QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **2**, 289–306 (2001)
ARTICLE NO. 21

Possible Jumps of Entropy for Interval Maps*

Michał Misiurewicz

Department of Mathematical Sciences,
IUPUI, 402 N. Blackford Street,
Indianapolis, IN 46202-3216
E-mail: mmisiure@math.iupui.edu

Submitted by J. Llibre

The paper deals with the question for which piecewise monotone interval maps topological entropy can jump up under small perturbations preserving the number of pieces of monotonicity. It turns out that for continuous transitive maps jumps cannot occur if the number of pieces of monotonicity is smaller than 6, while they can occur if this number is 6 or more. Additionally, unified and simple proofs of the fact that such jumps are impossible for unimodal and Lorenz-like maps of positive entropy are presented.

Key Words: entropy, interval maps

1. INTRODUCTION

For continuous piecewise monotone interval maps topological entropy is lower semi-continuous [15] but not upper semi-continuous [13]. This means that small perturbations cannot make it drastically lower, but can make it drastically larger. We will refer to the later phenomenon as a *jump* of the entropy. The perturbations that we consider keep the number of laps (pieces of monotonicity) constant. Similar results have been obtained for piecewise continuous piecewise monotone maps [20], [16], [14].

The results about the jumps of entropy allow us to check whether for a given map a jump can occur, but do not give straightforward answers whether within given classes there are maps at which there is a jump. Some results in this direction have been obtained in [13], where it is proven that there cannot be jumps for unimodal maps of positive entropy, and in [12] for Lorenz-like maps. In Section 2 we present a unified approach to those

* The author was partially supported by NSF grant DMS-9970543

results and show that they do not hold for other types of piecewise continuous piecewise monotone maps with two laps (we call them *bifragmental maps*).

In Section 4 we consider continuous interval maps with n laps and show that a jump of entropy cannot occur at a transitive map if $n \le 5$, but can if $n = 6$ and 7.

In order to simplify entropy estimates in this section, we prove in Section 3 that for a transitive continuous piecewise monotone map with slope bounded by m and strictly smaller than m on some interval the entropy is strictly smaller than m. The same also holds if "smaller" is replaced by "larger". These results are in fact interesting independently of Section 4. Although they are not surprising, the proofs involve quite strong theorems.

In Section 5 we show (under some additional mild conditions) that if a jump of entropy occurs at a transitive map then the perturbation realizing this jump can be also made transitive. Moreover, we show that the examples of jumps from Section 4 can be modified to get maps with number of laps 8 and greater.

2. BIFRAGMENTAL MAPS

For positive integers k, n, denote by $\beta(k, n)$ the (unique) solution to the equation

$$x^{-k} + x^{-n} = 1.$$

Let G be an oriented (directed) graph. Topological entropy $h(G)$ of G is equal to $\lim_{m\to\infty}(1/m)\log s_m$, where s_m is the number of loops of length at most m. A loop γ in G will be called *elementary* if it passes through each vertex at most once. If γ_1 and γ_2 are distinct elementary loops in G then we call them *linked* if they pass through some common vertex.

LEMMA 1. *Let G be an oriented graph. If there are no linked elementary loops in G then $h(G) = 0$. If there are linked elementary loops of length k and n in G then $h(G) \ge \log\beta(k, n)$.*

Proof. Suppose that there are no linked elementary loops in G. Then the number of loops of length at most m in G grows at not faster than linearly with m. Hence, the entropy of G is zero in this case.

Suppose now that G has linked elementary loops γ_1 and γ_2 of lengths k and n respectively. The number of loops of length at most m in G that are concatenations of copies of γ_1 and γ_2 is the same as the number of loops of length at most m in the graph H which consists only of two loops of length k and n and those loops pass through have one common vertex. It is easy to compute that $h(H) = \log\beta(k, n)$ (use the rome method, see [5] or [1]). Thus, $h(G) \ge \log\beta(k, n)$. ▮

Lemma 2. *Let $k \geq l$ and $n \geq m$ be positive integers. Then $\beta(l,m) \geq \beta(k,n)$.*

Proof. If $x > 1$ then we have

$$x^{-l} + x^{-m} > x^{-k} - x^{-n}.$$

Therefore the value of the function $x^{-l}+x^{-m}$ at $\beta(k,n)$ is larger than 1, and since the limit of this function at infinity is 0, we get $\beta(l,m) \geq \beta(k,n)$. ∎

Lemma 3. *Let $n \geq k > m$ be positive integers. Then $\beta(k-m, n+m) > \beta(k,n)$.*

Proof. If $x > 1$ then we have

$$x^{-(k-m)} + x^{-(n+m)} - x^{-k} - x^{-n} = (x^m - 1)(x^{-k} - x^{-(n+m)}) > 0.$$

Therefore the value of the function $x^{-(k-m)} + x^{-(n+m)}$ at $\beta(k,n)$ is larger than 1, and since the limit of this function at infinity is 0, we get $\beta(k-m, n+m) > \beta(k,n)$. ∎

According to [13] (see also [1]), a jump of entropy under perturbations of a continuous piecewise monotone map f can occur only if f has a periodic orbit of period n whose p elements are turning points and $h(f) < (p/n)\log 2$. Note that for weakly unimodal maps we have $p \leq 1$.

In the proofs of the following two theorems we will be using standard tools (P-graphs, P-basic intervals, f-covering, etc.) that can be found e.g. in [1].

Theorem 4. *Topological entropy as a function on the class of weakly unimodal maps, is continuous at all points at which it is positive.*

Proof. It is sufficient to show that whenever a weakly unimodal map f is P-monotone and P is a periodic orbit of period n then either $h(f) = 0$ or $h(f) \geq (1/n)\log 2$. Let G be the P-graph of f. This graph has $n-1$ vertices, and therefore by Lemma 1 it has entropy either zero or at least $\log\beta(k,m)$ for some positive integers $k, m \leq n-1$. By Lemma 2, $\beta(k,m) \geq \beta(n-1, n-1) = 2^{1/(n-1)}$. Since the entropies of G and f are the same, we see that either $h(f) = 0$ or $h(f) \geq (1/(n-1))\log 2$. ∎

For Lorenz-like maps, according to [14], a jump of entropy under perturbations of f can occur only if the orbits obtained by taking the left-hand and right-hand limits of f at the discontinuity point are periodic of periods k and n and $h(f) < \log\beta(k,n)$.

Theorem 5. *Topological entropy as a function on the class of Lorenz-like maps, is continuous at all points at which it is positive.*

Proof. It is sufficient to show that whenever a Lorenz-like map f is P-monotone and P is the union of two periodic orbits P_l and P_r of the discontinuity point (obtained when we take the left-hand and the right-hand limits at this point respectively) of periods k and n respectively, then either $h(f) = 0$ or $h(f) \geq \log \beta(k, n)$.

Let G be the P-graph of f. In this graph there are two elementary loops γ_l and γ_r, corresponding to P_l and P_r in the following natural way. Each element p of P_l (respectively P_r) is a right-hand (respectively left-hand) endpoint of a P-basic interval I which f-covers the P-basic interval J adjacent to $f(p)$ from the same side. Then $I \to J$ is an arrow of γ_l (respectively γ_r). In the case when p is the discontinuity point, we take the corresponding one-sided limit of f at it. The lengths of γ_l and γ_r are the same as the periods of P_l and P_r, that is k and n respectively.

The discontinuity point belongs to P_r (also to P_l, but at this moment this is irrelevant) and the largest element of P belongs to P_l (it is equal to the left-hand limit of f at the discontinuity point). Therefore there exists a P-basic interval whose left-hand endpoint belongs to P_r and the right-hand one to P_l. Hence both γ_l and γ_r pass through this interval. There are two cases possible. The first one is $\gamma_l \neq \gamma_r$. Then γ_l and γ_r are two linked elementary loops in G of lengths k and n and by Lemma 1 the entropy of G (and therefore the entropy of f) is at least $\log \beta(k, n)$. The second case is $\gamma_l = \gamma_r$. To complete the proof it remains to show that in this case either $h(f) = 0$ or $h(f) \geq (1/n) \log 2$ (note that $k = n$, so $\beta(k, n) = 2^{1/n}$).

Assume that $\gamma_l = \gamma_r$. If $[a, b]$ is a P-basic interval through which γ_r passes then the next vertex in the loop γ_r has $f(a)$ as its left-hand endpoint and the next vertex in the loop γ_l has $f(b)$ as its right-hand endpoint. Therefore $[f(a), f(b)]$ is the only P-basic interval f-covered by $[a, b]$. Hence, the loop γ_r is not linked with any other elementary loop. Now we can repeat the argument from the proof of Theorem 4, since there are at most $n - 1$ vertices of G through which γ_r does not pass (in fact, even at most $n - 2$ of them, since two are adjacent to the discontinuity point). ∎

We shall denote the space of all bifragmental maps from an interval I into itself by $R_2(I)$ (see [14]).

EXAMPLE 6. We give an example of a map $f \in R_2([0, 1])$ with positive entropy such that the function $h : R_2([0, 1]) \to \mathbf{R}$ is discontinuous at f. Let $P = \{0, 1/3, 2/3, 1\}$ and let f be P-monotone with $f(0) = 1/3$, $f(1/3) = 2/3$,
$\lim_{x \to 2/3-} f(x) = 1$, $\lim_{x \to 2/3+} f(x) = 2/3$ and $f(1) = 0$ (see Figure 1). Then the discontinuity point belongs to periodic orbits of periods 1 and 4, so under small perturbations the entropy of f can jump up to the level $\log \beta(1, 4)$. However, the P-graph of f consists only of two linked elementary loops of lengths 2 and 3 ($[1/3, 2/3] \to [2/3, 1] \to [1/3, 2/3]$ and

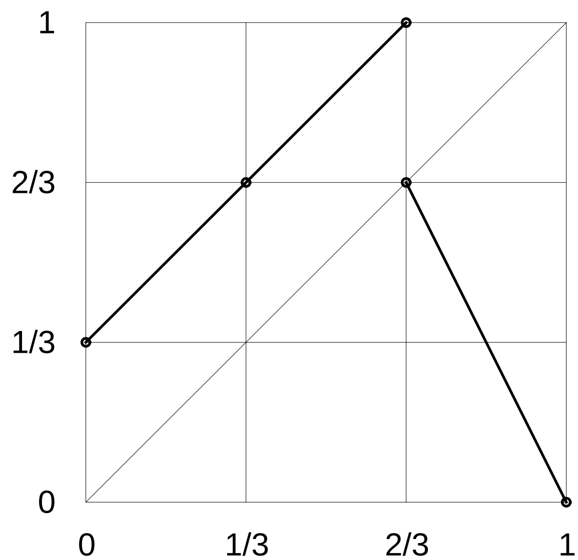

FIG. 1. The map from Example 6

$[1/3, 2/3] \to [2/3, 1] \to [0, 1/3] \to [1/3, 2/3]$), so $h(f) = \log \beta(2, 3)$. By Lemma 3, $\beta(1, 4) > \beta(2, 3)$, so the entropy can jump up at f. A perturbation of f with topological entropy $\log \beta(1, 4)$ is shown in Figure 2.

Remark 7. By taking $P = \{0, 1/6, 2/6, 3/6, 4/6, 5/6, 1\}$ and f P-monotone with $f(0) = 1$, $f(1/6) = 5/6$, $f(2/6) = 4/6$, $f(3/6) = 2/6$, $f(4/6) = 1/6$, $\lim_{x\to 5/6-} f(x) = 0$, $\lim_{x\to 5/6+} f(x) = 5/6$ and $f(1) = 3/6$ (see Figure 3), we get an analogous example with f decreasing on both pieces of continuity/monotonicity (we have $k = 1$, $n = 7$ and $h(f) = \log \beta(3, 5)$).

3. ENTROPY AND SLOPE

A piecewise smooth interval map is a map $f : I \to I$ for which there is a partition of the interval I into finitely many intervals I_i such that f can be extended to a smooth (C^r for a specified $r \geq 1$) map on the closure of each of I_i. By the slope of such map we will mean the absolute value of its derivative. It is well known (see [15], [1]) that if the slope is constant and equal to m (everywhere, except the finite number of points where it is not defined) then topological entropy is equal to $\log m$. It seems natural that if we make the slope strictly smaller than m or strictly larger

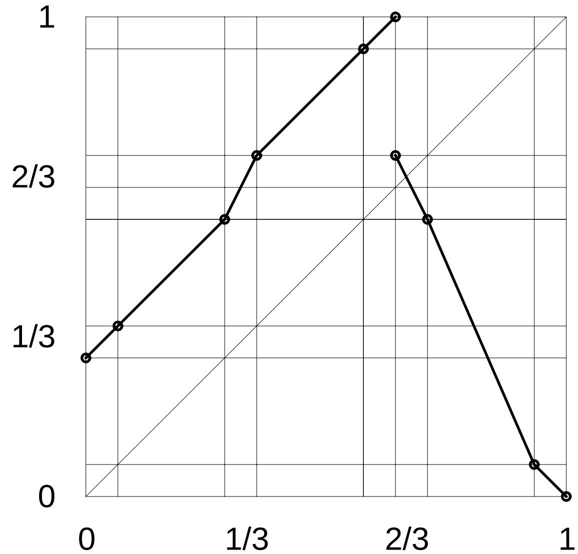

FIG. 2. After the perturbation - entropy jumped up

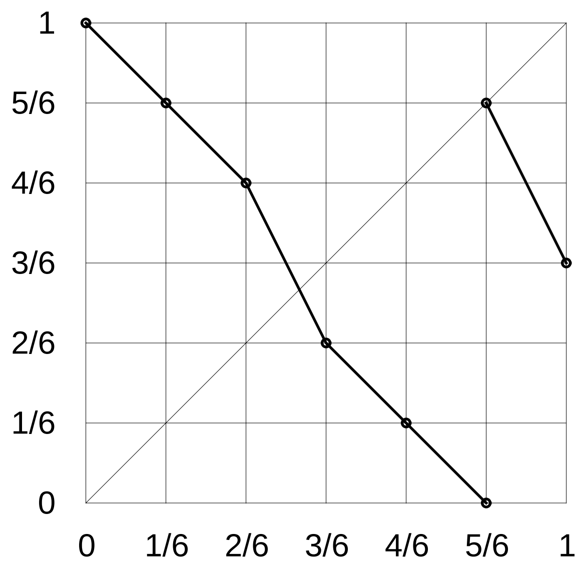

FIG. 3. The map from Remark 7

than m on some interval J then the entropy should become respectively strictly smaller or strictly larger than $\log m$. Of course, for this we have to make some assumption that makes J relevant for the global dynamics, since otherwise one can construct simple counterexamples. For instance, if J is contained in the set of wandering points, then it is irrelevant what slope the map has on it. A good assumption we can make is transitivity of the map.

Thus, we make in this section the following assumptions. The set I is a closed interval and $f : I \to I$ a continuous, piecewise monotone, piecewise C^2, transitive map. We will start by establishing some properties of f. We denote by λ the Lebesgue measure on I, and *acipm* stands for absolutely continuous (with respect to λ) invariant probability measure. The variation of a piecewise differentiable function η over I is $\mathrm{Var}_I(\eta) = \int_I \eta \, d\lambda$.

(1) If $\inf |f'| > 1$ then there exists an acipm [9], it is unique, ergodic, and its density is positive λ-a.e. [11].

(2) The topological entropy of f is positive [6].

(3) The map f is conjugate to a map with constant slope [17], [1].

(4) For any ergodic invariant probability measure μ we have $h_\mu(f) \le \int_I \log |f'| \, d\mu$ [19] (although this is stated in [19] for smooth maps, the proofs clearly works also for piecewise smooth ones).

(5) We have $h(f) = \lim_{n\to\infty} (1/n) \log \mathrm{Var}_I(f^n)$ [15].

(6) If f has constant slope m then $|f'|$ is piecewise monotone, so the results of [10] apply, and since the dimension of the acipm μ is 1, we get $h_\mu(f) = \int_I \log |f'| \, d\mu = \log m$. On the other hand, by (5), $h(f) = \log m$, and thus the acipm has maximal entropy.

Now we can prove the main result of this section.

THEOREM 8. *Let I be a closed interval and let $f : I \to I$ be a continuous, piecewise monotone, piecewise C^2, transitive map. Let $m > 1$.*

(a) Assume that $|f'| \le m$ and on a subinterval J of I we have $|f'| < m$. Then $h(f) < \log m$.

(b) Assume that $|f'| \ge m$ and on a subinterval J of I we have $|f'| > m$. Then $h(f) > \log m$.

Proof. (a) By (3), f is conjugate to a map g with constant slope, and by (1) this slope is larger than 1. The acipm $\tilde{\mu}$ for g, that exists by (1), has maximal entropy by (6). Thus, its image μ under the conjugacy has maximal entropy for f. By (1), $\tilde{\mu}$ is ergodic and positive on open nonempty sets, so the same holds for μ. In particular, $\mu(J) > 0$, so $\int_I \log |f'| \, d\mu < \log m$. Hence, by (4), we get $h(f) = h_\mu(f) < \log m$.

(b) By (1), f has an acipm ν, which is ergodic and has positive density λ-a.e. Thus, $\nu(J) > 0$, so $\int_I \log|f'|\,d\nu > \log m$. Take $\tilde{m}$ such that $\int_I \log|f'|\,d\nu > \log\tilde{m} > \log m$. By the Ergodic Theorem, if n is sufficiently large then there is a measurable set $A_n \subset I$ on which $(1/n)\sum_{i=0}^{n-1}\log|f'\circ f^i| \geq \log\tilde{m}$ and $\nu(A_n) \geq 1/2$. Since ν is absolutely continuous with respect to λ, there is $\varepsilon > 0$ such that $\lambda(A_n) \geq \varepsilon$ for all n. On A_n we have $|(f^n)'| \geq \tilde{m}^n$, and thus $\mathrm{Var}_I(f^n) \geq \varepsilon\tilde{m}^n$. Hence, by (5) we get $h(f) \geq \log\tilde{m} > \log m$. ∎

Remark 9. Peter Raith has pointed out that the above theorem holds also for piecewise C^1 piecewise monotone interval maps. In particular, the argument from the beginning of the proof of (a) can be replaced by the results of [7] and [8], and the proof of (b) can be replaced by an argument based on [18].

While in this paper Theorem 8 is used only for continuous piecewise linear maps, nevertheless it is interesting whether we need smoothness at all. One can replace conditions for $|f'|$ by analogous conditions on $|f(x) - f(y)|/|x-y|$ where x, y belong to the same interval of monotonicity and continuity of f.

Conjecture 10. With the above modifications, Theorem 8 holds for piecewise continuous piecewise monotone interval maps.

4. TRANSITIVE CONTINUOUS MAPS

Suppose that we want to construct a continuous piecewise monotone interval map with positive entropy at which the entropy as a function of a map with the same number of intervals of monotonicity is discontinuous. The simplest way to do it is to divide the interval into two invariant pieces, put a map with positive but small entropy on one of them and a map with zero entropy with a turning point which is also a fixed point on the other one. However, this is a kind of cheating. If we want to understand really when jumps of the entropy can occur, we have to study transitive maps. For them an example like the above does not exist, because if a map is transitive then there is no invariant proper interval.

Note that if f is transitive then there are no turning intervals, only turning points.

Let us denote the family of all continuous piecewise monotone interval maps with $s+1$ pieces of monotonicity (that is, of modality s) by $\mathcal{M}_s$ and the family of all transitive maps from $\bigcup_{s=1}^{\infty}\mathcal{M}_s$ by $\mathcal{T}$. We will use the following theorems (cf. Section 3).

Theorem 11 ([4], [6]). *If $f \in \mathcal{T}$ then $h(f) \geq (1/2)\log 2$.*

THEOREM 12 ([17]). *Every map $f \in \mathcal{T}$ is conjugate to a map of a constant slope. The logarithm of this slope is equal to $h(f)$.*

COROLLARY 13. *All cycles of a map from $\mathcal{T}$ are repelling.*

We will call a map $f \in \mathcal{T}$ a *(p,q)-map* if it has a q-cycle whose p elements are turning points and $(p/q)\log 2 > h(f)$. By [13], a transitive map $f \in \mathcal{M}_s$ is a point of discontinuity of the topological entropy as a map on $\mathcal{M}_s$ if and only if it is a (p,q)-map for some p,q. Note that if f is a (p,q)-map then $h(f) < \log 2$.

LEMMA 14. *Assume that f is a (p,q)-map. Then $p/q > 1/2$ and $q > 1$. Moreover, if $q = 3$ then $p = 3$.*

Proof. Since $f \in \mathcal{T}$, by Theorem 11 $h(f) \geq (1/2)\log 2$, and thus $p/q > 1/2$.

Assume that $q = 1$, that is f has a turning point c which is also a fixed point. We may assume that f has a local maximum at c. Let a be the left endpoint of the interval on which f is defined. Since the interval $[a,c]$ is not invariant, there is a point in $[a,c]$ whose image is to the right of c. Since c is a local maximum, the points close to c are mapped to the left of c. Thus, the point $d = \sup\{x \in [a,c) : f(x) = c\}$ is well defined and $d < c$. Each point from (d,c) is mapped to the left of c. However, the interval $[d,c]$ is not invariant, so there is a point $e \in (d,c)$ which is mapped to the left of d. This gives us a 2-horseshoe and therefore $f(f) \geq \log 2$, a contradiction. Hence, $q > 1$.

If $q = 3$ then f has a 3-cycle and therefore $h(f) \geq \lambda_3$, where λ_3 is the greatest zero of the polynomial $x^3 - 2x - 1$ (see [5], [1]). We have $x^3 - 2x - 1 = (x+1)(x^2 - x - 1)$, so $\lambda_3 = (1+\sqrt{5})/2$. Hence, $2^{p/3} > (1+\sqrt{5})/2$, that is $2^p > ((1+\sqrt{5})/2)^3 = 2 + \sqrt{5} > 4$. Hence, $p > 2$. ▮

LEMMA 15. *Assume that $f \in \mathcal{T}$. Then each endpoint of the interval on which f is defined is the image of a turning point of f under f or f^2.*

Proof. Let a and b be the endpoints of the interval on which f is defined. Assume that a is not the image of a turning point of f under f or f^2. Since f is transitive, it is onto, so $f^{-1}(a)$ is nonempty. If can contain only a and/or b (the turning points are excluded by our assumption). If it is equal to $\{a,b\}$ then $f^{-1}(b)$ has to contain a turning point, a contradiction. Thus either $f^{-1}(a) = \{a\}$ or $f^{-1}(a) = \{b\}$. In the latter case, by a similar argument we must have $f^{-1}(b) = \{a\}$. Hence, either $\{a\}$ or $\{a,b\}$ is a cycle and each of its points has only one inverse image.

By Corollary 13, all cycles of f are repelling. Thus, the cycle to which a belongs has a small backward invariant neighborhood. This contradicts transitivity of f. ▮

Now we state the first of the two main results of this section.

THEOREM 16. *If a transitive map from $\mathcal{M}_s$ is a point of discontinuity of the topological entropy as a function on $\mathcal{M}_s$ then $s \geq 5$.*

Proof. Assume that f is a transitive map from $\mathcal{M}_s$ and that it is a point of discontinuity of the topological entropy as a function on $\mathcal{M}_s$. Then f has to be a (p, q)-map for some p, q. If $s = 1$, this is impossible by Lemma 14. If $s = 2$ then by Lemma 14 we have $p = q = 2$, which contradicts Lemma 15.

The cases of $s = 3$ and $s = 4$ are more complicated. The reader is advised to draw a graph of f whenever it becomes too difficult to visualize it without a picture. We will call the interval on which f is defined $[a, b]$.

Consider the case $s = 3$. By Lemma 14 there are two possibilities: $p = q = 2$ and $p = q = 3$. The second one is ruled out by Lemma 15. Thus $p = q = 2$, that is, there are turning points $c < d$ such that $f(c) = d$ and $f(d) = c$. By Theorem 12, the third turning point, e, lies between c and d. We may assume that f has a local maximum at e. Then f has local minima at c and d. Therefore f attains its global minimum at d and its value is c. Hence, f is not onto, a contradiction.

Consider now the case $s = 4$. By Lemma 14 there are four possibilities for (p, q): $(2, 2)$, $(3, 3)$, $(3, 4)$ and $(4, 4)$. However, the last one is ruled out immediately by Lemma 15.

Let us start with $p = q = 2$. As in the case of $s = 3$, there must be at least one turning point in (c, d), where $c < d$ are turning points such that $f(c) = d$ and $f(d) = c$. If there is only one such turning point, we call it e and we may assume that f has a local maximum at e. Then f has local minima at c and d, and a local maximum at the fourth turning point e'. The only turning points involved in Lemma 15 can be e and e', so the image of one of them is b and $f(b) = a$. In particular, $f(b) < f(d)$, so e' has to be to the right of d. There is a fixed point z in (e, d), and by Corollary 13 it is repelling. Since f is transitive, there cannot exist a neighborhood of z such that every point of this neighborhood has only one preimage. Thus, $f(e') \geq z$. Then the intervals $[c, e]$ and $[e, z]$ f-cover $[d, e']$ and $[e', b]$, and the intervals $[d, e']$ and $[e', b]$ f-cover $[c, e]$ and $[e, z]$. This produces a 4-horseshoe for f^2, so $h(f) \geq \log 2$, a contradiction.

If there are two turning points $e < e'$ in (c, d), we have two possibilities. One is that f has a local minimum at c and a local maximum at d. Then each point in (c, d) has only one preimage, a contradiction as before. The second possibility is that f has a local maximum at c and a local minimum at d. Then none of the endpoints can be mapped to the other one, so we must have $f(e) = a$ and $f(e') = b$. However, this produces a 3-horseshoe on $[c, d]$, a contradiction. This completes the analysis of the case $s = 4$, $p = q = 2$.

If $p = q = 3$, we have turning points c, d, e such that $f(c) = d$, $f(d) = e$ and $f(e) = c$. We may assume that $c < d < e$. The fourth turning

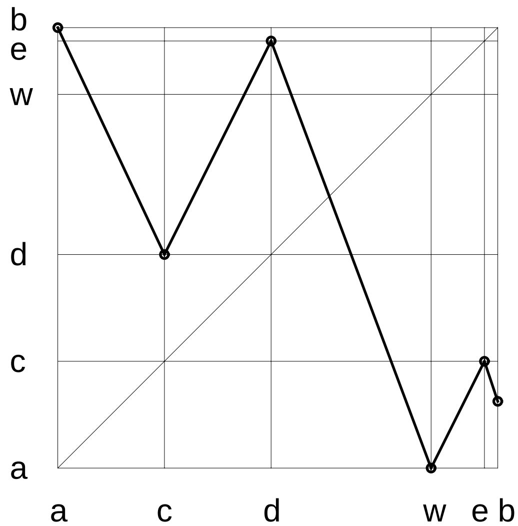

FIG. 4. The case $s = 4$, $p = q = 3$, $w \in [d, e]$

point, w, can be in any of four intervals: $[a, c]$, $[c, d]$, $[d, e]$, $[e, b]$. By Lemma 15 we have either $f(w) = a$ and $f(a) = b$ or $f(w) = b$ and $f(b) = a$. This eliminates the possibilities $w \in [a, c]$ and $w \in [c, d]$. If $w \in [e, b]$ then $f(w) = b$, which produces a 2-horseshoe on $[e, b]$, a contradiction. Thus we are left with the possibility of $w \in [d, e]$. In this case we have $a < c < d < w < e < b$ and $f(a) = b$, $f(c) = d$, $f(d) = e$, $f(w) = a$, $f(e) = c$, $f(b) < c$ (see Figure 4). By replacing f by a map conjugate to it, we may assume that $a = 0$, $c = 2$, $d = 4$, $w = 7$, $e = 8$, $b = 8 + \varepsilon$, where $2\varepsilon < c - f(b)$. Then the slope of f is larger than or equal to 2 everywhere, and therefore $f(f) \geq \log 2$, a contradiction.

Now we start to consider the last possibility: $p = 3$ and $q = 4$. That is, there is a 4-cycle P whose 3 elements are turning points. Let us check first how the pattern of P can look like. There are 4 patterns of period 4. In the cyclic notation (where (1342) means that the points are $1 < 2 < 3 < 4$ and 1 is mapped to 3 to 4 to 2 to 1) they are (1234), (1243), (1324) and (1342) (the other two differ from the listed ones by orientation only). However, patterns (1243) and (1342) give us maps that are at least 2-to-1, so their entropy is at least $\log 2$, and we can exclude them. The pattern (1234) has entropy $\log \lambda$, where λ is the largest zero of the polynomial $W(x) = x^3 - x^2 - x - 1$ (this can be verified easily by the rome method or obtained from the kneading sequence). We have $W'(x) = (3x + 1)(x - 1)$, so W is increasing on $[1,\infty)$. Since $W(2^{3/4}) < 5-(5/2+3/2+1) = 0$, we get $\log\lambda > (3/4)\log 2$, which excludes also this pattern.

Hence, P has pattern (1324) (it has entropy zero). This time we will use the following notation. The interval on which f is defined is $[0,5]$ and $P = \{1,2,3,4\}$ (we may make this assumption). Three of the points $1,2,3,4$ are turning points of f; the fourth turning point of f is w. The things we do not know are: which of the points $1,2,3,4$ is not a turning point, and where w is. Thus, we have to consider 20 cases, which we will denote as follows. Case ij means that i is not a turning point and $w \in (j, j+1)$.

If 3 is a turning point then $w \in (2,4)$. This eliminates 9 cases ij with $i \neq 3$ and $j \in \{0,1,4\}$. By similar reasons, cases 22, 23 and 32 are impossible. Similarly as before, by Lemma 15 we have either $f(w) = 0$ and $f(0) = 5$ or $f(w) = 5$ and $f(5) = 0$. This eliminates cases 12, 13, 30 and 31. In Cases 33 and 42 points from the interval $(2,3)$ (which contains the fixed point of f) have only one preimage, so f cannot be transitive. In cases 34 and 43 there is a 2-horseshoe, so they are also ruled out. In this way we have eliminated all possible cases. ∎

Now we are going to show that the above estimate is sharp.

THEOREM 17. *There exists a topologically mixing map from $\mathcal{M}_5$ which is a point of discontinuity of the topological entropy as a function on $\mathcal{M}_5$.*

Proof. Define $f : [0,72] \to [0,72]$ as a piecewise linear map whose graph is the union of segments connecting consecutive points (a "connect-the-dots" map): $(0,32)$, $(20,52)$, $(24,60)$, $(25,58)$, $(32,72)$, $(52,32)$, $(58,20)$, $(60,24)$, $(72,0)$ (see Figure 5).

This map belongs to $\mathcal{M}_5$ and is Markov with respect to the partition given by the first coordinates of the above points. Figure 6 shows its Markov graph.

A simple analysis of this Markov graph shows that the corresponding subshift of finite type is topologically mixing. Therefore f is also topologically mixing. The slope of f is 2 on the interval $[20,72]$ and smaller than 2 on the interval $[0,20]$. Thus, by Theorem 8, $h(f) < \log 2$. On the other hand, $\{24,60\}$ is a cycle of period 2 whose both elements are turning points. Therefore the topological entropy is discontinuous at f as a function on $\mathcal{M}_5$ (there are arbitrarily small perturbations of f with entropy at least $\log 2$). ∎

Remark 18. The example from the above theorem can be easily modified to a topologically mixing map from $\mathcal{M}_6$ which is a point of discontinuity of the topological entropy as a function on $\mathcal{M}_6$. We make modification on the interval $[0,20]$, connecting the dots $(0,136/3)$, $(10,32)$ and $(20,52)$. We also add to the set of points defining the Markov partition the fixed point

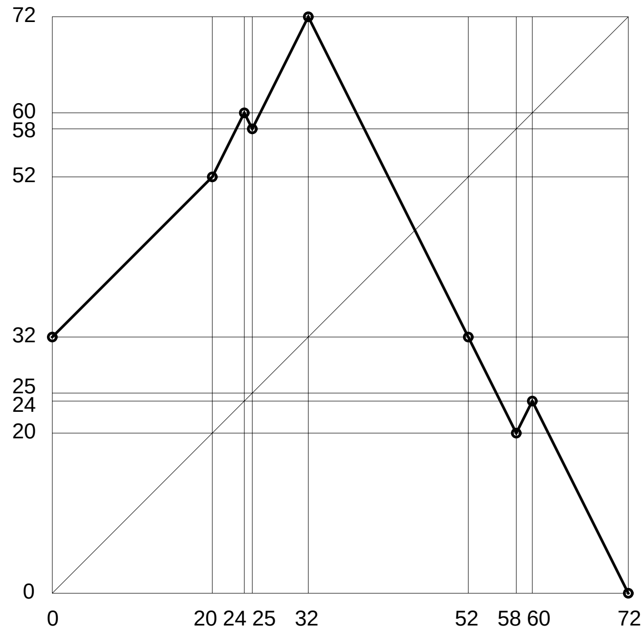

FIG. 5. The map from Theorem 17

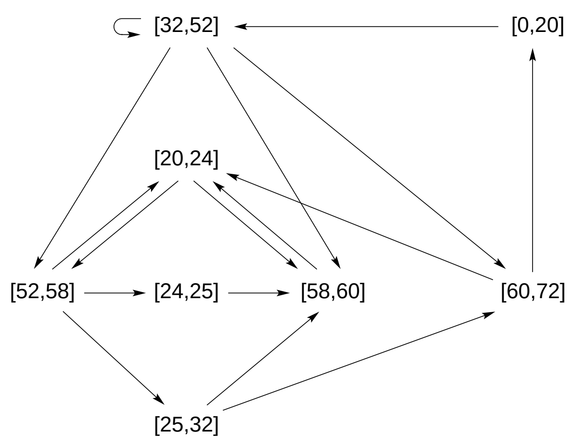

FIG. 6. Markov graph of the map from Theorem 17

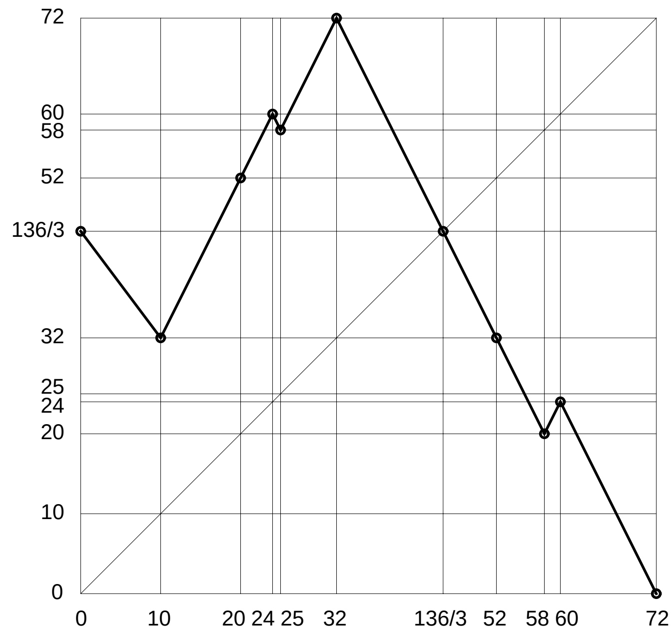

FIG. 7. The map from Remark 18

136/3. For the graph of this modified map see Figure 7 and for its Markov graph see Figure 8.

5. TRANSITIVITY OF PERTURBATIONS

The example from the preceding section (Theorem 17) does not solve two problems. The first one is what happens if instead of looking for a transitive map at which the entropy is discontinuous we restrict the entropy function to the set of transitive maps (of a given modality) and ask about its discontinuities. The second question is about similar examples of modality higher than 5.

We solve those problems in the following way. First we show that for a large class of examples like that from Theorem 17 the map is a discontinuity point for the entropy on the space of transitive (or even mixing) maps of a given modality. Then we modify the example from Theorem 17 to have any given modality larger than 5. The examples we get belong to the class mentioned above.

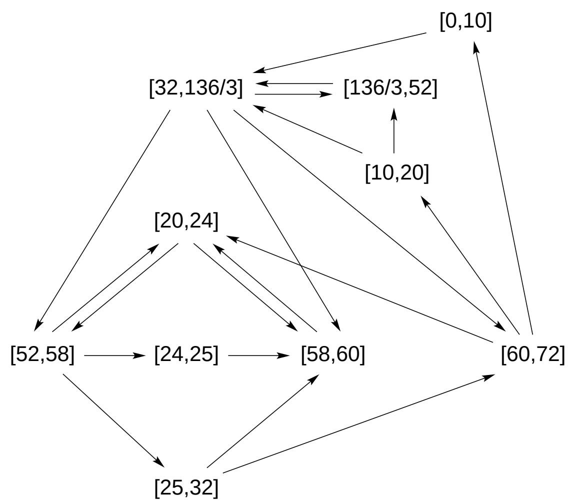

FIG. 8. Markov graph of the map from Remark 18

Denote by $\mathcal{T}_s$ the class of all topologically mixing maps from $\mathcal{M}_s$.

THEOREM 19. *Let $f \in \mathcal{T}_s$ have a cycle P of period q, containing p turning points and no endpoint of the interval K on which f is defined. Assume that no endpoint of K and no turning point of f is mapped to an element of P by any iterate of f. Assume also that $(p/q)\log 2 > h(f)$. Then the topological entropy as a function on $\mathcal{T}_s$ is discontinuous at f. More precisely, in $\mathcal{T}_s$ we have*

$$\limsup_{g \to f} \geq \frac{p}{q} \log 2.$$

Proof. We blow up the points of P and all its preimages under the iterates of f, that is replace those points by intervals with the sum of their length small. In such a way we get a slightly longer interval than the original one. However, we can later rescale it, and since f is uniformly continuous, if the sum of the lengths of new intervals is small and the images of new intervals are small, we get a small perturbation of f. Thus, we have to define our new map (call it g) on the new intervals in such a way that the images of those intervals are small (that is, less than some ε given before we blew up P; in particular the new intervals can be shorter than $\varepsilon/2$), $h(g) \geq (p/q)\log 2$ and $g \in \mathcal{T}_s$.

Take one of the new intervals, $I = [a, b]$. By the assumptions, none of the endpoints of I is an endpoint of K. Thus, by continuity, we know the

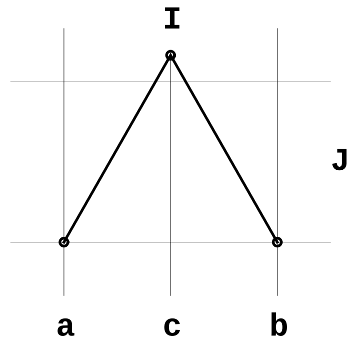

FIG. 9. How to define g on I

images of the endpoints of I. If J is the interval replacing the image of the point replaced by I, then both $g(a)$ and $g(b)$ are endpoints of J. If $g(a) \neq g(b)$ then we define g on I as the affine map with those two values (we connect the dots for the graph of g). In particular, this is the case if I replaces a point that does not belong to P. If $g(a) = g(b)$ then we choose $c \in (a, b)$, define $g(c)$ and connect three dots. We do it in such a way that $g(c) \notin J$, $J \subset g(I)$ and the length of $g(I)$ is less than ε (see Figure 9).

Two of the three required properties of g are obvious. The images of new intervals are sufficiently small (shorter than ε). The map g^q has a 2^p-horseshoe, and thus $h(g) \geq (p/q) \log 2$. It remains to show that g is topologically mixing.

Since f is topologically mixing, it is *locally eventually onto*, that is, for every open nonempty set U there is n such that $f^n(U)$ is equal to the whole domain of f (see [3], page 158). Let now U be a nonempty open subset of the domain of g and let φ be the map that collapses the new intervals back to points. If U is not contained in the union T of the new intervals then $\varphi(U)$ contains an open interval V. There is n such that $f^n(V)$ is the whole domain of f, and since the images of the interval $\varphi^{-1}(V)$ under the iterates of g are intervals, we see that $g^n(\varphi^{-1}(V))$ (and therefore also $g^n(U)$) is the whole domain of g. If U is contained in T then there is m such that $g^m(U)$ contains an open nonempty set that is not contained in T (the set of all points whose trajectories stay in T is a Cantor set), and by what we already proved some further image of U is the whole domain of g. Thus, g is also locally eventually onto, and therefore mixing. ▮

THEOREM 20. *For any $s \geq 5$ the topological entropy as a function on $\mathcal{T}_s$ is discontinuous.*

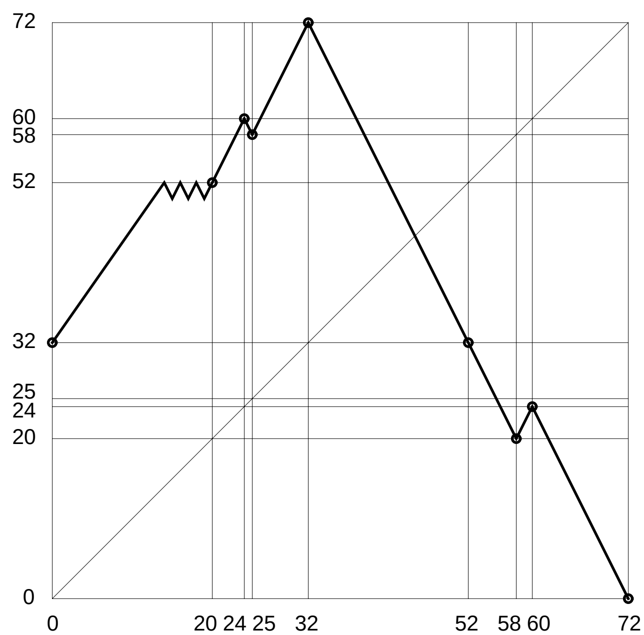

FIG. 10. Modification of the map from Theorem 17 – adding a "saw"

Proof. Examples from Theorem 17 and Remark 18 satisfy the assumptions of Theorem 19, and this proves discontinuity of the topological entropy as a function on $\mathcal{T}_s$ for $s = 5$ and 6.

If $s > 5$ is odd, we will modify the example from Theorem 17. Namely, we will add a small "saw" with slope 2 and $s - 5$ new turning points immediately to the left of 20 (see Figure 10); let us call the new map g. Our "saw" is mapped by g^2 to the right of 0 and after that each third iterate will map it to a larger and larger interval whose left endpoint is 0. We can choose the size of the "saw" in such a way that the slope at 0 remains strictly smaller than 2 (this guarantees us that $h(g) < \log 2$) and some image of the "saw" is $[0, 20]$. Then g is a Markov map and it is easy to see that it is topologically mixing (its Markov graph is obtained from the Markov graph of f by splitting some states, adding some arrows, but from every new vertex one can get to the unchanged portion og the graph and vice versa). The assumptions of Theorem 19 are satisfied by g, and this proves the theorem for s odd.

To prove it for $s > 6$ even, we make a similar construction, adding a "saw" to the left of 10 in the example from Remark 18. ∎

REFERENCES

1. L. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one* (Second Edition), Advanced Series in Nonlinear Dynamics **5**, World Scientific, Singapore, 2000.
2. Ll. Alsedà, J. Llibre, M. Misiurewicz and C. Tresser, *Periods and entropy for Lorenz-like maps*, Ann. Inst. Fourier **39** (1989), 929–952.
3. L. Block and W. A. Coppel, *Dynamics in one dimension*, Lecture Notes in Math. **1513**, Springer, Berlin, 1992.
4. L. Block and E. M. Coven, *Topological conjugacy and transitivity for a class of piecewise monotone maps of the interval*, Trans. Amer. Math. Soc. **300** (1987), 297–306.
5. L. Block, J. Guckenheimer, M. Misiurewicz and L.-S. Young, *Periodic points and topological entropy of one dimensional maps*, in Global theory of dynamical systems, Lecture Notes in Math. **819**, Springer, Berlin, 1980, pp. 18–34.
6. A. M. Blokh,, *On sensitive mappings of the interval*, Russian Math. Surveys **37:2** (1982), 203–204.
7. F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy II*, Israel J. Math. **38** (1981), 107–115.
8. F. Hofbauer, *An inequality for the Ljapunov exponent of an ergodic invariant measure for a piecewise monotonic map on the interval*, in Lyapunov exponents, Lecture Notes in Math. **1486**, Springer, Berlin, 1991, pp. 227–231.
9. A. Lasota and J. A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488.
10. F. Ledrappier and M. Misiurewicz, *Dimension of invariant measures for maps with exponent zero*, Ergod. Th. Dynam. Sys. **5** (1985), 595–610.
11. T.-Y. Li and J. A. Yorke, *Ergodic transformations from an interval into itself*, Trans. Amer. Math. Soc. **235** (1978), 183–192.
12. M. I. Malkin, *Continuity of entropy of discontinuous mappings of an interval* (in Russian), in Methods of the qualitative theory of differential equations, Gor'kov. Gos. Univ., Gorki, 1982, pp. 35–47.
13. M. Misiurewicz, *Jumps of entropy in one dimension*, Fundam. Math. **132** (1989), 215–226.
14. M. Misiurewicz and S. V. Shlyachkov, *Entropy of piecewise continuous interval maps*, in European Conference on Iteration Theory, ECIT 89, Word Scientific, Singapore, 1991, pp. 239–245.
15. M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. **67** (1980), 45–63.
16. M. Misiurewicz and K. Ziemian, *Horseshoes and entropy for piecewise continuous piecewise monotone maps* in From phase transitions to chaos, Word Scientific, Singapore, 1992, pp. 489–500.
17. W. Parry, *Symbolic dynamics and transformations of the unit interval*, Trans. Amer. Math. Soc. **122** (1966), 368–378.
18. P. Raith, *Hausdorff dimension for piecewise monotonic maps*, Studia Math. **94** (1989), 17–33.
19. D. Ruelle, *An inequality for the entropy of differentiable maps*, Bol. Soc. Brasil. Mat. **9** (1978), 83–87.
20. M. Urbański, *Invariant subsets of expanding mappings of the circle*, Ergod. Th. Dynam. Sys. **7** (1987), 627–654.