respect to a unitary smooth vector field ν normal to M. In this article we describe the bifurcation diagram of ν-principal configurations, where ν is parametrized in the space of 1-jets of normal vector fields which define an isolated umbilical point. Versal unfoldings of the nonlocally stable simple umbilical points are obtained.

Key Words: ν-principal configurations, bifurcations of umbilics, versality of umbilics.

1. INTRODUCTION

Let M be an immersed surface in $\mathbb{R}^4$ and ν a unitary smooth vector field normal to M. Let us consider the shape operator and the second fundamental form with respect to ν. We use this to define two orthogonal line fields, the ν-principal directions, whose singularities are the ν-umbilical points. A ν-principal configuration on M is the set formed by the umbilical points

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and the families of lines of maximal and minimal curvatures with respect to the normal vector field ν, integral curves of the ν-principal directions.

Ramírez-Galarza and Sánchez-Bringas [10] proved that for principal configurations of surfaces immersed in $\mathbb{R}^4$, there is a generic class of structurally stable pairs $(\mathcal{I}, \nu)$, where $\mathcal{I}$ is a local immersion of a compact surface M in $\mathbb{R}^4$ and ν a normal vector field defining an isolated umbilical point. Furthermore, the topological types of the corresponding principal configurations are the same types $D_i,\ i = 1, 2, 3$, of those which appear for the generic family of Darbouxian principal configurations of surfaces immersed in $\mathbb{R}^3$, described by Gutierrez and Sotomayor [12]. This generic class of pairs $(\mathcal{I}, \nu)$ is called Darbouxian as well. Despite these similarities between principal configurations in $\mathbb{R}^3$ and $\mathbb{R}^4$, we will point out a remarkable difference. The index of a ν-umbilical point is the index as a singularity of any of the two ν-principal directions. The famous Loewner conjecture states that any umbilical point of a smooth surface immersed in $\mathbb{R}^3$ must have index less than or equal to one. This conjecture has been proved to be true for analytic surfaces by several authors: H. Hamburger [8], G. Bol [2], T. Klotz [9], C. J. Titus [13], and H. Scherbel [11]. On the other hand, Gutierrez and Sánchez-Bringas [7] proved that, given any integer n, there exists an analytic surface immersed in $\mathbb{R}^4$ and an unitary smooth vector field normal to M having an isolated ν-umbilical point of index $n/2$.

The pairs $(\mathcal{I}, \nu)$ with umbilical points of index greater than one appear in the complement of the Darbouxian generic class of the corresponding surface, namely in the bifurcation set studied here.

Given a Darbouxian ν-umbilical point $p \in M$, it is possible to find other types of Darbouxian configurations at p by only varying the parameters of the normal vector fields of the surface. The goal of this paper is to describe how these configurations appear in the space $\mathcal{N}_1$ of parameters of 1-jets of normal vector fields which define isolated umbilical points at $p \in M$. In fact we obtain a bifurcation diagram of principal configurations by determining the bifurcation set of codimensions one and two in this space. We describe topologically this set and prove that the only singularity which belongs to it is of cusp type.

This work was inspired by the local study of singularities of positive quadratic differential equations, not necessarily coming from geometry, presented by Guíñez and Gutierrez in [5], [6]. They obtain versal unfoldings of arbitrary nonlocally stable simple singularities: D_{12}, of codimension one and $\tilde{D}_1$ of codimension two. For each differential equation of lines of curvature we consider the corresponding positive quadratic differential form [Remark 3] and we obtain here versal unfoldings of nonlocally stable simple umbilics parametrized in the space $\mathcal{N}_1$. It turns out that they are versal in the space of positive quadratic differential forms and different from those determined by Guíñez and Gutierrez [Remark 7]. Furthermore, this can be stated geometrically as follows: *the variation of parameters of the versal unfolding of this type of singularities in the space of positive quadratic differential equations, corresponds to the variation of parameters of the 1-jet of the normal vector field.*

This paper is organized as follows. Section 2 contains some generalities used in the sequel. In Section 3 we prove the main theorem which describe the bifurcation set of ν-principal configurations, where ν is parametrized in the space of 1-jets of normal vector fields which define an isolated umbilical point. In Section 4 we give versal unfoldings of the nonlocally stable simple umbilical points.

2. PRINCIPAL CONFIGURATIONS AND UMBILICAL POINTS IN $\mathbb{R}^4$

2.1. ν-Principal Configurations

Let M be a smooth oriented surface immersed in $\mathbb{R}^4$ with the Riemannian metric induced by the standard Riemannian metric of $\mathbb{R}^4$. For each $p \in M$ consider the decomposition $T_p\mathbb{R}^4 = T_pM \oplus (T_pM)^{\perp}$, where $(T_pM)^{\perp}$ is the orthogonal complement of T_pM in $\mathbb{R}^4$. Let $\bar{\nabla}$ be the Riemannian connection of $\mathbb{R}^4$. Given local vector fields V, W on M, let $\bar{V}, \bar{W}$ be some local extensions to $\mathbb{R}^4$. The Riemannian connection of M is well defined by the tangent component of the Riemannian connection of $\mathbb{R}^4 : \nabla_V W = \left(\bar{\nabla}_{\bar{V}}\bar{W}\right)^{\top}$. Let $\mathcal{X}(M)$ and $\mathcal{X}(M)^{\perp}$ be the space of smooth vector fields on M and the space of smooth vector fields normal to M, respectively. Consider the map

$$\alpha : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)^{\perp}, \;\; \alpha(V,W) = \bar{\nabla}_{\bar{V}}\bar{W} - \nabla_V W.$$

This map is well defined, symmetric and bilinear.

If $p \in M$ and $\nu \in (T_pM)^{\perp}$, $\nu \neq 0$, define the function

$$H_\nu : T_pM \times T_pM \to \mathbb{R}, \; H_\nu(V,W) = \langle \alpha(V,W), \nu \rangle.$$

Then, this function as well is symmetric and bilinear. *The second fundamental form* of M at p is the associated quadratic form,

$$II_\nu : T_pM \to \mathbb{R}, \;\; II_\nu(V) = H_\nu(V,V).$$

Recall *the shape operator*

$$S_\nu : T_pM \to T_pM, \;\; S_\nu(V) = -\left(\bar{\nabla}_{\bar{V}}\bar{\nu}\right)^{\top},$$

where $\bar{\nu}$ is a local extension to $\mathbb{R}^4$ of the normal vector field ν at p and $\top$ means the tangent component. This operator is bilinear, self-adjoint and for any $V, W \in T_pM$ satisfies the following equation

$$\langle S_\nu(V), W\rangle = H_\nu(V, W).$$

Therefore, the second fundamental form can be expressed by

$$II_\nu(V) = \langle S_\nu(V), V\rangle.$$

Thus for each $p \in M$, there exists an orthonormal basis of eigenvectors of S_ν in T_pM, for which the restriction of the second fundamental form to the unitary vectors takes its maximal and minimal values. The corresponding eigenvalues k_1, k_2 are called the *maximal* and *minimal ν-principal curvatures*, respectively. The point p is *ν-umbilical* if the ν-principal curvatures coincide. Let $\mathcal{U}_\nu$ be the set of ν-umbilical points in M. For any point $p \in M \setminus \mathcal{U}_\nu$ there are two ν-principal directions defined by the eigenvectors of S_ν, these fields of directions are smooth and integrable. Then they define two families of orthogonal curves, their integrals, which are called the *ν-curvature lines*, one maximal and the other minimal. The two orthogonal foliations with the ν-umbilical points as their singularities form the *ν-principal configuration* of M, $\mathcal{P}_\nu = (\mathcal{U}_\nu, \mathcal{L}_\nu, l_\nu)$. The differential equation which defines the ν-principal configuration is

$$S_\nu(c'(t)) = \lambda(c(t))\, c'(t),$$

where $c : (-\delta, \delta) \to M$ is a smooth curve determining a ν-curvature line.

In order to study the local principal configurations of a surface immersed in $I\!R^4$ in a neigborhood of a ν-umbilical point p, let us introduce a system of coordinates x, y, z, w, where the immersed surface can be seen as the graph of a differentiable function $f : U \subset I\!R^2 \to I\!R^2$. In this case the parametrization has the form

$$X(u, v) = (u, v, \varphi(u, v), \psi(u, v)).$$

We may assume that p is the origin of $I\!R^4$, the tangent plane at p is the xy-plane, and the vector field ν coincides with $\frac{\partial}{\partial z}$ at the origin. The differential equation of ν-lines of curvature in this system is

$$(f_\nu E - e_\nu F)\ du^2 + (g_\nu E - e_\nu G)\ du\, dv + (g_\nu F - f_\nu G)\ dv^2 = 0, \quad (1)$$

where $E = \langle X_u, X_u\rangle$, $F = \langle X_u, X_v\rangle$, $G = \langle X_v, X_v\rangle$ are the coefficients of the first fundamental form and $e_\nu = \langle X_{uu}, \nu\rangle$, $f_\nu = \langle X_{uv}, \nu\rangle$, $g_\nu = \langle X_{vv}, \nu\rangle$ are the coefficients of the second fundamental form with respect to the normal vector field ν.

The principal configuration near the origin defined by the 1-jet of the differential equation of ν-lines of curvature is determined by the 3-jet of the coordinate functions of the parametrization X and the 1-jet ν_1 of the normal vector field ν. So, consider the third order Taylor polynomials of φ and ψ around p,

$$\begin{aligned} X(u,v) = (u, v, \tfrac{k}{2}\left(u^2+v^2\right) + \tfrac{a}{6}u^3 + \tfrac{d}{2}u^2v + \tfrac{b}{2}uv^2 + \tfrac{c}{6}v^3 \ , \\ \tfrac{\alpha}{2}u^2 + \tfrac{\gamma}{2}v^2 + \tfrac{\delta}{6}u^3 + \tfrac{\epsilon}{2}u^2v + \tfrac{\zeta}{2}uv^2 + \tfrac{\eta}{6}v^3), \end{aligned} \tag{2}$$

where $k, ..., c, \alpha, ..., \eta \in I\!R.$ The uv term of ψ has been eliminated by a rotation of the xy-plane.

Assuming that the normal vector field ν defines an isolated umbilic at the origin, in this parametrization the 1-jet of it has the form

$$\nu_1 = \left(-ku, -kv, 1, mu + nv\right), \tag{3}$$

where k is the value of any of the principal curvatures with respect to ν at the origin and m, n are real numbers. Following the proof of lemma 2.4 in [10] for the case of this parametrization of the surface, it is not difficult to prove that for any pair $(m, n) \in I\!R^2$ there is a normal vector field ν with its first jet ν_1 as above. Then, the 1-jet of the differential equation of lines of curvature is written in terms of these parameters as

$$A_1\left(u,v\right)\ dv^2 + B_1\left(u,v\right)\ du\,dv + C_1\left(u,v\right)\ du^2 = 0, \tag{4}$$

where

$$\begin{aligned} A_1\left(u,v\right) &= d\,u + b\,v, \\ B_1\left(u,v\right) &= \left(a - b + \left(\alpha-\gamma\right)m\right)u + \left(-c + d + \left(\alpha-\gamma\right)n\right)v, \\ C_1\left(u,v\right) &= -d\,u - b\,v. \end{aligned}$$

2.2. Simple Umbilical Points

Let $\mathcal{I} : M \to I\!R^4$ be a smooth immersion of the surface M in $I\!R^4$. Consider the projective line bundle $\Pi : P\mathcal{I}(M) \to \mathcal{I}(M)$, where $P\mathcal{I}(M)$ is defined in the following way. Let $T(M) - \{\mathbf{0}\}$ be the tangent bundle without the zero section and identify any two elements $(x, \bar{v})$ and $(y, \bar{w})$ which satisfy $x = y$ and $\bar{v} = \lambda\bar{w}$ for $\lambda \neq 0$. The set of equivalence classes defined by this identification constitute the set $P\mathcal{I}(M)$.

Let $\mathcal{I}\left(x\right)$ be an umbilical point in $\mathcal{I}\left(M\right) \subset I\!R^4$. Then there is a neighborhood $V\left(\mathcal{I}\left(x\right)\right)$ with a canonical parametrization (2) which identifies $\mathcal{I}\left(x\right)$ with 0. In these terms the ν-curvature lines are defined by the equation (4).

The space $P\mathcal{I}\left(M\right)$ can be parametrized by two coordinate charts:

$$(u, v; q = du/dv)\,, \text{ and } \ (u, v; p = dv/du)\,.$$

Equation (4) defines in $P\mathcal{I}(M)$ a surface $S(\mathcal{I}, \nu)$. In the coordinate chart $(u, v; p)$ this surface is defined by:

$$\begin{aligned}
\mathcal{F}(u, v; p) &= (d\,u + b\,v)\ p^2 \\
&\quad + ((a - b + (\alpha - \gamma)\,m)\,u + (-c + d + (\alpha - \gamma)\,n)\,v)\,p \\
&\quad - (d\,u + b\,v) \\
&= A_1(u, v)p^2 + B_1(u, v)p - A_1(u, v) \\
&= 0.
\end{aligned}$$

Let $\mathcal{U}_{\mathcal{I},\nu}$ be the set of ν-umbilical points of $\mathcal{I}(M)$. Outside $\Pi^{-1}\mathcal{U}_{\mathcal{I},\nu}$, $S(\mathcal{I}, \nu)$ is a regular surface of $P\mathcal{I}(M)$, in fact it is a double covering of $\Pi^{-1}\mathcal{I}(M) - \mathcal{U}_{\mathcal{I},\nu}$. If $0 \in \mathbb{R}^4$ is an umbilical point, the real projective line $\Pi^{-1}(0)$ is contained in $S(\mathcal{I}, \nu)$.

Condition (T). The pair $(\mathcal{I}, \nu)$ satisfies the *transversality condition* in 0 if the curves defined by the zeros of the coefficients $A_1(u, v)$ and $B_1(u, v)$ of equation (4) intersect in 0 transversally.

This is equivalent to the condition that $S(\mathcal{I}, \nu)$ be regular along $\Pi^{-1}(0)$ and clearly, this property is independent of the parametrization.
Define the vector field $\mathcal{F}'$ in $P\mathcal{I}(M)$ by

$$\mathcal{F}' = \mathcal{F}_p\partial/\partial u + p\mathcal{F}_p\partial/\partial v + (-\mathcal{F}_u - p\mathcal{F}_v)\partial/\partial p. \tag{5}$$

This vector field has the following properties:

1. $\mathcal{F}'$ is tangent to $S(\mathcal{I}, \nu)$.
2. $\Pi_*(\mathcal{F}')$ vanishes only in the origin.
3. Let $(u, v; p_i) \in S(\mathcal{I}, \nu)$, then $\Pi_*(\mathcal{F}'(u, v; p_i))$ generates the principal line with the direction p_i.
4. The eigenvalues of the linear part of the vector field $\mathcal{F}'$ at $(0, 0, p_i)$ are:

(1) $\beta_1 = 0$,
(2) $\beta_2 = 2b - a - (\alpha - \gamma)m + 2(c - 2d - (\alpha - \gamma)n)p_i - 3bp_i^2$,
(3) $\beta_3 = (a - b + (\alpha - \gamma)m) + (3d - c + (\alpha - \gamma)n)p_i + 2bp_i^2$.

The singularities of $\mathcal{F}' \mid S(\mathcal{I}, \nu)$ in $\Pi^{-1}(0, 0)$ are the roots of the polynomial

$$f(p) = bp^3 + (2d - c + (\alpha - \gamma)\,n)\,p^2 + (a - 2b + (\alpha - \gamma)\,m)\,p - d. \tag{6}$$

The polynomial f defined by (6) will be called the *separatrix polynomial* of the pair $(\mathcal{I}, \nu)$. Its roots are the tool we use to describe the type of the principal configuration. In fact, in the generic Darbouxian class defined below, they are in direct correspondence with the tangent directions of the lines of curvature which contain the origin in their closure. These directions are called principal approximations and their number is determined by the sign of the discriminant Δ of the separatrix polynomial (6). See [10], Section 3.

Definition 1. The pair $(\mathcal{I}, \nu)$ satisfies the *discriminant condition* in an umbilical point, if the discriminant Δ of the separatrix polynomial (6) satisfies one of the following cases:

(i) $D_1 : \Delta > 0$ and $\beta_2\beta_3 < 0$ for the unique real root of $f(p)$.

(ii) $D_2 : \Delta < 0$ and $\beta_2\beta_3 < 0$ for two roots of $f(p)$ and $\beta_2\beta_3 > 0$ for the other.

(iii) $D_3 : \Delta < 0$ and $\beta_2\beta_3 < 0$ for the three roots of $f(p)$.

The cases D_1, D_2 and D_3 forms the Darbouxian class of $(\mathcal{I}, \nu)$.

These conditions determine the topological behavior near the umbilic in the following way: The previous construction gives a blowing up of the umbilic with the vector field $\mathcal{F}'$ tangent to the pull back of the lines of curvature. For the case D_1, we have only one singularity at the divisor $(0, 0, p)$ of saddle type, defining just one principal approximation. For the case D_2 we have three singularities along the divisor, two saddles and one node, defining two principal approximations. Finally, for the case D_3, we have three saddle type singularities along the divisor defining three principal approximations. See [10], Proposition 3.8. Obviously these conditions are invariant under rotations of the uv-plane.

Remark 2. If one of the roots of $f(p)$ is zero, the expression of the separatrix polynomial in this coordinate chart can be written as $f(p) = p\ f_2(p)$, and the relation between the discriminant Δ_2 of the quadratic factor $f_2(p)$ and Δ is given by $\Delta = -\rho^2\Delta_2$, where ρ is a function of the parameters.

Now, in order to define the simple umbilical points of a surface we need some preliminary concepts.

A C^∞ *quadratic differential form* on an oriented, connected, smooth surface M is defined by elements of the cotangent bundle in the form $\omega = \sum_{i=1}^{n} \varphi_i\psi_i$ where φ_i and ψ_i are differential 1-forms on M of class C^∞. For each point p in M,

$$\omega(p) = \sum_{i=1}^{n} \varphi_i(p)\,\psi_i(p) : T_pM \to I\!R$$

is the quadratic form on the tangent space T_pM defined by

$$\omega\left(p\right)\left(V\right)=\sum_{i=1}^{n}\varphi_i\left(p\right)\left(V\right)\cdot\psi_i\left(p\right)\left(V\right),$$

for any $V\in T_pM$. If $X:N\to M$ is a C^∞ diffeomorphism and ω is a C^∞ quadratic differential form on M, we denote by $X^*\left(\omega\right)$ the C^∞ quadratic differential form on N defined by

$$X^*\left(\omega\right)(q)(\xi)=\omega\left(X(q)\right)\left(dX_q(\xi)\right),$$

for q in N and ξ in T_qN. We denote also this form simply by ω^* when there is no confussion about the diffeomorphism X.

We say that ω is *positive* if for every point p in M the subset $\omega\left(p\right)^{-1}(0)$ of T_pM is either the union of two transversal lines (in this case p is called a *regular point* of ω) or all of T_pM (in this case p is called a *singular point* of ω).

Each positive quadratic differential form ω defines a *configuration*

$$C\left(\omega\right)=\{\mathit{Sing}\left(\omega\right),f_1\left(\omega\right),f_2\left(\omega\right)\},$$

where $\mathit{Sing}(\omega)$ is the set of singular points of ω and $f_1\left(\omega\right),\ f_2\left(\omega\right)$ are the transversal foliations on $M-\mathit{Sing}(\omega)$ whose tangent lines in each regular point p are given by the transversal lines of $\omega\left(p\right)^{-1}(0)$.

Let $Q\left(M\right)$ be the manifold made up of the pairs (p,α) such that $p\in M$ and $\alpha=\sum_{i=1}^{n}\phi_i\psi_i$, with ϕ_i and ψ_i in the cotangent space $(T_pM)^*$. Then every C^∞ quadratic differential form ω on M can be considered as a C^∞ section $\omega:M\to Q\left(M\right)$. With this representation, the usual derivative of ω at each p in M, $D\omega_p$, is a quadratic differential form on T_pM. See [4], p. 479.

Let ω be a positive quadratic differential form on M. A singular point p of ω is said to be *simple* if $D\omega_p$ is a positive quadratic differential form on T_pM. Every positive quadratic differential form ω on M with a simple singular point p may be expressed in an appropiate local chart $(u,v):(M,p)\to\left(I\!R^2,0\right)$ in the form

$$\begin{aligned}(u,v)^*\left(\omega\right)=\left(a_1u+a_2v+M_1\left(u,v\right)\right)dv^2+\left(b_1u+b_2v+M_2\left(u,v\right)\right)dudv\\+\left(c_1u+c_2v+M_3\left(u,v\right)\right)du^2,\end{aligned}$$

with $M_i\left(u,v\right)=\mathcal{O}(\left(u^2+v^2\right)^{1/2}),\ i=1,2,3,$ where

$$b_1^2-4a_1c_1>0,$$

and

$$(b_1^2 - 4a_1c_1)(b_2^2 - 4a_2c_2) - (b_1b_2 - 2a_1c_2 - 2a_2c_1)^2 > 0,$$

and this holds for every local chart at $p \in M$. See [4], Proposition 5.1.

Remark 3. The differential equation (1) of lines of curvature of a surface M immersed in $I\!\!R^4$ by $\mathcal{I} : M \to I\!\!R^4$ with normal umbilical vector field ν is defined by a positive quadratic differential form ω on M, and the configuration of ω corresponds to the ν-principal configuration of the surface M, i.e., $\{Sing(\omega), f_1(\omega), f_2(\omega)\}$ coincide with $(\mathcal{U}_\nu, \mathcal{L}_\nu, l_\nu)$.

By using the preceding inequalities and the canonical parametrization of the surface (2) it is easy to see that if p is an isolated generic ν-umbilical point of M, it will be *simple* if the following inequalities holds

$$4d^2 + (a - b + (\alpha - \gamma)m)^2 > 0, \tag{7}$$

$$((a - b)b + (c - d)d + (\alpha - \gamma)(bm - dn))^2 > 0. \tag{8}$$

The cases lying on the boundary of the open sets defined by the discriminant condition are interesting from the bifurcation point of view. They will be studied here.

DEFINITION 4. Let p be a simple umbilical point of a surface M immersed in $I\!\!R^4$ and f the separatrix polynomial of the pair $(\mathcal{I}, \nu)$, where $\mathcal{I} : M \to I\!\!R^4$ is the immersion at p with normal umbilical vector field ν. Then the point p is called

(a) *Darbouxian* if f has only simple roots and condition (T) holds.
(b) D_{12} if f has one simple and one double root.
(c) $\tilde{D}_1$ if f has a triple root.

3. BIFURCATION DIAGRAM

Consider an arbitrary local surface M immersed in $I\!\!R^4$ with an isolated umbilical point p. Generically any perturbation of the normal direction in $(T_pM)^\perp$ makes p non umbilic. We assume this direction fixed and only vary the first jet of the normal vector field with the condition of keeping the property of defining an isolated umbilic at p. Let $\mathcal{N}_1$ denote the set of the 1-jet of normal ν-umbilical vector fields at p. The normal form of $\nu \in \mathcal{N}_1$ given by (3) allows us to identify $\mathcal{N}_1$ with $I\!\!R^2$. Let (m, n) be the coordinates of an element of $\mathcal{N}_1$. Using the canonical parametrization (2) of the surface M around p and the inequalities (7) and (8), we see that for $d \neq 0$ the inequality (7) is satisfied for all $(m, n) \in \mathcal{N}_1$. Thus, to obtain the set of non simple points only remains to see where the second inequality (8) is not satisfied, namely the points $(m, n) \in \mathcal{N}_1$ which satisfy

$$n = \left(\frac{b}{d}\right) m + \frac{(a-b)\,b + (c-d)\,d}{(\alpha-\gamma)\,d}, \quad (\alpha-\gamma)\,d \neq 0. \tag{9}$$

Theorem 5. *Let $\nu \in \mathcal{N}_1$ with coordinates (m, n) and $d \neq 0,\ b \neq 0$. The bifurcation set of codimensions one and two, of the isolated ν-umbilical points in the space $\mathcal{N}_1$ is the line (9) and a real algebraic curve Γ which has two connected components. One of them has a singular cusp point and the other one is a smooth curve diffeomorphic to a line. The singular point of Γ is of type $\tilde{D}_1$ of codimension two and the regular points of Γ are of type D_{12} of codimension one except at the unique tangency point T to the line (9). See Figure 1.*

FIG. 1. Bifurcation set in the space $\mathcal{N}_1$.

Proof. The proof is presented in three steps. In the first one we obtain a convenient description of the set of points of type D_{12} and $\tilde{D}_1$ in the space $\mathcal{N}_1$. This set consists of a real algebraic curve Γ of degree four. In the second part we prove a lemma which describes the topological properties of the curve Γ stablished in the statement of the theorem. In the third part we analyse the types of principal configurations along Γ. We perform some symbolic computations in MATHEMATICA to obtain precise expressions in this proof.

First Part. We are going to determine the subset of $\mathcal{N}_1$ where we have bifurcations of type D_{12} and $\tilde{D}_1$, which is characterized by the multiplicity of the roots of the separatrix polynomial (6). For the first one there are one single and one double roots and for the second one there is just one triple root.

We shall determine the discriminant of (6) as a function of the parameters m, n in a proper coordinate chart in order to determine its locus, where the multiple roots appear.

Let $\mathcal{P}$ denote the set of separatrix polynomials written in the form

$$f(p) = p^3 + \lambda_2 p + \lambda_1.$$

We can identify $\mathcal{P}$ with $\mathbb{R}^2$ giving coordinates (λ_1, λ_2) to an element of $\mathcal{P}$.

Now, consider the mapping

$$F : \mathcal{N}_1 \to \mathcal{P},\ F\left(m, n\right) = \left(\lambda_1, \lambda_2\right), \tag{10}$$

defined by

$$\lambda_1 = \frac{1}{27b^3}\{(2d - c + (\alpha - \gamma)n)\left(2\left(2d - c + (\alpha - \gamma)n\right)^2\right. \tag{11}$$

$$\left. -9b\left(a - 2b + (\alpha - \gamma)m\right)\right) - 27b^2 d\},$$

and

$$\lambda_2 = \frac{1}{3b^2}\{3b\left(a - 2b + (\alpha - \gamma)m\right) - \left(2d - c + (\alpha - \gamma)n\right)^2\}. \tag{12}$$

The change of parameters defined by the mapping F is the composition $F = F_2 \circ F_1$, where $F_1 : \mathcal{N}_1 \to \mathbb{R}^4$ is defined by

$$F_1\left(m, n\right) = \left(c_0, c_1, c_2, c_3\right),$$

where $c_i,\ i = 0, 1, 2, 3$, are the coefficients of the terms of i-th order of the separatrix polynomial (6), namely

$$\begin{aligned} c_0 &= -d, \\ c_1 &= a - 2b + (\alpha - \gamma)m, \\ c_2 &= 2d - c + (\alpha - \gamma)n, \\ c_3 &= b, \end{aligned}$$

and $F_2 : \mathbb{R}^4 \to \mathcal{P}$ is defined by

$$F_2\left(c_0, c_1, c_2, c_3\right) = \left(\frac{c_2\left(2c_2^2 - 9c_1c_3\right) + 27c_0c_3^2}{27c_3^3}, \frac{3c_1c_3 - c_2^2}{3c_3^2}\right).$$

The function F_2 corresponds to the Vietà transformation which eliminates the quadratic term of the separatrix polynomial (6) after make it monic, yielding the form $f(p) = p^3 + \lambda_2 p + \lambda_1$. Then, the discriminant of (6) is

$$\Delta(\lambda_1, \lambda_2) = 4\lambda_2^3 + 27\lambda_1^2. \tag{13}$$

We see that the subset of $\mathcal{N}_1$ where multiple roots occur is given by the inverse image $\Gamma = F^{-1}(C)$ of the subset C of $\mathcal{P}$ where the discriminant vanishes,

$$C = \{(\lambda_1, \lambda_2) \in \mathcal{P} \mid \Delta(\lambda_1, \lambda_2) = 0\}. \tag{14}$$

Note that the mapping F given by (10) is an smooth covering almost everywhere, whose jacobian

$$J = \frac{1}{3b^3}(\gamma - \alpha)(a - 2b + (\alpha - \gamma)m),$$

vanishes only on the line $m = (a - 2b)/(\gamma - \alpha)$. So F is a local diffeomorphism outside its singular set

$$S_1(F) = \left\{(m, n) \in \mathbb{R}^2 \mid m = \frac{a - 2b}{\gamma - \alpha}\right\}.$$

Second Part. The subset Γ of $\mathcal{N}_1$ is defined by the polynomial equation $4\lambda_2^3 + 27\lambda_1^2 = 0$ with λ_1, λ_2 replaced by (11) and (12) respectively. Denote by $f(m, n) = 0$ this equation.

It is known that a real algebraic curve Γ defined by a polynomial equation of degree ≥ 2 in two variables consists of at most finitely many components. More precisely, when the curve is real non-singular each of its unbounded components are homeomorphic to a line and each of its bounded components are homeomorphic to a circle. On the other hand, if the curve Γ is real singular and $\sum$ is the set of its real singular points, then $\Gamma - \sum$ is a differentiable 1-manifold.

In this part we describe the topological properties of the curve Γ.

Lemma 5.1 *The real algebraic curve $\Gamma = F^{-1}(C)$, where F is the mapping given by (10), has two connected components. One of them has a singularity of cusp type and the other one is a smooth curve diffeomorphic to a line.*

Proof. Consider the translation of the mn-plane which takes the origin to

$$V = \frac{1}{\gamma - \alpha}\left(a - 2b - 3\left(bd^2\right)^{1/3}, 2d - c + 3\left(b^2 d\right)^{1/3}\right).$$

Then the algebraic set $\Gamma = F^{-1}(C)$ is expressed in the new system of coordinates (x, y) as

$$f(x,y) = f_2(x,y) + f_3(x,y) + f_4(x,y) = 0, \tag{15}$$

where $f_i(x,y)$, $i = 2,3,4$, are homogeneous polynomials of degree i given by

$$\begin{aligned}
f_2(x,y) &= 27(bd)^{2/3}\left(\sqrt[3]{b}x + \sqrt[3]{d}y\right)^2, \\
f_3(x,y) &= 2(\alpha-\gamma)\left(2\left(\sqrt[3]{b}x - \sqrt[3]{d}y\right)^3 + 9(bd)^{1/3}\left(\sqrt[3]{b}x - \sqrt[3]{d}y\right)xy\right), \\
f_4(x,y) &= -(\gamma-\alpha)^2 x^2y^2.
\end{aligned}$$

From this we see that the origin is a singular double point of the curve (15), and the line

$$\sqrt[3]{b}x + \sqrt[3]{d}y = 0$$

is a double tangent line (and the only one tangent) at the origin. Furthermore, the rotation of the xy-plane with angle $\theta = -\arctan\sqrt[3]{b/d}$ transforms (15) to $\tilde{f}(x,y) = 0$, which defines a curve whose tangent line at the origin is the horizontal axis, and such that

$$\frac{\partial^3\tilde{f}}{\partial x^3}(0,0) = \frac{24bd(\gamma-\alpha)}{\left[b^{2/3}+d^{2/3}\right]^{3/2}} \neq 0.$$

These properties imply that the curve Γ has a cusp point at V. See [3], p. 82.

Now in order to describe the connected components of Γ we will determine the intersection of the vertical lines $x = \textit{constant}$ with this curve, since $f(x,y) = 0$ is a cubic equation in y with coefficients defined by real functions in x. This can be seen by writing the polynomial equation (15) in the form

$$a_3(x)y^3 + a_2(x)y^2 + a_1(x)y + a_0(x) = 0, \tag{16}$$

where

$$\begin{aligned}
a_3(x) &= 4d(\gamma-\alpha), \\
a_2(x) &= 27\left(b^2d^4\right)^{1/3} + 6(\gamma-\alpha)\left(bd^2\right)^{1/3}x - (\gamma-\alpha)^2x^2, \\
a_1(x) &= 54bdx - 6(\gamma-\alpha)\left(b^2d\right)^{1/3}x^2, \\
a_0(x) &= 27\left(b^4d^2\right)^{1/3}x^2 - 4(\gamma-\alpha)bx^3.
\end{aligned}$$

Now, for $x = 0$ we have $a_3(0) = 4d(\gamma - \alpha)$, $a_2(0) = 27(b^2d^4)^{1/3}$ and $a_1(0) = a_0(0) = 0$. Thus, for $x = 0$, equation (16) reduces to

$$4d\,(\gamma - \alpha)\,y^3 + 27\left(b^2d^4\right)^{1/3} y^2 = 0,$$

which has the solutions $y_1 = y_2 = 0$, and $y_3 = -\frac{27\left(b^2d\right)^{1/3}}{4(\gamma-\alpha)} \neq 0$ (one simple and one double root).

Claim: The sign of the discriminant $\Delta\,(x)$ of (16) as a cubic equation in y behaves like the sign of x.

Therefore, for $x < 0$, then $\Delta < 0$ and there are three different real solutions. For $x = 0$, we have $\Delta = 0$ and there are one simple real solution and one double solution equal to zero as we have seen. And for $x > 0$, $\Delta > 0$ and there is only one real solution.

Now, since Cardano's formula gives differentiable solutions out of the locus of the discriminant, namely the line $x = 0$, it follows that $\Gamma - \{V\}$ consists of regular curves. This complete the topological description of Γ. See Figure 1.

To finish we will prove the claim in the following way: The discriminant of equation (16) depends on x as

$$\Delta\,(x) = 4b\,(\gamma - \alpha)\,xh\,(x)\,, \tag{17}$$

where

$$\begin{aligned} h(x) \;=\; & (\gamma-\alpha)^6x^6 - 27(bd^2)^{1/3}(\gamma-\alpha)^5x^5 + 324(b^2d^4)^{1/3}(\gamma-\alpha)^4x^4 \\ & -2187bd^2(\gamma-\alpha)^3x^3 + 8748(b^4d^8)^{1/3}(\gamma-\alpha)^2x^2 \\ & -19683(b^5d^{10})^{1/3}(\gamma-\alpha)x + 19683b^2d^4. \end{aligned}$$

From (17) we can see that the only possibility of multiple roots for (16) is where $x = 0$ or $h\,(x) = 0$.

We will prove that the function h is strictly positive for all $x \in I\!R$. Then by (17) the sign of $\Delta\,(x)$ and of x is the same assuming $b > 0$, $\gamma > \alpha$. (For the other cases the proof is analogous).

A straightforward computation shows that for $x_0 = 9(bd^2)^{1/3}/2(\gamma - \alpha)$, we have $h'\,(x_0) = 0$, and $h''\,(x_0) > 0$. Thus, x_0 is a critical point for which $h\,(x_0)$ is a local minimum. Furthermore, $h\,(x_0) > 0$. In fact, we will show that x_0 is the only critical point, and $h\,(x_0)$ is an absolute minimum.

For this, write $h'\,(x) = (x - x_o)\,h_1\,(x)$. Then, it is easy to find that

$$h_1\,(x) = 6\,(\gamma - \alpha)^2\,h_2\,(x)\,,$$

where

$$\begin{aligned} h_2(x) &= (\gamma-\alpha)^4 x^4 - 18\left(bd^2\right)^{1/3}(\gamma-\alpha)^3 x^3 + 135\left(b^2d^4\right)^{1/3}(\gamma-\alpha)^2 x^2 \\ &\quad -486bd^2(\gamma-\alpha)x + 729\left(b^4d^8\right)^{1/3}. \end{aligned}$$

It only remains to show that $h_2(x)$ has no real roots. From the Ferrari method for solving the quartic, we have that the solutions to the equation $h_2(x)=0$ are the solutions to the equation $(x^2+px+k)^2=0$, where

$$p = -\tfrac{9(bd^2)^{1/3}}{\gamma-\alpha}, \quad k = \tfrac{27(b^2d^4)^{1/3}}{(\gamma-\alpha)^2}.$$

Since the discriminant of the quadratic equation $x^2+px+k=0$ is

$$\Delta = -27\left(\frac{b^{1/3}d^{2/3}}{\gamma-\alpha}\right)^2 < 0,$$

the proof of the lemma is complete. ▌

Third Part. In this part we begin showing that the points of type D_{12} lie on $\Gamma - \{V,T\}$, where V is the cusp point of Γ,

$$V = \frac{1}{\gamma-\alpha}\left(a-2b-3\left(bd^2\right)^{1/3}, 2d-c+3\left(b^2d\right)^{1/3}\right),$$

and

$$T = \frac{1}{\gamma-\alpha}\left(\frac{ab-d^2}{b}, \frac{b^2-cd}{d}\right),$$

is the unique point in the intersection of the non simple line (9) and the curve Γ, as we can easily see by straightforward computations. Since it is the unique non simple point of Γ, we have that the points D_{12} and $\tilde{D}_1$ lie on $\Gamma - \{T\}$.

A direct computation shows that $F(V)=(0,0)$. We claim that there is no other point in $F^{-1}(0,0)$ different from V. Suppose there is a point V' in $F^{-1}(0,0)$ such that $V' \neq V$.

A straightforward computation shows that if $V_1 \in S_1(F)$, then $F(V_1) \neq (0,0)$. Therefore $V' \notin S_1(F)$. It follows that F is a local diffeomorphism at V'. This property of F and the fact that $(0,0)$ is a cusp point of the curve C implies that V' is also a cusp point of Γ but there is no other cusp point of Γ different from V, see Lemma 5.1.

Furthermore, the unique point of type $\tilde{D}_1$ on the space $\mathcal{P}$ is the origin with preimage $F^{-1}(0,0)=\{V\}$ as we have seen. Thus, the points of type

D_{12} in $\mathcal{N}_1$ are the simple points of $\Gamma - \{V\}$, namely, all the points along $\Gamma - \{V, T\}$.

Finally, the tangency of the curve Γ with the non simple line (9) at the point T stated in Theorem 5 is easily showed. Using the Implicit Function Theorem we can compute the slope of Γ at that point, which is b/d. This is the slope of the non simple line. The proof of Theorem 5 is now complete. ▮

Corollary 6. *The bifurcation diagram of the ν-principal configurations in the space $\mathcal{N}_1$ is given in Figure 2 for $d > 0,\ b > 0,$ and $\gamma > \alpha$.*

FIG. 2. Bifurcation diagram of principal configurations in $\mathcal{N}_1$.

Proof. Using the fact that the Darbouxian types of principal configurations define open components on the complement of the curves Γ and the non simple line (see [10], Theorem 3.9), it is enough to determine the type of principal configuration in any particular point of each component. The complement of these curves define five regions. See Figure 3.

So we make a clever choice of convenient points along the double tangent line to the cusp component of the curve Γ to test the regions 1, 2, and 3 of Figure 3 which becomes of type D_2, D_1, and D_3 respectively. Since region 2 is of type D_1 and region 3 is of type D_3, then regions 4 and 5 have to be of type D_2 if $\Delta < 0$ for these regions. This happens because any unfolding of a D_{12} simple umbilical point cannot deform a D_1 type into a D_3 type.

See [6]. The points used for this analysis are the following:

$$P_1 = \frac{1}{\rho}(-x_0, y_0),\ P_2 = \frac{1}{\rho}(x_0, -y_0),$$

$$P_3 = \frac{1}{\rho}(\frac{27}{4}x_0, -\frac{27}{4}y_0),$$

$$P_4 = \frac{1}{\rho}(0, -7y_0),\ P_5 = \frac{1}{\rho}(7x_0, 0),$$

where $x_0 = (bd^2)^{1/3}$, $y_0 = (b^2d)^{1/3}$ and $\rho = \gamma - \alpha$.

FIG. 3. Regions of Darbouxian Configurations.

For each of these test points P_i we obtain the corresponding normal vector field, the sign of the discriminant Δ of the separatrix polynomial f, which in each case is obtained by direct computations, the roots of f, and the eigenvalues of the vector field $\mathcal{F}'$ given by (5). Then the type of Darbouxian principal configuration is determined for the points $P_i,\ i = 1, 2, 3$ by the sign of these eigenvalues and the sign of Δ according to Definition 1 and for the points P_4 and P_5 only by the sign of Δ. ▮

Remark 7. The bifurcation diagram shown in Figure 2 will help us to determine versal unfoldings of the umbilical points of type D_{12} and $\tilde{D}_1$ in the space $\mathcal{N}_1$ (section 4). If for each differential equation we consider the corresponding positive quadratic differential form [Remark 3, section 2], then our unfoldings turn out to be versal in the space of positive quadratic differential forms. In [6] versal unfoldings of the general singular types D_{12} and $\tilde{D}_1$ in the space of positive quadratic differential forms are obtained. They are different to ours. However, for the case $\tilde{D}_1$ of codimension 2 it is shown, Remark 5.8, [6], that the bifurcation set of singularties of type D_{12} in that versal unfolding is a curve of cusp type, similar to the cusp component of the curve Γ which appears in our case. Nevertheless, those versal unfoldings do not define families of differential equations of lines of curvature, necessary in our case.

4. VERSAL UNFOLDINGS

In this section, we obtain versal unfoldings of the D_{12} and $\tilde{D}_1$ umbilical points.

Recall that two positive quadratic differential forms ω_1 and ω_2 are *equivalent* if there is an homeomorphism $h : M \to M$ such that $h(Sing(\omega_1)) = Sing(\omega_2)$ and maps leaves of $f_1(\omega_1)$ and $f_2(\omega_1)$ onto leaves of $f_1(\omega_2)$ and $f_2(\omega_2)$, respectively.

Since we have obtained the bifurcation diagram of the principal configurations on M defined in $\mathcal{N}_1$, then transversality to the bifurcation set provides us very good candidates for versal families. Let us begin with the D_{12} case.

Proposition 8. *Consider the surface parametrized by*

$$X(u,v) = \left(u, v, \frac{k}{2}\left(u^2+v^2\right) + \frac{1}{3}u^3 + \frac{1}{2}u^2 v + \frac{1}{2}uv^2, \frac{1}{2}u^2 + v^2\right).$$

Then the 1-parameter family of differential equations of lines of curvature

$$\vartheta_\lambda = (u+v)\ dv^2 + ((1-\lambda)u - 4v)\ du\,dv - (u+v)\ du^2 = 0,$$

defined by the family of normal vector fields

$$\nu_\lambda(u,v) = (-ku, -kv, 1, \lambda u + 5v),$$

is a versal unfolding of an umbilic of type D_{12} with $\lambda \in I\!\!R,\ 0 < \lambda < 5$.

Proof. From the translation defined in the proof of Lemma 5.1 we can see that for the parametrization X and the normal vector field ν_λ considered here, the line $n = 5$ becomes the horizontal axis in Figure 2. Now, because the line $n = 5$ represents the 1-parameter family ν_λ in the space $\mathcal{N}_1$ and this line intersects transversally the regular component of the curve Γ when $\lambda = 15/4$ and the non simple line when $\lambda = 5$, Corollary

6 implies that for the family ϑ_λ the umbilic is of type D_1, D_{12} or D_2 according to $0 < \lambda < 15/4$, $\lambda = 15/4$ or $15/4 < \lambda < 5$, respectively. This suggests the versality from the transversality of the family ϑ_λ to the bifurcation set at the value $\lambda = 15/4$. See [1], Section 5.3. Now, following the approach given in [6], we give a proof. For this, consider an arbitrary smooth family of equations of lines of curvature

$$\omega_s = a\,(u,v,s)\,dv^2 + 2b\,(u,v,s)\,dudv + c\,(u,v,s)\,du^2,$$

with parameter $s \in I\!R^k$ such that ω_o has at the origin a D_{12} singular point. Using the normal form obtained in Lemma 5.6 [6], it can be proved (Proposition 4.1, [6]) that for $\mid s \mid$ small this family ω_s is topologically equivalent to the family

$$\tilde{\omega}_s = v\;dv^2 + 2(B_1(s)\;u + B_2(s)\;v)\;du\;dv - v\;du^2.$$

Let us consider the real valued function $\psi : \mathcal{U} \subset I\!R^k \to I\!R$ defined in a neighborhood $\mathcal{U}$ of the origin of $I\!R^k$ by

$$\psi\,(s) = -\frac{1}{4}B_2^2(s) + \frac{1}{2}B_1(s) + \frac{7}{2},$$

and the family

$$\vartheta_\lambda = (u+v)\;\;dv^2 + ((1-\lambda)\,u - 4v)\;\;du\,dv - (u+v)\;\;du^2 = 0.$$

with parameter $\lambda \in I\!R,\ 0 < \lambda < 5$.

Then the unfolding induced by ψ from the family $(\vartheta_\lambda)_{\lambda \in I\!R}$ is

$$\bar{\omega}_s = \vartheta_{\psi(s)} = (u+v)dv^2 + ((1-\psi\,(s)\,u - 4v)dudv - (u+v)du^2.$$

Now, because the function ψ is continuous and $\psi\,(0) = \frac{15}{4}$, there exists a neighborhood of the origin (smaller than $\mathcal{U}$ if necessary) such that $\psi\,(s) > 0$ in that neighborhood. Since the discriminants of the separatrix polynomials of the families $\tilde{\omega}_s$ and $\bar{\omega}_s$ are

$$\tilde{\Delta} = 4\left(B_2^2(s) - 2B_1(s) + 1\right) \quad \text{and} \quad \bar{\Delta} = (3+\psi(s))^2(15 - 4\psi(s))$$

respectively, it follows that $\tilde{\Delta}$ and $\bar{\Delta}$ only differ by a positive factor. This implies that the family $\tilde{\omega}_s$ is topologically equivalent to $\bar{\omega}_s$. Therefore the family ϑ_λ is a versal unfolding of a singularity of D_{12} type. ▮

Next we are going to construct a versal unfolding of the nonlocally stable simple umbilical point of codimension 2 of type $\tilde{D}_1$. First, let

$$\Gamma_1(m,n) = 0,\ \Gamma_2(m,n) = 0$$

be the connected components of the curve Γ defined in Section 3 with Γ_2 being the regular component. Second, denote by $n_2 : \mathbb{R} \to \mathbb{R}$ the implicit function defined by $\Gamma_2(m, n_2(m)) = 0$. Finally, define

$$U_2 = \{(m,n) \in \mathbb{R}^2 \mid n > n_2(m)\}.$$

Then we have the following proposition.

Proposition 9. *Consider the surface parametrized by*

$$X(u,v) = \left(u, v, \frac{1}{2}k(u^2+v^2) + \frac{1}{3}u^3 + \frac{1}{2}u^2v + \frac{1}{2}uv^2, \frac{1}{2}u^2 + v^2\right).$$

Then, the two parameter family of differential equations of lines of curvature

$$\vartheta_{(m,n)} = (u+v)\ dv^2 + ((1-m)\,u + (1-n)v)\ du\ dv - (u+v)\ du^2 = 0,$$

defined by the family of normal vector fields

$$\nu_{(m,n)}(u,v) = (-ku, -kv, 1, mu+nv),$$

is a versal unfolding of an umbilic of type $\tilde{D}_1$ with $(m,n) \in U_2$.

Proof. First, the separatrix polynomial of the family $\vartheta_{(m,n)}$ has discriminant

$$\Delta(m,n) = -5 - 36m - 4m^2 - 4m^3 + 48n + 18mn + 4m^2n - 24n^2 - m^2n^2 + 4n^3.$$

Then the curve Γ of Lemma 5.1 is $\Delta(m,n) = 0$ for this family and Corollary 6 implies that for $(m,n) \in U_2$ the origin is an umbilical point of $\vartheta_{(m,n)}$ of type:

(a) D_1 if $\Delta(m,n) > 0$.
(b) D_2 if $\Delta(m,n) < 0$.
(c) D_{12} if $(m,n) \in \{\Gamma_1(m,n) = 0\}$ and $(m,n) \neq (-3,5)$.
(d) $\tilde{D}_1$ if $(m,n) = (-3,5)$.

See Figure 2.
Now, let

$$\omega_s = a(u,v,s)dv^2 + 2b(u,v,s)du\ dv + c(u,v,s)du^2$$

be an arbitrary family with parameter $s \in \mathbb{R}^k$ such that ω_0 has an umbilic point of type $\tilde{D}_1$ at the origin.

Using the normal form obtained in Lemma 5.6 [6], it can be proved (Proposition 4.1, [6]) that for $\mid s \mid$ small this family ω_s is topologically equivalent to the family

$$\tilde{\omega}_s = (A_1(s)u + v)dv^2 + (u - A_1(s)v)du\, dv + (C_1(s)u + (-1 + C_2(s))\, v)du^2.$$

Now, consider the function $\psi : \mathcal{U} \subset I\!\!R^k \to I\!\!R^2$ defined in a neighborhood $\mathcal{U}$ of the origin by

$$\psi(s) = (\psi_1(s), \psi_2(s)) = F^{-1}\left(C_1(s), C_2(s)\right),$$

where F is the map defined by (10).

Then, the unfolding induced by the family $\vartheta_{(m,n)}$ via ψ is given by

$$\tilde{\omega}_s = \vartheta_{\psi(s)} = (u+v)\, dv^2 + [(1-\psi_1(s))\, u + (1-\psi_2(s))\, v]\, du\, dv - (u+v)\, du^2,$$

with the separatrix polynomial

$$\bar{f}_s\,(u,v) = v^3 + (2-\psi_2(s))\, v^2 u - \psi_1(s) v u^2 - u^3,$$

whose discriminant depends on the parameter $s \in I\!\!R^k$ as

$$\begin{aligned}\bar{\Delta}_s \;=\; & -5 - 36\psi_1(s) - 4\psi_1^2(s) - 4\psi_1^3(s) + 48\psi_2(s) + 18\psi_1(s)\psi_2(s)\\ & + 4\psi_1^2(s)\psi_2(s) - 24\psi_2^2(s) - \psi_1^2(s)\psi_2^2(s) + 4\psi_2^3(s).\end{aligned}$$

Now, because the separatrix polynomial of the family $\tilde{\omega}_s$ is

$$\tilde{f}_s\,(u,v) = v^3 + C_2(s) v u^2 + C_1(s) u^3,$$

with discriminant

$$\tilde{\Delta}_s = 4C_2^3(s) + 27C_1^2(s),$$

we have that for every $s \in \mathcal{U} \subset I\!\!R^k$ there exists neighborhoods $\mathcal{U}_1, \mathcal{U}_2$ of the origin and of the point $(-3, 5)$ in $I\!\!R^2$ respectively such that the sign of $\tilde{\Delta}_s$ and of $\bar{\Delta}_s$ are the same.

This implies that the family $\tilde{\omega}_s$ is topologically equivalent to the family $\bar{\omega}_s$.

Therefore the 2-parameter family $\vartheta_{(m,n)}$ is a versal unfolding of an umbilic of type $\tilde{D}_1$. ▮

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