QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **2**, 79–92 (2001)
ARTICLE NO. 13

Markov Type Inequality for Functional Determinants

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Submitted by J.P. Françoise

We generalize the classical Markov inequality, relating the maximum of the polynomial and the maximum of its derivative on the interval, in two directions: first we replace polynomials by functions in "Bernstein classes" (studied in [4],[5]). Second, and more important generalization is that similar inequality is established for Wronskians of different order; of the system of polynomials (or function in a given Bernstein class).

1. INTRODUCTION

The classical Markov inequality [3] bounds the maximum modulus of the derivative of the polynomial f on the interval through the maximum modulus of the polynomial itself on the same interval :

$$\max_{x\in[-1,1]} |f'(x)| \leq d^2 \max_{x\in[-1,1]} |f(x)|, \tag{1}$$

where d is the degree of the polynomial f.

First of all, the mere existence of the bound of this form, with a certain constant, depending only on d, is immediate ; the two norms

$$||f||_1 = \max_{x\in[-1,1]} |f(x)| \quad and \quad ||f||_2 = ||f||_1 + \max_{x\in[-1,1]} |f'(x)|$$

on the finite -dimensional space of all polynomials of degree d, are equivalent. Hence $||f||_2 \leq K_d ||f||_1$, and $\max |f'(x)| \leq (K_d - 1) \max |f(x)|$.

[1]This research was supported by the Israel Science Foundation, Grant No. 117/99-1 and by the Minerva Foundation.

The same argument allows to get (nonsharp!) versions of the Bernstein inequality [1], relating the maximum of f on [-1,1] and on the cofocused ellipse, and of many similar results.

The constant d^2 in (1) is sharp and it is attended on f_d -a Chebyshev polynomial of degree d .

Notice also that (1) has no "pointwise" consequence : f' does not have to vanish at zeroes of f.

In this paper we suggest a generalization of 1 in two directions; first, instead of functions and their derivatives we consider Wronskians of various orders of a system of functions. Secondly , we allow these functions and Wronskians to be not necessary polynomials, but rather functions with known "Bernstein constants" (which is the ratio of the modulus of the function on two concentric disks in $\mathbb{C}$; such classes have been considered in [5, 6]).

These generalizations are motivated by applications to linear ODE's, some of which we shortly present in the last section.

Definition 1. Let $f_1, ..., f_n$ be analytic functions and $I = (i_1, \cdots, i_n)$ $i_1 \geqslant 0, \cdots, i_n \geqslant 0$ be a multi-index. We define

$$W_{n,I}(f_1, \cdots, f_n) := \ det \begin{pmatrix} f_1^{(i_1)} & \cdots & f_n^{(i_1)} \\ \vdots & & \vdots \\ f_1^{(i_n)} & \cdots & f_n^{(i_n)} \end{pmatrix} \quad , \quad where$$

$f^{(j)}$ denotes the j-th order derivative of f.

If $I = (0, 1, \cdots, n-1)$, $W_{n,I}$ is equal to the usual Wronskian, which we denote as W_n.

$$W_n(f_1, \cdots . f_n) := \ det \begin{pmatrix} f_1 & \cdots & f_n \\ & & \\ f_1' & & f_n' \\ \vdots & & \vdots \\ & & \\ f_1^{(n-1)} & & f_n^{(n-1)} \end{pmatrix} .$$

Our main result is the inequality of the form

$$\max_{|x| \le r} |W_{n,I}(x)| \le C \max_{|x| \le r} |W_n(x)|, \tag{2}$$

with $x \in \mathbb{C}$ and for any multi-index I.

Let us discuss shortly the structure of the inequality (2) and its relation to the classical Markov inequality (1).

For $n = 1$, $W = f$, and $W_I = f^{(i)}$; for f a polynomial we get the usual Markov inequality (for any order of the derivative; notice, that our constants are not sharp). However, for f in Bernstein classes we get a useful, although simple, new fact.

For $n \geqslant 2$ let us give a "geometric explanation" of (2) in the spirit of the first remark above. $W_n(f_1, ..., f_n) \equiv 0$ if and only if the functions $f_1, ..., f_n$ are linearly dependent. In this case any Wronskian $W_{n,I}$ also vanishes. Thus vanishing of W implies vanishing of any W_I . On the space V of n-tuples of polynomials $f_1, ..., f_n$ of fixed degrees, $\max_{|x| \leq r} |W(x)|$ vanishes exactly on the subset $\sum$, where $f_1, .., f_n$ (or , equivalently, the vectors of theire coefficients) are linearly dependent. Therefore, $\max_{|x| \leq r} |W_I(x)|$ also vanish on $\sum$, for any I.

Now these maxima are semialgebraic functions on V , so the existence of a certain inequality , bounding $\max |W_I|$ through $\max |W|$ is a general fact in the spirit of the Nullstellensatz (or Lojasiewicz inequality).

However , because of a nonlinear nature of the problem , and since we have to consider semialgebraic functions on V, and not just polynomials , to get in this way any specific inequality of the form (2) would be, probably, not easy.

Below we give an explicit proof of (2), which clarifies the algebraic structure of the Wronskians W_I and the relations between them. This proof extends directly to functions in Bernstein classes.

Also here no pointwise estimate of the form (2) is possible. This fact is closely related to the appearence of singularities in the linear ODE's , generating a given linear family of regular analytic function (see section (4) below).

2. THE POLYNOMIAL CASE

Theorem 2. *Let $f_1(x), ..., f_n(x)$ be polynomials, $x \in \mathbb{C}$. Then for any multy-index $I = (i_1, ..., i_n)$ the following inequality holds:*

$$\max_{|x| \leq r} |W_{n,I}(x)| \leq \gamma \max_{|x| \leq r} |W_n(x)| \ , \tag{3}$$

where γ depends only on the degrees of polynomials, multi-index I and radius r and may be effectively computed .

Remark 3. Unlike the classical Markov inequality the constant in inequality (3), which we obtain, is not sharp.

Remark 4. The replacement of the segment to the disk is not essential.Using the classical Bernstein inequality [1] the inequality on a disk reduces to the inequality on a segment.

The main tecnical point of the proof is that for some classes of functions one can estimate C-norm of the ratio of two functions through the ratio of C-norms of the nominator and the denominator, assuming, that the ratio of the functions also belongs to the same class. The simplest case is when both functions are polynomials and the one is a multiple of the other.

LEMMA 5 (Division Lemma). *Let f, g, and f/g be polynomials of degree n, m, and $n-m$ respectively, then*

$$\max_{|x|\le r} |\frac{f}{g}| \le \gamma(n,m)\ \frac{\max_{|x|\le r}|f|}{\max_{|x|\le r}|g|}\ ,$$

where γ depends only on n, m.

This is a special case of a “Polynomial Hironaka Division Theorem”, see e.g. [2].

Now we present the proof of Theorem 2.

Proof. For the case $n = 1$ we have

$$max_{|x|\le r}|f'(x)| \le c_1 max_{|x|\le r}|f(x)|\ ,$$

and similar inequalities for higher order derivaties.

This follows directly from the classical Markov inequality. Here and below c_i, β, μ, γ denote constants, depending only on the degrees of the polynomials involved and on r. To simplify the presentation we mostly do not compute them explicitly.

(a) Consider the case $n = 2$. This case contains all the main features of the general one, but is much simpler from the technical point of view.

Solving the differential equation

$$f_1 \cdot f_2' - f_1' f_2 = W_2$$

with respect to f_2, one obtains

$$f_2 = C \cdot f_1 + S \cdot f_1\ ,$$

where C is a constant and $S = \int W_2/f_1^2$. One has

$$f_2^{(k)} = C f_1^{(k)} + S \cdot f_1^{(k)} + \sum_{i=1}^{k} C_k^i (W_2/f_1^2)^{(i-1)} f_1^{(k-i)}\ .$$

Putting into $W_{2,I}$ these expressions for the derivative of f_2, one obtains

$$W_{2,I}(f_1,f_2) := \ det\begin{pmatrix} f_1^{(i_1)} & f_2^{(i_1)} \\ f_1^{(i_n)} & f_2^{(i_n)} \end{pmatrix} =$$

$$= \ det\begin{pmatrix} f_1^{(i_1)} & \sum_{j=1}^{i_1} C_{i_1}^{j}(W_2/f_1^2)^{(j-1)}\cdot f_1^{(i_1-j)} \\ f_1^{(i_2)} & \sum_{j=1}^{i_2} C_{i_2}^{j}(W_2/f_1^2)^{(j-1)}\cdot f_1^{(i_2-j)} \end{pmatrix}.$$

Denote $\hat{I} := max(i_1, i_2)$. We obtain

$$W_{2,I} = \ \frac{\phi(W_2,f_1)}{f_1^{2(\hat{I}-1)}} \ .$$

It is easy to see, that $\phi(W_2, f_1)$ is a linear operator with respect to W_2 and a homogeneous function of degree $2(\hat{I}-1)$ with respect to $(f_1, f_1', \cdots, f_1^{(\hat{I})})$. Direct computation, taking into account Markov inequality (1), yields the inequality

$$\max_{|x|\le r}|\phi| \le \beta\cdot \max_{|x|\le r}|W_2|\cdot(\max_{|x|\le r}|f_1|)^{2(\hat{I}-1)}, \tag{4}$$

where β is a constant, depending only on the multyindex I and the degrees of the polynomials f_1, f_2.

Since $W_{2,I}$ is also a polynomial this means, that $f_1^{2(\hat{I}-1)}$ divides ϕ. Therefore according to Lemma 5 ("division lemma")

$$\max_{|x|\le r}|W_{2,I}| \le c_2\frac{\max\limits_{|x|\le r}|\phi|}{\max_{|x|\le r}|f_1|^{2(\hat{I}-1)}} \ . \tag{5}$$

Finally, combining (4) with (5) one obtains the inequality

$$\max_{|x|\le r}|W_{2,I}(x)| \le \gamma_2 \ \max_{|x|\le r}|W_2(x)| \ ,$$

where $\gamma_2 = \beta\cdot c_2$.

(b) Consider now the general case ($n > 2$). We will represent $W_{n,I}$ in the same form as in the case $n = 2$, that is, as the fraction

$$W_{n,I}(x) = \ \frac{\phi(x)}{W_{n-1}^{2(\hat{I}-1)}(x)}, \tag{6}$$

where $\hat{I} = \max(i_1, \cdots, i_n)$, and estimate $|\phi(x)|$ in the same manner:

$$\max_{|x|\le r} |\phi| \le \beta \ \max_{|x|\le r} |W_n| (\max_{|x|\le r} |W_{n-1}|)^{2(\hat{I}-1)}, \tag{7}$$

where β depends only on $r, I,\ deg\ f_i,\ i = 1 \cdots n$. Since the fraction (6) is also a polynomial, we use the "division" lemma above, and bound $\max_{|x|\le r} |W_{n,I}|$ through the value of the fraction $\frac{\max_{|x|\le r} |\phi(x)|}{\max_{|x|\le r} |W_{n-1}^{2(\hat{I}-1)}|}$.

Next, taking onto account (7), one immediately comes to the desired inequality.

We compute β of (7),using induction with respect to Wronskian's order. Assume that the induction hypothesis holds, i.e., for any multi-index I one has

$$\max_{|x|\le r} |W_{n-1,I}| \le \gamma_{n-1} \cdot \max_{|x|\le r} |W_{n-1}| \ ,$$

where γ_{n-1} depends only on I, r and $deg\ f_i,\ i = 1 \cdots n-1$.

Consider the poynomial f_n as the solution of the linear non-homogeneous ODE (with respect to the unknown y)

$$det \begin{pmatrix} f_1 & \cdots & f_{n-1} & y \\ f_1' & \cdots & f_{n-1}' & y' \\ \vdots & & & \\ f_1^{(n-1)} & \cdots & f_{n-1}^{(n-1)} & y^{(n-1)} \end{pmatrix} = W_n \ .$$

The general solution of this equation is given by the formula

$$y = \sum_{i=1}^{n} C_i f_i \ + < F_{n-1} \circ b_n >_1 \ , \tag{8}$$

where C_i are arbitrary constants, and F_{n-1} is the fundamental matrix,

$$F_{n-1} = \begin{pmatrix} f_1 & \cdots & f_{n-1} \\ \vdots & & \\ f_1^{(n-2)} & \cdots & f_{n-1}^{(n-2)} \end{pmatrix},$$

b_n is an integral of the vector $F_{n-1}^{-1} \circ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ W_n/W_{n-1} \end{pmatrix}$, and $< >_1$ means the first component of an n-element vector. Putting the RHS of (8) into $W_{n,I}$ one has

$$W_{n,I} = \ det \begin{pmatrix} f_1^{(i_1)} & \cdots & f_{n-1}^{(i_1)} & < F_{n-1} \circ b_n >_1^{(i_1)} \\ \vdots & & & \\ f_1^{(i_n)} & \cdots & f_{n-1}^{(i_n)} & < F_{n-1} \circ b_n >_1^{(i_n)} \end{pmatrix} =$$

$$= \ det \begin{pmatrix} f_1^{(i_n)} & \cdots & f_{n-1}^{(i_1)} & < \sum_{k=0}^{i_1-1} \begin{pmatrix} k \\ i_1 \end{pmatrix} F_{n-1}^{(k)} \circ b_n^{(i_1-k)} >_1 \\ \vdots & & & \\ f_1^{(i_n)} & \cdots & f_{h-1}^{(i_n)} & < \sum_{k=0}^{i_1-1} \begin{pmatrix} k \\ i_n \end{pmatrix} F_{n-1}^{(k)} \circ b_n^{(i_n-k)} >_1 \end{pmatrix} . \quad (9)$$

The column $(< F_{n-1}^{(i_1)} \circ b_n >_1, \cdots < F_{n-1}^{(i_n)} \circ b_n >_1)$ is a linear combination of the other ones. It makes a zero contribution into the final result; only $b_n^{(k)}$ with $k > 1$ can enter the expression for $W_{n,I}$. One has

$$b_n^{(k)} = F_{n-1}^{-1} \circ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{W_n}{W_{n-1}} \end{pmatrix}^{(k-1)} = \sum_{i=0}^{k-1} \begin{pmatrix} i \\ k-1 \end{pmatrix} (F_{n-1}^{-1})^{(i)} \circ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{W_n}{W_{n-1}} \end{pmatrix}^{(k-1-i)} .$$

Putting this into (9) gives

$$W_{n,I} = \sum_{j=1}^{n} W_{n-1, I_{n-1,\hat{j}}} \cdot \sum_{k_1+k_2+k_3=i_j-1} \begin{pmatrix} k_2+k_3 \\ k_1+k_2+k_3 \end{pmatrix} \cdot \begin{pmatrix} k_2 \\ k_2+k_3 \end{pmatrix} \cdot l_{(k_1,k_2,k_)} \quad (10)$$

where

$$l_{(k_1,k_2,k_)} =< F_{n-1}^{(k_1)} \circ (F_{n-1}^{-1})^{(k_2)} \circ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{W_n}{W_{n-1}} \end{pmatrix}^{(k_3)} >_1 ,$$

where $I_{n-1,\hat{j}} = (i_1, \cdots, i_{j-1}, i_{j+1}, \cdots, i_n)$.

Consider first the matrix

$$S_{k_1,k_2} = F_{n-1}^{(k_1)} \circ (F_{n-1}^{-1})^{(k_2)}.$$

It is not difficult to check, that its elements have the form:

$$\frac{\sum_\alpha \sum_{\alpha_1,\cdots,\alpha_{k_2-1}} C_{\alpha,\alpha_1,\cdots,\alpha_{k_2-1}} \cdot W_{n-1,I_\alpha} \prod_{p=1}^{k_2-1} W_{n-1}^{(\alpha_p)}}{W_{n-1}^{k_2}}$$

with absolute constants $C_{\alpha,\alpha_1,\cdots,\alpha_{k_2-1}}$ and $I_\alpha = (\alpha_1, \cdots, \alpha_{n-1})$, where $\alpha_p \leq \max(k_1, k_2) + n, p = 1, \cdots, n-1$.

Taking into account the previous representation we reduce (10) to the form

$$W_{n,I} = \frac{\phi}{W_{n-1}^{2(\hat{I}-1)}} ,$$

where $\hat{I} = \max(i_i \cdots i_n)$. Now, taking into account the induction hypothesis

$$\max_{|x|\leq r} |W_{n-1,I}| \leq \gamma_{n-1} \max_{|x|\leq r} .|W_{n-1}| ,$$

one arrives to the inequality

$$\max_{|x|\leq r} |\phi| \leq \beta \max_{|x|\leq r} |W_n| \cdot (\max |W_{h-1}|)^{2(\hat{I}-1)} ,$$

where β is an "absolute" constant, which may be computed, via the formula (10). The proof is completed.

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3. THE CASE OF THE FUNCTIONS WITH KNOWN BERNSTEIN CONSTANTS

Definition 6. Let $\Omega \subseteq \mathbb{C}$ be a bounded domain, $K \subseteq \Omega$ a compact, and let f be analytic in Ω and continuous on $\bar{\Omega}$.

We call the ratio

$$B(f, K, \Omega) = \max_{\bar{\Omega}} |f| / \max_{K} |f| ,$$

the Bernstein constant of f on (K, Ω).

Let D_R denote the closed disk of radius $R > 0$, centered at $0 \in \mathbb{C}$.

Definition 7. Let $R > 0$, $0 < \alpha < 1$ and $K > 0$ be given and let f be holomorphic in a neighbourhood of D_R. We say that f belongs to the Bernstein class $B^1_{R,\alpha,K}$ if $\max_{D_R} |f| / \max_{D_{\alpha R}} |f| \leq K$ (or, in other words, if $B(f, \alpha D_R, D_R) \leq K$).

Example 8. One can check that if f is a polynomial of degree n, than for any R, α, as above $f \in B^1_{R,\alpha,1/\alpha^n}$.

Example 9. Let $y(x)$ be an algebraic function, given by an equation :

$$p_d(x)y^d + p_{d-1}(x)y^{d-1} + \cdots + p_1(x)y + p_0(x) = 0 ,$$

with $p_j(x)$ - polynomials in x of degree m. Let $\tilde{y}(x)$ be one of the branches of y and assume that $\tilde{y}$ is regular over D_{2R}. Then $\tilde{y}(x) \in B^1_{R,\alpha,K(\alpha,d,m)}$, for any α, where $K(\alpha, m)$ depends only on α and the degree m, see [5].

Theorem 10. *Let $f_1 \cdots f_n$ be regular functions in the disk $D(0, R)$. Suppose that for some α, $0 < \alpha < 1$,*

$$W_i = W_i(f_1, \cdots, f_i) \in B^1_{R,\alpha,K} \quad for \quad i = 1 \cdots n .$$

Then for any $r < R$ and for any multyindex $I = (i, \cdots, i_n)$ the following inequality holds:

$$\max_{|x| \leq r} |W_{n,I}(x)| \leq \gamma \cdot \max_{|x| \leq r} |W_n(x)| , \tag{11}$$

where $\gamma = \gamma(R, \alpha, r, K, I)$ depends only on parameters given in the brackets and may be effectively computed through them.

Instead of the “division lemma” we need here

Theorem 11 (Division Theorem). *Let f, g and f/g be holomorphic functions in the disk $D(0, R)$, and $g \in B^1_{R,\alpha,K_1}$. Let $r = \alpha R$. Then*

$$\max_{|x| \leq r} |\frac{f}{g}| \leq \mu \cdot \frac{\max_{|x| \leq R} |f|}{\max_{|x| \leq r} |g|}; \tag{12}$$

here μ can be effectively computed through R, α, K_1.

If in addition $f \in B^1_{R,\alpha,K}$, then the inequality (12) turns into inequality

$$\max_{|x|\le r} |\frac{f}{g}| \le K\mu \cdot \frac{\max_{|x|\le r} |f|}{\max_{|x|\le r} |g|} \ . \tag{13}$$

The proof of this theorem also follows from Hironaka division theorem (see [2]), taking into account that for functions in Bernstein classes the norms $\max_{|x|\le r}(\cdot)$ and $\max_{|x|\le R}(\cdot)$ are equivalent. Another proof can be found in [5].

To prove Theorem 10 we also need the following lemma.

Lemma 12. *Let f be a holomorphic function in the disk $D(0,R)$, $f \in B^1_{R,\alpha,K}$. Then for any $R' \le R$ and any $0 < \alpha' < 1$, $f \in B^1_{R',\alpha',K'}$; here $K' = K'(R, \alpha, R', \alpha', K)$ can be explicitely computed through the pharametres, given in the brackets.*

The proof as well as the explicit formula for K', can be founds in [5].

Now we present the proof of Theorem 10.

Proof. For any f analytic in Δ_R by the Cauchy inequality one has for $r < R$ that

$$\max_{|x|\le r} |f^{(i)}(x)| \le \ \frac{K}{2\pi(R-r)^{i+1}} \cdot \max_{|x|\le r} |f(x)| \ , \tag{14}$$

where $K = B(f, D_r, D_R)$ is the Bernstein constant of f on (D_R, D_r) . This replaces Markov inequality for polynomials. Here the dependence on R (or more accurately on $\frac{1}{R-r}$) enters. The proof for the Wronskians of the order $n > 1$ is essentially the same as in the polynomial case (Theorem 2) and is distinguished from this case by using "division theorem" instead of polynomial "division lemma" . We present here only the case $n = 2$.

Exactly as in polynomial case, solving the differential equation

$$f_1 \cdot f_2' - f_1' f_2 = W_2$$

with respect to f_2, and substituting into

$$W_{2,I} = \ det \begin{pmatrix} f_1^{(i_1)} & f_2^{(i_1)} \\ f_1^{(i_2)} & f_2^{(i_2)} \end{pmatrix} ,$$

we obtain

$$W_{2,I} = \frac{\phi(W_2, f_1)}{f_1^{2(\hat{I}-1)}} \ , \quad \text{where} \ \ \hat{I} := \max(i_1, i_2).$$

As above $\phi(W_2, f_1)$ is a linear operator with respect to W_2 and homogeneous function of degree $2(\hat{I}-1)$ with respect to $(f_1, f_1', \cdots, f_1^{(\hat{I})})$. Let us estimate $|\phi|$ on the disk with radius $r_0 = (R+r)/2$. Direct computation, taking into account (14) yields the inequality

$$\max_{|x|\le r_0} |\phi| \le \beta \cdot \max_{|x|\le r_0} |W_2| \cdot (\max_{|x|\le r_0} |f_1|)^{2(\hat{I}-1)}, \tag{15}$$

where

$$\beta = \frac{K_2}{(r-r_0)^{I+1}} \sum_{j=1}^{\hat{I}} \sum_{p=0}^{j-1} \begin{pmatrix} j \\ \\ \hat{I} \end{pmatrix} \begin{pmatrix} p \\ \\ j-1 \end{pmatrix} \frac{(j-p)!(j-p-1)^{j-p-1}}{(2\pi)^{j-p-1}} K_1^{j-p}. \tag{16}$$

Here K_1 and K_2 are the upper bounds for the Bernstein constants

$$B(f_1, D_{r_0}, D_R) \ \text{ and } \ B(W_2, D_{r_0}, D_R) \ ,$$

respectively, given by Lemma 12 through the values R, r, r_0, K. Since $W_{2,I}$ is also holomorhpic this means, that we may apply Theorem 11 to functions ϕ, $W_{2,I}$ and $f_1^{2(\hat{I}-1)}$. Therefore

$$\max_{|x|\le r} |W_{2,I}| \le \mu \cdot \frac{\max_{|x|\le r_0} |\phi|}{\max_{|x|\le r} |f_1|^{2(\hat{I}-1)}} \ , \tag{17}$$

where μ depends only on the values $r_0, \ \ r/r_0, \ \ B(f_1^{2(\hat{I}-1)}, D_r, D_{r_0})$. Finally, combining (15) with (17) and taking into account that

$$\max_{|x|\le r_0} |f_1| \le \hat{K}_1 \cdot \max_{|x|\le r} |f_1| \ \ \textit{and} \ \ \max_{|x|\le r_0} |W_2| \le \hat{K}_2 \cdot \max_{|x|\le r} |W_2|,$$

where

$$\hat{K}_1 = B(f_1, D_r, D_{r_0}) \ \ \textit{and} \ \hat{K}_2 = B(W_2, D_r, D_{r_0})$$

one obtains the inequality

$$\max_{|x|\le r} |W_{2,I}(x)| \le \beta \cdot \mu \cdot \hat{K}_1^{2(\hat{I}-1)} \cdot \hat{K}_2 \ \max_{|x|\le r} |W_2(x)| \ ,$$

where β is given by (16) , $\mu, \hat{K}_1, \hat{K}_2$ are effectively computed through the values R, α, K. ▮

4. APPLICATION: BERNSTEIN CONSTANT OF A LINEAR FAMILY OF FUNCTIONS

In this section we outline a typical application of the inequality of Theorem 10.

Let $f_1, \cdots, f_n$ be regular function in the disk $D(0, R)$. We want to bound the Bernstein constant (on a smaller concentric disks) of a linear combination $f = \sum_{i=1}^{n} \lambda_i f_i$, uniformly in $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$.

Knowing the Bernstein constants of f_i is not enough : indeed, for f_1 and $f_2 = f_1 + \epsilon g$, with g having a big Bernstein constant , $0 < \epsilon << 1$, the Bernstein constants of f_1 and f_2 may be small , while $g = (1/\epsilon)(f_2 - f_1)$.

The following theorem shows that an additional information on the Wronskians , given by the generalized Markov inequality , enough suffices.

THEOREM 13. *Let $f_1 \cdots f_n$ be regular functions in the disk $D(0, R)$. Suppose, that for some $r \le R$ and multi-indices I_i; $i = 1, \cdots, n$, where $I_i = (0, 1, i-1, i+1 \cdots n)$, the following inequality holds:*

$$\max_{|x|=r} |W_{n,I_i}(f_1 \cdots f_n)| \le A \max_{|x|=r} |W_n(f_1 \cdots f_n)|,$$

where A is a constant .

Then, for any $\tilde{R} < r$, $\alpha > 0$ and for any $\lambda = (\lambda_1 \cdots \lambda_n) \in \mathbb{C}^n$

$$f = \sum_{i=1}^{n} \lambda_i f_i \in B^1_{\tilde{R},\alpha,K},$$

with K depending only on $A, \tilde{R}$ and α and may be effectively computed through them.

We present here the outline of the proof.

Proof. First, linear combinations $f = \sum_{i=1}^{n} \lambda_i f_i$ are exactly solutions of the linear differential equation $W(f_1 \cdots f_n, y) = 0$,- which may be rewritten as

$$y^{(n)} + (W_{n,I_{n-1}}/W_n) y^{(k-1)} + \cdots + (W_{n,I_0}/W_n) y = 0 \ , \qquad (18)$$

where $I_i = (0, 1, \cdots, i-1, i+1, \cdot, n)$.

Using the assumptions of the theorem, one can show that W belongs to a certain fixed Bernstein class. Hence , the number of singularities of the coefficients of (18) i.e. the number of zeroes of W in $D(0,r)$ can be effectively bounded (see [4]).

Consequently , we can find a certain subring $\Re$ in the disk $D(0,r)$, of the size , effectively bounded from below , free of singularities of (18).

Now the assumptions of the Theorem 13 and the Theorem 11 allow us to estimate the coefficients of (18) in $\Re$.

This, in turn gives the possibility to bound the Bernstein constant of any solution in the ring $\Re$.

Finally, knowing that the solutions $f = \sum_{i=1}^{n} \lambda_i f_i$ of (18) are in fact regular over the whole disk $D(0,r)$, we can deduce from above their Bernstein constant on the whole disk (see Lemma 12). This completes the proof of the Theorem 13. ▮

Now by our main result, the Markov inequality for Wronskians, to check the conditions of Theorem 13 it is enough to show that the Wronskians $W_i(f_1,..,f_n)$, $i = 1,..,n$, belong to certain effectively known Bernstein classes. This information turns out to be easily available in many important situations.

For example, for functions

$$f_i(x) = R_i(x)^{\alpha_i} \cdot exp\ S_i(x),$$

where α_i are complex numbers and R_i, S_i are rational function , One can easily show , that any Wronskian has the same form. In turn, assuming that $|\alpha_i| \leq c$, and that the functions $S_i(x)$, represented as $S_i(x) = \frac{\sum_{j=0}^{d} t_{ij} x^j}{\prod_{j=1}^{d_0} (x - a_{ij})}$, satisfy $|t_{ij}| \leq T$, one can show that such functions belong on each disk to a certain Bernstein class, explicitly specified through c and T.

As a result we get an effective estimate of the Bernstein constant of the linear family

$$f(x) = \sum_{i=1}^{n} R_i(x)^{\alpha_i} \cdot exp\ S_i(x),$$

on each couple of concentric disks, in term of the degrees and of the same constant c and T only. Compare with [4].

This result can be extended as follows.

THEOREM 14. *Let $y_1, ..., y_n$ be solutions of n differential equations*

$$\dot{y} = A_1 y, \quad ..., \ \dot{y} = A_n y$$

respectively, where A_i $(i = 1, ..., n)$ are algebraic functions. Suppose that A_i are meromorfic branches of algebraic functions in the disk $D(0,R)$, i.e.

A_i are not ramified in $D(0,R)$ and their only singularities there are poles. Suppose that A_i are defined by polynomials of degree at most d. Suppose in addition that $|A_i(x)|$ are bounded by A on a certain subring $\Re$ of $D(0,R)$ of the size Δ.

Then if a linear combination $y_\lambda = \sum_{i=1}^n \lambda_i y_i$ is regular in the disk $D(0,R)$ its Bernstein constants in the disk $D(0,R/2)$ can be bounded effectively in terms of n, d, A, Δ.

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