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A Starting Condition Approach to Parameter Distortion in Generalized Renormalization

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We control parameter distortion in the generalized renormalization procedure provided a certain set of starting conditions is satisfied. This allows us to prove that for a C^∞–open set of unimodal families, almost all parameters inside an interval present either stochastic dynamics or a renormalization (in the classical sense). Moreover, easy consequences are that renormalization happens densely on this interval and stochastic behaviour with positive measure. A wide range use of this approach would rely mostly on proving that the starting conditions are satisfied for general families.

Key Words: One–dimensional dynamics, generalized renormalization, parameter distortion, quadratic family

1. INTRODUCTION AND RESULTS.

1.1. Introduction

The main purpose of this work is to discuss the distortion of parameter derivatives in the onset of *generalized renormalization* or *inducing* (names vary according to the authors), a concept which first appeared in the beginning of the 80's with a paper of Jacobson [5] and underwent many developments in the last two decades as a tool for the study of the *quadratic family* and, more generally, families of *unimodal* functions (see definitions in Subsection 1.4).

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The use of generalized renormalization has proven to be fruitful, but the main assertions were achieved only by enlargement to the complex setting, motivated, among others, by the lack of estimates on the distortion of parameter derivatives. The importance of this kind of estimates is that once they are obtained conclusions on the parameter space follow almost automatically from results on the configuration space.

With this in mind Jacobson proved that in the quadratic family parameters with stochastic behavior have positive Lebesgue measure, but at the expense of some excluded set of parameters for which distortion could not be controlled.

Here we intend to show that distortion control is possible even without parameter exclusions. The price to pay is to show that for some stage of the generalized renormalization procedure a set of conditions is satisfied, in order that for subsequent steps the induction works. This kind of approach resembles very much how the *decay of geometry* (see Subsection 1.6) has been proven for a class of unimodal functions (called *quasi–quadratic*), in the beginning of the 90's. First ([6], [7]) decay of geometry was obtained provided a certain *starting condition* was satisfied. Secondly, one had to show that for every map in this class the starting condition could be achieved for some stage of the generalized renormalization ([7], [4]).

In this analogy, the present work corresponds to the first part: we give a starting condition set of bounds on certain parameter derivatives that can be kept by induction for every subsequent step of the generalized renormalization. It is not clear yet in what generality this approach can be used or, in other words, what do we need to arrive at a step for which the starting condition is valid. However we are able to show its potentiality, by applying it to some families of unimodal functions specially constructed to satisfy the starting condition at the *first* step of the induction. These families, which we call *special*, contain a C^∞–open set of the space of families of quasi–quadratic functions, but the parameters are taken only near the usual last bifurcating value of unimodal families.

We prove that for these special families *almost all parameters are renormalizable or stochastic*, a result related to the one proven by Lyubich [9] for the quadratic family or, more generally, to families of functions which admit a *quadratic–like* extension to the complex plane. Moreover, we are able to recover a Jacobson–like assertion, showing that *parameters with stochastic behavior have positive measure*, and also to obtain that *renormalizable parameters are dense.*

All the undefined expressions of this introduction will be clarified along the text. In the remaining of Section 1 we tell what is generalized renormalization, who are the special families for which we apply our techniques (its existence is postponed to Subsection 2.2) and the results above men-

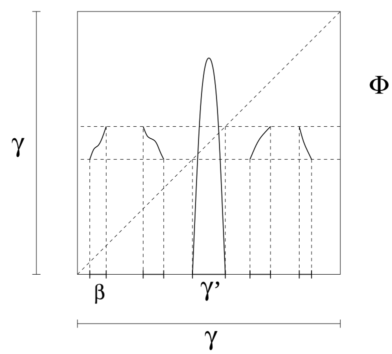

FIG. 1. Maps in which generalized renormalization is applied.

tioned. Moreover, we discuss a criterion for stochasticity due to Martens and Nowicki [12] which is also used by Lyubich in [9].

Up until this point generalized renormalization is performed for a *fixed parameter*. We show still in this Section how to do it for (one–parameter) families, although the procedure will only make sense after the control of parameter distortion. In the sequel, we precisely state the bounds on parameter and mixed derivatives we need to control distortion. These bounds are much more than what is necessary to applications, but the only set of inequalities we found that could be kept independently by induction.

We also prove in this Section the applications mentioned above, assuming that our special families satisfy the starting conditions and that the induction procedure is true. In between, we explain generalized renormalization in more detail, taking profit to introduce the technical notation to be used in the rest of the paper.

Section 2 is dedicated to show the starting conditions for the special families and also the existence of such families. Finally, in Section 3 the induction step is proven.

A comment about notation. We use the symbols "$\simeq$", "$\gtrsim$" and "$\lesssim$", meaning that equality or inequality is true up to a small multiplicative factor, whenever there is no danger of accumulation of small errors. In the same way C and c will denote sufficiently large constants, all independent of the remaining specific constants that appear along the text.

1.2. Generalized renormalization

Generalized renormalization is a concept whose precise definition may vary according to the specific needing or results one is looking for. Here we apply it for maps Φ of the following kind (see Figure 1). There is an interval γ which contains the (disconnected) domain of Φ and a central interval $\gamma' \subset \gamma$ such that $\Phi(\gamma') \subset \gamma$, $\Phi(\partial\gamma') \subset \partial\gamma$ and $\Phi|\gamma'$ is a *unimodal* function, i.e. a function with one and only one turning point c in its interval of definition. The remaining of $\mathrm{dom}(\Phi)$ is a union of pairwise disjoint intervals from a collection $\mathcal{B}$, all of them contained in the interior of γ (there could be nonempty open sets in the complement of the closure of this union, but here this will never be the case). For each $\beta \in \mathcal{B}$, we have that $\Phi(\beta) = \gamma'$ and $\Phi|\beta : \beta \to \gamma'$ is a diffeomorphism. We may suppose that Φ is C^∞ inside each connected component of its domain, although C^3 is enough to our considerations.

Let $1 \leq \mathrm{esc} \leq \infty$ be the first positive integer such that $\Phi^{\mathrm{esc}}(c) \in \gamma \setminus \gamma'$. The renormalized function $\Phi' = \mathrm{Ren}(\Phi)$ will be defined if and only if $\mathrm{esc} < \infty$ and $\Phi^{\mathrm{esc}}(c) \in \mathrm{int}(\beta)$, for some $\beta \in \mathcal{B}$. In this case we set

$$\gamma'' = \Phi^{-\mathrm{esc}}(\beta)$$

as the central interval of $\mathrm{dom}(\Phi')$ and $\Phi'|\gamma'' = \Phi|\beta \circ \Phi^{\mathrm{esc}}|\gamma''$. Besides γ'', the domain of Φ' is composed by the union of all preimages of γ'' under Φ taken *outside* γ'' and contained in γ'. This collection of intervals will be called $\mathcal{B}'$. For each $\beta' \in \mathcal{B}'$ there is a power of Φ which sends β' diffeomorphically onto γ'', and this will be exactly the definition of $\Phi'|\beta'$.

This scheme is described with more details in Subsection 1.13. For the moment, we observe that starting from a map $\Phi = \Phi_0$, having

$$\mathrm{dom}(\Phi_0) = \gamma_0 \cup \bigcup_{\beta \in \mathcal{B}_0} \beta \subset \gamma_{-1}$$

(where γ_0 corresponds to γ' and γ_{-1} corresponds to γ), we obtain a sequence of maps $\{\Phi_n\}_{n \geq 0}$, where $\Phi_n = \mathrm{Ren}(\Phi_{n-1})$, $n \geq 1$. The only *a priori* assumption is that for each n the critical value position implies that Φ_n is renormalizable in this sense. This procedure generates a decreasing nested sequence of intervals $\gamma_{-1} \supset \gamma_0 \supset \gamma_1 \supset \ldots$ and collections $\mathcal{B}_n$ of pairwise disjoint intervals inside γ_{n-1} for all $n \geq 0$.

For the sake of clarity and later use, we call

$$H_n = \Phi_n|\gamma_n ,$$

the *central branch* of γ_n, and for each $\beta \in \mathcal{B}_n$ we let $B : \beta \to \gamma_n$ be the diffeomorphism $B = \Phi_n|\beta$.

1.3. The quadratic family

This whole procedure arises naturally for families of unimodal functions of the interval, as for example the *quadratic family*. The quadratic family is the family of functions $(\phi_a)_a$, $\phi_a : \mathbb{R} \to \mathbb{R}$, where $\phi_a(x) = a - x^2$. Although the parameter a is intended to vary along the real line, there is

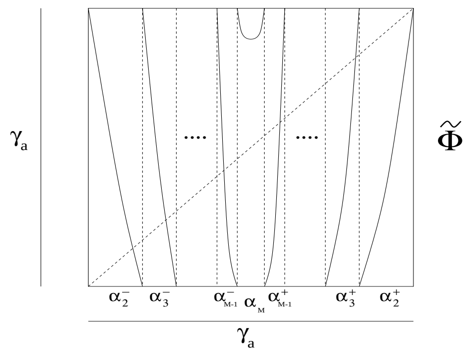

FIG. 2. First return map to γ_a.

a special range, the interval $[-\frac{1}{4}, 2]$, with the following property: "there is an interval I_a, symmetric with respect to the origin, such that $\phi_a(I_a) \subset I_a$ and $\phi_a(\partial I_a) \subset \partial I_a$". The interval I_a is defined by $[-x_a, x_a]$, where $-x_a$ is the leftmost fixed point of ϕ_a (by a parameter dependent affine change of coordinates the interval I_a could be sent onto $[0,1]$, and in this case the family would be written as $x \mapsto rx(1-x)$, with r varying between 1 and 4 as a runs through $[-\frac{1}{4}, 2]$).

For $a > 0$ we define the interval $\gamma_a = [-q_a, q_a]$, where q_a is the rightmost fixed point of ϕ_a and consider the first return map $\tilde{\Phi}$ in γ_a, as depicted in Figure 2. We call $\alpha_M (\equiv \gamma_0)$ and $\alpha_k^{\pm}$, $k = 2, \ldots, M-1$, the closure of the connected components of the domain of continuity of $\tilde{\Phi}$ and consider the (continuous) extensions of $\tilde{\Phi}$ to these intervals, giving the names $A_k^{\pm} = \tilde{\Phi}|\alpha_k^{\pm} = \phi_a^k|\alpha_k^{\pm}$ and $H_0 = \tilde{\Phi}|\alpha_M = \phi_a^M|\alpha_M$ (of course all the functions depend on a, but we omit the index to avoid a cumbersome notation).

To obtain a map Φ_0 like the ones described above, we look at the collection $\mathcal{B}_0$ of all preimages β of $\gamma_0 = \alpha_M$ under $\tilde{\Phi}$ which are taken outside γ_0. Then, for each $\beta \in \mathcal{B}_0$, $\Phi_0|\beta$ is defined as the power of $\tilde{\Phi}$ which sends β diffeomorphically onto γ_0. The interval γ_{-1} is γ_a and $\Phi_0|\gamma_0 = \tilde{\Phi}|\gamma_0$.

The same construction can be done for other unimodal families, as for example the *special families* we describe in the sequel. These families will be important here since for them our methods can be easily applied, as it was stressed in the introduction.

1.4. Special families

We start by some definitions. The Schwarzian derivative of a C^3 function ϕ is given by

$$S\phi(x) = \frac{D_3\phi(x)}{D\phi(x)} - \frac{3}{2}\left[\frac{D_2\phi(x)}{D\phi(x)}\right]^2 ,$$

except for the points where the derivative vanishes. We say that ϕ is *S–unimodal* in I if $S\phi \leq 0$ and ϕ is *unimodal* in I (i.e. ϕ has one and only one turning point in I), and *quasi–quadratic* if moreover the critical point is non–degenerate. It is easy to verify that the functions of the quadratic family are quasi–quadratic. We refer the reader to [13], Chap.IV, for the distortion properties resulting from the assumption of non–positive Schwarzian derivative.

Let $\epsilon_0 > 0$ (which will be made small accordingly), $3 > \lambda > 2$, $\tilde{I} \equiv \tilde{I}_a \equiv [-2, +2]$ and $\tilde{\phi}_a : \tilde{I} \to \tilde{I}$ satisfy:

- $a \in [2 - \epsilon_0, 2]$;
- $\tilde{\phi}_a(\tilde{I}) \subset \tilde{I}$ and $\tilde{\phi}_a(\partial\tilde{I}) \subset \partial\tilde{I}$, $\forall a$, i.e. $\tilde{I}$ is a *restrictive interval* for $\tilde{\phi}_a$;
- $(a, x) \mapsto \tilde{\phi}_a(x)$ is C^∞;
- $\tilde{\phi}_a(x) = -2 + \lambda(x + 2)$ for $x \in [-2, -x_\lambda]$, where $x_\lambda = \frac{2(\lambda-1)}{\lambda+1}$;
- $\tilde{\phi}_a(x) = -2 + \lambda(x - 2)$ for $x \in [x_\lambda, 2]$;
- $\tilde{\phi}_a(x) = a - \frac{\mu}{2}x^2$ for $x \in [-\frac{1}{5}, \frac{1}{5}]$, for some $\mu > \lambda$;
- $D\tilde{\phi}_a(x)$ is monotone in $\tilde{I}$ and $D_2\tilde{\phi}_a(x)$ is monotone (separately) in $\tilde{I} \cap \{x < 0\}$ and $\tilde{I} \cap \{x > 0\}$;
- $D_3\tilde{\phi}_a(x) \leq 0$ for $x \leq 0$ and $D_3\tilde{\phi}_a(x) \geq 0$ for $x \geq 0$.

In Subsection 2.2 we prove that such a family does exist, but in fact we have to restrict the possible range of λ's to certain values between 2.3 and 2.4. From the definition it is easy to verify that $(D\tilde{\phi}_a)^2 S\tilde{\phi}_a \leq 0$ in $\tilde{I}$, hence $\tilde{\phi}_a$ is always quasi–quadratic. We now say that a C^∞ family of quasi–quadratic functions $(\phi_a)_a$ for which $[-2, +2]$ is a *restrictive interval* (i.e. $\phi_a([-2, +2]) \subseteq [-2, +2]$ and $\phi_a(\{-2, +2\}) \subset \{-2, +2\}$) is *special* if it is C^∞ δ–near $(\tilde{\phi}_a)_a$ in $[-2, +2]$ for some $\delta > 0$ that we will specify later in Section 2. It is clear that there exist *open sets* of special families (among the families of maps having $[-2, +2]$ as a restrictive interval). In fact our results are essentially true for C^3 families which are C^3 near $(\tilde{\phi}_a)_a$, but we adopt infinite differentiability and C^∞ topology to avoid technical difficulties in Subsection 3.2.

The definition of $(\tilde{\phi}_a)_a$ resembles the family of unimodal functions studied in [1], where a Jacobson–like theorem is proven, but there the coefficient of the quadratic part varies with the parameter.

We can use the same notation as for the quadratic family. If $(\phi_a)_a$ is a special family, there is an interval γ_a whose boundary points are the two preimages of the rightmost fixed point of ϕ_a (equal to x_λ if $\phi_a = \tilde{\phi}_a$), and also the intervals $\alpha_M = \gamma_0$ and $\alpha_k^\pm$, $k = 2, \ldots, M-1$, which are the closure of the components where the return time to γ_a is constant. We can also suppose without loss of generality that 0 is the critical point for all a.

1.5. Results for special families

We say that ϕ_a is *renormalizable* if there is an interval $\hat{I} = \hat{I}_a \subset \mathrm{int}(I_a)$ containing 0 and a number $n > 1$ such that $\hat{I}_a$ is a restrictive interval for the n–th power of ϕ_a and $\phi_a^n|\hat{I}_a$ is a unimodal function. Let

$$\mathcal{R} = \{a \; ; \; \phi_a \text{ is renormalizable}\} \; .$$

We say that ϕ_a is *stochastic* if there is an absolutely continuous ϕ_a–invariant probability measure (a.c.p.i.m) and define

$$\mathcal{E} = \{a \; ; \; \phi_a \text{ is stochastic}\} \; .$$

As a Corollary of [11] (see also [13],Chap.III.5), if ϕ_a is renormalizable in an interval $\hat{I}_a$ then for almost all $x \in I_a$ there is $n = n(x)$ such that $\phi_a^n(x) \in \hat{I}_a$. If not, then according to [7] for almost all $x \in I_a$ we have $\omega(x) = [\phi_a^2(0), \phi_a(0)]$. If moreover ϕ_a is stochastic and ν_a is the corresponding a.c.p.i.m. then $\overline{\mathrm{supp}\mu} = [\phi_a^2(0), \phi_a(0)]$. We prove the following Theorem.

THEOREM 1. *If $(\phi_a)_a$ is a special family, then there is $\epsilon > 0$ such that ϕ_a is renormalizable or stochastic for almost all $a \in [2-\epsilon, 2]$.*

Despite being a ‘bifurcation–like’ result, it is of the same kind as the one which appears in [9] (and we make use as well of the result in [12], see Subsection 1.7), but there the quasi–quadratic families considered admit a *quadratic–like extension* (see [13], Chap.VI.1, for a definition) and no parameter restriction is made. As the quasi–quadratic families coming from the renormalizations are also in the same class, the analysis can go beyond any finite number of renormalizations, and the conclusion is that almost all finitely renormalizable parameters are stochastic. In the present case the analysis is no longer valid for the renormalizations, since neither they are in the same class as the original family nor all parameters can be embraced.

On the other hand, our result is valid for an open set of families (in the C^∞ topology), while the class of families in [9] has empty interior among C^∞ families.

Notwithstanding, besides the discussion above, the important fact to retain in this work is that control of parameter distortion can be achieved inductively provided some starting condition is satisfied (see Subsection 1.9). As the special families are constructed to satisfy these starting conditions, the theorems of this section can be derived as easy corollaries.

We will be also able to prove Jacobson's Theorem [5] for these special families, as a consequence of our methods and [12].

THEOREM 2. *If* $(\phi_a)_a$ *is a special family then* $Leb(\mathcal{E}) > 0$.

We say that ϕ_a is *Misiurewicz* if the critical point is neither recurrent nor attracted to a periodic orbit, and denote by $\mathcal{M}$ the set of Misiurewicz parameters. It turns out (see [14] or [13], Chap.V.3) that if $a \in \mathcal{M} \cap \mathcal{R}^c$ then ϕ_a is stochastic. These parameters however are rare, as stated in the following Theorem.

THEOREM 3. *If* $(\phi_a)_a$ *is a special family and* $\epsilon > 0$ *is sufficiently small, then*

$$Leb(\mathcal{M} \cap \mathcal{R}^c \cap [2-\epsilon, 2]) = 0 \ .$$

This result is a restricted version of a result obtained by Sands [16] for general S–unimodal real–analytic families.

Finally, another Corollary of our methods will be

THEOREM 4. *If* $(\phi_a)_a$ *is a special family then* $\mathcal{R}$ *is dense in* $[2-\epsilon, 2]$, *for* $\epsilon > 0$ *sufficiently small.*

This result goes, with the limitations exposed above as well, in the same direction as [4] and [8].

1.6. Decay of geometry

We add some data to the renormalization scheme and carry this information throughout the induction. Suppose that we have defined a sequence $(\Phi_n)_{n \geq 0}$, where $\Phi_n = \mathrm{Ren}(\Phi_{n-1})$, for all $n \geq 1$. Assume that the Schwarzian derivative of Φ_0 is non–positive in every component of its domain. Moreover, for each $\beta \in \mathcal{B}_0$, the corresponding diffeomorphism $B : \beta \to \gamma_0$ admits a diffeomorphic extension from $\hat{\beta} \supset \beta$ to γ_{-1} with negative Schwarzian derivative, which implies small distortion of DB in β, if the ratio $|\gamma_0|/|\gamma_{-1}|$ is small (see [13],Chap.IV).

For each $\beta \in \mathcal{B}_n$ define

$$p_n(\beta) = \frac{|\beta|}{\mathrm{dist}(\beta, \partial\gamma_{n-1})} \ , \ q_n(\beta) = \frac{|\beta|}{\mathrm{dist}(\beta, \gamma_n)} \ .$$

Let

$$r_n = \sup_a \frac{|\gamma_n|}{|\gamma_{n-1}|} \ , p_n = \sup_a \sup_{\beta \in \mathcal{B}_n} p_n(\beta) \ , \ q_n = \sup_a \sup_{\beta \in \mathcal{B}_n} q_n(\beta) \ .$$

The supremum is taken among the parameters a for which renormalization is defined up to the n–th stage, inside the parameter interval of definition of Φ_0 (see Subsection 1.8 below). We use the following Theorem.

THEOREM 5 ([6],[7]). *If r_0, p_0 and q_0 are small enough, then r_n, p_n and q_n tend to zero as $n \to \infty$, and in particular r_n do it exponentially. Although they may not decrease monotonically, given r, p and q sufficiently small (and greater than zero), if r_0, p_0 and q_0 are sufficiently small then $r_n \leq r$, $p_n \leq p$ and $q_n \leq q$ for all $n \geq 0$. Moreover, for each $\beta \in \mathcal{B}_n$ the diffeomorphism $B : \beta \to \gamma_n$ is extendible to a $\theta_n^{-1}|\beta|$–neighborhood of β as the same power j such that $B = \Phi_0^j|\beta$, diffeomorphically onto its image (which is contained in γ_{n-1}), where $\theta_n = \theta(q_n, r_n)$ and θ goes to zero as its arguments go to zero.*

The exponential decrease of r_n is called *decay of geometry* of the generalized renormalization. The growth of the extendibility neighborhood of the β's, together with a negative Schwarzian derivative of the iterates imply that the distortion of the $B : \beta \to \gamma_n$ is becoming smaller with n, or in other words these functions are becoming 'linear'.

We also obtain some expansion on the functions B.

THEOREM 6. *There is $\tau > 0$ such that if r_0, p_0 and q_0 are sufficiently small and $|DB| \geq 1 + \tau$ for all $\beta \in \mathcal{B}_0$ then $|DB| \geq 1 + \tau$ for all $\beta \in \mathcal{B}_n$, $n \geq 1$.*

This result cannot be explicitly used in our applications, but the expansion we get along iterates underlies the distortion control which is central in this work.

1.7. A sufficient condition for stochasticity

If $(\phi_a)_a$ is the quadratic family or one of the special families defined in Subsection 1.4 then the corresponding *initial map* Φ_0 we have defined satisfies the hypotheses in Subsection 1.6. The Schwarzian derivative property is immediate, by its invariance under compositions. Also, the construction of Φ_0 from $\tilde{\Phi}$ gives the extendibility property of the diffeomorphisms $B : \beta \to \gamma_0$, for $\beta \in \mathcal{B}_0$.

In addition, $r_0 = |\gamma_0|/|\gamma_{-1}|$ can be made as small as we wish (and also p_0 and q_0, which depend directly on r_0), provided that a is sufficiently near 2 (or equivalently M is sufficiently big), and moreover the expanding property of the functions $B : \beta \to \gamma_0$ is assured. This will be shown in Section 2 for the special families, although for the quadratic family the reasoning is analogous.

Therefore, for a near 2 we can apply Theorems 4 and 5, in particular to conclude that in these cases there is decay of geometry. In fact the decay of geometry turns out to be valid in a much larger extent, even if r_0 is not sufficiently small. This can be proven if one shows, for example, that there is a *quasi–symmetric* conjugacy between Φ_0 and a map $\tilde{\Phi}_0$ such that $\tilde{r}_0 = |\tilde{\gamma}_0|/|\tilde{\gamma}_{-1}|$ is sufficiently small (see [7]), or by direct means (see [3]).

We continue to suppose that $(\Phi_n)_{n\geq 0}$ is well defined. This is guaranteed if and only if ϕ_a is neither renormalizable nor Misiurewicz (since $|\gamma_n| \to 0$, by Theorem 5).

As in Subsection 1.2, for each $n \geq 0$ there is a number $\mathrm{esc}_n \geq 1$ which is the first integer such that $\Phi_n^{\mathrm{esc}_n}(0) \in \gamma_{n-1} \setminus \gamma_n$. Following [7], we say that Φ_n has a *central return* if $\mathrm{esc}_n > 1$, i.e. $\Phi_n(0) \in \gamma_n$, and a *non–central return* otherwise. Martens and Nowicki prove

THEOREM 7 ([12]). *If $(\phi_a)_a$ is a quasi–quadratic family and*

$$\sharp\{n \; ; \; esc_n = esc_n(a) > 1\} < +\infty$$

then ϕ_a is stochastic.

This result is used in [9] applied to families which admit a complex quadratic–like extension to a neighborhood of their original interval of definition. Here we use it for the special families constructed above, with real techniques, to show

THEOREM 8. *If $(\phi_a)_a$ is a special family, then for almost all $a \in \mathcal{R}^c$ either ϕ_a is Misiurewicz or $\sharp\{n; esc_n = esc_n(a) > 1\} < +\infty$.*

Clearly Theorem 8 implies Theorem 1, by Theorem 7, and the fact that Misiurewicz parameters are automatically stochastic.

1.8. Generalized renormalization in families

Let us now describe the renormalization procedure for families, instead of fixing a parameter. We start with a parameter interval J_0 where $\Phi_0 = \Phi_{0,a}$ is defined. If $(\phi_a)_a$ is the quadratic family or a special family then we choose $J_0 = A_M$, where

$$A_M = \{a \; ; \; \phi_a^j(0) \notin \gamma_a \; , \; \forall 0 < j < M \text{ and } \phi_a^M(0) \in \gamma_a\} \; .$$

Although the domain of Φ_0 may vary with a, each connected component of $\mathrm{dom}(\Phi_0)$ has a continuation defined for all $a \in J_0$. The next generation Φ_1 is defined only for the parameters a such that $H_0^{\mathrm{esc}_0}(0) \in \mathrm{int}(\beta)$ for some $\beta \in \mathcal{B}_0$. Therefore the family Φ_1 is defined for parameters in a *union of intervals* and inside each interval the connected components of the domain always admit a continuation. All connected components of $\mathrm{dom}(\Phi_1)$ shrink to a point as $H_0^{\mathrm{esc}_0}(0)$ approaches one of the boundary points of β.

Precisely speaking, we proceed by induction. Let $\Phi_n = (\Phi_{n,a})_{a\in J, J\in\mathcal{J}_n}$ be the n–th renormalization of $\Phi_0 = (\Phi_{0,a})_{a\in J_0}$, where $\mathcal{J}_n$ is a collection of intervals, each one inside some interval of the collection $\mathcal{J}_{n-1}$, for $n \geq 1$ (and $\mathcal{J}_0 = \{J_0\}$). For each $J \in \mathcal{J}_n$ there is a central interval $\gamma_n =$

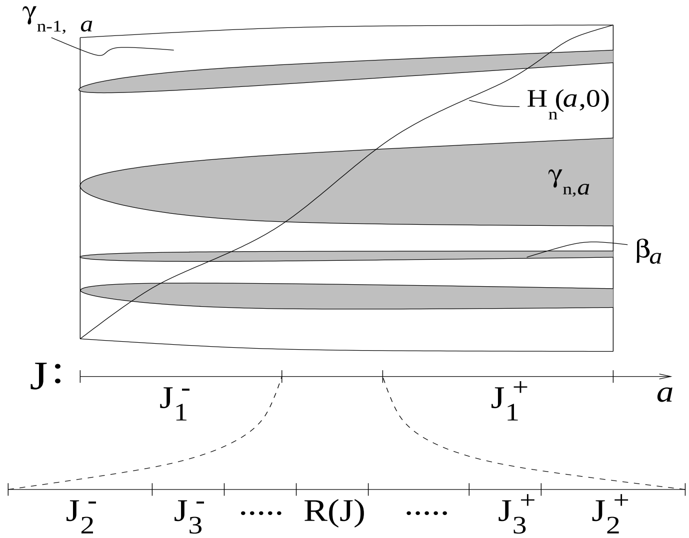

FIG. 3. Parameter dependence of the critical value and the domain of Φ_n.

$\gamma_n(J)$ which varies continuously with $a \in J$, a collection $\mathcal{B}_n = \mathcal{B}_n(J)$ of preimages of the central branch, varying continuously with $a \in J$, a–dependent diffeomorphisms $B : \beta \to \gamma_n$ and a central branch $H_n : \gamma_n \to \gamma_{n-1}$ with the property that $H_n(0)$ completely crosses γ_{n-1} as a varies inside J.

At this point of the exposition we may assume that Φ_n is well behaved in both space and parameter. For example, the central branch is approximately quadratic, the velocity of the critical value is almost constant and the elements $\beta = \beta_a$ move themselves with much smaller velocities. These assumptions are justified by induction and we show they are preserved under generalized renormalization. In fact, they are exactly what we want to prove! Moreover, without loss of generality we suppose that $D_2 H_n < 0$ and $|\mathrm{Im} H_n|$ grows with a (see Figure 3).

We divide the parameter interval $J \in \mathcal{J}_n$ in the following way. First, let $R = R(J)$ be the closed interval of parameters for which H_n is renormalizable (similarly to the interval $[-\frac{1}{4}, 2]$ for the quadratic family). It is easy to see that $a \in R$ if and only if $\mathrm{esc} = \mathrm{esc}(a) = +\infty$ (see for example the explicit descriptions in Subsection 1.13). Secondly, let $\sigma(a) = -$ if $0 \notin \mathrm{Im} H_n$ and $\sigma(a) = +$ otherwise, and define $J^\pm = \{a \in J \ ; \ \sigma(a) = \pm, \mathrm{esc}_n(a) < +\infty\}$, in such a way that $J = J^- \cup R \cup J^+$. For each $k \geq 1$ define

$$J_k^\pm = \{a \in J^\pm \ ; \ \mathrm{esc}_n(a) = k\}.$$

Therefore $\{R, \{J_k^\pm\}_{k\geq 1}\}$ defines a partition of J (see Figure 3) and as a varies along $J_k^\pm$ the point $H_n^k(0)$ crosses the left connected component of $\gamma_{n-1} \setminus \gamma_n$ (except for J_1^+, where $H_n(0)$ crosses the right one).

The next step is to define the collection $\mathcal{J}_{n+1}(J)$ of the elements of $\mathcal{J}_{n+1}$ contained in J. Each $J' \in \mathcal{J}_{n+1}(J)$ can be written as

$$J' = J'(\sigma, k, \beta) = \{a \in J; \sigma(a) = \sigma, \mathrm{esc}_n(a) = k, H_n^k(0) \in \beta_a\}$$

for some $\sigma = \pm$, $k \geq 1$ and $\beta \in \mathcal{B}_n$. Of course $J'(\sigma, k, \beta) \subset J_k^\sigma$ and if β belongs to the right component of $\gamma_{n-1} \setminus \gamma_n$ then $J'(+, k, \beta)$ is not empty only if $k = 1$.

1.9. Parameter distortion in generalized renormalization

The main contribution of this work is to add and control some new data about Φ_n related to its derivatives involving space and parameter. We establish the following notation. For fixed $n \geq 0$ and $J \in \mathcal{J}_n$, each $\beta = (\beta_a)_{a\in J}$ (or $\gamma_n = (\gamma_{n,a})_{a\in J}$) is a continuous family of intervals. Note that β is used to denote both the family and a particular $\beta = \beta_a$ if a is fixed (we believe that this ambiguity will not cause confusion). The function $B : \beta \to \gamma_n$ (resp. $H = H_n : \gamma_n \to \gamma_{n-1}$) can be regarded as a two–variable real function in x and in a, with domain $\{(a, x)\ ;\ a \in J, x \in \beta_a\}$ (resp. $\{(a, x)\ ;\ a \in J, x \in \gamma_{n,a}\}$), such that $B(a, \beta_a) = \gamma_{n,a}$, for a fixed $a \in J$. However, we keep the old notation when dealing with compositions: for example, $B \circ H$ means $B(a, H(a, x))$, H^2 means $H(a, H(a, x))$ and $B^{-1} = B^{-1}(a, x)$ is the function such that $B(a, B^{-1}(a, x)) = x$. Moreover, whenever we write $B(x)$ we mean $B(a, x)$, if there is no reason for doubt. The partial derivatives are written as B_x, B_a, H_{xa}, etc. If j is an integer, H_x^j will mean $(H^j)_x$ and $(H_x)^j$ is H_x to the j–th power.

We will say that the sub-family $\Phi_n = (\Phi_{n,a})_{a\in J}$, for $J \in \mathcal{J}_n$, $n \geq 0$, *satisfies the inequalities* $(P_0)_{\delta_0}$ if the central branch $H = H_n : \gamma_n \to \gamma_{n-1}$ obeys

$$|H_{xx}| > 0\ ,\ |H_a| > 0$$

and

$$|\gamma_n| \cdot \left|\frac{H_{xxx}}{H_{xx}}\right|\ ,\ |\gamma_n| \cdot \left|\frac{H_{ax}}{H_a}\right|\ ,\ |J| \cdot \left|\frac{H_{aa}}{H_a}\right|\ ,\ |J| \cdot \left|\frac{H_{xxa}}{H_{xx}}\right| < \delta_0$$

for all $x \in \gamma_n = \gamma_{n,a}$ and all $a \in J$.

Let us see what these conditions imply. Integrating $\frac{\partial}{\partial x} \log |H_{xx}|$ we get, for sufficiently small δ_0,

$$1 - C\delta_0 \leq \frac{H_{xx}}{2S} \leq 1 + C\delta_0$$

for all $x \in \gamma_n$ and for some constant $S = S_a$, where C is a universal constant. By integration of this last inequality we obtain

$$(1 - C\delta_0) \cdot 2Sx \le H_x \le (1 + C\delta_0) \cdot 2Sx \ ,$$

$$(1 - C\delta_0) \cdot Sx^2 \le H(x) - H(0) \le (1 + C\delta_0) \cdot Sx^2 \ ,$$

that is, H is nearly quadratic for all $a \in J$, but the curvature $S = S_a$ may (in principle) vary with a. By integration of $\frac{\partial}{\partial x} \log |H_a|$ we get

$$1 - C\delta_0 \le \frac{H_a}{v} \le 1 + C\delta_0$$

for all $x \in \gamma_n$ and for some constant $v = v_a$. In other words, the velocity of $H(a,x)$ with respect to a is approximately equal to the velocity of $H(a,0)$, for any $x \in \gamma_n$. Similarly, using the other two inequalities we show that v_a and S_a are almost constants along $a \in J$, justifying the definition of constants $v_n = v_n(J)$ and $S_n = S_n(J)$, which approximate the values of $H_a(a,x)$ and $H_{xx}(a,x)$ for all $a \in J$ and $x \in \gamma_n = \gamma_{n,a}$.

Let us observe that generalized renormalization preserves the sign of the product $v_n \cdot S_n$. In Subsection 1.8 we supposed it to be negative, and it is indeed negative in the way we have defined the quadratic family and the special families.

We now define a second set of derivative inequalities. We say that $\Phi_n = (\Phi_{n,a})_{a\in J}$, $J \in \mathcal{J}_n$, $n \ge 0$, *satisfies the inequalities* $(P_1)_{\delta_1}$ if for all $\beta \in \mathcal{B}_n = \mathcal{B}_n(J)$ and the corresponding function $B : \beta \to \gamma_n$ the quotients

$$\left|\frac{B_a}{B_x H_a}\right| \ , |\gamma_n| \cdot \left|\frac{B_{xx}}{(B_x)^2}\right| \ , |\gamma_n| \cdot \left|\frac{B_{xa}}{(B_x)^2 H_a}\right| \ ,$$

$$|\gamma_n|^2 \cdot \left|\frac{B_{xxx}}{(B_x)^3}\right| \ , |\gamma_n|^2 \cdot \left|\frac{B_{xxa}}{(B_x)^3 H_a}\right| \ , |\gamma_n| \cdot \left|\frac{B_{aa}}{(B_x)^2 (H_a)^2}\right|$$

are smaller than δ_1 for all $x \in \beta_a$ and $a \in J$ such that $|\mathrm{Im} H_n| \ge \frac{1}{8}|\gamma_{n-1}|$ or $\mathrm{Im} H_n \cap \mathcal{U}(\beta) \ne \emptyset$ for a certain neighborhood $\mathcal{U}(\beta)$ of β (for the induction we only need $\mathrm{Im} \cap \beta \ne \emptyset$, the $\mathcal{U}(\beta)$ is used in Subsection 1.14). Here H_a means $H_a(a,0)$, not $H_a(a,x)$ since x lies inside β (or else $v_n = v_n(J)$, if inequalities $(P_0)_{\delta_0}$ are already satisfied). The condition

$$|\gamma_n| \cdot \left|\frac{B_{xx}}{(B_x)^2}\right| < \delta_1$$

assures us small distortion in β, since for $x_1, x_2 \in \beta$

$$|\log |B_x(x_1)| - \log |B_x(x_2)|| \le \int_{\gamma_n} \left|\frac{B_{xx}(B^{-1}y)}{[B_x(B^{-1}y)]^2}\right| dy < \delta_1 \ .$$

This quotient, in particular, can be controlled in two ways: by direct calculation (as for the others) or by extendibility properties, with the use of Köebe's Lemma (see [13],Chap. IV). It stands as an open question if there is a method other than direct calculation to control the remaining quotients.

The condition

$$\left| \frac{B_a}{B_x H_a} \right| < \delta_1$$

can be easily understood if we translate it into the form

$$|B_a^{-1}| < \delta_1 |H_a(0)| \ ,$$

that is to say, the velocity of β_a is much smaller than the velocity of $H_n^{\rm esc}(0)$. This is valid for values of a which include the moment when β is crossed by $H_n^{\rm esc}(0)$.

Let us remark that the final goal is to keep under control the velocity of the critical value: it has to be almost constant along any interval $J \in \mathcal{J}_n$, $n \geq 0$. This corresponds to the quotient in H_{aa}. But it turns out that to control this quotient we need also the others involving the function H and, as a consequence, all the ones involving the functions B.

THEOREM 9. *If $\Phi_0 = (\Phi_{0,a})_{a \in J_0}$ satisfies inequalities $(P_0)_{\delta_0}$ and $(P_1)_{\delta_1}$ for sufficiently small $\delta_0, \delta_1 > 0$, then $\Phi_n = (\Phi_{n,a})_{a \in J}$ also satisfies the inequalities for all $J \in \mathcal{J}_n$, $n \geq 0$, provided r_m, p_m and q_m are sufficiently small for all $m \geq 0$ (which is the same as requiring r_0, p_0 and q_0 sufficiently small, by Theorem 5).*

The proof of Theorem 9 is the content of Section 3. The following Theorem, which will be proven in Subsection 2.1 , guarantees that Theorem 9 can be used for the special families: they satisfy the *starting condition* in the hypotheses of Theorem 9.

THEOREM 10. *Let $(\phi_a)_a$ be a special family. Given $\delta_0, \delta_1 > 0$, if M is large enough then $\Phi_0 = (\Phi_{0,a})_{a \in A_M}$ satisfies inequalities $(P_0)_{\delta_0}$ and $(P_1)_{\delta_1}$.*

As we have already observed, also r_0 (and hence p_0 and q_0) can be made sufficiently small by choosing M large. The conclusion is that Theorem 9 can be applied for the special families. Unfortunately, the same cannot be said for the quadratic family: H_{xx} and H_a have a fixed amount of distortion along $x \in \gamma_0$ and $a \in A_M$, for arbitrarily large M. Inequalities $(P_1)_{\delta_1}$, however, can be proven in the same way as for the special families.

1.10. Infinitely many central returns are rare: proof of Theorem 8.

We assume that $(\phi_a)_a$ is a special family and $J_0 = A_M$, with M sufficiently large, so that by Theorem 10 $(\Phi_{0,a})_{a\in J_0}$ satisfies properties $(P_0)_{\delta_0}$ and $(P_1)_{\delta_1}$ for sufficiently small δ_0 and δ_1. The constants δ_0 and δ_1 will be chosen in the proof of Theorem 9, and M is taken accordingly. Therefore Theorem 9 is applied, and in particular a good control of the critical value velocity is attained.

To use this consequence we first remark the following fact: "*for any* $J \in \mathcal{J}_n$*,* $n \geq 0$*,* $|\gamma_{n-1,a}|$ *is approximately constant*". This can be easily proven by induction, using the quadratic approximation of the central branch and the fact that the distance between each $\beta \in \mathcal{B}_n$ and the boundary of γ_{n-1} is much larger than β. For each $J \in \mathcal{J}_n$, $n \geq 0$, let

$$\mathrm{Cent}(J) = \{a \in J \ ; \ H_n(0) \in \gamma_n\} \ .$$

By the above considerations,

$$\frac{|\mathrm{Cent}(J)|}{|J|} \leq C r_n \ ,$$

for some universal constant $C > 0$.

Let $\mathcal{C}$ be the parameter subset such that if $a \in \mathcal{C}$ then ϕ_a is not renormalizable and the critical point is recurrent (shortly, $\mathcal{C} = \mathcal{R}^c \setminus \mathcal{M}$), corresponding exactly to the set of parameters for which $(\Phi_n)_{n\geq 0}$ is well defined. We restrict ourselves to

$$\mathcal{C} \cap A_M = \bigcap_{n\geq 0} \bigcup_{J\in\mathcal{J}_n} J \ ,$$

where $J_0 = A_M$ and $\mathcal{J}_0 = \{J_0\}$. For each $J \in \mathcal{J}_n$, $n \geq 0$, the collection $\mathcal{J}_{n+1}(J) = \{J' \in \mathcal{J}_{n+1} \ ; \ J' \subset J\}$ can be decomposed into two parts:

$$\mathcal{J}_{n+1}(J) = \mathcal{J}_{n+1}^{\mathrm{cent}}(J) \cup \mathcal{J}_{n+1}^{\mathrm{ext}}(J) \ ,$$

where $J' \in \mathcal{J}_{n+1}^{\mathrm{cent}}(J)$ if and only if $J' = J'(\sigma, k, \beta)$ for some $k > 1$, $\sigma = \pm$ and $\beta \in \mathcal{B}_n$, and $J' \in \mathcal{J}_{n+1}^{\mathrm{ext}}(J)$ if and only if $J' = J'(\sigma, 1, \beta)$ for some $\sigma = \pm$ and $\beta \in \mathcal{B}_n$.

Hence for any $a \in \mathcal{C} \cap A_M$ there is a sequence $J_0 \supset J_1 \supset \ldots \supset J_n \supset \ldots$ converging to a (since $|J_n| \lesssim \frac{1}{2}|J_{n-1}|$, $\forall n \geq 1$), such that for each $n \geq 1$ either $J_n \in \mathcal{J}_n^{\mathrm{cent}}(J_{n-1})$ or $J_n \in \mathcal{J}_n^{\mathrm{ext}}(J_{n-1})$, thus defining a function $\theta_a : \mathbf{N} \to \{\mathrm{cent}, \mathrm{ext}\}$. According to Theorem 7, if $\theta_a^{-1}(\{\mathrm{cent}\})$ contains only finitely many elements then $a \in \mathcal{E}$. We will prove that the set of $a \in A_M$ such that $\theta_a^{-1}(\{\mathrm{cent}\})$ is infinite has Lebesgue measure zero, consequently proving Theorem 8.

For this purpose we use the so–called Borel–Cantelli Lemma: "*if* $\{I_i\}_{i\geq 0}$ *is a collection of intervals* $I_i \subset [0,1]$ *such that* $\sum_{i\geq 0} |I_i| < \infty$ *then the set of*

$x \in [0,1]$ *such that* x *belongs to infinitely many* I_i*'s has Lebesgue measure zero*".

For us, $\theta_a^{-1}(\{\text{cent}\})$ is infinite if and only if a belongs to infinitely many intervals of the set

$$\bigcup_{n\geq 0}\bigcup_{J\in\mathcal{J}_n}\mathcal{J}_{n+1}^{\text{cent}}(J)\ .$$

For any $J\in\mathcal{J}_n$, $n\geq 0$, the intervals of the collection $\mathcal{J}_{n+1}^{\text{cent}}(J)$ are pairwise disjoint and contained in the interval $\text{Cent}(J)$. Hence

$$\frac{1}{|J|}\sum\{|J'|\ ;\ J'\in\mathcal{J}_{n+1}^{\text{cent}}(J)\}\leq Cr_n\ ,$$

as we remarked above. Since $\sum\{|J|;J\in\mathcal{J}_n\}\leq|J_0|$ we get

$$\sum_{n\geq 0}\sum_{J\in\mathcal{J}_n}\{|J'|;J'\in\mathcal{J}_{n+1}^{\text{cent}}(J)\}\leq C|J_0|\sum_{n\geq 0}r_n\ .$$

By Theorem 5 r_n decreases exponentially and therefore the sum is finite.

1.11. Abundance of stochasticity: proof of Theorem 2

Let $n\geq 0$, $J\in\mathcal{J}_n$, $S_n=S_n(J)$ and $v_n=v_n(J)$. As H_n is almost quadratic and the velocity of the critical value is almost constant, we can estimate the size of the renormalization interval $R(J)$:

$$\frac{2}{|S_n\cdot v_n|}<|R(J)|<\frac{3}{|S_n\cdot v_n|}.$$

But we want to express $|R(J)|$ as a fraction of $|J|$. Note that if $a=\sup J$, then

$$S_n(\frac{1}{2}|\gamma_n|)^2\simeq|\gamma_{n-1}|.$$

On the other hand,

$$v_n\simeq\frac{|\gamma_{n-1}|}{|J|},$$

taking into account that $|\gamma_{n-1}|$ is almost constant. Hence

$$|R(J)|<r_n^2|J|.$$

As $r_n<1$, $\forall n\geq 0$, and $(r_n)_n$ decreases exponentially, there is $\lambda<1$ such that $r_n^2<\lambda^n$, $\forall n\geq 0$. Then, for some $C(\lambda)>0$,

$$\text{Leb}(A_M\setminus\mathcal{R})\geq|A_M|\cdot\prod_{n=1}^{\infty}(1-\lambda^n)\geq|A_M|\cdot\exp\{-C(\lambda)\sum_{n=1}^{\infty}\lambda^n\}>0.$$

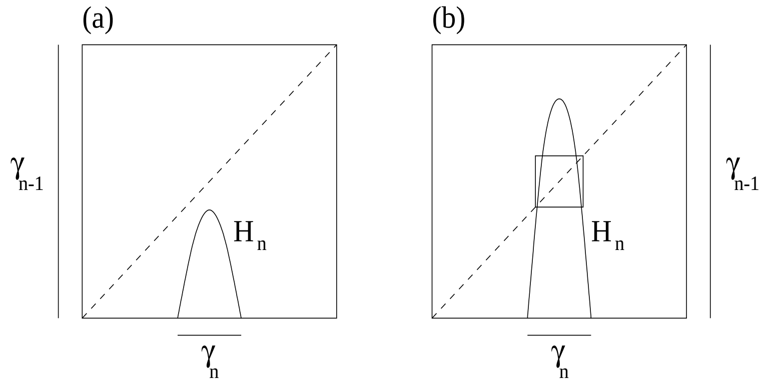

FIG. 4. Two possibilities for the position of the critical value.

By Theorem 1, almost all parameters in $A_M \setminus \mathcal{R}$ are stochastic, proving Theorem 2.

Observe that without any additional cost if one shows that the series $\sum_{n\geq 0} r_n^2$ is dominated by r_0^2, this would imply $\mathrm{Leb}(\mathcal{R} \cap A)M) \lesssim |A_M|^2$, since $r_0 \lesssim 2|A_M|^{1/2}$ (see Subsection 2.1, just before Lemma 12), and therefore we would have $\mathrm{Leb}(\mathcal{R}\cap[2-\epsilon, 2]) \leq C\epsilon^2$ (a slightly weaker estimate was obtained by Tsujii in [17] using other methods). This is the best superior bound possible, since $|R(A_M)|$ is of the order of $|A_M|^2$.

1.12. Denseness of renormalization: proof of Theorem 4

If $\mathcal{R}$ was not dense in $A_M = J_0$ then there would be an infinite sequence $J_0 \supset J_1 \supset J_2 \supset \cdots J_n \cdots$ of nested intervals, $J_n \in \mathcal{J}_n$, $\forall n \geq 0$, such that $\bigcap_{n\geq 0} J_n$ is an interval. On the other hand, as the velocity of the critical value is almost constant for every $J \in \mathcal{J}_n$, $n \geq 0$, then it is easy to see that $|J_{n+1}| << \frac{1}{2}|J_n|$, implying that the intersection must be a single point.

1.13. Generalized renormalization in detail: notation

We explain a bit more how $\Phi_{n+1} = \mathrm{Ren}(\Phi_n)$ is obtained, taking profit to establish the notation which will be used in Section 3. We will suppose that Φ_n is well behaved, for example that it satisfies inequalities $(P_0)_{\delta_0}$ and $(P_1)_{\delta_1}$ for small $\delta_0, \delta_1 > 0$. In particular, $H = H_n : \gamma_n \to \gamma_{n-1}$ is almost quadratic. We suppose without loss of generality that $D_2 H_n < 0$.

There are essentially two cases to consider, as shown in Figure 4: (a) *beneath renormalization* or $\sigma(a) = -$ and (b) *beyond renormalization* or $\sigma(a) = +$. Observe that situation (b) *always* occurs after (a) in the parameter line, since in our setting $S_n \cdot v_n < 0$.

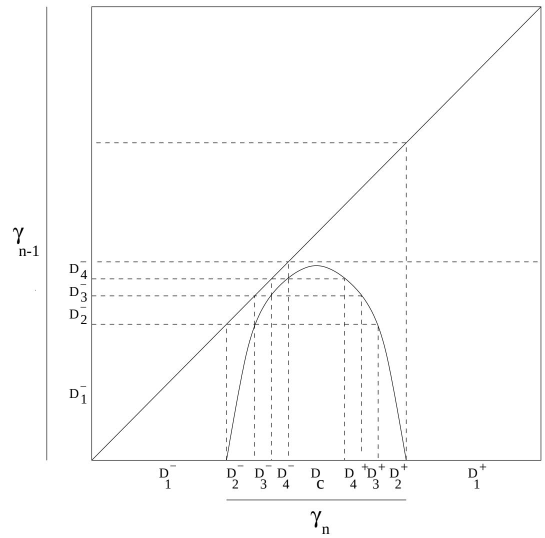

FIG. 5. Fundamental domains beneath renormalization.

Beneath renormalization. We establish a division of the configuration space into fundamental domains. Define first D_1^- as the left connected component of $\gamma_{n-1} \setminus \gamma_n$ and D_1^+ as the right one. Then by induction define D_t^- (resp. D_t^+) as the left (resp. right) connected component of $H^{-1}(D_{t-1}^-)$, for $t \geq 2$. This definition must stop at $t = esc$, since $H(0) \in D_{\text{esc}}^-$. The extremal points of $D_t^\pm$ are defined by $D_t^\pm = [d_{t-1}^\pm, d_t^\pm]$ for $t \geq 1$ and we also define a central domain $D_c = [d_{\text{esc}}^-, d_{\text{esc}}^+]$ (see Figure 5 with $esc = 4$).

Let $\mathcal{P} = \mathcal{P}_{n+1}$ be the set of preimages of elements of $\mathcal{B}_n$ under H. That is to say, for every $\pi \in \mathcal{P}$ there is $1 \leq t \leq \text{esc}$ such that $H^t(\pi) = \beta$, for $\beta \in \mathcal{B}_n$. It means that $\pi \subset D_{t+1}^+ \cup D_{t+1}^-$ if $t < \text{esc}$ and $\pi \subset D_c$ if $t = \text{esc}$.

To this element $\pi \in \mathcal{P}$ we associate the diffeomorphism $P : \pi \to \gamma_n$ given by $P = B \circ H^t$, where $B : \beta \to \gamma_n$ is the diffeomorphism associated to $\beta = H^t(\pi)$. Now it is clear that $\mathcal{B}_{n+1}$ is the set of preimages of γ_{n+1} under compositions $P_j \circ \cdots \circ P_1$ for $\pi_1, \ldots, \pi_j \in \mathcal{P}$. The interval γ_{n+1} itself can be defined as $\gamma_{n+1} = H^{-1}(\pi^*)$, where $\pi^* \in \mathcal{P}$ is the element in D_{esc}^- to which $H(0)$ belongs.

If $x \in \gamma_n$ then there is $t \geq 1$ such that $H^t(x) \in \gamma_{n-1} \setminus \gamma_n$. These iterates can be decomposed, in general, as

$$H^t = F_E \circ F_S \circ F_0 \circ F_H \ .$$

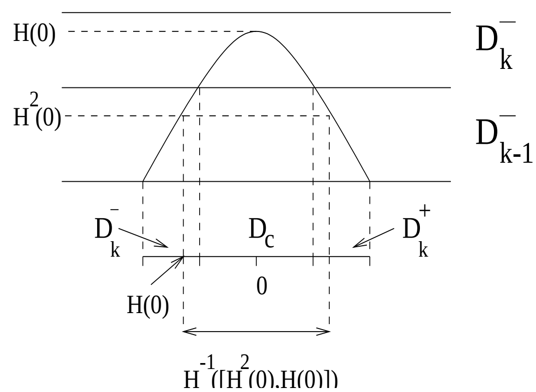

FIG. 6. Detail of the critical region beneath renormalization.

This decomposition will be made in the same way for both beneath and beyond renormalization. In the case we treat here, beneath renormalization, $F_H = \mathrm{Id}$. If $x \in H^{-1}([H^2(0), H(0)])$ (see Figure 6) or esc $= 1$ then $F_0 = H$, otherwise $F_0 = \mathrm{Id}$. Observe that $|H^i(x)|$ and $|DH(H^i x)|$ are growing with i. Let i_0 be the first integer such that $|DH(H^{i_0}(F_0 x))| \geq 4$ and define $F_S = H^{i_0}$. It may happen that $|DH(F_0 x)| \geq 4$, in this case set $F_S = \mathrm{Id}$. Finally, F_E is defined by the remaining powers of H.

Beyond renormalization. Let x^- be the leftmost fixed point of H and $x^+ \neq x^-$ such that $H(x^+) = x^-$. As in the other case, we divide the space outside $[x^-, x^+]$ into fundamental domains. Let D_1^- (resp. D_1^+) be the left (resp. right) connected component of $\gamma_{n-1} \setminus \gamma_n$. By induction, define D_t^- and D_t^+ as the left and right connected components of $H^{-1}(D_{t-1}^-)$, for $t \geq 2$. Differently from before, this induction never stops, and the D_t^- and D_t^+ accumulate, respectively, on x^- and x^+. For $t \geq 1$, let $D_t^\pm \equiv [d_{t-1}^\pm, d_t^\pm]$ (see Figure 7).

Also inside $[x^-, x^+]$ we make subdivisions. Let $[p^-, p^+] = H^{-1}(\{x \geq x^+\})$, q^+ be the rightmost fixed point of H and $q^- \neq q^+$ be such that $H(q^-) = q^+$. We divide the interval $[x^-, q^-]$ (resp. $[q^+, x^+]$) into intervals $\omega_k^- = [q_k^-, q_{k-1}^-]$ (resp. $\omega_k^+ = [q_{k-1}^+, q_k^+]$), for $k \geq 1$, such that the functions $H^k : \omega_k^\pm \to [q^-, q^+]$ are diffeomorphisms. In addition, we divide $[q^-, p^-]$ (resp. $[p^+, q^+]$) into intervals α_k^- (resp. α_k^+), for $k \geq 2$, such that $H(\alpha_k^\pm) = \omega_{k-1}^+$ (analogously to the definitions in Subsections 1.3 and Subsection 1.4, but we keep the same notation since there is no danger of confusion).

Let $\mathcal{B}_{n+1}^H \subset \mathcal{B}_{n+1}$ be the set of *direct* preimages of γ_{n+1} under powers of H. These preimages are located in the gaps of the Cantor set

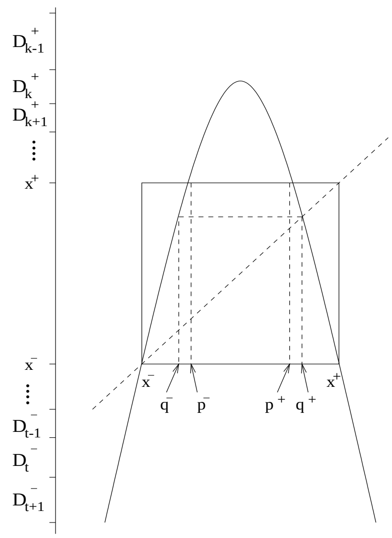

FIG. 7. Fundamental domains beyond renormalization.

$\Lambda_H \equiv \bigcap_{i\geq 0} H^{-i}([x^-, x^+])$ and if $\tilde{\beta}_H \in \mathcal{B}^H_{n+1}$ then its corresponding diffeomorphism $\tilde{B}_H : \tilde{\beta}_H \to \gamma_{n+1}$ is given by $\tilde{B}_H = H^i$ for some $i \geq 1$.

As in the previous case we also produce a set $\mathcal{P} = \mathcal{P}_{n+1}$ of preimages of γ_n, taking all possible preimages of the elements of $\mathcal{B}_n$ under iterates of H. These preimages spread around in the domains $D^\pm_t$, inside the central gap $[p^-, p^+]$ and between the gaps of the Cantor set Λ_H.

The set $\mathcal{B}_{n+1}$ is the union of $\mathcal{B}^H_{n+1}$ with all the preimages of $\{\gamma_{n+1}\}\cup\mathcal{B}^H_{n+1}$ under all possible compositions $P_j \circ \cdots \circ P_1$, for $\pi_1, \ldots, \pi_j \in \mathcal{P}$. Therefore to each $\tilde{\beta} \in \mathcal{B}_{n+1}$ the diffeomorphism $\tilde{B} : \tilde{\beta} \to \gamma_{n+1}$ is given either by the composition $\tilde{B}_H \circ P_{j-1} \circ \cdots \circ P_1$ or by the composition $P_j \circ P_{j-1} \circ \cdots \circ P_1$, for some $\tilde{\beta}_H \in \mathcal{B}^H_{n+1}$, $\pi_1, \ldots, \pi_j \in \mathcal{P}$, $j \geq 1$.

If $x \in \gamma_n$, either $x \in \Lambda_H$ or there is $t \geq 1$ such that $H^t(x) \in \gamma_{n-1} \setminus \gamma_n$. The latter is the only possibility if $x \in \tilde{\beta}$ for some $\tilde{\beta} \in \mathcal{B}_{n+1}$. We decompose the orbit as $H^t = F_E \circ F_S \circ F_0 \circ F_H$, where this time $F_S = \mathrm{Id}$. If x belongs to $[x^-, q^-]$ or $[q^+, x^+]$ then $F_H = H^i$, where i is the first integer such that $H^i(x) \in [p^-, p^+]$, otherwise $F_H = \mathrm{Id}$. If $F_H(x) \in [p^-, p^+]$ then $F_0 = H$, otherwise $F_0 = \mathrm{Id}$. And $F_E = H^i$ if $F_S \circ F_0 \circ F_H(x) \in D^\pm_{i+1}$, for some $i \geq 1$, otherwise $F_S \circ F_0 \circ F_H(x) \in D^+_1$ and $F_E = \mathrm{Id}$.

Notation remark. For coherence, we define the set $\mathcal{P}_0$ as the set

$$\{\alpha^-_2, \alpha^-_3, \ldots, \alpha^-_{M-1}, \alpha_M, \alpha^+_{M-1}, \ldots, \alpha^+_3, \alpha^+_2\}\,,$$

in the sense of the definitions in Subsections 1.3 and 1.4.

1.14. Misiurewicz are rare: proof of Theorem 3

We only sketch the main ideas involved and let the details to the reader, using the notation introduced until now.

As we have already observed before, ϕ_a is Misiurewicz if and only if ϕ_a is not renormalizable and Φ_{n+1} is not defined for some $n \geq 0$. This will happen if and only if $H_n^{\text{esc}}(0) \in \Gamma_n$, where

$$\Gamma_n = \left(\bigcup_{\beta \in \mathcal{B}_n} \text{int}(\beta) \right)^c .$$

The set Γ_n (which clearly depends on a) admits a decomposition

$$\Gamma_n = \Gamma_n^{-1} \cup \Gamma_n^0 \cup \Gamma_n^1 \cup \ldots \cup \Gamma_n^n$$

defined in the following way. For each $n \geq 0$, let

$$\Gamma_{-1}^{-1} = \bigcup_{k=1}^{M-1} \partial \alpha_k^{\pm} ,$$

$$\tilde{\Gamma}_n^n = \bigcap_{j \geq 1} \bigcup_{\{\pi_1, \ldots, \pi_j\} \in \mathcal{P}_n^j} (P_j \circ \cdots \circ P_1)^{-1} (\gamma_{n-1} \setminus \text{int}(\gamma_n)) ,$$

$$\Lambda_H = \bigcap_{i \geq 0} H_n^{-i} \left([x^-, x^+] \right) ,$$

($\Lambda_H = \emptyset$ if $\sigma = -$), and

$$\Gamma_n^n = \tilde{\Gamma}_n^n \cup \Lambda_H .$$

By induction on n, for each $-1 \leq m < n$ define

$$\Gamma_n^m = \bigcup_{j \geq 0} \bigcup_{\{\pi_1, \ldots, \pi_j\} \in \mathcal{P}_n^j} (P_j \circ \cdots \circ P_1)^{-1} \left(\bigcup_{i \geq 1} H_n^{-i} (\Gamma_{n-1}^m) \right) .$$

The sets $\tilde{\Gamma}_n^n$, $n \geq 1$ and Λ_H are called *dynamically defined Cantor sets* (see [15], Chap. 4), the former with infinitely many branches. They clearly have measure zero since for each $j \geq 1$ a hole with a definite fraction of the total remaining measure is suppressed. This is guaranteed by the non–positive Schwarzian derivative and the resulting distortion properties.

The suppressed fraction is given by, up to a constant factor, the ratio $|\gamma_{n-1}|/|\gamma_n|$. The sets Γ_n have as well measure zero, for they are formed by countable preimages of the Γ_m^m, $0 \leq m \leq n$.

To each point x in $\tilde{\Gamma}_n^n$ it is naturally assigned a *code*, which is given by the sequence $\underline{\pi} = \{\pi_1, \pi_2, \ldots\} \in \mathcal{P}_m^\infty$ such that $x \in \pi_1$ and $P_j \circ \cdots \circ P_1(x) \in \pi_{j+1}$ for all $j \geq 1$. As each $\pi \in \mathcal{P}_n$ has a continuation $\pi = \pi_a$ defined for all $a \in J$, then also $x \in \tilde{\Gamma}_n^n$ has a continuation $x = x_a$ defined for all $a \in J$. The same idea can be applied to the Cantor set Λ_H.

Moreover, a code can be also assigned to each point $y \in \Gamma_n$, by determining how it is obtained by successive preimages of a point $x \in \Gamma_m^m$ for some $m \leq n$. Anyway, as a varies along J the escaping image of the critical point $H_n^{\mathrm{esc}}(0)$, $\mathrm{esc} = \mathrm{esc}(a)$ crosses a countable intertwined union of (pre-)images of dynamically defined Cantor sets Γ_m^m in $\gamma_{n-1} \setminus \gamma_n$. It turns out that this crossing is always *transverse*. This can be proven directly in the same way as we calculate the quotients $|B_a/B_xH_a|$ for each $\beta \in \mathcal{B}_n$.

We want to show that

$$M(J) = \{a \in J; H_n^{\mathrm{esc}}(0) \in \Gamma_n\}$$

has Lebesgue measure zero. However, the informations we have above are, at least in principle, not enough to prove it. We illustrate the idea of the proof by showing that the set

$$M^*(J) = \{a \in J; H_n(0) \in \tilde{\Gamma}_n^n\}$$

has Lebesgue measure zero. The ideas can be easily applied to $M(J)$, remarking in addition that $|H_a^{\mathrm{esc}}(0)| \geq |H_a(0)|$ (see Lemma 62).

If we admit Theorem 9 to be valid, then inequalities $(P_1)_{\delta_1}$ are satisfied for some $\delta_1 > 0$ small. In particular, this implies that

$$\left|\frac{B_a}{B_xH_a}\right| < \delta_1$$

for all $x \in \beta_a$ and $a \in J$ such that $|\mathrm{Im}H_n| \geq \frac{1}{8}|\gamma_{n-1}|$ or $\mathrm{Im}H_n \cap \mathcal{U}(\beta) \neq \emptyset$ for a certain neighborhood $\mathcal{U}(\beta)$ of β. The neighborhood $\mathcal{U}(\beta)$ can be specified at this point (see Subsection 3.6). If $\beta = (P_j \circ \cdots \circ P_1)^{-1}(\gamma_n)$, for some collection $\{\pi_1, \ldots, \pi_j\} \subset \mathcal{P}_n$, then $\mathcal{U}(\beta) = (P_j \circ \cdots \circ P_1)^{-1}(\gamma_{n-1})$. As remarked above, by distortion properties,

$$\frac{|\beta|}{|\mathcal{U}(\beta)|} \geq C^{-1}\frac{|\gamma_n|}{|\gamma_{n-1}|}$$

for some universal constant $C > 0$. If we take the parameter intervals

$$J(\beta) = \{a \in J; H_n(0) \in \beta_a\} \subset \tilde{J}(\beta) = \{a \in J; H_n(0) \in \mathcal{U}(\beta_a)\}$$

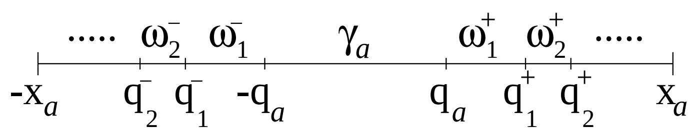

FIG. 8. Partition of I_a.

then

$$\frac{|J(\beta)|}{|\tilde{J}(\beta)|} \geq C^{-1} \inf_{a \in \tilde{J}(\beta)} \frac{|\gamma_{n,a}|}{|\gamma_{n-1,a}|} \; .$$

The right hand side of this last inequality constitutes in fact an inferior bound for the proportion suppressed after each step $j+i$, $i \geq 0$, inside $\tilde{J}(\beta)$. The result is a zero Lebesgue measure set.

2. THE STARTING CONDITION

2.1. Proof of Theorem 10.

In this Section we prove that inequalities $(P_0)_{\delta_0}$ and $(P_1)_{\delta_1}$ are satisfied for $(\Phi_{0,a})_{a\in A_M}$ with δ_0, δ_1 small, provided that M is sufficiently large. For this purpose, we will need to evaluate all the mixed derivatives listed in Subsection 1.9. Some problems arising here and their solutions give us a clue of the proof of Theorem 9, postponed to Section 3.

We have already defined the intervals $\gamma_a = [q_a^-, q_a^+] \subset I_a = [x_a^-, x_a^+]$, where $\phi_a(q_a^\pm) = q_a^+$ and $\phi_a(x_a^\pm) = x_a^-$, a partition $\{\alpha_M, \{\alpha_k^\pm\}_{k=2,\dots,M-1}\}$ of γ_a, as well as the diffeomorphisms $A_k^\pm : \alpha_k^\pm \to \gamma_a$ and the unimodal function $H : \alpha_M \to \gamma_a$. As we wish to study these functions, we also need some terminology for iterates outside γ_a. Let

$$\omega_1^\pm = \omega_{1,a}^\pm = \phi_a^{-1}(\gamma_a) \cap \{\pm x > 0\}$$

and by induction define, for all $k \geq 1$,

$$\omega_{k+1}^\pm = \phi_a^{-1}(\omega_k^-) \cap \{\pm x > 0\}$$

(see Figure 8). We remark that $a \in \mathrm{cl}(A_M)$ if and only if $\phi_a(0) \in \omega_{M-1}^+$. With these definitions, we have

$$\alpha_M = \phi_a^{-1}(\omega_{M-1}^+) \; , \; \alpha_k^\pm = \phi_a^{-1}(\omega_k^+) \cap \{\pm x > 0\} \; .$$

For each $\omega_k^\pm$, $k \geq 1$, there is a diffeomorphism $W_k^\pm : \omega_k^\pm \to \gamma_a$, where $W_k^\pm = \phi_a^k | \omega_k^\pm$. To avoid confusion, we define $F(a, \cdot) \equiv \phi_a$, so that now F_a

must be understood as the derivative of F with respect to the parameter. In this way, we have

$$H = W^{+}_{M-1} \circ F|\alpha_M \ , \ A^{\pm}_k = W^{\pm}_{k-1} \circ F|\alpha^{\pm}_k \ .$$

Observe that by our construction, F is nearly quadratic in $\alpha^-_{M-1} \cup \alpha_M \cup \alpha^+_{M-1}$, if M is large (it might be exactly quadratic if $\phi_a = \tilde{\phi}_a$, see Subsection 1.4).

As $(\phi_a)_a$ is C^3 near $(\tilde{\phi}_a)_a$, we assume that there is $\delta > 0$ and $C > 0$ such that

1. $|F_a - 1|, |F_{xx} + \mu| < \delta$ if $x \in [-\frac{1}{5}, +\frac{1}{5}]$;
2. $|F_a|, |F_{xx}|, |F_{ax}|, |F_{xxx}|, |F_{aa}|, |F_{xxa}| < C$ if $x \in \gamma_a$;
3. $|F_a|, |F_{xx}| < \delta$ if $x \notin \gamma_a$;
4. $|F_{ax}|, |F_{xxx}|, |F_{aa}|, |F_{xxa}| < \delta$ if $x \notin \gamma_a$ or $x \in [-\frac{1}{5}, +\frac{1}{5}]$.

Lemma 11. *Let $W = W^{\pm}_k$, for $k \geq 1$. Then*

1.the absolute values of the quotients

$$\frac{W_a}{W_x} \ , \ \frac{W_{xx}}{(W_x)^2} \ , \ \frac{W_{xa}}{(W_x)^2} \ , \ \frac{W_{aa}}{(W_x)^2} \ , \ \frac{W_{xxx}}{(W_x)^3} \ , \ \frac{W_{xxa}}{(W_x)^3}$$

are smaller than δ, for all $x \in \omega^{\pm}_k$;
2.$W_x \simeq -|\gamma_a| \cdot |\omega^+_k|^{-1}$.

Proof. The proof is straightforward, using the Appendix and the uniform expansion of nearly λ along the iterates. ∎

Before continuing, we observe that

$$|\alpha_M| \lesssim \left(\frac{8}{\mu}|\omega^+_{M-1}|\right)^{1/2} \ ,$$

$|\omega^+_{M-1}|$ is almost constant for $a \in A_M$ (by Lemma 11) and $|A_M| \simeq |\omega^+_{M-1}|$.

Lemma 12. *For all $x \in \alpha_M$ and $a \in A_M$,*

1.$|H_a| \simeq |\gamma_a| \cdot |\omega^+_{M-1}|^{-1}$;
2.$|H_{xx}| \simeq \mu|\omega^+_{M-1}|^{-1}$;
3.the absolute values of the quotients

$$|\alpha_M| \cdot \frac{H_{xxx}}{H_{xx}} \ , \ |\alpha_M| \cdot \frac{H_{xa}}{H_a} \ , \ |A_M| \cdot \frac{H_{aa}}{H_a} \ , \ |A_M| \cdot \frac{H_{xxa}}{H_{xx}}$$

are smaller than δ.

Proof. 1) Write $W = W^+_{M-1}$. Then $H = W^+_{M-1} \circ F$ and $H_a = F_a W_x + W_a$. By Lemma 11, $|W_a| < \delta |W_x|$, hence $|H_a| \simeq |W_x|$; 2) As $H_{xx} = W_{xx}(F_x)^2 + W_x F_{xx}$, we deduce that $|H_{xx}| \simeq |F_{xx}| \cdot |W_x|$, since

$$\frac{W_{xx}(F_x)^2}{W_x F_{xx}} < \frac{1}{\mu} \left| \frac{W_{xx}}{(W_x)^2} \right| \cdot (F_x)^2 |W_x| < 2\delta \ ,$$

(use Lemma 11 and $|F_x| \lesssim \left(2\mu |\omega^+_{M-1}|\right)^{1/2}$, for $x \in \alpha_M$); 3) Appendix plus the previous estimates; note that M does not need to be very large. ∎

LEMMA 13. *Let $A = A^\pm_k$, $2 \le k \le M-1$. There is $C > 0$ such that for all $x \in \alpha^\pm_k$ and $a \in A_M$,*

1. $\left| \frac{A_a}{A_x H_a} \right| < \frac{|\omega^+_{M-1}|}{|F_x|}$,
2. $\left| \frac{A_{xx}}{(A_x)^2} \right| < \delta + \frac{C}{|A_x F_x|}$,
3. $\left| \frac{A_{xa}}{(A_x)^2 H_a} \right| < C|\omega^+_{M-1}| \cdot \left(\frac{1}{|F_x|} + \frac{1}{|A_x F_x|} \right)$,
4. $\left| \frac{A_{aa}}{(A_x)^2 (H_a)^2} \right| < C \frac{|\omega^+_{M-1}|^2}{(F_x)^2}$,
5. $\left| \frac{A_{xxx}}{(A_x)^3} \right| < \delta + C \left(\frac{1}{|A_x F_x|} + \frac{1}{|(A_x)^2 F_x|} \right)$,
6. $\left| \frac{A_{xxa}}{(A_x)^3 H_a} \right| < C|\omega^+_{M-1}| \left(\frac{1}{|F_x|} + \frac{1}{|A_x F_x|} + \frac{1}{|A_x (F_x)^2|} + \frac{1}{|(A_x)^2 F_x|} \right)$,

where $F_x = F_x(a, x)$ and $H_a = H_a(a, 0)$.

Proof. Appendix and evaluations on F. ∎

COROLLARY 14. $\left| \frac{A^\pm_{k,a}}{A^\pm_{k,x} H_a} \right| < |\omega^+_{M-1}|^{1/2}$, *for all $k = 2, \dots, M-2$.*

The functions $A^\pm_i$ are uniformly expanding for $i = 2, \dots, M-2$, but for $i = M-1$ there may be contracting regions near the boundary of α_M, for low values of $a \in A_M$.

LEMMA 15. $|A^\pm_{k,x}| \ge 2^{\frac{k}{2}}$, *for all $k = 2, \dots, M-2$.*

Proof. We do the proof for $(\tilde\phi_a)_a$ and sufficiently large M. With small effort one easy generalizes to any special family $(\phi_a)_a$ δ–C^3 near $(\tilde\phi_a)_a$, for δ sufficiently small. We use that $2 < \lambda < 3$ and $\mu < 2\lambda$ (see Subsection 2.2, just after Lemma 31).

Consider $F(a, x) = \tilde\phi_a(x)$ and $\hat F(a, x) = a - \frac{\mu}{2} x^2$. By the definition of $(\tilde\phi_a)_a$, $\hat F$ coincides with F for small x. Moreover, as $|D_2 F(x)| \le \mu$ for every $x \in I$, then $\hat F(a, x) \le F(a, x)$, $\forall x \in I$.

Let q_k be the right endpoint of ω^+_k, for $k \ge 1$. For the family $(\tilde\phi_a)_a$, these points do not depend on the parameter. We have

$$|DA^\pm_k(x)| \ge \lambda^{k-1} DF(a, F^{-1}(a, q_{k-1}))$$

for all $2 \leq k \leq M-1$. But $|DF(a,x)|$ must be greater than $\frac{\lambda}{x_\lambda}|x|$, i.e. greater than $\lambda|x|$ (since $x_\lambda < 1$), hence $|DA_k^\pm(x)| \geq \lambda^k |F^{-1}(a,q_{k-1})|$.

On the other hand,

$$|F^{-1}(a,q_{k-1})| \geq |\hat{F}^{-1}(a,q_{k-1})| = \sqrt{\frac{2}{\mu}(a-q_{k-1})}\ .$$

We compare this last value with $|\hat{F}^{-1}(2,q_{k-1})| = \sqrt{\frac{2}{\mu}(2-q_{k-1})}$. As $k \leq M-2$ and $a \in \omega_{M-1}^+$, then $a - q_{k-1} \geq |\omega_k^+| + |\omega_{k+1}^+| + \ldots + |\omega_{M-2}^+|$ and $2 - q_{k-1} \geq |\omega_k^+| + |\omega_{k+1}^+| + \ldots$, so

$$\frac{\sqrt{2-q_{k-1}}}{\sqrt{a-q_{k-1}}} \leq \sqrt{1+2\lambda^{k-M+1}}\ .$$

Therefore, as $2 - q_k = \lambda^{-k}(2-x_\lambda) > \lambda^{-k}$,

$$|DA_k^\pm(x)| \geq 2^{\frac{k}{2}} \sqrt{\frac{(\lambda/2)^k}{1+2\lambda^{k-M+1}}}\ .$$

We claim that $(\frac{\lambda}{2})^k > 1+2\lambda^{k-M+1}$ for all $2 \leq k \leq M-2$, if M is sufficiently large, and then the Lemma follows. Note that $(\frac{\lambda}{2})^2 > 1+2\lambda^{3-M}$ and $(\frac{\lambda}{2})^{M-2} > 1+2\lambda^{-1}$, if M is big. As $t \mapsto (\frac{\lambda}{2})^t$ and $t \mapsto 1+2\lambda^{t-M+1}$ are convex functions, they do not intersect and the claim is valid. ∎

Lemma 16. *If $x \in \alpha_{M-1}^\pm$ and $dist(F(x), \omega_{M-1}^+) \geq \frac{1}{16}|\omega_{M-2}^+|$ then we have $|DF(x)| > \frac{1}{32}|\omega_{M-2}^+|^{1/2}$ and $|A_{M-1,x}^+| > \frac{1}{16}|\omega_{M-2}^+|^{-1/2}$.*

Proof. If M is big then $\alpha_{M-1}^\pm \subset [-\frac{1}{5}, \frac{1}{5}]$, where $F(x) \simeq a - \frac{\mu}{2}x^2$ and $|DF(x)| \simeq \mu|x|$. Let x' be the nearest point to x such that $F(x) \in \partial\omega_{M-1}^+$ and Max be the maximum value of $|DF|$ in $\alpha_{M-1}^- \cup \alpha_M \cup \alpha_{M-1}^+$. Then

$$|x| \geq |x - x'| \geq (\mathrm{Max})^{-1}\frac{1}{16}|\omega_{M-2}^+|\ .$$

On the other hand

$$\mathrm{Max} \lesssim \mu\sqrt{\frac{2}{\mu}(|\omega_{M-1}^+| + |\omega_{M-2}^+|}\ ,$$

hence

$$|DF(x)| \gtrsim \frac{1}{16}\sqrt{\frac{\mu}{2(1+\lambda^{-1})}}|\omega_{M-2}^+|^{1/2}\ .$$

As $A^{\pm}_{M-1,x} = (W^{+}_{M-2,x} \circ F) \cdot F_x$ and $|W^{+}_{M-2,x}| \simeq 2x_\lambda |\omega^{+}_{M-2}|^{-1}$ then

$$|A^{+}_{M-1,x}| > \frac{1}{16}\sqrt{\frac{\mu}{1+\lambda^{-1}}}\ |\omega^{+}_{M-2}|^{-1/2} .$$

A rough estimate on λ and μ based on their definition in the Appendix implies the Lemma. ▮

To distinguish between low and high values of $a \in A_M$ we define

$$d = d(a) = |\mathrm{Im} F \cap \omega^{+}_{M-1}| \cdot |\omega^{+}_{M-1}|^{-1} .$$

By Lemma 11, the distortion of $W^{+}_{M-1,x}$ is small, so

$$\frac{|\mathrm{Im} H|}{|\gamma_a|} \simeq d.$$

LEMMA 17. *If $d \geq 2^{-8}$ then, for all $x \in \alpha^{\pm}_{M-1}$ and M large enough we have $|F_x| > \frac{1}{16}|\omega^{+}_{M-1}|^{1/2}$ and $|A^{\pm}_{M-1,x}| \geq 4|\omega^{+}_{M-1}|^{-1/2}$.*

Proof. If $d \geq 2^{-8}$ then for all $x \notin \alpha_M$

$$|F_x| > \sqrt{2\mu \cdot 2^{-8}|\omega^{+}_{M-1}|} > \frac{1}{16}\sqrt{2\mu}|\omega^{+}_{M-1}|^{1/2} .$$

As $A^{\pm}_{M-1} = W^{+}_{M-2} \circ F$, then

$$|A^{\pm}_{M-1,x}| \gtrsim 2x_\lambda |\omega^{+}_{M-2}|^{-1} \cdot |\omega^{+}_{M-1}|^{1/2} > 2\lambda |\omega^{+}_{M-1}|^{-1/2} ,$$

since $|\omega^{+}_{M-2}| \simeq \lambda^{-1}|\omega^{+}_{M-1}|$. ▮

COROLLARY 18. *There is a constant $C > 0$ such that in any one of the following hypotheses (compatible with $a \in A_M$): a) $i = 2, \ldots, M-2$ and $x \in \alpha^{\pm}_i$, b) $i = 2, \ldots, M-1$, $d \geq 2^{-8}$ and $x \in \alpha^{\pm}_i$, c) $i = M-1$, $x \in \alpha^{\pm}_{M-1}$ and dist$(F(x), \omega^{+}_{M-1}) \geq \frac{1}{16}|\omega^{+}_{M-2}|$, if we call $A = A^{\pm}_i$ then*

1. $\left|\frac{A_a}{A_x H_a}\right| < C|\omega^{+}_{M-1}|^{1/2}$,
2. $\left|\frac{A_{xx}}{(A_x)^2}\right| < \delta + \frac{C}{|A_x|}|\omega^{+}_{M-2}|^{-1/2}$,
3. $\left|\frac{A_{xa}}{(A_x)^2 H_a}\right| < C|\omega^{+}_{M-1}|^{1/2}$,
4. $\left|\frac{A_{aa}}{(A_x)^2 (H_a)^2}\right| < C|\omega^{+}_{M-1}|$,
5. $\left|\frac{A_{xxx}}{(A_x)^3}\right| < \delta + \frac{C}{|A_x|}|\omega^{+}_{M-2}|^{-1/2}$,

6. $\left|\frac{A_{xxa}}{(A_x)^3 H_a}\right| < \frac{C}{|A_x|}$.

Proof. Appendix and the previous Lemmas. ▮

LEMMA 19. *For all $i = 2, \ldots, M-1$ and $x \in \alpha_i^{\pm}$,*

$$\left|\frac{A_{i,a}^{\pm}}{A_{i,x}^{\pm} H_a}\right| < \frac{1}{4}\frac{1}{|A_{i,x}^{\pm}|}.$$

Proof. Appendix plus the facts that $F_a \simeq 1$ for $x \in \alpha_i^{\pm}$, i large, and M is large enough. ▮

Having already the estimates for the A_i's, we can now consider their compositions. The next lemma is the easiest situation.

LEMMA 20. *Let $d = d(a) \geq 2^{-8}$. Given $\delta_1 > 0$, then*

$$\left|\frac{B_a}{B_x H_a}\right| < \delta_1$$

for every $\beta \in \mathcal{B}_0$ and $x \in \beta$, if M is sufficiently large.

Proof. Write $B = P_j \circ \ldots \circ P_1$, where $P_i = A_{k_i}^{\sigma_i}$, $\sigma_i \in \{+,-\}$ and $k_i \in 2, \ldots, M-1$. Then, using the Appendix,

$$\left|\frac{B_a}{B_x H_a}\right| = \left|\sum_{i=1}^{j} \frac{P_{i,a}}{P_{i,x} H_a} \cdot \frac{1}{P_{i-1,x} \ldots P_{1,x}}\right| < C|\omega_{M-1}^{+}|^{1/2} < \delta_1,$$

if M is sufficiently large, using Corollary 18 with hypothesis b) and Lemma 15. ▮

Now we look at the more difficult situation where $d = d(a) < 2^{-8}$. Remark that we are only interested in the case where $\beta \cap \mathrm{Im} H \neq \emptyset$ (to be precise, $\mathcal{U}(\beta) \cap \mathrm{Im} H \neq \emptyset$, for some neighborhood $\mathcal{U}(\beta)$ of β), following the requirements of inequalities $(P_1)_{\delta_1}$ (Subsection 1.9). First we state a simple Lemma.

LEMMA 21. $|\alpha_2^{\pm}| > 3^{-2}|\gamma_a|$.

Proof. As $A_2^{\pm}(\alpha_2^{\pm}) = \gamma_a$, $A_2^{\pm} = F^2|\alpha_2^{\pm}$ and $|DF| \lesssim \lambda < 3$ the Lemma follows. ▮

If $d = d(a) < 2^{-8}$ and $\beta \cap \mathrm{Im} H \neq \emptyset$ then in particular $\beta \subset \alpha_2^{+}$, by Lemma 21. More than that, we have

$$\mathrm{dist}(\beta, q_a) \lesssim d.$$

As a consequence, the first iterates of $x \in \beta$ are done inside α_2^+, near q_a. For $x \in \beta$, define $n_0 = n_0(x) \geq 1$ as the minimal number such that $(A_2^+)^{n_0}(x) \notin \alpha_2^+$.

Let $\alpha^+ = \alpha^+(M)$ (resp. $\alpha^- = \alpha^-(M)$) be the interval contained in α_{M-1}^+ (resp. α_{M-1}^-) such that if $x \in \alpha^+$ (resp. $x \in \alpha^-$) then $\mathrm{dist}(F(x), \omega_{M-1}^+) < \frac{1}{16}|\omega_{M-2}^+|$ (in particular, such an x must satisfy $A_{M-1}^{\pm}(x) \in \alpha_2^+$). As remarked above, this is the region of possible 'loss' of derivative.

Assuming that $P_j \circ \cdots \circ P_1(\beta) = \gamma_0 \equiv \alpha_M$, where $P_i = A_{k_i}^{\sigma_i}$, $\sigma_i \in \{+,-\}$, $k_i \in 2, \ldots, M-1$, $i = 1, \ldots, j$, we propose the following decomposition of orbits. For $x \in \beta$, let $1 < i_1 < i_2 < \ldots < i_r < j$ be the maximal sequence such that

$$P_{i_l-1} \circ P_{i_l-2} \circ \cdots \circ P_1(x) \in \alpha^+ \cup \alpha^+ ,$$

for all $l = 1, \ldots, r$.

By the definitions above, $i_1(x) > n_0(x)$. Lemma 22 below says that eventual losses when the orbit of $x \in \beta$ visits the 'bad' region $\alpha^+ \cup \alpha^-$ are compensated by the $n_0(x)$ first iterates. This kind of compensation is used only for the last visit to the bad region. The previous visits are compensated by iterates in the way stated by Lemma 23. From Lemma 25 on both Lemmas are put together to control inequalities $(P_1)_{\delta_1}$.

Lemma 22. *For any $l = 1, \ldots, r$,*

$$|P_{1,x}P_{2,x}\ldots P_{n_0,x}|^{-1} \cdot \left|\frac{P_{i_l,a}}{P_{i_l,x}H_a}\right| < C\left(\frac{11}{10}\right)^{-n_0} |\omega_{M-1}^+|^{1/2}$$

and

$$|P_{1,x}P_{2,x}\ldots P_{n_0,x}|^{-1} \cdot |P_{i_l,x}|^{-1} < C\left(\frac{11}{10}\right)^{-n_0} |\omega_{M-1}^+|^{1/2} .$$

Proof. By Lemma 13,

$$\left|\frac{P_{i_l,a}}{P_{i_l,x}H_a}\right| < C\frac{|\omega_{M-1}^+|^{1/2}}{\sqrt{d}}.$$

On the other hand, for any $y, z \in \alpha_2^{\pm}$ we have $|A_{2,x}^{\pm}(y)| < (1+\delta)|A_{2,x}^{\pm}(z)|$, for some $\delta > 0$ small (since $D\tilde{\phi}_a$ is monotone and ϕ_a is C^3 near $\tilde{\phi}_a$). Observe that $P_i = A_2^+$ for all $i = 1, \ldots, n_0$. We have, for $x \in \beta$,

$$|P_{n_0,x}P_{n_0-1,x}\ldots P_{2,x}P_{1,x}(x)| \cdot |x - q^+| > C^{-1} ,$$

since $P_{n_0} \circ \cdots \circ P_1(x) \notin \alpha_2^+$ and the distortion of the derivative of $P_{n_0} \circ \cdots \circ P_1$ is controlled in $[x, q^+]$, by the expansivity for each iterate. We get

(assuming n_0 even, without loss of generality)

$$\left|P_{\frac{n_0}{2},x}\dots P_{2,x}P_{1,x}\right|\cdot(1+\delta)^{n_0/4}>C^{-1}d^{-1/2}.$$

Hence

$$|P_{n_0,x}\dots P_{1,x}|>C^{-1}d^{-1/2}\cdot\left(\frac{2^{1/4}}{(1+\delta)^{1/4}}\right)^{n_0},$$

by Lemma 15. ∎

The following Lemma says that $|P_{i_l,x}|^{-1}$ can be compensated by the subsequent iterates before i_{l+1}.

LEMMA 23. *There is $i<i_{l+1}-i_l$ such that*

$$|P_{i_l,x}|^{-1}\cdot|P_{i_l+1,x}P_{i_l+2,x}\dots P_{i_l+i,x}|^{-1}<C\left(\frac{11}{10}\right)^{-i}|\omega^+_{M-2}|^{1/2}.$$

Proof. The idea is the following. If $|P_{i_l,x}|$ is small it means that $y=P_{i_l-1}\circ\cdots\circ P_1(x)$ is very near α_M. But this implies that $P_{i_l}(y)$ is in α_2^+ and very near q^+. The subsequent iterates, all done near q^+, compensate the loss of derivative coming from $|P_{i_l,x}|$.

First we put $P_{i_l,x}=A^{\pm}_{M-1,x}$ in relation with $|P_{i_l}(y)-q^+|$:

$$|A^{\pm}_{M-1,x}|\simeq\frac{|\gamma_a|}{|\omega^+_{M-2}|}|F_x(y)|>\sqrt{2\mu}\frac{|\gamma_a|}{|\omega^+_{M-2}|}\left(\frac{\mathrm{dist}(F(y),\omega^+_{M-1})}{|\omega^+_{M-2}|}\right)^{1/2},$$

hence, by the small distortion of W^+_{M-2},

$$|P_{i_l}(y)-q^+|^{-1/2}>|\omega^+_{M-2}|^{-1/2}|A^{\pm}_{M-1,x}|^{-1}.$$

Choose the first integer i satisfying

$$|(A_2^+)^i(P_{i_l}(y))-q^+|>\frac{1}{32}.$$

As in the proof of Lemma 22, using the Mean Value Theorem and bounded distortion properties to obtain

$$|D(A_2^+)^i(P_{i_l}(y))|\cdot|P_{i_l}(y)-q^+|>C^{-1}$$

and

$$|D(A_2^+)^{i/2}(P_{i_l}(y))|\cdot(1+\delta)^{i/4}>C^{-1}|P_{i_l}(y)-q^+|^{-1/2}$$

and the Lemma follows. ▮

In the decomposition of orbits, add the following definitions: $i_0 = 0$, $\tilde{i}_0 = n_0$ and $\tilde{i}_l = i_l + i$, where i is given by the previous Lemma, for each $l \geq 1$.

For the next Lemmas, let $\beta \in \mathcal{B}_0$ be given by $\beta = (P_j \circ \cdots \circ P_1)^{-1}(\gamma_0)$ and let $\tilde{B} = P_{\tilde{j}} \circ \cdots \circ P_1|\beta$, where $1 \leq \tilde{j} \leq j$.

All the results are valid for $x \in \beta$ and $a \in A_M$, with the condition that $\beta \cap \mathrm{Im} H \neq \emptyset$. In fact, this restriction in the parameter is only needed when $d = d(a) < 2^{-8}$ and when Lemma 22 must be used.

Let us remark at this point that the same idea underlies the proof of Theorem 9, in Section 3. Every loss of derivative must be compensated, if possible, by subsequent iterates, otherwise by the first iterates, which is only guaranteed with a restriction on the parameter. It is to be noted, however, that this restriction on the parameter is not a problem when one passes to the next stage of the induction (only preimages of γ_n intersected by H are used to generate preimages of γ_{n+1}).

For some of the proofs we will use the following simple Corollary of Lemma 22. It means that when the orbit of β hits γ_0 all previous losses in the derivative are already compensated, and a definite expansion is indeed obtained. This kind of reasoning is also behind the proof of Theorem 6, which we do in Section 3. This Corollary gives us, indeed, the assumption of Theorem 6.

Corollary 24. *For every $x \in \beta$ and $1 \leq i < j$*

$$|P_{j,x} \ldots P_{i+1,x}| \geq \left(\frac{11}{10}\right)^{j-i} ,$$

if M is sufficiently large.

Lemma 25. *Given $\delta_1 > 0$, if M is sufficiently large then*

$$\left|\frac{\tilde{B}_a}{\tilde{B}_x H_a}\right| < \delta_1$$

for all $x \in \beta$, $\beta \in \mathcal{B}_0$, and a such that $d = d(a) \geq 2^{-8}$ or $\mathrm{Im}H \cap \beta \neq \emptyset$.

Proof. In view of Lemma 20 we can suppose $d < 2^{-8}$. We consider two cases: $i = i_l$ or $i_l < i < i_{l+1}$, for some $l \geq 0$. If $i = i_l$, $l \geq 1$, we write

$$\frac{P_{i,a}}{P_{i,x}H_a} \cdot \frac{1}{P_{i-1,x} \ldots P_{1,x}} = \frac{P_{i,a}}{P_{i,x}H_a} \cdot \frac{1}{P_{1,x} \ldots P_{\tilde{i}_0,x}} \times$$
$$\times \prod_{s=1}^{l-1} \frac{1}{P_{i_s,x}} \cdot \frac{1}{P_{i_s+1,x} \ldots P_{\tilde{i}_s,x}} \prod_{s=0}^{l-1} \frac{1}{P_{\tilde{i}_s+1,x} \ldots P_{i_{s+1}-1,x}}$$

which, by Lemmas 22 and 23, is bounded by

$$\left(C|\omega^+_{M-2}|^{1/2}\right)^l \left(\frac{11}{10}\right)^{-i} .$$

If $i_l < i < i_{l+1}$ for some $l \geq 0$ then the reasoning is analogous, but one has to use that

$$|P_{i_l,x}|^{-1} \cdot |P_{1,x} \dots P_{\tilde{i}_0,x}|^{-1} < C \left(\frac{11}{10}\right)^{-\tilde{i}_0} |\omega^+_{M-1}|^{1/2} ,$$

from the second inequality of Lemma 22. ∎

Lemma 26.

$$\frac{|\gamma_0|}{|P_{j,x} \dots P_{\tilde{j}+1,x}|} \cdot \left| \frac{\tilde{B}_{xx}}{(\tilde{B}_x)^2} \right| < \delta_1 .$$

Proof. Write, using the Appendix,

$$\frac{\tilde{B}_{xx}}{(\tilde{B}_x)^2} = \sum_{i=1}^{\tilde{j}} \frac{P_{i,xx}}{P_{\tilde{j},x} \dots P_{i+1,x}(P_{i,x})^2} .$$

By Lemma 13,

$$|\gamma_0| \left| \frac{P_{i,xx}}{(P_{i,x})^2} \right| < |\gamma_0| \left(\delta + \frac{C}{|P_{i,x}F_x|} \right) .$$

We have two cases: $i = i_l$, for some $l \geq 1$ or $i_l < i < i_{l+1}$, for some $l \geq 0$. If $i = i_l$ then the i–th term is bounded by

$$|\gamma_0| \frac{1}{|P_{j,x} \dots P_{i+1,x}|} \left(\delta + \frac{C}{(F_x)^2} \cdot |\omega^+_{M-2}| \right)$$
$$< \delta |\gamma_0| 2^{-j+i} + C|\gamma_0| \cdot |\omega^+_{M-2}| \frac{1}{|P_{j,x} \dots P_{\tilde{i}_l+1,x}|} \cdot \frac{1}{|P_{\tilde{i}_l,x} \dots P_{i+1,x}|(F_x)^2} ,$$

using Corollary 24. The last fraction is bounded by a constant, since $|\omega^+_{M-2}|(F_x)^{-2}$ is bounded by $3|P_{i_l} - q^+|^{-1}$ and the fifth inequality of the proof of Lemma 23 applies, by the definition of $\tilde{i}_l$. Therefore for $i = i_l$ the i–th term is bounded by $C|\gamma_0|(\frac{11}{10})^{-j+\tilde{i}_l}$.

If $i_l < i < i_{l+1}$ then the i–th term is bounded by

$$|\gamma_0| \cdot \frac{C}{|P_{j,x} \dots P_{i+1,x}|} < C \left(\frac{11}{10}\right)^{-j+i} |\gamma_0| ,$$

by Corollaries 18 and 24.

Therefore the Lemma follows provided $|\gamma_0|$ is sufficiently small, i.e. M is sufficiently large. ▮

We omit the proof of the following Lemma, as they follow the same ideas of Lemmas 25 and 26. Lemmas 25, 26 and 27 are stated for $\tilde{B}$, since their quotients appear in the proof of Lemmas 27, 28, 29 and 30.

Lemma 27.

$$\frac{|\gamma_0|}{|P_{j,x} \ldots P_{\tilde{j}+1,x}|} \cdot \left| \frac{\tilde{B}_{xa}}{(\tilde{B}_x)^2 H_a} \right| < \delta_1 \ .$$

Lemma 28.

$$|\gamma_0| \cdot \left| \frac{\tilde{B}_{aa}}{(\tilde{B}_x)^2 (H_a)^2} \right| < \delta_1 \ .$$

Lemma 29.

$$|\gamma_0|^2 \cdot \left| \frac{\tilde{B}_{xxx}}{(\tilde{B}_x)^3} \right| < \delta_1 \ .$$

Lemma 30.

$$|\gamma_0|^2 \left| \frac{\tilde{B}_{xxa}}{(\tilde{B}_x)^3 H_a} \right| < \delta_1 \ .$$

2.2. Existence of special families

We show here that the family $(\tilde{\phi}_a)_a$ referred in Subsection 1.4 does actually exist. This task will be split into two parts: find a suitable λ and then define the family.

We consider only the negative axis and force $\tilde{\phi}_a$ to be an even function. Take two (continuous, piecewise linear) functions, which later will be related to $D\tilde{\phi}_a$, called f_λ and g_λ, defined in the following way:

1. $f_\lambda(x) = \lambda$, $\forall x \in [-2, -x_\lambda + \frac{2}{10}]$,
2. $f_\lambda(x) = -\mu x$, $\forall x \in [-x_\lambda + \frac{2}{10}, 0]$,
3. $g_\lambda(x) = f_\lambda(x)$, $\forall x \in [-2, -x_\lambda + \frac{1}{10}] \cup [-x_\lambda + \frac{3}{10}, 0]$,
4. $g_\lambda(x) = -\tilde{\mu} x + d$, $\forall x \in [-x_\lambda + \frac{1}{10}, -x_\lambda + \frac{3}{10}]$,

where $\mu = \mu(\lambda) = \frac{\lambda}{x_\lambda - \frac{1}{5}}$ and $\tilde{\mu} > 0$, $d > 0$ are such that $g_\lambda(-x_\lambda + \frac{1}{10}) = \lambda$ and $g_\lambda(-x_\lambda + \frac{3}{10}) = -\mu(-x_\lambda + \frac{3}{10})$ (note that $\tilde{\mu} < \mu$).

Lemma 31. *If $\lambda \in [2.36, 2.37]$ then*

$$\int_{-2}^{0} g_\lambda(x)dx < 4 < \int_{-2}^{0} f_\lambda(x)dx.$$

Proof. The functions $\int_{-2}^{0} g_\lambda$ and $\int_{-2}^{0} f_\lambda$ grow with λ, for $\lambda > 2$. By straightforward calculation, $\int_{-2}^{0} f_\lambda = 4$ for $\lambda = 2.359\ldots$ and $\int_{-2}^{0} g_\lambda = 4$ for $\lambda = 2.374\ldots$. ∎

Now pick a fixed λ as in Lemma 31 (for example $\lambda = 2.365$) and call $f = f_\lambda$, $g = g_\lambda$ (this implies, in particular, $\mu \in [3.8, 3.9]$). Let φ be a positive even C^∞ bump function with support in $[-1, +1]$ and $\int_{-1}^{+1} \varphi(x)dx = 1$ and define the associated family $\varphi_\delta(x) = \frac{1}{\delta}\varphi(\frac{x}{\delta})$. Observe that if

$$\psi_\delta(x) = \int_{-\infty}^{x} \varphi_\delta(y)dy$$

then, by the symmetry of φ, $\int_{-1}^{+1} \psi_\delta(x)dx$ is independent of δ and always equal to 1. Let $x_i = -x_\lambda + \frac{i}{10}$, $i = 1, 2, 3$, and consider the following two families of functions,

$$h_{f,\delta}(x) = -\mu\varphi_\delta(x - x_2),$$

$$h_{g,\delta}(x) = -\tilde{\mu}\varphi_\delta(x - x_1) - (\mu - \tilde{\mu})\varphi_\delta(x - x_3),$$

for $\delta < \frac{1}{10}$. Observe that $h_{f,\delta} \stackrel{\delta\to 0}{\longrightarrow} D_2 f$ and $h_{g,\delta} \stackrel{\delta\to 0}{\longrightarrow} D_2 g$ in the sense of distributions.

Let $\tilde{\phi}_{f,\delta}$ and $\tilde{\phi}_{g,\delta}$ be defined by $D_3\tilde{\phi}_{f,\delta} = h_{f,\delta}$ and $D_3\tilde{\phi}_{g,\delta} = h_{g,\delta}$, with constants of integration given by $\tilde{\phi}_{f,\delta}(-2) = -2 = \tilde{\phi}_{g,\delta}(-2)$, $D\tilde{\phi}_{f,\delta}(-2) = \lambda = D\tilde{\phi}_{g,\delta}(-2)$ and $D_2\tilde{\phi}_{f,\delta}(-2) = 0 = D_2\tilde{\phi}_{g,\delta}(-2)$.

Some properties are verified: i) $\int_{-2}^{0} D_2\tilde{\phi}_{f,\delta}(x)dx = \int_{-2}^{0} D_2\tilde{\phi}_{g,\delta}(x)dx = -\lambda$, since φ is even; in other words, we have $D\tilde{\phi}_{f,\delta}(0) = D\tilde{\phi}_{g,\delta}(0) = 0$; ii) $D\tilde{\phi}_{f,\delta}(x) = D\tilde{\phi}_{g,\delta}(x) = \lambda$ for all $x \in [-2, -x_\lambda]$; iii) $D\tilde{\phi}_{f,\delta}(x) = D\tilde{\phi}_{g,\delta}(x) = -\mu x$, for all $x \in [-x_\lambda + \frac{4}{10}, 0]$; iv) $\int_{-2}^{0} D\tilde{\phi}_{f,\delta}(x)dx \stackrel{\delta\to 0}{\longrightarrow} \int_{-2}^{0} f(x)dx$ and $\int_{-2}^{0} D\tilde{\phi}_{g,\delta}(x)dx \stackrel{\delta\to 0}{\longrightarrow} \int_{-2}^{0} g(x)dx$.

By property iv) and Lemma 31, we can choose δ_0 sufficiently small in such a way that

$$\int_{-2}^{0} D\tilde{\phi}_{g,\delta}(x)dx < 4 < \int_{-2}^{0} D\tilde{\phi}_{f,\delta}(x)dx$$

and then let $\tilde{\phi}_f \equiv \tilde{\phi}_{f,\delta_0}$, $\tilde{\phi}_g \equiv \tilde{\phi}_{g,\delta_0}$.

Finally, take the family of convex combinations

$$\tilde{\phi}_t = (1-t)\tilde{\phi}_g + t\tilde{\phi}_f.$$

Then $\tilde{\phi}_t$ satisfies, for all $t \in [0,1]$,

1. $D_3\tilde{\phi}_t(x) \leq 0$, for $x \leq 0$;
2. $D\tilde{\phi}_t(x) = \lambda$ for $x \in [-2, -x_\lambda]$ and $D\tilde{\phi}_t(x) = -\mu x$ for $x \in [-x_\lambda + \frac{4}{10}, 0]$;
3. $\tilde{\phi}_t(0) < 2$ for $t = 0$ and $\tilde{\phi}_t(0) > 2$ for $t = 1$, hence equal to 2 for some $0 < t_0 < 1$.

By an affine coordinate change in the parameter $t \mapsto a$, given by $a(t) = \tilde{\phi}_t(0)$, we have

$$\tilde{\phi}_a(x) = a - \frac{\mu}{2}x^2$$

for $x \in [-x_\lambda + \frac{4}{10}, 0] \supset (-\frac{1}{5}, \frac{1}{5})$.

3. PARAMETER DISTORTION BY INDUCTION

3.1. Coordinates, extendibility and subordination.

In this Section, we aim at proving Theorem 9. First we introduce coordinate changes both in space and parameter to obtain uniform estimates along the induction, independent of n. Recalling the notation presented in Section 1, we look at some $J \in \mathcal{J}_n$ and the correspondent sub-family $\Phi_n = (\Phi_{n,a})_{a\in J}$, with its central branch $H = H_n : \gamma_n \to \gamma_{n-1}$ and its preimages of the central branch $B : \beta \to \gamma_n$, $\beta \in \mathcal{B}_n$. Let $a_0 = a_0(J)$ be such that $H(a_0, 0) = 0$, $S = S(J)$ be the "mean" curvature of H and $v = v(J)$ the "mean" velocity of the critical value, as defined in Subsection 1.9. Then define $w = -Sx$ and $b = -Sv(a - a_0)$ and let T be the expression of $H = H(a, x)$ in these new coordinates, i.e.

$$T(b, w) = -SH(a_0 - S^{-1}v^{-1}b, -S^{-1}w)\ .$$

It is easy to see that such a coordinate change does not affect inequalities $(P_0)_{\delta_0}$, and the same is true for the new expressions of the functions $B : \beta \to \gamma_0$. The main feature of the particular coordinate change above is that T is near the quadratic family $b - w^2$ (for b varying along the interval $-SvJ + Sva_0$ and w along $S\gamma_{n,a}$), in the sense that $|T_{ww} + 2| \leq \delta_0$ and $|T_b - 1| \leq \delta_0$.

However it will be easier to keep the old notation, with H, a and x instead of T, b and w, i.e. from now on we suppose that the coordinate change is already done. Moreover, based on Theorem 5 we assume that there are small numbers $r > 0$, $p > 0$ and $q > 0$ such that $r_n \leq r$, $p_n \leq p$ and $q_n \leq q$, for all $n \geq 0$.

First we state two simple quantitative facts which are consequence of the linear coordinate change above.

Lemma 32. *In the new coordinates, $|J| \simeq |\gamma_{n-1}|$ and if $H(0) \in \gamma_n$ then $|\gamma_n| > r^{-1}$.*

Proof. The first assertion follows from $H_a \simeq 1$ and that $|\gamma_{n-1}|$ is approximately constant for $a \in J$, as remarked in Subsection 1.10. If $H(0) \in \gamma_n$ then $\left(\frac{1}{2}|\gamma_n|\right)^2 \simeq \frac{1}{2}|\gamma_{n-1}|$, which implies $|\gamma_n| \gtrsim 2r^{-1} > r^{-1}$. ∎

Now we recall the definition of fundamental domains in Subsection 1.13, in the case where $\mathrm{esc} = \mathrm{esc}_n > 1$ (or in other words $H(0) \in \gamma_n$, a central return). The following Lemma is a standard application of the distortion properties of H, which has negative Schwarzian derivative (this follows from the inequalities $(P_0)_{\delta_0}$, together with the monotonicity of its derivative). We skip the proof and refer the reader to [13], Chap.IV, for standard techniques.

Lemma 33. *Let $\beta \in \mathcal{B}_n$, $\beta \subset D_1^-$, $t \geq 2$ and $\eta \subset D_t^{\pm}$ such that $H^{t-1}(\eta) = \beta$. Then*

$$\frac{|\eta|}{dist(\eta, d_{t-1}^{\pm})} \leq 10 p_n(\beta)\ ,\ \frac{|\eta|}{dist(\eta, d_t^{\pm})} \leq q_n(\beta).$$

From the statement of Theorem 5 we know that for every $\beta \in \mathcal{B}_n$ there is a concentric $\theta_n^{-1}|\beta|$–neighborhood of β where $B : \beta \to \gamma_n$ can be extended diffeomorphically as the same power of the original return map $\Phi : \gamma_{-1} \to \gamma_{-1}$, where $\theta_n = \theta_n(r_n, q_{n-1})$.

This fact is simple to prove by induction, so let us show it for $n+1$. Following the notation of Subsection 1.13, for each $\tilde{\beta} \in \mathcal{B}_{n+1}$ the corresponding diffeomorphism $\tilde{B} : \tilde{\beta} \to \gamma_{n+1}$ is given either by $P_j \circ \cdots \circ P_1$ or by $\tilde{B}_H \circ P_{j-1} \circ \cdots \circ P_1$, where $P_i : \pi_i \to \gamma_n$, $i = 1, \dots, j$, π_i a preimage of some $\beta_i \in \mathcal{B}_n$ under iterates of H (hence $P_i = B_i \circ H^{t_i}$, where $B_i : \beta_i \to \gamma_n$) and $\tilde{B}_H$ is an iterate of H landing on γ_{n+1}.

In the first case, $\tilde{B}$ is extendible to $\tilde{B} : (P_j \circ \cdots \circ P_1)^{-1}(\gamma_n) \to \gamma_n$. As $\tilde{\beta} = (P_j \circ \cdots \circ P_1)^{-1}(\gamma_{n+1})$ and $|\gamma_{n+1}| \cdot |\gamma_n|^{-1} \leq r_{n+1}$ then the domain has been at least enlarged by $C^{-1} r_{n+1}$, where C is a universal constant, by the distortion properties coming from the negative Schwarzian derivative ([13], Chap.IV).

In the other case the extendibility is determined by a (universal) constant factor of the codomain extendibility of $\tilde{B}_H$, by two reasons: first, $P_{j-1} \circ \cdots \circ P_1(\tilde{\beta})$ must be an element of $\mathcal{B}_{n+1}^H$, the set of preimages of γ_{n+1} under iterates of H, each one placed inside a gap of the Cantor set Λ_H, so that $\tilde{B}_H$ extends (in the codomain) at most to the boundary of γ_n by one side and at most to the critical point by the other; second, $P_{j-1} \circ \cdots \circ P_1$ extends in such a way that its codomain is the whole γ_n, hence containing the possible extension of the domain of $\tilde{B}_H$. Now the codomain extendibility of $\tilde{B}_H$ is at least $|p^+ - p^-| \cdot |\gamma_{n+1}|^{-1}$, which is approximately greater than $q_n^{-1/2}$, by Lemma 33. Once more the negative Schwarzian derivative implies our claim.

We can define for each $\beta \in \mathcal{B}_n$ a neighborhood $\mathcal{U}(\beta)$ to which $B : \beta \to \gamma_n$ is extendible with small distortion. Here "small distortion" means that there is some small $\delta > 0$ such that $B_x(x_1)/B_x(x_2) \leq 1+\delta$ for any $x_1, x_2 \in \beta$. This neighborhood is still much greater than β, by a factor $C^{-1}\theta_n^{-1}$, for a universal constant $C > 0$.

If $\beta_1, \beta_2 \in \mathcal{B}_n$ then we say (following [6]) that β_1 is *subordinated* to β_2 if $\beta_1 \subset \mathcal{U}(\beta_2)$, and that β_1 and β_2 are *independent* if neither β_1 is subordinated to β_2 nor β_2 is subordinated to β_1.

Observe that the $\theta_n^{-1}|\beta|$–neighborhood of a preimage $\beta \in \mathcal{B}_n$ is always contained in the connected component of $\gamma_{n-1} \setminus \gamma_n$ to which β belongs (this can be seen in the arguments above). Therefore to each $\beta \in \mathcal{B}_n$, $t \geq 2$ and $\eta \subset D_t^{\pm}$ such that $H^{t-1}(\eta) = \beta$ we also assure a $\theta_n^{-1}|\eta|$–neighborhood of extendibility contained in $D_t^{\pm}$ and a neighborhood $\mathcal{U}(\eta)$ where this extension has small distortion. This allows us to define the notion of subordination and independence for a pair η_1, η_2 inside *the same* fundamental domain $D_t^{\pm}$.

The following Lemma is valid as well for η_1, η_2 as above.

LEMMA 34. *If $\beta_1 \in \mathcal{B}_n$ is subordinated to $\beta_2 \in \mathcal{B}_n$ then*

$$\frac{|\beta_1|}{dist(\beta_1, \beta_2)} \lesssim q \,.$$

Proof. As $\mathcal{U}(\beta_2)$ is mapped with small distortion on a neighborhood of γ_n and the image of β_1 is also a preimage of γ_n, say β_3, which satisfies $|\beta_3| \leq q_n \mathrm{dist}(\beta_3, \gamma_n)$, the Lemma follows. ∎

3.2. Iterates near the saddle–node bifurcation

For some estimates in the parameter region "beneath renormalization" ($a \in J^-$ or $\sigma(a) = -$) we will proceed in a slightly different way according to the value of k. First we take k_0 such that if $k \leq k_0$ and $a \in J_k^-$ then $|H(a,0)| \geq 2$, maximal with this property. The value of k_0 is big if $|\gamma_n|$ is big, so that it can be controlled directly by r (see Lemma 32). Then we define the following partition of J^-: $J_a = J_1^-$, $J_b = \bigcup_{k=2}^{k_0} J_k^-$, $J_c = \bigcup_{k=k_0+1}^{k_1} J_k^-$, $J_d = \bigcup_{k>k_1} J_k^-$, where k_1 will be defined below.

Let $a_s = \sup_k J_k^-$, corresponding to the lower boundary point of the renormalization interval $R(J)$, which is the parameter where the saddle–node bifurcation occurs. We can assume that $a_s = -\frac{1}{4}$ by a linear parameter change of coordinates very near the identity, in view of the δ_0–proximity of H to the *true* quadratic family. Analogously we can suppose that x_0, the saddle–node point for a_s, is equal to $-\frac{1}{2}$. The higher order differential estimates of H are invariant under these changes, although not the absolute values of H_{xx} and H_a. But the factor of the linear change of coordinates is near 1, for example between $1 - 10\delta_0$ and $1 + 10\delta_0$, so that for all practical purposes the new values of H_{xx} and H_a are still very near respectively to -2 and 1 and do not affect the estimates.

According to [18] (see also [2]) there are points $x_l < x_0 < x_r$ and a parameter value $\tilde{a} < a_s$ such that for all $a \in [\tilde{a}, a_s]$ and $x \in [x_l, x_r]$ the map H is the time–one map of the flow determined by a vector field X. In fact, x_l, x_r and $\tilde{a}$ can be chosen fixed with respect to any family H sufficiently near the quadratic family. Hence the constant δ_0, which determines this proximity, is chosen accordingly to the choice of x_l, x_r and $\tilde{a}$ for the quadratic family. Moreover, if δ_0 is sufficiently small, all the bounds on the derivatives of X can be supposed to be uniform over all such H. We can write

$$\begin{aligned} H(a,x) &= x - A(a_s - a) - B(x - x_0)^2 \\ &+ D(a_s - a)(x - x_0) + E(a_s - a)^2 + \psi(a,x) \end{aligned}$$

where $A, B > 0$ and

$$\frac{\psi(a,x)}{|(a,x) - (a_s,x_0)|^2} \longrightarrow 0$$

as (a,x) tends to (a_s,x_0). Analogously,

$$\begin{aligned} X(a,x) &= -\tilde{A}(a_s - a) - \tilde{B}(x - x_0)^2 \\ &+ \tilde{D}(a_s - a)(x - x_0) + \tilde{E}(a_s - a)^2 + \tilde{\psi}(a,x) \end{aligned}$$

where $\tilde{A}, \tilde{B} > 0$ and

$$\frac{\tilde{\psi}(a,x)}{|(a,x) - (a_s,x_0)|^2} \longrightarrow 0$$

as (a,x) tends to (a_s,x_0).

We choose x_l and x_r sufficiently near x_0 and $\tilde{a}$ sufficiently near a_s so that the following two Lemmas are satisfied.

Lemma 35. $|X(a,x)| \geq \frac{1}{2}\tilde{A}(a_s - a)$ *for all* $x \in [x_l, x_r]$ *and* $\tilde{a} < a < a_s$.

Proof. Straightforward, using the expressions above. ▮

LEMMA 36. *There is $C > 0$ such that $|H(a,x) - x| \leq C|X(a,y)|$ for all $v \in [Hx, x]$, $x \in [H^{-1}x_l, x_r]$ and $\tilde{a} < a < a_s$.*

Proof. If x_l, x_r and $\tilde{a}$ are conveniently chosen, then

$$|H(a,x)| \leq 2A(a_s - a) + 2B(x - x_0)^2$$

and

$$|X(a,x)| \geq \frac{1}{2}\tilde{A}(a_s - a) + \frac{1}{2}\tilde{B}(x - x_0)^2$$

implying

$$|H(a,x) - x| \leq \tilde{C}|X(a,x)|.$$

Let $M_x = \min\{|X(a,y)|; v \in [Hx, x]\}$. There are only three possibilities for the point where M_x is attained: 1. $M_x = |X(a,x)|$, 2. $M_x = |X(a, Hx)|$, and 3. $M_x = |X(a, y_0)|$ for some $y_0 \in (Hx, x)$. If 1. is valid then the Lemma is proven with $C = \tilde{C}$. In case 2., we observe that $X(a, Hx) = DH(x) \cdot X(a,x)$, hence $|X(a,x)| \leq 2|X(a, Hx)|$ if $x_l \approx x_0 \approx x_r$ and $\tilde{a} \approx a_s$, proving the Lemma with $C = 2\tilde{C}$. Finally in case 3. we observe that

$$|X(a, y_0)| < |X(a,x)| < |X(a, H^{-1}y_0)|$$

since $D_2X < 0$. By the previous argument

$$|X(a, y_0)| > \frac{1}{2}|X(a, H^{-1}y_0)| > \frac{1}{2}|X(a,x)|$$

and the Lemma is also proven with $C = 2\tilde{C}$. ▮

From now on we fix x_l, x_r and $\tilde{a}$ and define a new parameter value a_1 between $\tilde{a}$ and a_s. The parameter a_1 will be sufficiently near a_s to validate the following assertions and will define the number k_1 of the beginning of this section in the following way:

$$k_1 = \min\{k\ ;\ J_k^- \subset [a_1, a_s]\}.$$

Define

$$\Gamma(a) = \int_{x_r}^{x_l} \frac{1}{X(a,x)} dx$$

so that

$$x_l = \Psi_{\Gamma(a)}(a, x_r),$$

where Ψ is the flow of X. We will also use the notation $\Psi(t,a,x) \equiv \Psi_t(a,x)$ for this flow.

For $a_s > a > a_1$ the function Γ is C^∞ and strictly increasing (see [2]). Therefore for each l bigger than some l_0 there is a unique a_l^* satisfying $\Gamma(a_l^*) = l$. If we write $\Gamma(a) = l + \sigma_l(a)$ then $\sigma_l : [a_l^*, a_{l+1}^*] \to [0,1]$ is an increasing C^∞ diffeomorphism onto $[0,1]$. So we can define the inverse maps

$$a_l = \left(\sigma_l|_{[a_l^*,a_{l+1}^*]}\right)^{-1} : [0,1] \longrightarrow [a_l^*, a_{l+1}^*].$$

The proof of the following Lemma is contained in [2].

Lemma 37.

1.$\lim_{l\to\infty} l(a_s - a_l^*)^{1/2} = M$, *so that for large* l

$$\frac{1}{2}\frac{M^2}{l^2} \le (a_s - a_l^*) \le 2\frac{M^2}{l^2};$$

2.for $a \in [a_l^*, a_{l+1}^*]$ *and large* l

(i)$C^{-1}l^3 \le D_1\Gamma(a) \le Cl^3$;

(ii)$C^{-1}l^5 \le D_2\Gamma(a) \le Cl^5$.

Let $x \in [x_l, x_r]$ and let j be the integer such that $H^{l+1}x_l < H^j x \le H^l x_r$. We are interested in the derivatives of H^j. For that, define first the function

$$\tau(a,z,x) = \int_z^x \frac{1}{X(a,u)}du$$

which satisfies $x = \Psi(\tau(a,z,x),a,z)$ and for each $0 \le i \le l$ the function

$$t_i(a,x) = \tau(a, H^i(a,x_r), x).$$

The number $t_i(a,x)$ is positive and smaller than one if $H^{i+1}x_r < x \le H^i x_r$. In this case $i + j = l$ and

$$H^j(a,x) = \Psi(-\sigma_l(a) + t_i(a,x), a, x_l). \tag{1}$$

This expression will be used for derivatives of H^j involving the parameter, but only when i is such that $H_x(a, H^i x_r) < 1$. The other cases can be recovered if we write

$$H^j(a,x) = \Psi(\tau(a, H^{-l+i}(a,x_l), x), a, x_l) \tag{2}$$

and work with estimates on H^{-1} instead of H.

For the first parameter derivative of $t_i(a,x)$ we apply the chain rule

$$\frac{d}{da}t_i(a,x) = \\ = \int_x^{H^i(a,x_r)} \frac{1}{X(a,u)^2}\frac{\partial X}{\partial a}(a,u)du - \frac{1}{X(a,H^i x_r)}\frac{d}{da}H^i(a,x_r). \quad (3)$$

LEMMA 38. *Let i be such that $H_x(a, H^s x_r) < 1$ for all $s \le i$. Then* $\left|\frac{d}{da}H^i(a,x_r)\right| < 2l$ *and* $\left|\frac{d^2}{da^2}H^i(a,x_r)\right| \le Cl^3$.

Proof. As, for each $1 < s \le i$

$$\frac{d}{da}H^s(a,x_r) = H_a(a,H^{s-1}) + H_x(a,H^{s-1})\frac{d}{da}H^{s-1}(a,x_r)$$

then, using the hypothesis and $H_a \simeq 1$,

$$\left|\frac{d}{da}H^s(a,x_r)\right| < 2 + \left|\frac{d}{da}H^{s-1}(a,x_r)\right|.$$

The result follows by induction. For the other derivative the reasoning is analogous. ∎

LEMMA 39. *There is $C > 0$ such that for all i such that $H_x(a,x) < 1$, $x \in [H^{i+1}x_r, H^i x_r]$ and $a > a_1$ then*

$$\left|\frac{d}{da}t_i(a,x)\right| \le Cl^3.$$

If $i = 0$ the derivative is simply bounded by C.

Proof. The minimum of $|X(a,x)|$ in $[H^{i+1}x_r, H^i x_r]$ is attained for some u_0. By Lemma 36, $|H^i x_r - H^{i+1}x_r| \le C|X(a,u_0)|$. So the first term of the R.H.S. of Equation (2) is bounded by Cl^2, according to Lemma 35 and Lemma 37.1.. Also by these Lemmas and Lemma 38, the second term is bounded by Cl^3. For $i = 0$ observe only that $|X(a,x)| \ge C^{-1}$ for all $x \in [Hx_r, x_r]$, $a > a_1$. ∎

LEMMA 40. *There is $C > 0$ such that for all i such that $H_x(a,x) < 1$, $x \in [H^{i+1}x_r, H^i x_r]$ and $a > a_1$ then*

$$\left|\frac{d^2}{da^2}t_i(a,x)\right| \le Cl^5.$$

Proof. Write

$$\frac{d^2}{da^2}t_i(a,x) = \frac{d}{da}\left[\Delta(a,x,H^i(a,x_r)) - \frac{1}{X(a,H^i x_r)}\frac{d}{da}H^i(a,x_r)\right]$$

where

$$\Delta(a,x,y) = \int_x^y [X(a,u)]^{-2}\frac{\partial X}{\partial a}(a,u)du.$$

and proceed as in Lemma 39. ∎

Lemma 41. *There is $C > 0$ such that for $a > a_1$*

1.$C^{-1} \leq |H^j_x| \leq Cl^2$ for all $x \in [x_l, x_r]$;
2.$C^{-1} \leq |H^j_x| \leq C$ for all $x \in [Hx_r, x_r]$;
3.$|H^j_a| \leq Cl^3$ for all $x \in [x_l, x_r]$;
4.$|H^j_a| \geq C^{-1}l^3$ for all $x \in [Hx_r, x_r]$;
5.$|H^j_{aa}| \leq Cl^6$ for all $x \in [x_l, x_r]$;
6.$|H^j_{xx}| \leq C|H^j_x|^2$ for all $x \in [x_l, x_r]$;
7.$|H^j_{xxx}| \leq C|H^j_x|^3$ for all $x \in [x_l, x_r]$;
8.$|H^j_{xa}| \leq C|H^j_x|l^3$ for all $x \in [x_l, x_r]$;
9.$|H^j_{xxa}| \leq C|H^j_x|^2 l^3$ for all $x \in [x_l, x_r]$.

Proof. The proof is carried on with the help of the formula in the Appendix. We use Equation (1), assuming that $x \in [H^{i+1}x_r, H^i x_r]$, for i such that $H_x \circ H^i x_r < 1$. Otherwise a similar procedure can be done using Equation (2) for H^j and adapting Lemmas 39 and 40 for H^{-1} and x_l instead of H and x_r.

1. $|H^j_x| \leq C\min\{|X(a,x)|^{-1}; x \in [x_l, x_r]\}$, according to (A.1) (see Appendix). So $|H^j_x| \leq Cl^2$ using Lemmas 35 and 37.1. As $|X| \geq C^{-1}$ in $[Hx_l, H^{-1}x_l]$ and $|X| \leq C$ in $[x_l, x_r]$ the other inequality follows easily;

2. $|X| \geq C$ in $[Hx_r, x_r]$;

3. the first term of (A.4) is bounded by Cl^3, by Lemmas 37.2(i) and 39. The function Ψ is evaluated in a bounded domain, since $|-\sigma_l + t_i| \leq 2$ and $\lambda \in [a_1, a_s]$, so $\frac{\partial\Psi}{\partial a}$ is bounded by some constant C (if one wants to be more precise, it is enough to observe that $\theta = \frac{\partial\Psi}{\partial a}$ satisfies the differential equation $\theta' = b_{a,x}(t) + d_{a,x}(t)\theta$, where $b_{a,x}(t) = X_a(a, \Psi(t,a,x))$ and $d_{a,x}(t) = X_x(a,\Psi(t,a,x))$);

4. by Lemma 39, if $i = 0$ then $|\frac{\partial t_i}{\partial a}| \leq C$, and $|\dot{\sigma}_l| \geq C^{-1}l^3$ by Lemma 37.2(i). As $|X \circ H^j| \geq C^{-1}$ and $|\frac{\partial\Psi}{\partial a}| \leq C$ the result follows;

5. by Lemmas 37.2(ii) and 40, the dominant term of (A.6) is

$$H_a^j \cdot X_x \circ H^j \cdot \left[-\dot{\sigma}_l + \frac{\partial t_i}{\partial a} \right] ,$$

which is smaller than Cl^6 by item (3), Lemma 37.2(i) and Lemma 39;

6. using (A.2), with $|X \circ H^j| > C^{-1}$;

7. using (A.3);

8. $|H_{xa}^j| \le C|X|^{-1} + C|X|^{-1} \cdot |H_a^j| + |X|^{-2} \le C|X|^{-1}(l^3 + |X|^{-1})$ by (A.5); but $|X|^{-1} \le Cl^2$, so $|H_{xa}^j| \le C|X|^{-1}l^3 \le C|H_x^j|l^3$;

9. $|H_{xxa}^j| \le C|X|^{-3} + C|X|^{-2} \cdot |H_a^j| \le C|H_x^j|^2 l^3$, by (A.5).

▮

Lemma 42. *Let $x \in [-2, H(0)] \cup [-H(0), 2]$ and F_S be the least power of H such that $F_S(x) \le -2$. Then*

$$|F_{S,x}|^{-1} \ , \ \left| \frac{F_{S,xx}}{(F_{S,x})^2} \right| \ , \ \left| \frac{F_{S,xxx}}{(F_{S,x})^3} \right| \ \le \ C_1; ,$$

$$|F_{S,a}| \ , \ |F_{S,x}| \ , \ \left| \frac{F_{S,xx}}{F_{S,x}} \right| \ , \ \left| \frac{F_{S,xa}}{F_{S,x}} \right| \ , \ \left| \frac{F_{S,xxa}}{(F_{S,x})^2} \right| \ \le \ C_2 ,$$

$$|F_{S,aa}| \ \le \ C_3 ,$$

where $(C_1, C_2, C_3) = (C, Cl^3, Cl^6)$ if $a \in J_d$ and $(C_1, C_2, C_3) = (C, C, C)$ if $a \in J_c$. Moreover if $x \in [H^2(0), H(0)]$ then $|F_{S,a}| \ge C^{-1}l^3$ if $a \in J_d$, $|F_{S,a}| \ge C^{-1}$ if $a \in J_c$ and $|F_{S,a}|$, $|F_{S,xx}| \le C$ in both cases.

Proof. When $a \in J_d$ it suffices to divide the orbit of x into three parts: to the right of x_r, inside $[x_l, x_r]$ and to the left of x_l. The first and third parts have all the derivatives bounded by constants, the second is dealt with Lemma 41 and the composition of the three gives the result. For $a \in J_c$ we bound all the derivatives by some constant which may depend on the choice of a_1. ▮

Remark. The constant $C > 0$ that appear in the lemmas of this subsection depends mostly on the estimates for F_S when $a \in J_c \cup J_d$ and is directly affected by the choice of a_1 (as well as x_l and x_r). However this constant can be 'killed' by the small constants r, p, q, δ_0 and δ_1 (and even $|H_x^s|$ for $s \ge k_0$) since by their dependence on ϵ (or M) they can be made small *after* the choice of a_1, x_l and x_r.

3.3. Horseshoe estimates

Consider now the parameter region "beyond renormalization", i.e. $\sigma = +$ (or else $a \in J^+$), recalling the definitions given in Subsection 1.13. There we have defined a partition of the interval $[x^-, x^+]$: (a) the central interval

$[p^-, p^+]$, which is the preimage of $\{x \geq x^+\}$; (b) the point $q^+ \in (p^+, x^+)$, fixed point of H, and its preimage $q^- \in (x^-, p^-)$; (c) a partition $\{\omega_i^+ = [q_i^+, q_{i-1}^+]\}_{i\geq 1}$ (resp. $\{\omega_i^- = [q_{i-1}^-, q_i^-]\}_{i\geq 1}$) of $[q^+, x^+]$ (resp. $[x^-, q^-]$) such that $H^i(\omega^{\pm}) = [q^-, q^+]$; and (d) a partition $\{\alpha_i^+ = [\tilde{q}_i^+, \tilde{q}_{i-1}^+]\}_{i\geq 1}$ (resp. $\{\alpha_i^- = [\tilde{q}_{i-1}^-, \tilde{q}_i^-]\}_{i\geq 2}$) of $[p^+, q^+]$ (resp. $[q^-, p^-]$) such that $H^i(\alpha_i^{\pm}) = \omega_{i-1}^+$. Observe that $\tilde{q}_1^+ = q^+$ and $\tilde{q}_1^- = q^-$.

The following two Lemmas are easily proven by straightforward calculations.

Lemma 43. *Let $x \neq y$ be such that $H(a,x) = H(a,y)$. Then $|x|/|y| \leq 1 - 2\delta_0$, if δ_0 is small.*

Lemma 44. *$|x^+| \geq 2(1-\delta_0)$ if δ_0 is small.*

Define $d = d_a = H(a,0) - x_a^+$ and for $x \in [x^-, x^+] \setminus [p^-, p^+]$ let $l = l(x) \geq 1$ be such that $H^l x \in [p^-, p^+]$. With these hypothesis we state the following Lemmas.

Lemma 45. *If $d \geq \frac{1}{2}$ then $|H_x^j| \geq \left(\frac{4}{3}\right)^j$ for all $1 \leq j \leq l$.*

Proof. For all $x \notin [p^-, p^+]$, $|H_x| \gtrsim 2d^{1/2} > \frac{4}{3}$. ▮

Lemma 46. *If $d < 1/2$ there is $C > 0$ such that the following assertions are valid.*

1. $|H_x^j| \geq C^{-1} d^{1/2} (4/3)^j$, *for all* $1 \leq j \leq l$;
2. $|H_x^l| \geq (4/3)^l$;
3. $\sum_{i=0}^{j} |H_x^i|^{-1} \leq C d^{-1/2}$;
4. $\sum_{i=1}^{j} |H_x^i \circ H^{j-i}|^{-1} \leq C d^{-1/2}$, *for all* $1 \leq j \leq l$;
5. $\sum_{i=1}^{l} |H_x^i \circ H^{l-i}|^{-1} \leq C$.

Proof. Immediate consequence of Lemma 47, that we state in the sequel. ▮

Lemma 47. *Let $d < 1/2$ and $0 \leq n_0 < n_1 < n_2 < \ldots < n_s \leq l$ be such that $H^{n_t} \in [q^-, q^+]$ and $H^j \notin [q^-, q^+]$ if $j \neq n_t$, for all $t = 0, \ldots, s$. Then, for all t,*

1. $|H_x \circ H^{n_t}| \geq \frac{3}{2} d^{1/2}$.
2. $|H_x^j \circ H^{n_t}| \geq 2^{j-1} |H_x \circ H^{n_t}|$, *for all* $2 \leq j \leq n_{t+1} - n_t$;
3. $|H_x^{n_{t+1}-n_t} \circ H^{n_t}| \geq \left(\frac{4}{3}\right)^{n_{t+1}-n_t}$;
4. $|H_x^j| \geq 2^j$, *for* $j \leq n_0$.

Proof. For $0 \leq t \leq s$, the point H^{n_t} of the orbit belongs to $[q^-, p^-] \cup [p^+, q^+]$, hence to some $\alpha_i^{\pm}$, for $i \geq 2$. If $t < s$ then $H^{n_{t+1}}$ is the first return to $[q^-, q^+]$, and in this case $n_{t+1} - n_t = i$. The Lemma follows from the estimates below.

i. $2(1+\frac{d}{4}-c\delta_0) \le x^+ \le 2(1+d+c\delta_0)$ and $1+\frac{d}{4}-c\delta_0 \le q^+ \le 1+2d+c\delta_0$, for some constant $c>0$;
ii. if $x \le q_1^-$ or $x \ge q_1^+$ then $|H_x| \ge 3$;
iii. if $d < 1/2$ and δ_0 is small then $|H_x| \le 5$ for all $x \in [x^-, x^+]$;
iv. as a consequence of the previous item, $x^+ - q_i^+ \ge 5^{-i}$, for all $i \ge 0$;
v. let $x \in \alpha_i^{\pm}$, $i \ge 2$; then $|H_x| \gtrsim |H_x(\tilde{q}_{i-1}^+)| \simeq 2\sqrt{d + x^+ - q_{i-2}^+} \ge 2(x^+ - q_{i-2}^+)^{1/2} \ge 10 \times 5^{-i/2}$. Hence, for $x \in \alpha_i^{\pm}$, $i \ge 2$,

$$|H_x^i| \ge 10 \times 5^{-i/2} \times 3^{i-2} \times 2 > \left(\frac{4}{3}\right)^i$$

using that $|H_x| \ge 2$ for $x \in [q_1^-, q_0^+] \cup [q_0^+, q_1^+]$.

∎

LEMMA 48. *Let $d < 1/2$, $x \in [x^-, x^+] \setminus [p^-, p^+]$ and l be the first integer such that $H^l x \in [p^-, p^+]$. Then there is $C > 0$ such that*

1. $\left|\frac{H_a^j}{H_x^j}\right| \le Cd^{-1/2}$ *for all* $1 \le j \le l$;
2. $\left|\frac{1}{H_x^{l-j} \circ H^j} \frac{H_{xx}^j}{(H_x^j)^2}\right| \le Cd^{-1/2}$ *for all* $1 \le j \le l$;
3. $\left|\frac{1}{H_x^{l-j} \circ H^j} \frac{H_{xa}^j}{(H_x^j)^2}\right| \le Cd^{-1}$ *for all* $1 \le j \le l$;
4. $\left|\frac{H_{aa}^l}{(H_x^l)^2}\right| \le Cd^{-3/2}$;
5. $\left|\frac{H_{xxx}^l}{(H_x^l)^3}\right| \le Cd^{-1}$;
6. $\left|\frac{H_{xxa}^l}{(H_x^l)^3}\right| \le Cd^{-3/2}$;

Proof. The proof is a straightforward application of the Appendix, the properties of H, the preceding Lemmas and also items (1), (2) and (3), which are used to prove (3), (4), (5) and (6). ∎

LEMMA 49. *Let $d \ge \frac{1}{2}$, $x \in [x^-, x^+] \setminus [p^-, p^+]$ and $j \ge 1$ such that $H^j x \in [x^-, x^+]$. Then*

1. $\left|\frac{H_a^j}{H_x^j}\right| < 2d^{-1/2}$;
2. $\left|\frac{H_{xx}^j}{(H_x^j)^2}\right| < 2d^{-1}$;
3. $\left|\frac{H_{xa}^j}{(H_x^j)^2}\right| < 3d^{-1}$;
4. $\left|\frac{H_{aa}^j}{(H_x^j)^2}\right| < 10d^{-3/2}$;
5. $\left|\frac{H_{xxx}^j}{(H_x^j)^3}\right| < 25d^{-3/2}$;

6. $\left|\frac{H^j_{xxa}}{(H^j_x)^3}\right| < 25d^{-3/2}$;

Proof. If $d \geq \frac{1}{2}$ all iterates are expanded by $M \equiv 2d^{1/2}$. We use the Appendix, together with the following facts: $jM^{-j} \leq \frac{3}{2}$ for all $j \geq 1$ and $d < \frac{1}{2}|\gamma_{n-1}|$, hence $|\gamma_n|^{-1} < M^{-1}$. ▮

3.4. Estimates on expanding regions

In this subsection we will be dealing with the properties of powers of H taken far away from the origin. To be more specific, we consider as hypothesis an initial point x_0 outside $[x^-, x^+]$ if $\sigma = +$, or outside $[-2, 2] \cup H^{-1}([H^2(0), H(0)])$ if $\sigma = -$ (see Figures 6 and 7). We denote $x_t = H^t(x_0)$ for $0 \leq t \leq j$, where $x_j \in D_1^-$ and $j \geq 1$. The following two lemmas relate x_t and x_{t+1} for subsequent use in Lemma 52.

Lemma 50. *If $\sigma = +$ then $|x_t|^2 \gtrsim |x_{t+1}|$ for all $0 \leq t \leq j-1$.*

Proof. As $x_t < 0$ for all $t \geq 1$, $a + |x_{t+1}| \simeq |x_t|^2$ together with $a > 0$ implies the Lemma. ▮

Lemma 51. *If $\sigma = -$ then $|x_0|^2 \gtrsim \frac{2}{3}|x_1|$ and $|x_t|^2 \gtrsim \frac{8}{9}|x_{t+1}|$ for all $1 \leq t \leq j-1$.*

Proof. For all $t \geq 0$

$$\frac{|x_{t+1}|}{|x_t|^2} \simeq \frac{|H(0)| + |x_t|^2}{|x_t|^2} = 1 + \frac{|H(0)|}{|x_t|^2}.$$

We consider two cases: if $|H(0)| \geq 2$ we use $|x_t| \gtrsim |H(0)|$ and if $|H(0)| < 2$ we use $|x_t| \geq 2$. In both cases

$$\frac{|H(0)|}{|x_0|^2} \lesssim \frac{1}{2}$$

and, for $t \geq 1$,

$$\frac{|H(0)|}{|x_t|^2} \leq \frac{|H(0)|}{|x_1|^2} \lesssim \frac{|H(0)|}{|x_0|^4} \lesssim \frac{1}{8},$$

proving the Lemma. ▮

In what follows we consider an additional hypothesis. We suppose that $x_j \in \beta$ for some $\beta \in \mathcal{B}_n$ and let η be such that $x_0 \in \eta$ and $H^j(\eta) = \beta$.

Lemma 52. *Following the notation above,*

$$|\beta| \cdot \sum_{t=1}^{j} \frac{1}{H_x^{j-t}(x_t)[H_x(x_{t-1})]^2} < q.$$

Proof. First we verify that the sum is geometric, with the dominant term given by $t = j$. Dividing the $(t+1)$–term by the t–term we obtain, approximately,

$$\frac{[H_x(x_{t-1})]^2}{H_x(x_t)} \simeq \frac{4|x_{t-1}|^2}{2|x_t|} .$$

By Lemmas 50 and 51 (in fact 51 gives the worst estimate) this quotient is greater than $\frac{8}{5}$ for $t \geq 2$ and greater than $\frac{6}{5}$ for $t = 1$, characterizing the geometric growth with t. The dominant term is given by $t = j$ and is approximately equal to

$$\frac{1}{4}\frac{1}{|x_{j-1}|^2} \lesssim \frac{2}{9}\frac{1}{|x_j|}.$$

The Lemma follows since $|\beta| \cdot |x_j|^{-1} \leq q$. ∎

Remark. As a Corollary of the proof of Lemma 52 we also have

$$|\beta|^2 \cdot \sum_{t=1}^{j} \frac{1}{[H_x^{j-t}(x_t)]^2 \cdot [H_x(x_{t-1})]^4} < q^2$$

and, for $1 \leq s < j$,

$$|H^s(\eta)| \cdot \sum_{t=1}^{s} \frac{1}{H_x^{s-t}(x_t) \cdot [H_x(x_{t-1})]^2} < q,$$

using also Lemma 33 for this inequality. Moreover we observe that by distortion properties we have $|H^s(\eta)| \simeq |\beta| \cdot |H_x^{j-s}(x_s)|^{-1}$.

We call $F_E = H^j$ and, using the Appendix together with the estimates above and the induction hypotheses, we obtain the following Lemma.

Lemma 53.

1. $\left|\frac{F_{E,a}}{F_{E,x}}\right| \lesssim \frac{1}{3}$;
2. $|\beta| \cdot \left|\frac{F_{E,xx}}{(F_{E,x})^2}\right| < 2q$;
3. $|\beta| \cdot \left|\frac{F_{E,xa}}{(F_{E,x})^2}\right| < 2q$;
4. $|\beta| \cdot \left|\frac{F_{E,aa}}{(F_{E,x})^2}\right| < 2q$;
5. $|\beta|^2 \cdot \left|\frac{F_{E,xxx}}{(F_{E,x})^3}\right| < 4q^2$;
6. $|\beta|^2 \cdot \left|\frac{F_{E,xxa}}{(F_{E,x})^3}\right| < 5q^2$.

Proof. Observe also that $|\beta| \cdot |\gamma_n|^{-1} \cdot |F_{E,x}|^{-1} \lesssim q$. ∎

The following Lemma is about the expansion in the critical value's orbit when $\sigma = +$ and $d < 1/2$.

LEMMA 54. *If $\sigma = +$ and $d < \frac{1}{2}$ then $|H_x^{esc-1}(H(0))| \geq \frac{1}{2}\frac{|\gamma_{n-1}|}{|\gamma_n|}d^{-1} \geq \frac{1}{2}r^{-1}d^{-1}$, where d was defined in Subsection 3.3.*

Proof. Write $k = \text{esc}$ to avoid a cumbersome notation. Let $\hat{x} \in D_{k+1}^+$ be such that $|H_x^k(\hat{x})| \cdot |D_{k+1}^+| = |D_1^-|$. We have

$$|H_x^{k-1}(H(0))| \geq |H_x^{k-1}(\hat{x})| = \frac{|D_1^-|}{|D_{k+1}^+|} \cdot |H_x(H^{k-1}\hat{x})|^{-1}.$$

We have $|D_{k+1}^+| < d$ and $|H_x(H^{k-1}\hat{x})| < |H_x(d_1^-)|$. Moreover, $|D_1^-| \simeq \frac{1}{2}|\gamma_{n-1}|$ and also $|H_x(d_1^-)| \lesssim |\gamma_n|$, so that

$$|H_x^{k-1}(H(0))| > \frac{1}{2}d^{-1}\frac{|\gamma_{n-1}|}{|\gamma_n|} .$$

∎

3.5. Tools to prove differential conditions

We set now all the notational convention needed to the following two sections, in such a way that all the statements and symbols have to be addressed to this introduction or to previous definitions (mainly Subsection 1.13).

Let $\beta^* \in \mathcal{B}_n$ be the preimage of the central branch which contains $H^{\text{esc}}(0)$, $\text{esc} = \text{esc}(a)$, and let $B^* : \beta^* \to \gamma_n$ be its associated function. We write $H' = B^* \circ F_E^* \circ F_S^* \circ H : \gamma_{n+1} \to \gamma_n$ if $\sigma = -$, and $H' = B^* \circ F_E^* \circ H : \gamma_{n+1} \to \gamma_n$ if $\sigma = +$, according to the decomposition of orbits explained in Subsection 1.13. Here $\gamma_{n+1} = H^{-1}(\eta^*)$, where η^* is the connected component of $H^{-\text{esc}+1}(\beta^*)$ to the left of the origin if $\sigma = -$, to the right of the origin if $\sigma = +$ (and $\eta^* = \beta^*$ if $\text{esc} = 1$). The function associated to η^* is denoted by $G^* : \eta^* \to \gamma_n$ and $|G_w^*| \simeq |\eta^*|^{-1} \cdot |\gamma_n|$, by usual distortion properties.

For each $\pi \in \mathcal{P}_{n+1}$ in γ_n there is an associated diffeomorphism $P : \pi \to \gamma_n$ given by the composition $P = B \circ F_E \circ F_S \circ F_0$ if $\sigma = -$ or $P = B \circ F_E \circ F_0 \circ F_H$ if $\sigma = +$, where $B : \beta \to \gamma_n$ is the function associated to an element $\beta \in \mathcal{B}_n$ contained in D_1^-.

Moreover, for each $\tilde{\beta}_H \in \mathcal{B}_{n+1}^H$ the associated function $\tilde{B}_H : \tilde{\beta}_H \to \gamma_{n+1}$ is a power of H whose corresponding orbit is done inside $[x^-, x^+]$. This kind of preimage only occurs for $\sigma = +$ as it has been stressed in Subsection 1.13.

For each $\tilde{\beta} \in \mathcal{B}_{n+1}$ there are two ways of writing its associated function $\tilde{B} : \tilde{\beta} \to \gamma_{n+1}$: $\tilde{B} = P_j \circ P_{j-1} \circ \cdots \circ P_1$ or $\tilde{B} = \tilde{F} \circ P_{j-1} \circ \cdots \circ P_1$, for some $j \geq 1$, where each $P_i : \pi_i \to \gamma_n$ is associated to $\pi_i \in \mathcal{P}_{n+1}$, for all $i = 1, \ldots, j$. The second decomposition is allowed only if $\sigma = +$. By the definition of $\pi_i \in \mathcal{P}_{n+1}$ there is an element $\beta_i \in \mathcal{B}_n$ such that π_i is sent onto β_i by some power of H. To be more specific, we can formally write, taking together cases $\sigma = +$ and $\sigma = -$ and in the same way as described before,

$$P_i = B_i \circ F_{E,i} \circ F_{S,i} \circ F_{0,i} \circ F_{H,i},$$

where each one of the functions, except B_i, can be the identity, depending on the sign of σ or the position of π_i. We call $\eta_i = (F_{0,i} \circ F_{H,i})(\pi_i)$, $\xi_i = F_{H,i}(\pi_i)$ and $G_i = B_i \circ F_{E,i} \circ F_{S,i} : \eta_i \to \gamma_n$.

Following Subsection 3.1 we can also define a concentric $C^{-1}\theta_n^{-1}|\eta_i|$–neighborhood $\mathcal{U}(\eta_i)$ of η_i such that $G : \eta_i \to \gamma_n$ is extendible with δ–distortion to $\mathcal{U}(\eta_i)$, so that we can use the notion of subordination for the pair (η_i, η^*) (η_i and η^* will be automatically independent if η_i does not belong to the same fundamental domain that η^* belongs).

For $1 \leq i \leq j-1$ define the compositions $A_i = P_i \circ P_{i-1} \circ \cdots \circ P_1$, $A_j \equiv \tilde{B}$, and $\tilde{B}_i = P_j \circ \cdots \circ P_i$ or $\tilde{B}_i = \tilde{F} \circ P_{j-1} \circ \cdots \circ P_i$, according to the case considered, and the intervals $\tilde{\beta}_i \equiv \tilde{B}_i^{-1}(\gamma_{n+1})$. In the first case, $\tilde{\beta}_i \subset \tilde{\pi}_i \equiv \tilde{B}_i^{-1}(\gamma_n) \subset \pi_i$. Also the definition $W_i = F_{H,i+1} \circ A_i$ will be important for our purposes.

The following Lemma collects a few small tricks which will be used in the proof of many other Lemmas.

Lemma 55.

1. $|\eta^*| \cdot |\gamma_n|^{-1} < 1$;

2. $|\eta^*| \cdot |\gamma_n|^{-2} < p$ *(and other assertions involving β_i's, η_i's, etc)*;

3. $|\gamma_{n+1}| \cdot \frac{1}{2}|W_i|^{-1} \lesssim 1$ *for* $i = 0, \ldots, j-1$;

4. $|B_w| \simeq |\gamma_n| \cdot |\beta|^{-1}$ *for* $\beta \in \mathcal{B}_n$;

5. if $F_{S,i} \neq Id$ *then* $|F_S(\eta_i)| < 10q$, *if* $F_S^* \neq Id$ *then* $|\eta^*| < 10q$ *and* $|\gamma_{n+1}|^2 < 40q$.

Proof. 1. By induction, $|B_w^*| > 1$ and F_E^*, $F_E^* \circ F_S^*$ are expanding; 2. for $\sigma = -$ use Lemma 33, the expansion of F_E^* and the geometry; for $\sigma = +$ analogous, taking into account that now $\mathrm{dist}(\eta^*, \partial\gamma_{n-1})$ is always smaller than $|\mathrm{Im}H|$; 3. for W_i does not belong to γ_{n+1}; 4. by the small distortion property of the preimages of the central branch; 5. if $F_{S,i}$ is non–trivial, $F_{S,i}(\eta_i) \subset F_{E,i}^{-1}(D_1^-)$ and we apply Lemma 33 to the fact that $|F_{E,i}^{-1}(D_1^-)|$ is not much bigger than one; the other two inequalities follow similar reasonings. ▮

LEMMA 56.

1. $C^{-1}l^3 \le |H'_a| \cdot |G^*_x|^{-1} \le Cl^3$ *if* $a \in J_d$;
2. $C^{-1} \le |H'_a| \cdot |G^*_x|^{-1} \le C$ *if* $a \in J_c$;
3. $1 \le |H'_a| \cdot |G^*_x|^{-1} < \frac{3}{2}$ *if* $a \in J_b$;
4. $|H'_a| \cdot |G^*_x|^{-1} \simeq 1$ *if* $a \in J_a$.

Proof. We write, using the Appendix:

$$\frac{H'_a}{G^*_x} = H_a + \frac{F^*_{S,a}}{F^*_{S,x}} + \frac{1}{F^*_{S,x}} \cdot \frac{F^*_{E,a}}{F^*_{E,x}} + \frac{1}{F^*_{S,x}F^*_{E,x}} \cdot \frac{B^*_a}{B^*_x}.$$

Lemma 42 says that $C^{-1} \le |F^*_{S,x}| \le C$ for $a \in J_c \cup J_d$, $C^{-1}l^3 \le |F^*_{S,a}| \le Cl^3$ if $a \in J_d$, $C^{-1} \le |F^*_{S,a}| \le C$ if $a \in J_c$ and $F^*_S = \mathrm{Id}$ if $a \in J_a \cup J_b$. Moreover $|F^*_{E,a}| \cdot |F^*_{E,x}|^{-1}| \lesssim \frac{1}{3}$ by Lemma 53 and $|B^*_a| \cdot |B^*_x|^{-1} < \delta_1$ by induction, and the Lemma follows. ▮

LEMMA 57. *If* $\sigma = +$ *then* $\frac{2}{3} \lesssim |H'_a| \cdot |G^*_x|^{-1} \lesssim 1$.

Proof. By an analogous development as in the proof of Lemma 56, but now paying attention to signals. ▮

COROLLARY 58. *If* $\sigma = +$ *and* $d < \frac{1}{2}$ *then* $|H'_a| \ge \frac{1}{4}\frac{|\gamma_{n-1}|}{|\gamma_n|}d^{-1}$.

Proof. Combine Lemmas 57 and 54. ▮

LEMMA 59. *If* $\sigma = -$, *let* $G = B \circ F_E \circ F_S$ *at a point* x *that does not belong to* $H^{-1}[H^2(0), H(0)]$ *or, if* $\sigma = +$, *let* $G = B \circ F_E$ *at a point* x *that does not belong to* $[x^-, x^+]$, *where* $B : \beta \to \gamma_n$ *is the function associated to some* $\beta \in \mathcal{B}_n$. *Then*

1. $\left|\frac{G_a}{G_x H'_a}\right| < C\frac{|\eta^*|}{|\gamma_n|}$;
2. $\left|\frac{G_{xx}}{(G_x)^2}\right| < 2\delta_1|\gamma_n|^{-1}$;
3. $\left|\frac{G_{xxx}}{(G_x)^3}\right| < 2\delta_1|\gamma_n|^{-2}$;
4. $\left|\frac{G_{xa}}{(G_x)^2 H'_a}\right| < C\delta_1\frac{|\eta^*|}{|\gamma_n|^2}$;
5. $\left|\frac{G_{aa}}{(G_x)^2 (H'_a)^2}\right| < C\delta_1\frac{|\eta^*|^2}{|\gamma_n|^3}$;
6. $\left|\frac{G_{xxa}}{(G_x)^3 H'_a}\right| < C\delta_1\frac{|\eta^*|}{|\gamma_n|^3}$.

Proof. With the Appendix and almost all the previous Lemmas, provided q is sufficiently small with respect to δ_1. ▮

As an immediate application of Lemma 59 we can prove the differential conditions of $H' = G^* \circ H$, since G^* satisfies the hypothesis of Lemma 59.

LEMMA 60. *$|H'_{xx}| \simeq 2|G^*_x \circ H|$ in γ_{n+1}.*

Proof. We write

$$H'_{xx} = G^*_{xx}(H_x)^2 + G^*_x H_{xx}.$$

As $|H_x|^2 \simeq 4|x|^2 \lesssim |\gamma_{n+1}|^2$, $|H_{xx}| \simeq 2$ and $|G^*_x| \simeq |\gamma_n| \cdot |\eta^*|^{-1}$ then

$$\left| \frac{G^*_{xx}(H_x)^2}{G^*_x H_{xx}} \right| \lesssim \frac{1}{2} |\gamma_{n+1}|^2 \cdot \frac{|\gamma_n|}{|\eta^*|} \cdot \left| \frac{G^*_{xx}}{(G^*_x)^2} \right|.$$

By Lemma 59 and since $|\gamma_{n+1}|^2 \lesssim 4|\eta^*|$ we conclude that $|H'_{xx}| \simeq |H_{xx}| \cdot |G^*_x|$ and the Lemma is proven. ▮

LEMMA 61. *For $x \in \gamma_{n+1}$*

1. $\left|\frac{xH'_{xxx}}{H'_{xx}}\right| < \delta_0$;

2. $\left|\frac{xH'_{xa}}{H'_a}\right| < \delta_0$;

3. $|\gamma_n| \cdot \left|\frac{H'_{aa}}{(H'_a)^2}\right| < \delta_0$;

4. $|\gamma_n| \cdot \left|\frac{H'_{xxa}}{H'_{xx}H'_a}\right| < \delta_0$.

Proof. We use the Appendix, Lemma 59 for G^* and many other lemmas. ▮

In Subsection 1.14 we used the following Lemma.

LEMMA 62. *Whenever esc > 1 then $|H^{esc}_a(0)| \geq H_a(0)|$.*

Proof. We write $H^{\text{esc}} = F_E \circ F_S \circ H$ if $\sigma = -$ and $H^{\text{esc}} = F_E \circ H$ if $\sigma = +$. In the latter case, $H^{\text{esc}}_a = F_{E,a} + F_{E,x}$, with $F_{E,x} < 0$ and $|F_{E,a}| \cdot |F_{E,x}|^{-1} \lesssim \frac{1}{3}$, by Lemma 53. But $|F_{E,x}| \geq 4$ and the Lemma follows in this case. In the former case, $H^{\text{esc}}_a = F_{E,a} + F_{E,x}(F_{S,a} + F_{S,x}H_a)$. Since in this case all x–derivatives are positive on iterates of the critical value and $H_a \simeq 1$, this number is positive. The Lemma follows since $F_{E,a} > H_a$ (use the Appendix and the fact that in expanding regions the derivatives are bigger than one). ▮

3.6. Mean expansion

The main goal of this subsection is to obtain some properties for the x–derivative of the functions P_i that enter in the composition of $\tilde{B}$. This is will be the main ingredient to control distortion of mixed derivatives.

Let Y be the distance between the center of $H(\tilde{\beta})$ and $H(0)$, let $Z = |\text{Im}H|$ and $X = Z - Y$. Define $\omega = X/Z$ and $\tau = X/Y$. The variables X, Y, Z, ω and τ are functions of $\tilde{\beta}$.

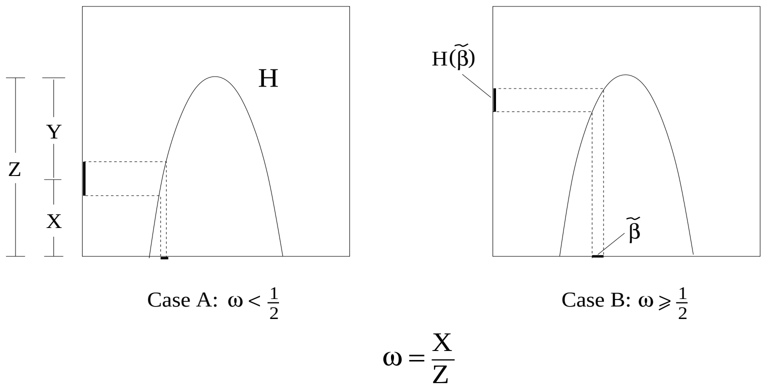

FIG. 9. Two cases for the position of $\tilde{\beta}$.

We say that $\tilde{\beta}$ is in *Case A* if $\omega < \frac{1}{2}$ and in *Case B* otherwise (see Figure 9).

Here we determine once and for all the neighbourhood $\mathcal{U}(\tilde{\beta})$ of $\tilde{\beta}$ referred to in Subsection 1.9. Writing $\tilde{B} = P_j \circ \cdots \circ P_1$ or $\tilde{B} = \tilde{F} \circ P_{j-1} \circ \cdots \circ P_1$, as in the beginning of the previous Subsection, we observe in particular that, in Case A, $P_1 = B_1 \circ H$, and therefore $\mathcal{U}(\tilde{\beta}) \subset \pi_1$. In Case B it will be useful to note that there are basically two cases to consider: (a) $\mathrm{dist}(\tilde{\beta}, \partial\gamma_n) \geq \frac{2}{15} > \frac{1}{8}$ and (b) otherwise. The latter situation also obliges $P_1 = B_1 \circ H$ and $\mathcal{U}(\tilde{\beta}) \subset \pi_1$.

Lemma 63. *In Case B, if $ImH' \cap \mathcal{U}(\tilde{\beta}) \neq \emptyset$ then*

$$|ImH \cap \eta^*| > \frac{1}{8}|\eta^*|.$$

Proof. If $\mathrm{dist}(\tilde{\beta}, \partial\gamma_n) \geq \frac{2}{15}$ then the Lemma is immediate. Otherwise $\mathcal{U}(\beta) \subset \pi_1$ and as G^* has small distortion,

$$\frac{|\mathrm{Im}H \cap \eta^*|}{|\eta^*|} \simeq \frac{|\mathrm{Im}H'|}{|\gamma_n|}\ .$$

Since $\mathrm{Im}H' \cap \pi_1 \neq \emptyset$ then $|\mathrm{Im}H' \cap \gamma_n| \geq \mathrm{dist}(\pi_1, \partial\gamma_n)$. But $\mathrm{dist}(\pi_1, \partial\gamma_n) \simeq \mathrm{dist}(\tilde{\beta}, \partial\gamma_n) > (1 - \frac{\sqrt{2}}{2})|\mathrm{Im}H|^{1/2} > \frac{1}{8}|\gamma_n|$ and the Lemma follows. ▮

Lemma 64. *In Case A, if $ImH' \cap \pi_1 \neq \emptyset$ then*

$$|\gamma_{n+1}| \gtrsim \omega^{1/2}|\eta^*|^{1/2}.$$

Proof. As $\mathrm{dist}(\beta_1, \partial\gamma_{n-1}) \simeq \omega|\mathrm{Im}H|$ and $|H_x| \lesssim |\gamma_n|$ then

$$\mathrm{dist}(\pi_1, \partial\gamma_n) \gtrsim |\gamma_n|^{-1}\omega|\mathrm{Im}H|$$

Therefore

$$|\gamma_{n+1}| \gtrsim 2\left[\mathrm{dist}(\pi_1, \partial\gamma_n)\frac{|\eta^*|}{|\gamma_n|}\right]^{1/2} \gtrsim \omega^{1/2}|\eta^*|^{1/2}.$$

▮

LEMMA 65. *In Case A, if* $ImH' \cap \pi_1 \neq \emptyset$ *then*

$$|P_{1,x}| \gtrsim \frac{1}{2}p^{-1}\omega^{-1}.$$

Proof. In Case A, $P_1 = B_1 \circ H$, hence $|P_{1,x}| = |B_{1,x}| \cdot |H_x| \gtrsim \frac{|\gamma_n|}{|\beta_1|} \cdot \frac{\sqrt{2}}{2}|\gamma_n|$. But $|\beta_1| < p\omega|\gamma_n|^2$ by Lemma 55. ▮

LEMMA 66. *For all* $i = 1, \ldots, j$, *if* η_i *is subordinated to* η^* *then* $|P_{i,x}| \geq |(G_i \circ H)_x| \geq q^{-1/2}p^{-1/2}$.

Proof. As $P_i = G_i \circ H \circ F_{H,i}$ and $F_{H,i}$ is an expansion (if it is not the identity), by Lemmas 45 and 46, then $|P_{i,x}| \geq |(G_i \circ H)_x| \geq q^{-1/2}p^{-1/2}$. We have $|G_{i,x}| \simeq |\gamma_n| \cdot |\eta_i|^{-1}$ and since $\mathrm{dist}(\eta_i, \eta^*) \gtrsim q^{-1}|\eta_i|$, then $|H_x| \gtrsim 2q^{-1/2}|\eta_i|^{1/2}$. Hence

$$|(G_i \circ H)_x| > 2q^{-1/2}|\gamma_n| \cdot |\eta_i|^{-1/2} > q^{-1/2}p^{-1/2}$$

by Lemma 55. ▮

LEMMA 67. *If* η^* *is subordinated to* η_i *or* η^* *and* η_i *are independent then*

$$|P_{i,x}| \geq |(G_i \circ H)_x| > p^{-1}\frac{|\eta^*|^{1/2}}{|\gamma_n|}.$$

Proof. In both cases $\mathrm{dist}(\eta_i, \eta^*) >> |\eta^*|$, so that $|H_x| >> |\eta^*|^{1/2}$. Hence

$$|(G_i \circ H)_x| >> \frac{|\eta^*|^{1/2}}{|\gamma_n|} \cdot \frac{|\gamma_n|^2}{|\eta_i|}$$

and the Lemma follows using Lemma 55. ▮

Lemma 68. *Let $\tilde{x}$ be the point in $\xi_i = F_{H,i}(\pi_i)$ at the greatest distance from the origin. Then*

$$|(G_i \circ H)_x(\tilde{x})| \gtrsim \max_{x\in\xi_i} |(G_i \circ H)_x(x)|.$$

Proof. As the distortion of G_i is small any eventual big non–uniformity of $(G_i \circ H)_x$ is due to H. In other words, as $|G_{i,x}|$ is almost constant in η_i and $|H_x|_{\xi_i}|$ attains its maximum at $\tilde{x}$, the Lemma follows. ▮

Lemma 69. *Let $x \in \xi_i$ and let*

$$\hat{\omega} = \hat{\omega}(Hx) = |\eta_i|^{-1} \cdot dist(Hx, \text{boundary point of } \eta_i \text{ nearest } \eta^*).$$

Then $|P_{i,x}(F_{H,i}^{-1}(x))| \geq |(G_i \circ H)_x(x)| \geq p^{-1/2}\hat{\omega}^{1/2}$.

Proof. We have

$$\frac{1}{|\xi_i|}\int_{\xi_i} |(G_i \circ H)_x| dx = \frac{|\gamma_n|}{|\pi_i|} \gtrsim 2p^{-1/2}$$

hence $|(G_i \circ H)_x(\tilde{x})| > \frac{3}{2}p^{-1/2}$, where $\tilde{x}$ is defined as in Lemma 68. Write

$$|(G_i \circ H)_x(x)| = |(G_i \circ H)_x(\tilde{x})| \cdot \left|\frac{G_{i,x}(Hx)}{G_{i,x}(H\tilde{x})}\right| \cdot \left|\frac{H_x(x)}{H_x(\tilde{x})}\right| > \frac{|x|}{|\tilde{x}|}p^{-1/2}.$$

It is easy to see that $|\frac{x}{\tilde{x}}| > \hat{\omega}^{1/2}$, proving the Lemma. ▮

Corollary 70.

1.If $\tilde{B} = P_j \circ \cdots \circ P_1$ then $|P_{j,x}| > (4p)^{-1/2}$ for all $x \in P_j^{-1}(\gamma_{n+1})$;
2.if $\tilde{B} = \tilde{F} \circ P_{j-1} \circ \cdots \circ P_1$ then $|P_{j-1,x}| > (10p)^{-1/2}$ for all $x \in P_{j-1}^{-1}(\tilde{\beta}_H)$;
3.more generally, if $F_{H,i+1} \neq Id$ for some $i = 1, \ldots, j-1$ then $|P_{i,x}| > (10p)^{-1/2}$ for all $x \in P_i^{-1}(\pi_{i+1})$.

Proof. Use Lemma 69, with $\hat{\omega} \simeq \frac{1}{2}$ in the first case and $\hat{\omega} > \frac{1}{10}$ in the second and third cases. ▮

Remark. The conclusion for the $P_{i,x}$'s in the Corollary 70 are exactly the same for the corresponding $(G_i \circ H)_x$'s.

Lemma 71. *For all $i = 1, \ldots, j$*

$$\left|\frac{P_{i,a}}{P_{i,x}H'_a}\right| < C\left(\frac{|\gamma_n|}{|\gamma_{n-1}|} + \frac{|\eta^*|}{|\gamma_n|} \cdot |W_{i-1}|^{-1}\right),$$

or even smaller than $C\frac{|\eta^*|}{|\gamma_n|}W_{i-1}^{-1}$ *if* $F_{H,i} = Id$.

Proof. Write $P_i = G_i \circ H \circ F_{H,i}$ and use the Appendix with Lemmas 56, 57, 59, 48 and 49. If $F_{H,i} \neq \mathrm{Id}$ but $d \geq \frac{1}{2}$ then also in this case the bound can be taken as $C\frac{|\eta^*|}{|\gamma_n|}W_{i-1}^{-1}$. ▮

LEMMA 72. *In Case A*

$$\left|\frac{P_{1,a}}{P_{1,x}H'_a}\right| \leq C(r+p).$$

Proof. It is enough to apply Lemma 71 after observing that in Case A, $|W_0|^{-1} < 2p^{1/2}|\eta^*|^{-1/2}$, and since $|\eta^*|^{1/2} \cdot |\gamma_n|^{-1} < p^{1/2}$, by Lemma 55. ▮

LEMMA 73. *In Case B, if* $ImH' \cap \mathcal{U}(\tilde{\beta}) \neq \emptyset$ *then*

$$\left|\frac{P_{i,a}}{P_{i,x}H'_a}\right| < C\frac{|\gamma_{n+1}|}{|\gamma_n|}$$

for all $i = 1, \dots, j$.

Proof. By Lemma 63, $|W_{i-1}| > \frac{1}{3}|\eta^*|^{1/2}$ for all $i = 1, \dots, j$, hence by Lemmas 63 (implying $|\gamma_{n+1}| > \frac{2}{3}|\eta^*|^{1/2}$) and 71 we obtain the inequality. ▮

LEMMA 74. *In Case A, if* $ImH' \cap \pi_1 \neq \emptyset$ *then*

$$|P_{1,x}|^{-1} \cdot \left|\frac{P_{i,a}}{P_{i,x}H'_a}\right| < p$$

for any $i = 1, \dots, j$.

Proof. In Case A, $|P_{1,x}| \leq 2p\omega$, by Lemma 65. On the other hand, as $|W_{i-1}|^{-1} < |\gamma_{n+1}|^{-1} < \omega^{-1/2}|\eta^*|^{1/2}$, by Lemma 64, then the expression above is smaller than

$$2p\omega C(r + \frac{|\eta^*|^{1/2}}{|\gamma_n|}\omega^{-1/2}),$$

which is smaller than p if r and p are sufficiently small. ▮

LEMMA 75. *Let* $1 \leq i \leq j-1$. *If* $|P_{i,x}| \leq 2$ *for some* $x_0 \in P_i^{-1}(\pi_{i+1})$ *then*

1. $|(P_{i+1} \circ P_i)_x| \geq \frac{1}{8}p^{-2}$,

2. $|P_{i,x}|^{-1} \cdot \left|\frac{P_{i+1,a} \circ P_i}{(P_{i+1,x} \circ P_i)H'_a}\right| \leq p$, *for all* $x \in P_i^{-1}(\pi_{i+1})$.

Proof. Let $x \in P_i^{-1}(\pi_{i+1})$ be such that $|P_{i,x}(x)| \leq 2$ and $z = F_{H,i}(x)$. It means, by Lemma 69, that $\hat{\omega}(Hz) \leq 4p$. As the distortion of G_i is small, then we have $\operatorname{dist}(\pi_{i+1}, \partial\gamma_n) \cdot |\gamma_n|^{-1} \lesssim \hat{\omega}$. It is not difficult to see that in fact $\operatorname{dist}(x, \partial\gamma_n) \cdot |\gamma_n|^{-1} \simeq \hat{\omega}$ for all $x \in \pi_{i+1}$. Let us estimate $P_{i+1,x}$ for $x \in \pi_{i+1}$ as a function of $\hat{\omega}$, if $\hat{\omega} < 5p$. First, we note that in this case $H(\pi_{i+1}) = \beta_{i+1}$.

Let $\hat{Y}$ be the distance between the center of β_{i+1} and $T(0)$, $\hat{Z} = Z = |\mathrm{Im}H|$ and $\hat{X} = \hat{Z} - \hat{Y}$. We have $|H_x \circ P_i| \gtrsim 2\hat{Y}^{1/2}$ restricted to $P_i^{-1}(\pi_{i+1})$ and $|B_{i+1,x} \circ H \circ P_i| \simeq |\gamma_n| \cdot |\beta_{i+1}|^{-1} \geq |\gamma_n| p^{-1} \hat{X}^{-1}$. But $\frac{1}{2}|\gamma_n| \simeq \sqrt{\hat{Z}} = \sqrt{\hat{Y} + \hat{X}}$ implies

$$|P_{i+1,x}| = |B_{i+1,x} \cdot H_x| \geq 4p^{-1} \left(\frac{\hat{Y}}{\hat{X}}\right)^{1/2} \sqrt{1 + \frac{\hat{Y}}{\hat{X}}}$$

for all $x \in \pi_{i+1}$. We claim that $\hat{Y}\hat{X}^{-1} \geq (10\hat{\omega})^{-1}$, which implies, by Lemma 69, the first assertion of the Lemma.

To prove the claim, observe only that $Y \geq \left[\sqrt{Z}\,(1 - 4\hat{\omega})\right]^2$.

To prove the second assertion, note that $|P_i| \simeq \frac{1}{2}|\gamma_n|$, so by Lemma 71,

$$\left|\frac{P_{i+1,a} \circ P_i}{(P_{i+1,x} \circ P_i) H'_a}\right| \leq 2C \frac{|\eta^*|}{|\gamma_n|^2},$$

since $F_{H,i} = \mathrm{Id}$. On the other hand, according to Lemma 66, as there is $x \in \pi_i$ such that $|P_{i,x}| \leq 2$ then η_i cannot be subordinated to η^*, hence by Lemma 67

$$|P_{i,x}| \geq p^{-1} \frac{|\eta^*|^{1/2}}{|\gamma_n|}$$

and the result follows, since $|\eta^*|^{1/2} \cdot |\gamma_n|^{-1} < p^{1/2}$ and p is small. ∎

Remark. The first assertion of Lemma 75 can be stated for $(G_i \circ H)$ instead of P_i in the following way. If $|(G_i \circ H)_x| \leq 2$ for some x_0 such that $G_i \circ H(x_0) \in \pi_{i+1}$ then

$$|(G_{i+1} \circ H)_x \cdot (G_i \circ H)_x| \geq \frac{1}{8} p^{-2},$$

where $G_{i+1} \circ H$ is taken at $G_i \circ H(x_0)$ (in fact, by the proof of Lemma 75 we see that $G_{i+1} \circ H = P_{i+1}$).

Corollary 76. *For all* $1 \leq i \leq j$ *and* $0 \leq i_0 \leq j - i$ *let* $\Delta_{i,i_0} \equiv \left|\prod_{t=i}^{i+i_0} P_{t,x} \circ A_{t-1}\right|$.

1.if $|P_{i+i_0,x}| \geq 2$ *then* $\Delta_{i,i_0} \geq 2^{i_0}$;
2.if $|P_{i+i_0,x}| < 2$ *then* $\Delta_{i,i_0} \geq 2^{i_0-1}p^{-1}|\eta^*|^{1/2}|\gamma_n|^{-1}$;
3.if $|P_{i+i_0,x}| \geq \frac{1}{4}p^{-1/2}$ *then* $\Delta_{i,i_0} \geq \frac{1}{4} \cdot 2^{i_0-1}p^{-1/2}$;
4.in any case $\Delta_{i,i_0} \geq 2^{i_0-1}p^{-1}|\eta^*|^{1/2}|\gamma_n|^{-1}$.

Proof. According to Lemma 75(1) if $|P_{t,x}| < 2$ then $|P_{t+1,x}| \geq 2$, moreover $|P_{t,x} \cdot P_{t+1,x}| \geq \frac{1}{8}p^{-2} \geq 2^2$. Hence (1) is immediate. If $|P_{i+i_0,x}| < 2$ then by Lemmas 66 and 67 we must have $|P_{i+i_0,x}| \geq p^{-1}|\eta^*|^{1/2}|\gamma_n|^{-1}$. As $|P_{i+i_0-1,x}| \geq 2$ then by item (1) we must have $\Delta_{i,i_0-1} \geq 2^{i_0-1}$ and item (2) follows. If $|P_{i+i_0,x}| \geq \frac{1}{4}p^{-1/2}$ then in particular $|P_{i+i_0,x}| \geq 2$. There are two possibilities: if $|P_{i+i_0-1,x}| \geq 2$ then $\Delta_{i,i_0-1} \geq 2^{i_0-1}$ and item (3) follows, otherwise $|P_{i+i_0-1,x}| < 2$ implies $|P_{i+i_0-2,x}| \geq 2$ and $\Delta_{i,i_0-2} \geq 2^{i_0-2}$; moreover, $|P_{i+i_0-1,x} \cdot P_{i+i_0,x}| \geq \frac{1}{8}p^{-2} \geq p^{-1/2}$ and item (3) follows as well. Finally, similar reasonings and the fact that the estimate of Lemma 67 is always worse than the estimate of Lemma 66 imply item (4). ∎

Remark. The Lemma is also valid if we define

$$\Delta_{i,i_0} \equiv \left| \prod_{t=i}^{i+i_0} (G_t \circ H)_x(W_{t-1}) \right|$$

and consider $|(G_{i+i_0} \circ H)_x|$ instead of $|P_{i+i_0,x}|$, according to the Remark following Lemma 75.

Corollary 77. $|\tilde{B}_x| \geq \frac{4}{3}$.

Proof. Just apply Corollary 70 and 76(3), for the two types of decomposition of $\tilde{B}$, as well as the results of Subsection 3.3. ∎

3.7. Derivatives of the P_i's

To prove the Lemmas below it is enough to carefully apply the Appendix to $P_i = G_i \circ H \circ F_{H,i}$, with the help of the Lemmas proven in the preceding subsections. The inequalities involving $\tilde{F}$ are always consequences of Lemmas 48 (for $d < \frac{1}{2}$) and 49 (for $d \geq \frac{1}{2}$) together with $|\gamma_{n+1}| \cdot \frac{1}{2}d^{-1/2} < q^{1/2}$.

Lemma 78. *For all* $i = 1, \ldots, j$

$$|\gamma_{n+1}| \cdot \left| \frac{P_{i,xx}}{(P_{i,x})^2} \right| < 4\frac{|\gamma_n|}{|\gamma_{n-1}|} + 2\delta_1 \frac{|\gamma_{n+1}|}{|\gamma_n|} + 2|(G_i \circ H)_x|^{-1},$$

but the first term of the R.H.S can be omitted if $F_{H,i} = Id$. *Moreover,*

$$|\gamma_{n+1}| \cdot \left| \frac{\tilde{F}_{xx}}{(\tilde{F}_x)^2} \right| < C\frac{|\gamma_{n+1}|}{d^{1/2}} < Cq^{1/2}.$$

LEMMA 79. *For all $i = 1, \dots, j$*

$$|\gamma_{n+1}|^2 \left| \frac{P_{i,xxx}}{(P_{i,x})^3} \right| < \frac{\delta_1}{4}(1 + |(G_i \circ H)_x|^{-1} + |(G_i \circ H)_x|^{-2}).$$

Moreover,

$$|\gamma_{n+1}|^2 \cdot \left| \frac{\tilde{F}_{xxx}}{(\tilde{F}_x)^3} \right| < Cq.$$

LEMMA 80. *For all $i = 1, \dots, j$*

$$|\gamma_{n+1}| \cdot \left| \frac{P_{i,xa}}{(P_{i,x})^2 H'_a} \right| < \Theta^{xa}_{G,i} + \Theta^{xa}_{H,i} \ ,$$

where

$$\Theta^{xa}_{G,i} = (\delta_1 + \delta_0) C \frac{|\eta^*|}{|\gamma_n|^2} \left(1 + |(G_i \circ H)_x|^{-1} \right)$$

and, if $F_{H,i} \neq Id$,

$$\Theta^{xa}_{H,i} = C(\frac{|\gamma_n|}{|\gamma_{n-1}|} + \frac{|\eta^*|}{|\gamma_n|})(\delta_1 \frac{|\gamma_{n+1}|}{|\gamma_n|} + |(G_i \circ H)_x|^{-1}) \ ,$$

otherwise $\Theta^{xa}_{H,i} = 0$. Moreover,

$$|\gamma_{n+1}| \cdot \left| \frac{\tilde{F}_{xa}}{(\tilde{F}_x)^2 H'_a} \right| < Cq^{1/2}$$

LEMMA 81. *For all $i = 1, \dots, j$*

$$|\gamma_{n+1}|^2 \cdot \left| \frac{P_{i,xxa}}{(P_{i,x})^3 H'_a} \right| < \Theta^{xxa}_{G,i} + \Theta^{xxa}_{H,i} \ ,$$

where

$$\Theta^{xxa}_{G,i} = (\delta_1 + \delta_0) C \frac{|\eta^*|}{|\gamma_n|^2} \left(\frac{|\gamma_{n+1}|}{|\gamma_n|} + |(G_i \circ H)_x|^{-1} \right)$$

and

$$\Theta^{xxa}_{H,i} = 1 + |(G_i \circ H)_x|^{-1} + |(G_i \circ H)_x|^{-2}$$

if $F_{H,i} \neq Id$, otherwise $\Theta^{xxa}_{H,i} = 0$. Moreover,

$$|\gamma_{n+1}|^2 \cdot \left| \frac{\tilde{F}_{xxa}}{(\tilde{F}_x)^3 H'_a} \right| < Cq.$$

LEMMA 82.

$$|\gamma_{n+1}| \cdot \left| \frac{P_{i,aa}}{(P_{i,x})^2 (H'_a)^2} \right| < \Theta^{aa}_{G,i} + \Theta^{aa}_{H,i} \ ,$$

where

$$\Theta^{aa}_{G,i} = (\delta_1 + \delta_0) C \frac{|\eta^*|^2}{|\gamma_n|^3} (1 + |(G_i \circ H)_x|^{-1} + |W_{i-1}|^{-1})$$

and

$$\Theta^{aa}_{H,i} = (C\delta_1) \frac{|\eta^*|^2}{|\gamma_n|^3} + C|(G_i \circ H)_x|^{-1} \ .$$

Moreover,

$$|\gamma_{n+1}| \cdot \left| \frac{\tilde{F}_{aa}}{(\tilde{F}_x)^2 (H'_a)^2} \right| < C q^{1/2} \ .$$

3.8. Proof of the differentiable conditions

Now we are ready to prove the induction for the quotients involving the derivatives of $\tilde{B}$. The estimates are valid for every $x \in \tilde{\beta}$ and we always assume that a is such that $\mathrm{Im} H' \cap \mathcal{U}(\tilde{\beta}) \neq \emptyset$ (even if sometimes this condition is not used).

From now on we define

$$\Delta^{-1}_{i,i+i_0} \equiv \left| \prod_{t=i}^{i+i_0} (G_t \circ H) \right|^{-1} .$$

LEMMA 83. *For all $i = 1, \ldots, j$,*

$$\left| \frac{A_{i,a}}{A_{i,x} H'_a} \right| < \delta_1.$$

Proof. Based on the Appendix we write, if $\tilde{B} = P_j \circ \cdots \circ P_1$,

$$\frac{A_{i,a}}{A_{i,x} H'_a} = \frac{P_{1,a}}{P_{1,x} H'_a} + \sum_{t=2}^{i} (A_{t-1,x})^{-1} \frac{P_{t,a}}{P_{t,x} H'_a}.$$

This expression holds for $i < j$ or for $i = j$ if $\tilde{B} = P_j \circ \cdots \circ P_1$. If $i = j$ and $\tilde{B} = \tilde{F} \circ P_{j-1} \circ \cdots \circ P_1$ we change the term in $t = j$ by

$$(A_{j-1,x})^{-1} \frac{\tilde{F}_a}{\tilde{F}_x H'_a},$$

which is smaller than $Cp^{1/2}$, using Corollaries 70 and 76 for $A_{j-1,x}$ and for the rest Lemma 48 and Corollary 58, if $d < 1/2$, or else Lemmas 49 and 57 if $d \geq 1/2$. Returning to the first expression, we suppose first that we are in Case B. Using Lemma 73 and Corollary 76(4) we obtain the bound $C(r+p)$.

In Case A, $C(r+p)$ bounds the first term, by Lemma 72. Next, we use conveniently Lemmas 74 and 75(2), in the following way. For $t > 2$, if $|P_{t-1,x}| \geq 2$ then by Corollary 76(1) $|P_{t-1,x} \cdots P_{2,x}|^{-1} \leq 2^{-t+2}$, so that by Lemma 74

$$\left| (A_{t-1,x})^{-1} \frac{P_{t,a}}{P_{t,x} H'_a} \right| \leq 2^{-t+2} p.$$

Otherwise, by Lemma 75(1) we have $|P_{t-2,x}| \geq 2$, and so $|P_{t-2,x} \cdots P_{1,x}|^{-1}$ is smaller or equal than 2^{-t+2}, by Corollary 76(1). Now we can use Lemma 75(2) to obtain exactly the same expression as above. The Lemma follows if p and r are sufficiently small with respect to δ_1. ∎

Lemma 84. *For all $i = 1, \ldots, j-1$ or for $i = j$ and $\tilde{B} = P_j \circ \cdots \circ P_1$,*

$$|\gamma_{n+1}| \cdot \left| \frac{A_{i,xx}}{(A_{i,x})^2} \right| < 2 \sum_{t=1}^{i} \Delta_{t+1,i}^{-1} + \Delta_{t,i}^{-1}.$$

Proof. By the Appendix

$$\frac{A_{i,xx}}{(A_{i,x})^2} = \sum_{t=1}^{i} \frac{A_{t,x}}{A_{i,x}} \cdot \frac{P_{t,xx}}{(P_{t,x})^2}$$

and the Lemma immediately follows using Lemma 78. We use also that

$$|\gamma_{n+1}| \cdot \left| \frac{\tilde{F}_{xx}}{(\tilde{F}_x)^2} \right| < C \frac{|\gamma_{n+1}|}{d^{1/2}} < C q^{1/2}.$$

∎

Corollary 85.

$$|\gamma_{n+1}| \cdot \left| \frac{\tilde{B}_{xx}}{(\tilde{B}_x)^2} \right| < \delta_1$$

Proof. We take $i = j$ in the previous Lemma, if $\tilde{B} = P_j \circ \cdots \circ P_1$, and use Corollaries 70 and 76. If $\tilde{B} = \tilde{F} \circ P_{j-1} \circ \cdots \circ P_1$ the procedure is analogous. ∎

LEMMA 86.

$$|\gamma_{n+1}|^2 \left| \frac{\tilde{B}_{xxx}}{(\tilde{B}_x)^3} \right| < \delta_1$$

Proof. Suppose $\tilde{B} = P_j \circ \cdots \circ P_1$, without loss of generality. Using the Appendix we write

$$|\gamma_{n+1}|^2 \frac{\tilde{B}_{xxx}}{(\tilde{B}_x)^3} = \sum_{i=1}^{j} |\gamma_{n+1}|^2 \frac{P_{i,xxx}}{(P_{i,x})^3} \frac{1}{(\tilde{B}_{i+1,x})^2} +$$
$$+3 \sum_{i=2}^{j} |\gamma_{n+1}| \frac{P_{i,xx}}{(P_{i,x})^2} \cdot |\gamma_{n+1}| \frac{A_{i-1,xx}}{(A_{i-1,x})^2} \cdot \frac{1}{\tilde{B}_{i+1,x} \tilde{B}_{i,x}}.$$

The first sum is bounded by

$$\frac{\delta_1}{4} \sum_{i=1}^{j} (\Delta_{i+1,j}^{-1} + \Delta_{i,j}^{-1})^2 ,$$

which is smaller than $6\delta_1 p$, following the remark just after Corollary 76. By Lemmas 79 and 84 the second sum is bounded by

$$12 \sum_{i=2}^{j} \sum_{t=1}^{i-1} (\Delta_{i,j}^{-1} + \Delta_{i+1,j}^{-1})(\Delta_{t+1,j}^{-1} + \Delta_{t+1,j}^{-1}) ,$$

which is smaller than, for example, $3000p$, using Corollary 76 and the Remark following it. The Lemma follows if p is small. ▮

LEMMA 87. *For all $i = 1, \ldots, j-1$ or for $i = j$ and $\tilde{B} = P_j \circ \cdots \circ P_1$,*

$$|\gamma_{n+1}| \cdot \left| \frac{A_{i,xa}}{(A_{i,x})^2 H_a'} \right| < \frac{\delta_1}{5} \left(1 + \sum_{t=2}^{i} \Delta_{t,i}^{-1} \right) + Ci\Delta_{1,i}^{-1}.$$

Proof. By the Appendix

$$\frac{A_{i,xa}}{(A_{i,x})^2} = \sum_{t=1}^{i} \frac{P_{t,x}}{A_{i,x}} \cdot \frac{P_{t,xa}}{(P_{t,x})^2} + \sum_{t=2}^{i} \frac{A_{t-1,a}}{A_{t-1,x}} \cdot \frac{P_{t,xx}}{(P_{t,x})^2} \cdot \frac{1}{P_{i,x} \cdots P_{t+1,x}}$$

where $P_{i,x} \cdots P_{t+1,x} \equiv 1$ if $t = i$. Let us first analyse the second term of the sum, multiplied by $|\gamma_{n+1}| \cdot |H_a'|^{-1}$. By Lemmas 83 and 78 it is bounded by

$$\frac{\delta_1}{20}\left(1+\sum_{t=2}^{i}\Delta_{t,i}^{-1}\right).$$

For the first term of the sum we use Lemma 80 and Corollary 76 (and its Remark) a few times to get, if p is small,

$$\Delta_{t+1,i}^{-1}\Delta_{1,t-1}^{-1}\Theta_{G,t} < 2^{-i}p$$

and

$$\Delta_{t+1,i}^{-1}\Delta_{1,t-1}^{-1}\Theta_{H,t} < p\Delta_{t+1,i}^{-1} + C\Delta_{1,i}^{-1} \ ,$$

and the Lemma follows. ▌

Corollary 88.

$$|\gamma_{n+1}|\cdot\left|\frac{\tilde{B}_{xa}}{(\tilde{B}_x)^2H_a'}\right| < \delta_1$$

Proof. If $\tilde{B} = P_j\circ\cdots\circ P_1$ it is enough to apply Lemma 87 with $i=j$ and use Corollary 76. If $\tilde{B} = \tilde{F}\circ P_{j-1}\circ\cdots\circ P_1$ it is a combination of Lemma 87 with $i=j-1$, Corollaries 70(2) and 76 and Lemma 80. ▌

Lemma 89.

$$|\gamma_{n+1}|^2\left|\frac{\tilde{B}_{xxa}}{(\tilde{B}_x)^3H_a'}\right| < \delta_1$$

Proof. We write

$$|\gamma_{n+1}|^2\frac{\tilde{B}_{xxa}}{(\tilde{B}_x)^3H_a'} = S_1+S_2+S_3+S_4+S_5$$

where

$$S_1 = \sum_{i=1}^{j}|\gamma_{n+1}|^2\frac{P_{i,xxa}}{(P_{i,x})^3H_a'}\cdot\frac{1}{(\tilde{B}_{i+1,x})^2A_{i-1,x}} \ ,$$

$$S_2 = \sum_{i=2}^{j}|\gamma_{n+1}|^2\frac{P_{i,xxx}}{(P_{i,x})^3}\cdot\frac{A_{i-1,a}}{A_{i-1,x}H_a'}\cdot\frac{1}{(\tilde{B}_{i+1,x})^2} \ ,$$

$$S_3 = 2\sum_{i=2}^{j}|\gamma_{n+1}|\frac{P_{i,xx}}{(P_{i,x})^2}\cdot|\gamma_{n+1}|\frac{A_{i-1,xa}}{(A_{i-1,x})^2H_a'}\cdot\frac{1}{\tilde{B}_{i+1,x}\tilde{B}_{i,x}} \ ,$$

$$S_4 = \sum_{i=2}^{j} |\gamma_{n+1}| \frac{P_{i,xa}}{(P_{i,x})^2 H'_a} \cdot |\gamma_{n+1}| \frac{A_{i-1,xx}}{(A_{i-1,x})^2} \cdot \frac{1}{(\tilde{B}_{i+1,x})^2 A_{i,x}} \ ,$$

$$S_5 = \sum_{i=2}^{j} |\gamma_{n+1}| \frac{P_{i,xx}}{(P_{i,x})^2} \cdot |\gamma_{n+1}| \frac{A_{i-1,xx}}{(A_{i-1,x})^2} \cdot \frac{A_{i-1,a}}{A_{i-1,x} H'_a} \cdot \frac{1}{\tilde{B}_{i+1,x} \tilde{B}_{i,x}} \ .$$

Special attention must be paid to S_1 and S_4, since in these cases one needs to distinguish between $F_{H,i} = \mathrm{Id}$ and $F_{H,i} \neq \mathrm{Id}$. If $F_{H,i} \neq \mathrm{Id}$,

$$|S_1| < \sum_{i=1}^{j} (\Delta^{-1}_{i+1,j})^2 \Delta^{-1}_{1,i-1} + \Delta^{-1}_{1,j} \Delta^{-1}_{i+1,j} + \Delta^{-1}_{1,j} \Delta^{-1}_{i,j}.$$

By Corollaries 70(3) and 76(3), $\Delta^{-1}_{1,i-1} \leq p^{1/2}$. The remaining Δ's are well controlled by Corollaries 70(1) and 76(3) and after all $|S_1|$ can be bound by a multiple of p. If $F_{H,i} = \mathrm{Id}$ then the 'problematic' term

$$(\delta_1 + \delta_0) C \frac{|\eta^*|}{|\gamma_n|^2} \sum_{i=1}^{j} (\Delta^{-1}_{i+1,j})^2 \Delta^{-1}_{1,i-1}$$

is also bounded by a multiple of p (for p small, of course, to kill constants), since by Corollary 76(4) $\Delta^{-1}_{1,i-1} \leq p|\gamma_n| \cdot |\eta^*|^{-1/2}$. For S_4 we proceed similarly. The sums S_2, S_3 and S_5 are easier, and also the case where $\tilde{B} = \tilde{F} \circ P_{j-1} \circ \cdots \circ P_1$. ∎

LEMMA 90.

$$|\gamma_{n+1}| \cdot \left| \frac{\tilde{B}_{aa}}{(\tilde{B}_x)^2 (H'_a)^2} \right| < \delta_1$$

Proof. By the Appendix

$$\frac{\tilde{B}_{aa}}{(\tilde{B}_x)^2} = S_1 + S_2 + S_3 \ ,$$

where

$$S_1 = \sum_{i=1}^{j} |\gamma_{n+1}| \frac{P_{i,aa}}{(P_{i,x})^2 (H'_a)^2} \cdot \frac{1}{\tilde{B}_{i+1,x} (A_{i-1,x})^2} \ ,$$

$$S_2 = 2 \sum_{i=2}^{j} |\gamma_{n+1}| \frac{P_{i,xa}}{(P_{i,x})^2 H'_a} \cdot \frac{A_{i-1,a}}{A_{i-1,x} H'_a} \cdot \frac{1}{\tilde{B}_{i+1,x} A_{i-1,x}} \ ,$$

$$S_3 = \sum_{i=2}^{j} |\gamma_{n+1}| \frac{P_{i,xx}}{(P_{i,x})^2} \cdot \left(\frac{A_{i-1,a}}{A_{i-1,x} H'_a} \right)^2 \cdot \frac{1}{\tilde{B}_{i+1,x}} \ .$$

The sums S_2 and S_3 and part of S_1 are treated in the same way as in the preceding Lemmas, with an eventual distinction between $F_{H,i} = \mathrm{Id}$ and $F_{H,i} \neq \mathrm{Id}$. The only new problem comes from S_1, where we have to bound

$$\sum_{i=1}^{j} C(\delta_1 + \delta_0) \frac{|\eta^*|^2}{|\gamma_n|^3} |W_{i-1}|^{-1} \Delta_{i+1,j}^{-1} (\Delta_{1,j-1}^{-1})^2 \ .$$

In Case B, $|W_{i-1}|^{-1} < 5|\eta^*|^{-1/2}$, hence using Corollary 76(4)

$$\frac{|\eta^*|^2}{|\gamma_n|^3} |W_{i-1}|^{-1} (\Delta_{1,i-1}^{-1})^2 < 5p^{5/2} \ .$$

In Case A we use $\Delta_{1,1}^{-1} \leq 2p\omega$ (by Lemma 65) and $\gamma_{n+1} \gtrsim \omega^{1/2} |\eta^*|^{1/2}$ (by Lemma 64), hence $|W_{i-1}|^{-1} \lesssim 2\omega^{-1/2} |\eta^*|^{-1/2}$. Then, as $\Delta_{1,i-1}^{-1} = \Delta_{1,1}^{-1} \Delta_{2,i-1}^{-1}$,

$$\frac{|\eta^*|^2}{|\gamma_n|^3} |W_{i-1}|^{-1} (\Delta_{1,i-1}^{-1})^2 < 8p^{9/2} \omega^{3/2} \ .$$

As $\omega < 1$, $\sum_{i=1}^{j} \Delta_{i+1,j}^{-1} < Cp^{1/2}$ (by Corollaries 70(2) and 76(3)) and p is small with respect to δ_1, the Lemma follows. The case where $\tilde{B} = \tilde{F} \circ P_{j-1} \circ \cdots \circ P_1$ is analogous. ▮

APPENDIX: GLOSSARY OF FORMULA

A.1. ITERATES NEAR THE SADDLE–NODE

The formula below are used in the context of Subsection 3.2. We use the following notational conventions: H_x^j means the x–partial derivative of the function H^j defined by induction by $H^j(a,x) = H(a, H^{j-1}(a,x))$. We consider $x \in [H^{i+1}(a, x_r), H^i(a, x_r)]$, for i such that $H_x(a, H^i(a, x_r)) < 1$ (the remaining cases are similar). In this case, we write $H^j(a,x) = \Psi(-\sigma_l(a) + t_i(a,x), a, x_l)$, where Ψ is the flow of the field X and the functions σ_l and t_i are explained in the referred Subsection. This expression is used for H_a^j and H_{aa}^j. When there is no argument it means that the derivative is taken in (a,x). For the field X, X itself means $X(a,x)$ and $X \circ H^j$ means $X(a, H^j(a,x))$. Finally, Ψ is evaluated always at $(-\sigma_l(a) + t_i(a,x), a, x_l)$

(as well as its partial derivatives).

$$H^j_x = \frac{X \circ H^j}{X} \tag{A.1}$$

$$H^j_{xx} = \frac{X \circ H^j}{X^2}\left(X_x \circ H^j - X_x\right) \tag{A.2}$$

$$\begin{aligned} H^j_{xxx} = \tfrac{X\circ H^j}{X^3}\big[(X_x \circ H^j)^2 & + 2(X_x)^2 - 3X_x \cdot X_x \circ H^j + \\ & + X \circ H^j \cdot X_{xx} \circ H^j - X \cdot X_{xx}\big] \end{aligned} \tag{A.3}$$

$$H^j_a = X \circ H^j \cdot \left[-\dot{\sigma}_l + \frac{\partial t_i}{\partial a}\right] + \frac{\partial \Psi}{\partial a} \tag{A.4}$$

$$H^j_{xa} = \frac{1}{X}\left(X_a \circ H^j + H^j_a \cdot X_x \circ H^j\right) - \frac{X \circ H^j}{X^2} X_a \tag{A.5}$$

$$\begin{aligned} H^j_{xxa} &= -2\frac{X \circ H^j}{X^3} X_a \left(X_x \circ H^j - X_x\right) + \\ &+ \frac{1}{X^2}\left\{\left[X_a \circ H^j + H^j_a \cdot X_x \circ H^j\right] \cdot \left[X_x \circ H^j - X_x\right]\right\} + \\ &+ \frac{1}{X^2}\left\{X \circ H^j \cdot \left[X_{xa} \circ H^j + H^j_a \cdot X_{xx} \circ H^j - X_{xa}\right]\right\} \end{aligned} \tag{A.6}$$

$$\begin{aligned} H^j_{aa} &= \frac{\partial^2 \Psi}{\partial a^2} + \left[-\ddot{\sigma}_l + \frac{\partial^2 t_i}{\partial a^2}\right] \cdot X \circ H^j + \\ &+ 2\left[-\dot{\sigma}_l + \frac{\partial t_i}{\partial a}\right] \cdot \left[X_a \circ H^j + H^j_a \cdot X_x \circ H^j\right] \end{aligned} \tag{A.7}$$

A.2. COMPOSITIONS OF DIFFEOMORPHISMS.

Let $G_i \equiv F_i \circ \ldots \circ F_1$, $G = G_r$ for $r > 1$ and $Q_i = F_r \circ \ldots \circ F_i$.

$$\frac{G_a}{G_x} = \sum_{i=1}^{r} \frac{F_{i,a}}{G_{i,x}} \tag{A.8}$$

$$\frac{G_{xx}}{(G_x)^2} = \sum_{i=1}^{r} \frac{F_{i,xx}}{Q_{i+1,x}(F_{i,x})^2} \tag{A.9}$$

$$\frac{G_{xa}}{(G_x)^2} = \sum_{i=1}^{r} \frac{F_{i,xa}}{F_{i,x}G_{r,x}} + \sum_{i=2}^{r} \frac{F_{i,xx}}{Q_{i+1,x}(F_{i,x})^2} \cdot \frac{G_{i-1,a}}{G_{i-1,x}} \tag{A.10}$$

$$\frac{G_{aa}}{(G_x)^2} = \sum_{i=1}^{r} \frac{F_{i,aa}}{G_{i,x}G_{r,x}} + \\ +2\sum_{i=2}^{r} \frac{F_{i,xa}}{F_{i,x}G_{r,x}} \cdot \frac{G_{i-1,a}}{G_{i-1,x}} + \sum_{i=2}^{r} \frac{F_{i,xx}}{Q_{i+1,x}(F_{i,x})^2} \cdot \left(\frac{G_{i-1,a}}{G_{i-1,x}}\right)^2 \tag{A.11}$$

$$\frac{G_{xxx}}{(G_x)^3} = \sum_{i=1}^{r} \frac{F_{i,xxx}F_{i,x}}{(Q_{i+1,x})^2(F_{i,x})^4} + \\ +3\sum_{i=2}^{r} \frac{F_{i,xx}}{Q_{i+1,x}(F_{i,x})^2} \cdot \frac{1}{Q_{i,x}} \frac{G_{i-1,xx}}{(G_{i-1,x})^2} \tag{A.12}$$

$$\frac{G_{xxa}}{(G_x)^3} = \sum_{i=1}^{r} \frac{F_{i,xxa}}{G_{r,x}Q_{i+1,x}(F_{i,x})^2} + \\ + \sum_{i=2}^{r} \frac{F_{i,xxx}F_{i,x}}{(Q_{i+1,x})^2(F_{i,x})^4} \cdot \frac{G_{i-1,a}}{G_{i-1,x}} + \sum_{i=2}^{r} \frac{F_{i,xa}}{F_{i,x}G_{r,x}} \cdot \frac{1}{Q_{i,x}} \frac{G_{i-1,xx}}{(G_{i-1,x})^2} + \\ +2\sum_{i=2}^{r} \frac{F_{i,xx}}{Q_{i+1,x}(F_{i,x})^2} \cdot \left[\frac{2}{Q_{i,x}} \frac{G_{i-1,xa}}{(G_{i-1,x})^2} + \frac{G_{i-1,a}}{G_{i-1,x}} \cdot \frac{1}{Q_{i,x}} \frac{G_{i-1,xx}}{(G_{i-1,x})^2}\right] \tag{A.13}$$

A.3. CRITICAL COMPOSITIONS.

Let $H' = G^* \circ H$. Then

$$H'_a = G^*_a + G^*_x H_a \tag{A.14}$$

$$H'_{xx} = G^*_{xx}(H_x)^2 + G^*_x H_{xx} \tag{A.15}$$

$$H'_{xxx} = G^*_x H_{xxx} + G^*_{xxx}(H_x)^3 + 3H_x H_{xx} G^*_{xx} \tag{A.16}$$

$$H'_{xa} = H_{xa}G^*_x + H_x G^*_{xa} + H_x H_a G^*_{xx} \tag{A.17}$$

$$H'_{aa} = G^*_x H_{aa} + G^*_{aa} + G^*_{xa} H_a + G^*_{xx} (H_a)^2 \tag{A.18}$$

$$\begin{aligned} H'_{xxa} &= G^*_x H_{xxa} + G^*_{xxa} (H_x)^2 + G^*_{xxx} H_a (H_x)^2 + \\ &+ 2G^*_{xx} H_{xa} H_x + G^*_{xa} H_{xx} + G^*_{xx} H_a H_{xx} \end{aligned} \tag{A.19}$$

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