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Finite Cyclicity of Finite Codimension Nondegenerate Homoclinic Loops with Real Eigenvalues in $\mathbb{R}^3$*

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In this paper we study homoclinic loops in $\mathbb{R}^3$ which are nondegenerate in the sense of Šil'nikov ([21]) and with real principal eigenvalues in 1 : 1 resonance, i.e. homoclinic loops which have the *strong inclination property* and which are tangent to the principal eigenvectors. We are interested here in the higher codimensional cases. It is known that the dynamics of such systems is given by a 1-dimensional map. Using the ideas exposed in [5], we are able to show that, as for the "nontwisted" loops (cf. [17]), this 1-dimensional map admits a nice asymptotic expansion allowing to treat homoclinic loop bifurcations of arbitrarily high codimension and to exhibit an explicit bound for the number of isolated periodic solutions generated under small perturbations. The computations of the bound rely on derivation-division algorithms and Khovanskiĭ's fewnomials theory.

Key Words: Homoclinic, Bifucations, Cyclicity, Khovanskiĭ's theory

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1. INTRODUCTION

In studying families of vector fields depending on parameters it is natural to pay a special interest to generic families depending on a small number of parameters as these are more likely to be encountered in applications. There is no general pattern for the organization of bifurcations occuring in generic k-parameter families, but certain bifurcations can be studied for arbitrary codimensions. Examples of these are given by the Hopf bifurcation. The codimension 1 Hopf bifurcation is concerned with the appearance or disappearance of a limit cycle from a singular point as a pair of eigenvalues crosses the imaginary axis. The Hopf bifurcation of codimension k occurs when there are additional degeneracies at the level of higher order terms in the normal form. Similarly the homoclinic loop bifurcation through a hyperbolic saddle has been studied in the plane for arbitrary codimensions [15]. Another example is the study of the cusp of order n in the plane [12, 13], i.e. the study of the bifurcation of a singular point of nilpotent type and multiplicity 2 in the plane, the codimension being decided by higher degree terms in the normal form.

One strong motivation to study the higher order bifurcations (local or global) is that they are the organizing centers for the bifurcation diagrams of many multi-parameter families occuring in modelling (for instance [10]).

The complete studies mentioned above have been possible because of the powerful techniques that have been developed for analyzing vector fields in the plane. In these problems the most important and difficult question is the control of the number of limit cycles (isolated periodic solutions). The technique to analyse this is to "compute" the Poincaré first return map on a section. For instance in the case of the homoclinic loop in the plane this return map is calculated as the composition of a transition map (*Dulac map*) in the neighborhood of the saddle point with a regular transition far from the point. When the codimension increases it is necessary to be more precise in the calculations. In that respect the Dulac map is calculated in suitable C^k coordinates in which the system and its perturbation have a nice normal form. The normalizing coordinates are not unique. When looking to more complicated situations than the homoclinic loop we can use the freedom on the choice of the normalizing coordinates to simplify the regular transition map(s). This operation allows to determine the intrinsic generic properties of these maps determining the codimension and the bifurcation diagrams.

For most other classes of bifurcations a complete study is still out of reach. However, in the planar case it is highly believed that the number of limit cycles appearing by perturbation of a polycycle in a generic k-parameter family is uniformly bounded. This is the Hilbert-Arnold problem which we can state as: "Prove that, for any n, the bifurcation number

$B(n)$ is finite, where $B(n)$ is the maximum cyclicity of nontrivial polycycles occuring in generic n-parameter families". The equivalent of Hilbert-Arnold problem is obviously false in dimensions greater than 2, the obvious counter-example being given by Šil'nikov's example of a homoclinic loop in $\mathbb{R}^3$ through a saddle point with a pair of complex eigenvalues: under adequate conditions on the eigenvalues a perturbation of the homoclinic loop leads to horseshoes and chaos.

However not all bifurcations in $\mathbb{R}^3$ lead to such wild behaviour. It is an interesting question to identify genericity conditions which ensure that the dynamics created in a bifurcation remains under control and that its complexity can be measured in terms of the codimension of the bifurcation. The work of this paper is within that general framework.

More precisely in this paper we consider some homoclinic bifurcations in $\mathbb{R}^3$ from the point of view desribed above. We specialize to homoclinic bifurcations through a saddle point with real eigenvalues. We consider here nondegenerate bifurcations in the sense of Šil'nikov. A homoclinic loop through a saddle point with two negative and one positive eigenvalues is nondegenerate in the sense of Šil'nikov if

1. it is tangent to the principal eigenvectors;
2. the stable manifold together with its tangent space approaches the strong stable manifold along the homoclinic orbit, i.e in a tubular neighborhood of the homoclinic loop the strongly stable manifold is part of the adherence of the stable manifold. (This is often referred to as the "strong inclination property".)

In such systems there exists an attractor which sits in a topological 2-dimensional invariant manifold and contains all the bifurcating dynamics. Hence all bifurcating periodic solutions have period 1 or 2 (in a tubular neighborhood of the homoclinic loop) and there is no chaotic behaviour [6, 18, 19, 17]. The attractor can be either an annulus or a Möbius band. In the first case we say that the loop is *non twisted* and all periodic orbits have period one. In the second case the loop is *twisted.*

The codimension 1 case, when the sum of the principal eigenvalues does not vanish was studied by Šil'nikov in [20, 21] and leads, under perturbation, to a unique periodic orbit. In 1987, Yanagida [22] showed that resonant bifurcation (when the sum of the two principal eigenvalues vanishes) could lead to the birth of periodic curves of periodic 2. In 1990 Chow, Deng and Fiedler [2] studied the codimension 2 case by means of the Lyapunov-Schmidt reduction.

Nondegenerate homoclinic loops (in the sense of Šil'nikov) with the two principal eigenvalues in 1:1 resonance were later studied in the non twisted case by the following method [17]: a suitable reduction to normal form allows the exact calculation of the transition map (Dulac map) in a neighbourhood of the saddle point, and its composition with a C^k–diffeomorphism gives the first return map. The use of a derivation-division algorithm allows to bound the number of fixed points. The method provides a bound for the number of isolated periodic solutions generated under perturbation of the higher codimension homoclinic loops, i.e. what we call *the finite cyclicity property* of the loop and allows to show the finite cyclicity property for all finite codimensions. The optimality of the bound is still an open question which is not addressed here.

The study of the twisted case for small codimension was done in [5] in the case where the Möbius band is sufficiently differentiable. It was done by projecting the dynamics on the band. In the present paper we extend the result to arbitrary finite codimension. Since the Möbius band is not sufficiently differentiable, we do not project the dynamics on the band. We exhibit a bound for the number of isolated periodic solutions generated under perturbation of a twisted homoclinic loop of arbitrary finite codimension (the finite cyclicity property).

As we need to study periodic solutions of period 2, it is natural to consider the 2-return map (the second iterate of the Poincaré map). In fact we work with an equivalent displacement map defined on a 2-dimensional section. The domain of the displacement map shrinks to a point as $\lambda \to 0$ ($\lambda = 0$ corresponds to the unperturbed case). The first step is a blow up (a method first introduced by Jebrane and Mourtada in [9]). The effect of the blow-up is to stretch the domain to a quasi-rectangular domain. The existence of the invariant manifold allows to reduce the map to a 1-dimensional map via a global use of the implicit function theorem. Finally we reduce the problem to that of giving a bound for the number of zeros of a 1-dimensional map $V_\lambda(t)$ on the unit interval I. The map $V_\lambda(t)$ is analytic everywhere except at the endpoints of the interval. Hence we divide our study in three regions. Near the end points the function has an asymptotic expansion with generalized monomials which are totally ordered. This allows to bound the number of zeros of $V_\lambda(t)$ in a neigborhood of each of the endpoints. In the middle interval the number of zeros is obviously uniformly bounded since the map $V_\lambda(T)$ is analytic on a compact domain, depending on λ in a compact parameter space. The special form of the funtion $V_\lambda(t)$ allows to find an explicit bound for the number of zeros. This comes from the fact that the function is very close to a Pfaffian function, the number of zeros of which can be studied by the theory of fewnomials of Khovanskii. The knowledge of the asymptotic expansion of $V_\lambda(t)$ near the endpoints of I and of the Pfaffian function allows to define the codimension of the loop. Although in the paper the codimension of the loop is defined rather soon, the full justification of the definition can only come from the study of $V_\lambda(t)$ in the different regions. The strategy of proof is simple. However in practice it involves long and complicated calculations as the number of terms and functions appearing in $V_\lambda(t)$ is very large. A great part of the work is concerned with grouping terms adequately in the form of a term times a quantity of the form $1+O(t)$ or $(1+O(1-t))$ near the endpoints, and of the form $1+O(\lambda)$ in the middle region.

The paper is divided into two parts. The first part contains preliminaries, the definition of codimension and of the function $V_\lambda(t)$. In the second part, we prove the finite cyclicity property of twisted nondegenerate homoclinic loops of finite codimension.

2. THE ASYMPTOTIC EXPANSION OF THE 1-DIMENSIONAL MAP.

2.1. Setting and Framework of the Problem

Let $\mathfrak{X}_\lambda$ be a p–parameter family of C^∞–vector fields on $\mathbb{R}^3$ which has for $\lambda = 0$ a homoclinic loop Γ_0 through a saddle point at the origin (Figure 1). We consider families $\mathfrak{X}_\lambda$ for which the origin is a hyperbolic strongly 1-resonant saddle, i.e. the set of eigenvalues of the linearization of $\mathfrak{X}_0$ at the origin of $\mathbb{R}^3$ is $\{\nu_1(0), -\nu_2(0), -\mu(0)\}$ and is such that $0 < \nu_1(0) = \nu_2(0) = 1 < \mu(0)$ and $\mu(0) \notin \mathbb{Q}$ (the only resonance comes from $\nu_2(0) = \nu_1(0)$). The resonant monomial u is given by $u = xy$.

Since $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^p$ is hyperbolic, we take a small neighborhood Λ of $\lambda = 0$ such that the saddle point has eigenvalues $\nu_1(\lambda) > 0 > -\nu_2(\lambda) > -\mu(\lambda)$. There exists a C^N–change of coordinates and a rescaling of time such that the system defining the family can be written in the neighborhood of the singular point in the following way (cf. Theorem 3 in [7])

$$\begin{aligned}
\dot{x} &= x \\
\dot{u} &= u\left(\alpha_1(\lambda) + \sum_{i=1}^{K} \alpha_{i+1}(\lambda)u^i\right) \\
\dot{z} &= z\left(-\mu(\lambda) + \sum_{i=1}^{K} \beta_i(\lambda)u^i\right),
\end{aligned} \tag{1}$$

where $u = xy$, $\alpha_1(\lambda) = 1 - \nu_2(\lambda)/\nu_1(\lambda)$. We can suppose (after scaling) that the normal form is valid in a ball of radius 2.

The first return map (the Poincaré map) is the composition of two maps: a local transition map Δ_λ between two sections to the stable and unstable manifolds which is defined in a neighborhood U_0 of the singularity, and a regular map $\mathcal{R}_\lambda$ defined far from the singularity by the flow near Γ_0. The local transition map Δ_λ, called the *Dulac map* as in the planar case, is calculated using the normal form coordinates.

Definition 1. Let the origin be a saddle point with three real eigenvalues $-\mu < -\nu_2 < 0 < \nu_1$. The homoclinic loop Γ_0 is **nondegenerate in the sense of Šil'nikov** if it satisfies the following two properties:

(i) Γ_0 approaches the origin along the principal stable eigenvector (i.e. the eigenvector of the eigenvalue $-\nu_2$);

(ii) the stable manifold together with its tangent space approaches the strong stable manifold along the homoclinic orbit, i.e. in a tubular neighborhood of the invariant 1-manifold, the strong stable manifold is part of the adherence of the stable manifold. (This property is often referred to as the "strong inclination property".)

Let U be a sufficiently small tubular neighborhood of Γ_0. For all $\lambda \in \Lambda \subseteq \mathbb{R}^p$ with Λ a neighborhood of $0 \in \mathbb{R}^p$, let $\Sigma_1 = \{y = 1\}$ be a transversal of $\mathfrak{X}_0$ intersecting the local stable manifold of the origin, and let $\mathrm{T}_1 = \{x = 1\}$ be a transversal of $\mathfrak{X}_0$ intersecting the local unstable manifold of the origin (cf. Figure 1). (x, y, z) provides natural parametrizations (x, z) of Σ_1 and (Y_1, Z_1) of T_1 (cf. Figure 1). We denote by $\mathcal{P}_\lambda = (\mathcal{P}_{1,\lambda}, \mathcal{P}_{2,\lambda})$ the first return map on T_1.

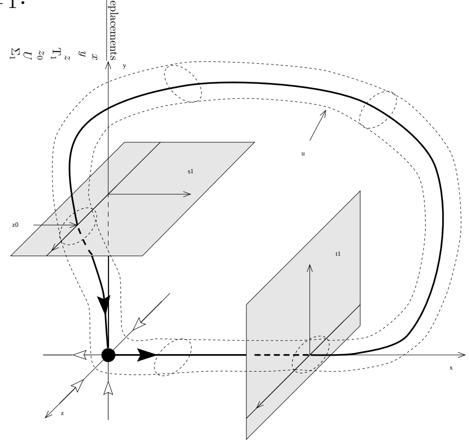


Figure 1. The homoclinic loop Γ_0.

The regular transition map $\mathcal{R}_\lambda(Y_1, Z_1)$ from T_1 to Σ_1 is a C^K–orientation preserving diffeomorphism

$$\mathcal{R}_\lambda(Y_1, Z_1) = \begin{pmatrix} C_0(\lambda) + \displaystyle\sum_{i+j>0}^{K} C_{ij}(\lambda) Y_1^i Z_1^j \\ D_0(\lambda) + \displaystyle\sum_{i+j>0}^{K} D_{ij}(\lambda) Y_1^i Z_1^j \end{pmatrix} + \check{R}_\lambda(Y_1, Z_1), \qquad (2)$$

where $C_{10}(0)D_{01}(0)-C_{01}(0)D_{10}(0)>0$ (orientation preserving), $C_0(0)=0$ but $D_0(0)=z_0$ need not vanish, and $\check{R}_\lambda(Y_1,Z_1)$ is C^K and K-flat at $Y_1=0=Z_1$.

Lemma 2. *System* (1) *is nondegenerate if* $C_{10}(0)\neq 0$. *The loop is* twisted (*resp.* nontwisted) *if* $C_{10}(0)<0$ (*resp.* $C_{10}(0)>0$). *The stable manifold for the twisted case is illustrated in Figure* 2.

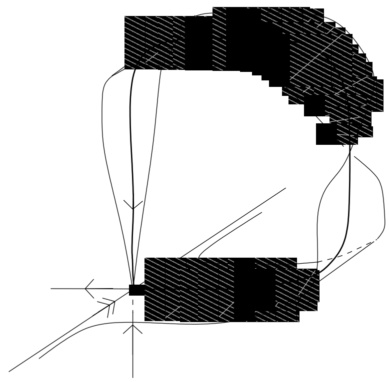


Figure 2. The invariant stable manifold in the twisted case.

Definition 3.

1. Let $\{\mathfrak{X}_\lambda\}_{\lambda\in\Lambda}$ be a family of C^K vector fields on $\mathbb{R}^3$ such as in our framework. We say that Γ_0 has **finite cyclicity** in the family $\{\mathfrak{X}_\lambda\}_{\lambda\in\Lambda}$ if there exists $N\in\mathbb{N}$, $\epsilon>0$ and a neighborhood Λ_0 of λ_0 in Λ such that for all $\lambda\in\Lambda_0$, the number $n(\epsilon,\lambda)$ of isolated periodic orbits γ of $\mathfrak{X}_\lambda$ with $\mathrm{dist}_H(\gamma,\Gamma)\le\epsilon$ is less than N, where dist_H is the Hausdorff distance on compact sets.

2. Let

$$n(\epsilon,\Lambda_0)=\sup_{\lambda\in\Lambda_0}\{n(\epsilon,\lambda)\}.$$

The **cyclicity of $\mathbf{\Gamma_0}$** in the family $\{\mathfrak{X}_\lambda\}_{\lambda\in\Lambda}$ is the minimum integer $n(\epsilon,\Lambda_0)$ when ϵ and the diameter of Λ_0 go to 0. We note it $Cycl(\Gamma_0,\mathfrak{X}_\lambda)$.

3. We say that Γ_0 has **absolute finite cyclicity** if there exists a finite upper bound to all $n(\epsilon,\Lambda_0)$ in any family $\{\mathfrak{X}_\lambda\}_{\lambda\in\Lambda}$ and sufficiently small ϵ and we note it $Cycl(\Gamma_0)$.

2.2. The Return Map $\mathcal{P}_\lambda$

In this section we recall the asymptotic expansion of the local transition map Δ_λ described in [17]. We also define *generalized monomials* as they appear in the expansions.

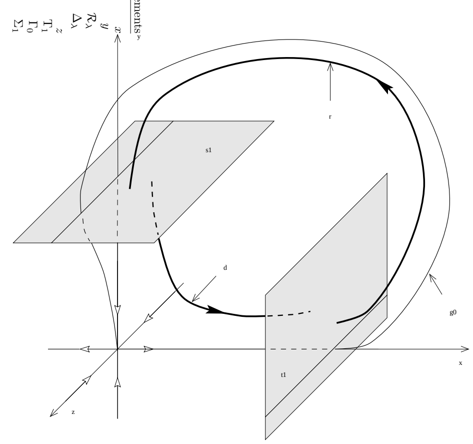


Figure 3. The maps $\mathcal{R}_\lambda$ and Δ_λ such that $\mathcal{P}_\lambda \stackrel{def}{=} \Delta_\lambda \circ \mathcal{R}_\lambda$.

The transition maps for planar systems have been thoroughly studied. Roussarie for instance uses generalized monomials which are well-ordered and behave adequately under differentiation (cf. [15], [14], and [16]). These monomials have the form $x^i\omega^j(x,\lambda)$ where

$$\alpha_1(\lambda) = 1 - \frac{\nu_2(\lambda)}{\nu_1(\lambda)}$$

$$\omega(x,\lambda) = \begin{cases} \dfrac{x^{-\alpha_1(\lambda)} - 1}{\alpha_1(\lambda)} & \text{if} \quad \alpha_1(\lambda) \neq 0 \\ -\ln(x) & \text{if} \quad \alpha_1(\lambda) = 0 \end{cases} .$$

The generalized monomials have the property that for all $k > 0$:

$$\lim_{\alpha_1(\lambda)\to 0} x^k\omega^j(x,\lambda) = -x^k \ln^j(x),$$

and this holds uniformly on $[0, X]$ for any fixed $X > 0$.

We will need to differentiate several times asymptotic expansions in which appear monomials of the form $x^k\omega^j$. To write the result of the differentiation in a simple form, it is useful to introduce the following concepts.

DEFINITION 4.

1. ([14]). Let $K \in \mathbb{N}$, $\psi(x,\lambda)$ a C^K–function on $]0,\epsilon[\times\Lambda_0$ such that $\psi(0,0)=0$, and a positive continuous function $\xi(x,\lambda)$ with $\xi(0,\lambda)=0$. We say that $\psi(x,\lambda)$ is $I_0^K\big(\xi(x,\lambda)\big)$ if for every $n\in\mathbb{N}$ such that $n\le K$, we have

$$\lim_{x\to 0}\xi^n(x,\lambda)\frac{\partial^n\psi(x,\lambda)}{\partial x^n}=0$$

uniformly on Λ_0.

2. Let $\psi(x,\lambda)\in I_0^K\big(\xi(x,\lambda)\big)$. We say that $\psi(x,\lambda)\in J_0^K\big(\xi(x,\lambda)\big)$ if for every $n\in\mathbb{N}$ such that $n\le K$, we have

$$\lim_{\lambda\to 0}\frac{\partial^n\psi(x,\lambda)}{\partial x^n}=0$$

uniformly on $[0,X]$ for all fixed X.

The generalized monomials $x^k\omega(x,\lambda)$ are $I_0^K\big(\rho(x,\lambda)\big)$, where $\rho(x,\lambda)=x^{1+\alpha_1(\lambda)}\omega(x,\lambda)$.

LEMMA 5 ([**9**]). *Let $f(x,\lambda)$ be a C^K–function on $[0,x_0[\times\Lambda$ such that $f(0,\lambda)=0$. Then there exists a C^K–function $g(x,\lambda)$ with $g(0,\lambda)=0$ and such that for all $a>0$, we have*

$$\omega\big(ax(1+f),\lambda\big)=\Big[1+O(\alpha_1(\lambda))\Big]\omega(x,\lambda)+g(x,\lambda)-\ln(a)\Big[1+O\big(\alpha_1(\lambda)\big)\Big].$$

The Dulac map $\Delta_\lambda=(\Delta_{1,\lambda},\Delta_{2,\lambda})$ from Σ_1 to T_1 has the following form [17]

$$\begin{aligned}\Delta_\lambda(x,z)&=\begin{bmatrix} x+\sum_{i=1}^{K}\alpha_i(\lambda)x^i\omega(x,\lambda)\big(1+\psi_i(x,\lambda)\big)+\phi_{1,K}(x,\lambda)\\ zx^{\mu(\lambda)}\big(1+\varphi_{2,K}(x,\lambda)\big)\end{bmatrix}\\ &=\begin{pmatrix}Y_1\\ Z_1\end{pmatrix},\end{aligned}\qquad(3)$$

where $\psi_i(x,\lambda)$ are $I_0^{K-i}\big(\rho(x,\lambda)\big)$, $\varphi_{2,K}(x,\lambda)$ is $I_0^K\big(\rho(x,\lambda)\big)$, and $\phi_{1,K}(x,\lambda)$ is C^K and K–flat at $x=0$.

Note that the Dulac map is not one to one for points of the form $(0,z)$. The inverse $\Delta_\lambda^{-1}(Y_1,Z_1)$ of the Dulac map $\Delta_\lambda(x,z)$ is computed by writing

Equation (3) for the system with reverse time. In fact, for points of T_1 with $Y_1 > 0$, it is of the form

$$\Delta_\lambda^{-1}(Y_1, Z_1) = \begin{bmatrix} Y_1 + \sum_{i=1}^{K} \overline{\alpha}_i(\lambda) Y_1^i \overline{\omega}(Y_1, \lambda)\big(1 + \overline{\psi}_i(Y_1, \lambda)\big) + \overline{\phi}_{1,K}(Y_1, \lambda) \\ Z_1 Y_1^{-\mu(\lambda)}\big(1 + \overline{\varphi}_{2,K}(Y_1, \lambda)\big) \end{bmatrix} \tag{4}$$
$$= \begin{pmatrix} x \\ z \end{pmatrix}$$

where functions $\overline{\psi}_i(Y_1, \lambda)$ are $\overline{I}_0^{K-i}\big(\overline{\rho}(Y_1, \lambda)\big)$, $\overline{\varphi}_{2,K}(Y_1, \lambda)$ are $\overline{I}_0^{K}\big(\overline{\rho}(Y_1, \lambda)\big)$, and $\overline{\phi}_{1,K}(Y_1, \lambda)$ is C^K and K–flat at $Y_1 = 0$. Moreover

$$\overline{\omega}(Y_1, \lambda) = \begin{cases} \dfrac{Y_1^{-\overline{\alpha}_1} - 1}{\overline{\alpha}_1} & \text{if} \quad \overline{\alpha}_1 \neq 0 \\ -\ln(Y_1) & \text{if} \quad \overline{\alpha}_1 = 0 \end{cases} \tag{5}$$

where

$$\overline{\alpha}_1(\lambda) = -\frac{\alpha_1}{1 - \alpha_1} = -\alpha_1\big(1 + O(\alpha_1)\big)$$

and $\overline{\alpha}_i(\lambda) = -\alpha_i(\lambda) + \overline{p}_i(\lambda)$ with $\overline{p}_i(\lambda)$ some polynomial in the $\alpha_{i'}$ with $i' < i$. Note that although $\Delta_{2,\lambda}^{-1}(Y_1, Z_1)$ is not defined at $Y_1 = 0$, however $Z_1 Y_1^{-\mu(\lambda)}$ is bounded on $\Delta_\lambda(\Sigma_1)$, the region of T_1 where N–curves can appear.

As mentioned in [4], a change of coordinates

$$(x, y, z) \to (\overline{x}, \overline{y}, \overline{z})$$

tangent to the identity preserving the type of the normal form (1) generates a pair of maps f_λ^1 and f_λ^2 such that

$$f_\lambda^1 \circ \Delta_\lambda \circ f_\lambda^2 = \overline{\Delta}_\lambda, \tag{6}$$

where $\overline{\Delta}_\lambda$ is the Dulac map expressed in the $(\overline{x}, \overline{y}, \overline{z})$ coordinates.

Lemma 6. *In Equation* (6)*, the maps* f_λ^1 *and* f_λ^2 *have the following form:*

$$f_\lambda^1(Y, Z) = \left(Y\left(1 + \sum_{i=1}^{K} a_i^1(\lambda) Y^i\right), \, Z\left(1 + \sum_{i=1}^{K} b_i^1(\lambda) Y^i\right)\right),$$

and

$$f_\lambda^2(x,z) = \left(x\left(1+\sum_{i=1}^{K} a_i^2(\lambda)x^i\right), z\left(1+\sum_{i=1}^{K} b_i^2(\lambda)x^i\right)\right). \tag{7}$$

Proof. Let

$$f_\lambda^1(Y,Z) = \left(Y\left(1+\sum_{i+j=1}^{K} a_{ij}^1(\lambda)Y^iZ^j\right), Z\left(1+\sum_{i+j=1}^{K} b_{ij}^1(\lambda)Y^iZ^j\right)\right)$$

and

$$f_\lambda^2(x,z) = \left(x\left(1+\sum_{i+j=1}^{K} a_{ij}^2(\lambda)x^iz^j\right), z\left(1+\sum_{i+j=1}^{K} b_{ij}^2(\lambda)x^iz^j\right)\right). \tag{8}$$

We want to find coefficients $a_{ij}^\ell(\lambda)$ and $b_{ij}^\ell(\lambda)$ such that the relation $f_\lambda^1 \circ \Delta_\lambda \circ f_\lambda^2 = \overline{\Delta}_\lambda$ holds. Looking at the coefficients of $x^{j\mu}$ in $f_\lambda^1 \circ \Delta_\lambda \circ f_\lambda^2$, one obtains that $a_{ij}^1(\lambda) = a_{ij}^2(\lambda) = 0$ and $b_{ij}^1(\lambda) = b_{ij}^2(\lambda) = 0$ for $j > 0$. ∎

2.3. Geometric Preliminaries

The nondegeneracy hypotheses impose important geometric constraints on the bifurcating dynamics.

Definition 7. Let Γ_0 be a homoclinic loop of $\mathfrak{X}_0(x)$. Fix U a small tubular neighborhood of Γ_0. Assume $\overline{\Gamma} \subseteq U$ with Γ some orbit of $\mathfrak{X}_\lambda(x)$ intersecting a section of U N–times.

1. If Γ is an homoclinic loop then it is called an **N-homoclinic loop**.
2. If Γ is a periodic curve then it is called an **N–periodic curve**.
3. An **N-curve** is either an N-homoclinic loop or an N–periodic curve.

As long as U is chosen small enough, the above definitions are independent of the choice of U.

Facts 8. *In our framework, we have the following facts.*

1. There exists a $C^{[\mu]}$–Möbius band depending on λ and containing the bifurcating dynamics (*cf. [18] and [19]*).

2. If there is a 2-curve on the Möbius band then there is one and only one 1–periodic curve that coexists with the 2-curve.

3. The cyclicity of Γ_0 *is bounded by 1 plus the number of 2-curves bifurcating from* Γ_0*.*

4. Denote by $\beta(\lambda) = \big(C_0(\lambda), D_0(\lambda)\big)$ *the first intersection of* W^u *with* Σ_1 (*cf. Figure* 4)*. A necessary condition for the existence of periodic solutions is* $C_0(\lambda) > 0$*.*

5. All fixed points $(Y_1, Z_1) \in \mathrm{T}_1$ *of the 2-return map satisfy, for* λ *sufficiently small,* $R_{1,\lambda}(Y_1, Z_1) \in \big[0, C_0(\lambda)\big]$*.*

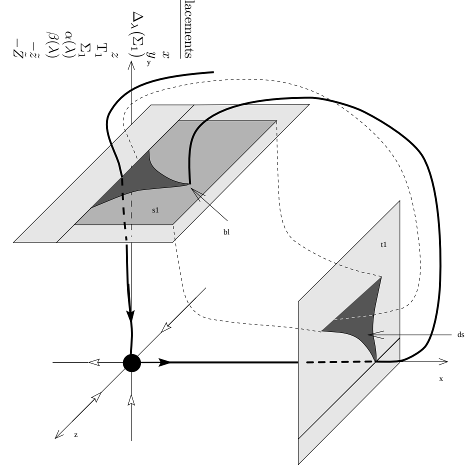


Figure 4. Parts of the bifurcated 1-homoclinic loops in $\mathbb{R}^3$.

2.4. Main Result

The different bifurcations that can occur will be described by their *type* (a 4−tuple) from which we can give a bound for the cyclicity.

The codimension k associated to a type will describe the minimal number ℓ of parameters such that the bifurcation occurs in an $l-$family. The knowledge of k is sufficient to give a bound for the cyclicity, but this bound is not explicit. Particular cases can be studied easily.

Before we state the main result, we define the type and the codimension of nondegenerated homoclinic loop.

Definition 9. [17] The generalized monomials $\{1, x^{i+j\mu}\omega^\ell(x,\lambda) \mid 1 \leq i+j \leq K, 0 \leq \ell \leq i,$ and $\ell \leq 1$ if $j = 0\}$ are totally ordered with respect to flatness at $x = 0$ in the following way

$$x^{i+j\mu}\omega^\ell(x,\lambda) \prec x^{i'+j'\mu}\omega^{\ell'}(x,\lambda) \iff \begin{cases} i+j\mu < i'+j'\mu & \text{or} \\ i+j\mu = i'+j'\mu & \text{and } \ell > \ell'. \end{cases} \tag{9}$$

We will only be working with monomials of the form $(i, 0, \ell)$ and $(i, j, 0)$.

DEFINITION 10. Let $k(i_1, i_2, j, \ell)$ denote the number of generalized monomials of the form $(i, 0, \ell)$ and $(i, j, 0)$ and of order lower than $x^{i_1+ji_2+j\mu}\omega^\ell(x,\lambda)$.

Here are examples of orders (depending on the value of μ).

$$1 \prec x\omega \prec x \prec x^\mu \prec x^2\omega \prec x^2 \prec x^{1+\mu} \prec x^{2\mu} \prec x^3\omega \prec \cdots \quad \text{if } 1 < \mu < 1.5, \tag{10}$$

$$1 \prec x\omega \prec x \prec x^\mu \prec x^2\omega \prec x^2 \prec x^{1+\mu} \prec x^3\omega \prec x^3 \prec \cdots \quad \text{if } 1.5 < \mu < 2, \tag{11}$$

$$1 \prec x\omega \prec x \prec x^2\omega \prec x^2 \prec x^\mu \prec x^3\omega \prec x^3 \prec x^{1+\mu} \prec \cdots \quad \text{if } 2 < \mu < 3. \tag{12}$$

In Equations (10) and (11) $k(2,0,0,1) = 4$, and in Equation (12) $k(2,0,0,1) = 3$.

DEFINITION 11. Let Γ_0 be a nondegenerated loop in $\mathbb{R}^3$ for which the return map is the composition of the two maps given in (2) and (3).

1. Γ_0 is nondegenerate of **finite codimension** if it is not degenerate in the sense of Šil'nikov [21] and one of the following generic conditions holds:

(i) $\alpha_1(0) = 0$ and $C_{10}(0) \neq -1$, we say that Γ_0 is of type $(1,0,0,0)$.

(ii) $\exists\, I_1$ such that $C_{10}(0) = -1$, $\alpha_i(0) = C_{i0}(0) = 0$ for all $i < I_1$, $C_{ij}(0)D_{\ell 0}(0) = 0$ for all $i + j\ell + j\mu < I_1$, and $\alpha_{I_1}(0) \neq 0$, we say that Γ_0 is of type $(I_1, 0, 0, 1)$.

(iii) $\exists\, I_1$ such that $C_{10}(0) = -1$, $\alpha_i(0) = C_{i0}(0) = 0$ for all $i < 2I_1 + 1$, $C_{ij}(0)D_{\ell 0}(0) = 0$ for all $i + j\ell + j\mu < 2I_1 + 1$, $\alpha_{2I_1+1}(0) = 0$, and $C_{2I_1+1,0}(0) \neq 0$, we say that Γ_0 is of type $(2I_1 + 1, 0, 0, 0)$.

(iv) $\exists\, I_1, I_2, J$, with $J > 0$, such that $C_{10}(0) = -1$, $\alpha_i(0) = C_{i0}(0) = 0$ for all $i < I_1 + I_2J + J\mu$, $C_{ij}(0)D_{\ell 0}(0) = 0$ for all $i + j\ell + j\mu < I_1 + I_2J + J\mu$, and $C_{I_1J}(0)D_{I_20}(0) \neq 0$, we say that Γ_0 is of type $(I_1, I_2, J, 0)$.

2. Let Γ_0 be of finite type. If (I_1, I_2, J, L) is the type of Γ_0, then Γ_0 is said to be of **codimension** k with $k = k(I_1, I_2, J, L)$.

Table 1.

Type and value of $\boldsymbol{n}$ (given in (13)) together with the conditions for small codimensions (assuming the existence of the loop).

<table>
<tr><th>codim
values of μ</th><th>3</th><th>4</th><th>5</th></tr>
<tr><td rowspan="2">$1 < \mu < 1.5$</td><td rowspan="2">$(0,0,1,0)$
$n = 3$
$\alpha_1 = 0$
$C_{10} = -1$
$C_{01}D_0 \neq 0$</td><td rowspan="2">$(2,0,0,1)$
$n = 5$
$\alpha_1 = 0$
$C_{10} = -1$
$C_{01}D_0 = 0$
$\alpha_2 \neq 0$</td><td>$(1,0,1,0)$; $n = 5$
$\alpha_1 = \alpha_2 = 0$
$C_{10} = -1, \quad C_{01} = 0$
$C_{11}D_0 \neq 0$</td></tr>
<tr><td>$(0,1,1,0)$; $n = 5$
$\alpha_1 = \alpha_2 = 0$
$C_{10} = -1 , \quad D_0 = 0$
$C_{01}D_{10} \neq 0$</td></tr>
<tr><td rowspan="2">$1.5 < \mu < 2$</td><td rowspan="2">$(0,0,1,0)$
$n = 3$
$\alpha_1 = 0$
$C_{10} = -1$
$C_{01}D_0 \neq 0$</td><td rowspan="2">$(2,0,0,1)$
$n = 5$
$\alpha_1 = 0$
$C_{10} = -1$
$C_{01}D_0 = 0$
$\alpha_2 \neq 0$</td><td>$(1,0,1,0)$; $n = 5$
$\alpha_1 = \alpha_2 = 0$
$C_{10} = -1, \quad C_{01} = 0$
$C_{11}D_0 \neq 0$</td></tr>
<tr><td>$(0,1,1,0)$; $n = 5$
$\alpha_1 = \alpha_2 = 0$
$C_{10} = -1, \quad D_0 = 0$
$C_{01}D_{10} \neq 0$</td></tr>
<tr><td>$2 < \mu < 3$</td><td>$(2,0,0,1)$
$n = 5$
$\alpha_1 = 0$
$C_{10} = -1$
$\alpha_2 \neq 0$</td><td>$(2,0,0,1)$
$n = 5$
$\alpha_1 = \alpha_2 = 0$
$C_{10} = -1$
$C_{01}D_0 \neq 0$</td><td>$(3,0,0,1)$; $n = 7$
$\alpha_1 = \alpha_2 = 0$
$C_{10} = -1$
$C_{01}D_0 = 0$
$\alpha_3 \neq 0$</td></tr>
<tr><td>$\mu > 3$</td><td>$(2,0,0,1)$; $n = 5$
$\alpha_1 = 0$
$C_{10} = -1, \quad \alpha_2 \neq 0$</td><td>$(3,0,0,1)$; $n = 7$
$\alpha_1 = \alpha_2 = 0$
$C_{10} = -1, \quad \alpha_3 \neq 0$</td><td>$(3,0,0,1)$; $n = 7$
$\alpha_1 = \alpha_2 = \alpha_3 = 0$
$C_{10} = -1, \quad C_{30} \neq 0$</td></tr>
</table>

Table 1.

Type and value of $\boldsymbol{n}$ (given in (13)) together with the conditions for small codimensions (assuming the existence of the loop).

codim / values of μ	6	7	8
$1 < \mu < 1.5$	$(0,0,2,0)$ $n = 5$ $\alpha_1 = \alpha_2 = 0$ $C_{10} = -1$ $C_{01}D_0 = 0$ $C_{01}D_{10} = 0$ $C_{11}D_0 = 0$ $C_{02}D_0 \neq 0$	$(3,0,0,1)$ $n = 7$ $\alpha_1 = \alpha_2 = 0$ $C_{10} = -1$ $C_{01}D_0 = 0$ $C_{11}D_0 = 0$ $C_{01}D_{10} = 0$ $C_{02}D_0 = 0$ $\alpha_3 \neq 0$	$(3,0,0,0)$ $n = 7$ $\alpha_1 = \alpha_2 = \alpha_3 = 0$ $C_{10} = -1$ $C_{01}D_0 = 0$ $C_{11}D_0 = 0$ $C_{01}D_{10} = 0$ $C_{02}D_0 = 0$ $C_{30} \neq 0$
$1.5 < \mu < 2$	$(3,0,0,1)$ $n = 7$ $\alpha_1 = \alpha_2 = 0$ $C_{10} = -1$ $C_{01}D_0 = 0$ $C_{11}D_0 = 0$ $C_{01}D_{10} = 0$ $\alpha_3 \neq 0$	$(3,0,0,0)$ $n = 7$ $\alpha_1 = \alpha_2 = \alpha_3 = 0$ $C_{10} = -1$ $C_{01}D_0 = 0$ $C_{11}D_0 = 0$ $C_{01}D_{10} = 0$ $C_{30} \neq 0$	
$2 < \mu < 3$	$(3,0,0,0)$ $n = 7$ $\alpha_1 = \alpha_2 = \alpha_3 = 0$ $C_{10} = -1$ $C_{01}D_0 = 0$ $C_{30} \neq 0$		

Remark 12.

1. The codimension $k(I_1, I_2, J, L)$ depends on $I_1 + JI_2$, J, L. Conversely, from $k(I_1, I_2, J, L)$ and μ we can recover $I_1 + JI_2$, J, L.
2. Unfortunately, Definition 11 can only be justified *a posteriori* from the proof of Theorem 14 below. Indeed the conditions above concern precisely the coefficients of a function $V_\lambda(t)$ appearing in (35) below and whose zeros are in $1:1$ correspondence with the fixed points of the $2-$return map.

PROPOSITION 13. *Conditions* (i)–(iv) *are intrinsic.*

Proof. Using Lemma 6 we can simplify the expression of $\mathcal{R}_\lambda$ so that for the first nonvanishing $C_{i'0}$, i' is odd. Moreover, for each $j > 0$, the first nonvanishing C_{ij} is intrinsic; the first nonvanishing D_{i0} is intrinsic. Also the first nonvanishing α_i is intrinsic. Indeed, the action of maps as the f_λ^i of Lemma 6 allows to simplify the expression of $\mathcal{R}_\lambda$. In the case $C_{10}(0) < 0$, we can choose f_λ^i such that

$$f_\lambda^2 \circ \mathcal{R}_\lambda \circ f_\lambda^1(Y_1, Z_1) = \begin{pmatrix} C_0(\lambda) + \sum_{i=1}^{K} C_{i,0}(\lambda) Y_1^i + \sum_{\substack{0<i+j<K \\ j>0}} C_{ij}(\lambda) Y_1^i Z_1^j \\ D_0(\lambda) + \sum_{i+j>0}^{K} D_{ij}(\lambda) Y_1^i Z_1^j \end{pmatrix} + \hat{R}_\lambda(Y_1, Z_1),$$

where, if there exists i' such that $C_{i'0}(0) \neq 0$, the minimum of such i' is odd. ▮

The finite cyclicity property can be stated in the following way.

THEOREM 14. *If* Γ_0 *is of type* (I_1, I_2, J, L)*,* $I = I_1 + JI_2$ *and*

$$n = 2(I + [J\mu]) + 1. \tag{13}$$

Then

$$Cycl(\Gamma_0) \leq A(n) \left[\frac{n(4n^2 + 16n + 37) + 3}{4}\right] = n^3 + 4n^2 + 9n + \left[\frac{n+3}{4}\right].$$

In particular, $A(n)$ *depends only on* $k(I_1, I_2, J, L)$*.*

For all values of $\mu > 1$, the condition to have a homoclinic loop of codimension 1 is $\alpha_1 \neq 0$. If $\alpha_1 = 0$ and $C_{10} \neq -1$, then Γ_0 is of codimension ≥ 2. For all higher codimensions, the conditions depend on the value of μ.

In table 1, we give the conditions for small codimensions together with the type.

From now on we will assume that Γ_0 has finite codimension k, i.e. that there exist I_1, I_2, J and L such that Γ_0 is of type (I_1, I_2, J, L).

2.5. New Parametrization on T_1

Following the ideas introduced by Jebrane and Mourtada in [9] and the techniques used in [5], we look for a "good" parametrization of the transversals Σ_1 and T_1. In the new parametrization the asymptotic expansion of the displacement map will be easier to compute.

The idea of the new parametrization is to change Y_1 to Y so that the first coordinate of R_λ becomes linear in Y. Namely, let $(Y_1, Z_1) \in \mathrm{T}_1$. We note by (x, z) its image on Σ_1 by the diffeomorphism $\mathcal{R}_\lambda$, i.e. $(x, z) = \mathcal{R}_\lambda(Y_1, Z_1)$. Then from Equation (2) we have the following.

$$\mathcal{R}_\lambda(Y_1, Z_1) = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} C_0(\lambda) \\ D_0(\lambda) \end{pmatrix} + \begin{pmatrix} f_{1,\lambda}(Z_1) \\ f_{2,\lambda}(Z_1) \end{pmatrix} + \begin{pmatrix} r_{1,\lambda}(Y_1, Z_1) \\ r_{2,\lambda}(Y_1, Z_1) \end{pmatrix}, \tag{14}$$

where $f_{i,\lambda}(0) = 0 = r_{i,\lambda}(0, Z)$. Let

$$\Theta(Y_1, Z_1) = \begin{pmatrix} -r_{1,\lambda}(Y_1, Z_1) \\ Z_1 \end{pmatrix}, \tag{15}$$

and set $(Y, Z) = \Theta(Y_1, Z_1)$ as the new parameterization of T_1. Since by hypothesis $C_{10}(0) \neq 0$, we have

$$Jac_{(Y_1,Z_1)}(Y, Z)(0, 0) = \begin{vmatrix} -C_{10}(\lambda) & 0 \\ 0 & 1 \end{vmatrix} = -C_{10}(\lambda) > 0,$$

for all $\lambda \in \Lambda$. We can thus inverse Equation (15). We obtain a solution $(Y_1, Z_1) = \Theta^{-1}(Y, Z)$, where

$$\Theta^{-1}(Y, Z) = \begin{pmatrix} \displaystyle\sum_{\substack{1 \le i+j \le K \\ i>0}} \eta_{ij}(\lambda) Y^i Z^j + Y \cdot a_{K,\lambda}(Y, Z), \\ \\ Z \end{pmatrix} = \begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} \tag{16}$$

in which $a_{K,\lambda}(Y, Z)$ is C^{K-1} and $(K-1)$–flat at $(0, 0)$.

LEMMA 15. *The coefficients $\eta_{ij}(\lambda)$ in Equation (16) are given below.*

1. $\eta_{10}(\lambda) = \big(-C_{10}(\lambda)\big)^{-1}$;

2. $\eta_{ij}(\lambda) = C_{ij}(\lambda)\eta_{10}^{i+1}(\lambda) + P_{ij}(\lambda)$, where $P_{ij}(\lambda)$ is a polynomial in $C_{10}^{-1}(\lambda)$ and the $C_{\ell m}(\lambda)$ with $(l, m) \prec (i, j)$.

Proof. We simply substitute Equation (15) in Equation (16).

1. For the coefficient of Y, we obtain the relation

$$-\eta_{10}(\lambda)C_{10}(\lambda) = 1.$$

2. Using induction, we obtain that the coefficient of $Y_1^I Z_1^J$ is given by the relation

$$(-1)^I C_{10}^I(\lambda)\eta_{IJ}(\lambda) + P'_{ij}(\lambda) - \eta_{10}(\lambda)C_{IJ}(\lambda) = 0.$$

∎

Let us note by $\tilde{R}_\lambda(Y,Z) = \mathcal{R}_\lambda \circ \Theta^{-1}(Y,Z)$ the expression of $\mathcal{R}_\lambda$ in the new parameterization (15):

$$\tilde{R}_\lambda(Y,Z) = \begin{pmatrix} C_0(\lambda) \\ D_0(\lambda) \end{pmatrix} + \begin{pmatrix} f_{1,\lambda}(Z) - Y \\ f_{2,\lambda}(Z) + r_{2,\lambda}\big(Y_1(Y,Z),Z\big) \end{pmatrix}, \tag{17}$$

where every function is C^K in (Y,Z,λ).

Consider the displacement map

$$\tilde{\delta}_\lambda(Y,Z) = \begin{pmatrix} \tilde{\delta}_{1,\lambda}(Y,Z) \\ \tilde{\delta}_{2,\lambda}(Y,Z) \end{pmatrix} = \tilde{G}_\lambda(Y,Z) - \tilde{\Delta}_\lambda^{-1}(Y,Z) \tag{18}$$

$$\begin{pmatrix} \tilde{G}_\lambda(Y,Z) \overset{\mathrm{d}ef}{=} \mathcal{R}_\lambda \circ \Delta_\lambda \circ \mathcal{R}_\lambda \circ \Theta^{-1}(Y,Z) \\ \tilde{\Delta}_\lambda^{-1}(Y,Z) \overset{\mathrm{d}ef}{=} \Delta_\lambda^{-1} \circ \Theta^{-1}(Y,Z) \end{pmatrix}.$$

The map $\tilde{\delta}_\lambda(Y,Z)$ has, for small values of the parameter, the same number of zeros as the 2-return map.

Let

$$X = C_0(\lambda) + f_{1,\lambda}(Z) - Y = \tilde{R}_{1,\lambda}(Y,Z). \tag{19}$$

Then from equations (3) and (17), we have that

$$\Delta_\lambda \circ \tilde{R}_\lambda(Y,Z)$$
$$= \begin{bmatrix} X + \displaystyle\sum_{i=1}^{K} \alpha_i X^i \omega(X,\lambda)\big(1+\psi_{i,\lambda}(X)\big) + \phi_{1,K,\lambda}(X) \\ X^\mu \left(\displaystyle\sum_{i+j=0}^{K} D_{ij}(\eta_{10}Y)^i Z^j \big((1+O(\lambda)+\varphi_{ij,\lambda}(X,Z)\big) + \phi_{2,K,\lambda}(X,Z) \right) \end{bmatrix}$$

where $D_{00}(\lambda) = D_0(\lambda)$, every coefficient is a function in λ, $\varphi_{ij,\lambda}(X,0)$ is $I_0^K(X)$, $\varphi_{ij,\lambda}(0,Z)$ and $\phi_{i,K,\lambda}(X,Z)$ are C^K and K-flat.

Since the map $(Y, Z) \to (X, Z)$ in Equation (19) is a diffeomorphism, we can work with either system of coordinates.

2.6. The Blow-up

It is still out of reach to control directly expansions with terms of the form $X^i \omega^j(X)$ and of the form $Y^j \overline{\omega}^m(Y)$. The main step in bounding the number of zeros of Equation (18) is to use an adequate blow-up that will allow us to:

- extend in a "bijective" way the function $\tilde{\Delta}_\lambda^{-1}(Y, Z)$ to $Y = 0$.
- bring the domain of the functions to "square-like" domains.
- divide the study of this dynamics in several regions in order to avoid to have simultaneously X and Y small.

We will blow-up the variables X and Z (this will induce a blow-up of the Y variable).

Before introducing the blow-up, we notice that the system in normal form (1) is invariant under coordinate changes of the form

$$(x, y, z) = (x, y, Az),$$

so we can assume that either $z_0 = 0$ or $z_0 = 1/2$. (We choose $z_0 = 1/2$ instead of $z_0 = 1$ because this allows to work in the region $|z| < B < 1$). Also, from Equation (3), we have

$$\Delta_{2,\lambda}(x, z) = z x^{\mu(\lambda)} \left[1 + \varphi_{2,K,\lambda}(x)\right],$$

where, by Fact 8.5., $x \in [0, C_0(\lambda)]$ and, since we are working in a small neighborhood of $(0, z_0) \in \Sigma_1$, $z \in [z_0 - \epsilon_0, z_0 + \epsilon_0]$ with $\epsilon_0 > 0$ as small as we want. We have that

$$|Z| = |\Delta_{2,\lambda}(x, z)| \leq \big(|z_0| + \epsilon_0\big) x^{\mu(\lambda)} [1 + \epsilon_1] \leq x^{\mu(\lambda)},$$

where $C_0(\lambda) \leq \epsilon_1$ for all $\lambda \in \Lambda$, i.e. N–curves intersect transversals T_1 and Σ_1 in specific regions which we call *domains of interest*.

This suggests the blow-up $(X, Z) = \Phi(s, t)$, where

$$\Phi(s, t) = \begin{pmatrix} tC_0(\lambda) \\ sC_0^{\mu(\lambda)}(\lambda) \end{pmatrix} = \begin{pmatrix} X \\ Z \end{pmatrix} \tag{20}$$

where, by Fact 8.5., $t \in [0, 1]$.

This blow-up has two important consequences.

1. In the blow-up coordinates, the point corresponding to Γ_0 has coordinates $(s, t) = (s_0, 0)$, where $s_0 = z_0\big(-C_{10}(0)\big)^{-\mu}$. Indeed, it is clear,

from the construction of the blow-up, that $t = 0$ corresponds to the z–axis on Σ_1. Its inverse image by $\mathcal{R}_\lambda$ yields the upper bound of the coordinate Y of the domain of interest on T_1. This is reflected in the first line of Equation (17):

$$Y = C_0(\lambda) + f_{1,\lambda}(Z) - X = C_0(\lambda)(1 - t) + f_{1,\lambda}(Z).$$

On the other hand,

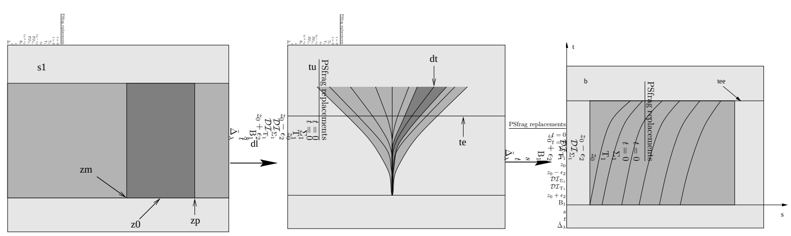
$$\begin{aligned} \Delta_{2,\lambda}^{-1}\big(Y_1(Y,Z),Z\big) &= \tilde{\Delta}_{2,\lambda}^{-1}(Y,Z) \\ &= \tilde{\Delta}_{2,\lambda}^{-1}\big(tC_0(\lambda), sC_0^{\mu(\lambda)}(\lambda)\big) \end{aligned} \tag{21}$$

Then if we evaluate (21) at $|\lambda| = 0 = t$

$$\begin{aligned} \Delta_{2,\lambda}^{-1}\big(Y_1(Y,Z),Z\big)\Big|_{|\lambda|=0=t} &= s_0\big(-C_{10}(0)\big)^{\mu} \\ &= \tilde{R}_{2,\lambda}\big(tC_0(\lambda), sC_0^{\mu(\lambda)}(\lambda)\big)\Big|_{|\lambda|=0=t} \\ &= z_0. \end{aligned}$$

Equation (21) also implies that $s\big(-C_{10}(0)\big)^{\mu} \in \big[z_0 - \epsilon_0, z_0 + \epsilon_0\big]$.

2. Geometrically, this blow-up acts on T_1 by separating curves with different asymptotic behavior at $Y = 0$ allowing an extension, in the domain of interest on Σ_1, of the diffeomorphism $\tilde{\Delta}_\lambda(x,z)$ to the values with $x = 0$ (cf. Figure 5 for the case $z_0 = 1/2$): the domain of interest on Σ_1 ($\mathcal{DI}_{\Sigma_1}$) is illustrated in Figure 5.a ; the image of $\mathcal{DI}_{\Sigma_1}$ by Δ_λ, noted $\mathcal{DI}_{\mathrm{T}_1}$ is illustrated in Figure 5.b; finally, we illustrate $\mathcal{DI}_{\mathrm{T}_1}$ in the blown-up coordinates (s,t).



(a) Domain of interest on Σ_1: $\mathcal{DI}_{\Sigma_1}$.

(b) Domain of interest on T_1: $\mathcal{DI}_{\mathrm{T}_1}$.

(c) Blow-up of the domain of interest on T_1.

Figure 5. Effect of the blow-up on $\tilde{\Delta}_\lambda(\Sigma_1)$ in the case $z_0 = 1/2$.

2.7. Dividing the Study in Two Regions

Equation (18) has the same number of zeros as the 2-return map. Hence we may get an explicit bound for the cyclicity (given in Theorem 14) by studying the zeros of Equation (18) in the blown-up coordinates for $t \in [0,1]$. It is convenient to divide the study into the following three regions:

$$t \in [0,\epsilon] \quad , \quad t \in [\epsilon, 1-\epsilon] \quad \text{, and} \quad t \in [1-\epsilon, 1].$$

In this section we will show that it is sufficient to only consider the values $t \in [0, 1-\epsilon]$, i.e. only the two regions $t \in [0,\epsilon]$ and $t \in [\epsilon, 1-\epsilon]$. The intuitive geometric reason is an argument of symmetry: what occurs near $t = 1$ is similar to what occurs near $t = 0$. First we need to develop the different terms in Equation (18).

Let

$$U(t) = C_0(\lambda)(1-t) \quad ; \quad X(t) = C_0(\lambda)t. \tag{22}$$

Definition 16. The notation $\mathcal{O}_{A,\lambda}$ is used to denote a function which is at least $J_0^{K-2(I_1+JI_2+[J\mu]+1)}$. ($A$ is a multi-index numbering such functions.)

For all $0 < \epsilon < 1$ we have from (19) and (20) that in the blown-up coordinates and for $t \in [0, 1-\epsilon]$,

$$\begin{aligned} Y = C_0(\lambda)(1-t) + f_{1,\lambda}\big(C_0^{\mu}(\lambda)s\big) &= C_0(\lambda)(1-t)\left(1 + \frac{f_{1,\lambda}\big(C_0^{\mu}(\lambda)s\big)}{C_0(\lambda)(1-t)}\right) \\ &= U(t)\big(1 + \mathcal{O}_{1,\lambda}(s,t)\big), \end{aligned}$$

and from (16)

$$\begin{aligned} Y_1 &= \sum_{i+j=1}^{K} \eta_{ij} C_0^{i+j\mu}(1-t)^i s^j \big(1 + \mathcal{O}_{2,i,j,\lambda}(s,t)\big) + C_0^K \mathcal{O}_{3,K,\lambda}(s,t) \\ &= \eta_{10} U(t)\big(1 + \mathcal{O}_{4,\lambda}(s,t)\big). \end{aligned} \tag{23}$$

Lemma 17. *Let $\overline{\omega}(Y_1,\lambda)$ be defined as in* (5). *Then*

$$\begin{aligned} Y_1^i \overline{\omega}(Y_1,\lambda) = {} & \eta_{10}^{i-\overline{\alpha}_1(\lambda)} U^i(t) \overline{\omega}\big(U(t),\lambda\big)\big(1 + \mathcal{O}_{5,i,\lambda}(s,t)\big) \\ & + \eta_{10}^i U^i(t)\Big(\overline{\omega}(\eta_{10},\lambda)\big(1 + \mathcal{O}_{6,\lambda}(s,t)\big) + \mathcal{O}_{7,\lambda}(s,t)\Big). \end{aligned} \tag{24}$$

Proof. From Equation (23) we have

$$\overline{\omega}(Y_1,\lambda) = \overline{\omega}\big(U(t),\lambda\big)\big(\eta_{10}^{-\overline{\alpha}_1(\lambda)} + \mathcal{O}_{8,\lambda}(s,t)\big) + \overline{\omega}\Big(\eta_{10}\cdot\big(1+\mathcal{O}_{9,\lambda}(s,t)\big),\lambda\Big).$$

▮

Lemma 18. *Let $\Phi(s,t)$ be defined in Equation (20).*

$$\tilde{\Delta}_\lambda^{-1}\circ\Phi(s,t) = \begin{pmatrix} \displaystyle\sum_{\substack{1\le i+j\le K\\(i>0)}} \eta_{ij}C_0^{i+j\mu}(1-t)^i s^j\big(\zeta_i(\lambda)+\mathcal{O}_{10,i,j,\lambda}(s,t)\big) \\ \eta_{10}^{-\mu}(1-t)^{-\mu}s\big(1+\mathcal{O}_{11,\lambda}(s,t)\big)\end{pmatrix}$$
$$+\begin{pmatrix}\displaystyle\sum_{i=1}^{K}\overline{\alpha}_i\eta_{10}^{i-\overline{\alpha}_1}U^i(t)\overline{\omega}\big(U(t),\lambda\big)\big(1+\mathcal{O}_{5,i,\lambda}(s,t)\big)+C_0^K\overline{\phi}_{3,K,\lambda}(s,t)\\ 0\end{pmatrix} \tag{25}$$

where $\zeta_i(\lambda) = 1+\eta_{10}^i\overline{\omega}(\eta_{10},\lambda)$ and $\overline{\phi}_{3,K,\lambda}$ is C^K and K–flat at 0.

Proof. We substitute Equation (24) in the first component of Equation (4). ▮

Lemma 19. *Let $X(t)$ be defined in (22), Φ in (20) and $\tilde{G}_\lambda$ in (18).*

$$\tilde{G}_\lambda\circ\Phi(s,t) = \begin{pmatrix}C_0\\D_0\end{pmatrix} + \begin{pmatrix}C_{10}\\D_{10}\end{pmatrix}\sum_{i=1}^{K}\alpha_i X^i(t)\omega\big(X(t),\lambda\big)\Big(1+\mathcal{O}_{12,i,\lambda}(t)\Big)$$
$$+\sum_{i+j=1}^{K}\begin{pmatrix}C_{ij}\\D_{ij}\end{pmatrix}X^{i+j\mu}(t)\Big(D_0+f_{2,\lambda}(C_0^\mu s)+r_{2,\lambda}\big(\tilde{Y}_1\circ\Phi(s,t),C_0^\mu s\big)\Big)^j$$
$$+C_0^K\begin{pmatrix}\phi_{3,K,\lambda}(s,t)\\ \phi_{4,K,\lambda}(s,t)\end{pmatrix}, \tag{26}$$

where $\phi_{i,K,\lambda}$ are C^K and K–flat at 0.

Proof. The result comes from the fact that $C_{10}(\lambda)\neq 0$,

$$\alpha_\ell X^\ell\omega(X,\lambda)(1+f)+\alpha_\ell X^{\ell+k}\omega^j(X,\lambda)(1+g) = \alpha_\ell X^\ell\omega(X,\lambda)(1+h),$$

and if $\ell\le k$

$$\alpha_\ell X^\ell\omega(X,\lambda)(1+f)+*\alpha_\ell\alpha_k X^{\ell+k}\omega^j(X,\lambda)(1+g) = \alpha_\ell X^\ell\omega(X,\lambda)(1+h)$$

where, if f and g are I_0^K, h is I_0^K. ∎

Let us denote

$$\delta_{1,\lambda}(s,t) \stackrel{\text{def}}{=} \tilde{\delta}_{1,\lambda} \circ \Phi(s,t),$$

where $\tilde{\delta}_{1,\lambda}$ is defined as the first line of (18).

PROPOSITION 20. *There exists $\epsilon > 0$ sufficiently small such that for each intersection point (s,t) of a 2-periodic orbit with the transversal T_1 with $t \in (1-\epsilon, 1]$, the t-coordinate of the second intersection point necessarily belongs to $[a, 1-\epsilon]$ with $a \in [0,\epsilon]$. Hence the number of 2–periodic orbits is bounded by the number of fixed points of P_λ^2 with t-coordinate in $[0, 1-\epsilon]$.*

Proof. We are looking for orbits of period 2. Any such orbit generates two fixed points of the 2-return map. Also, when there exists an orbit of period 2, the orbit of period 1 exists.

Let M^2 be the 2-dimensional invariant manifold containing all the bifurcating dynamics, $t_1(\lambda)$ and $t_2(\lambda)$ be the t-coordinates of the intersection points of the orbit of period 2 with T_1, and $t_0(\lambda)$ be the one with the orbit of period 1. It was shown in [17] that the intersection of M^2 with Σ_1 (and thus with T_1) is a graph, thus $t_1(\lambda) < t_0(\lambda) < t_2(\lambda)$.

To show the lemma, it is then sufficient to show that, whenever an orbit of period 2 exists, then $t_1(\lambda)$ and $t_2(\lambda)$ cannot be both close to 1 for $\lambda \in \Lambda$.

Let us first look at Equation (18). We have that

$$\begin{aligned}\frac{\delta_{1,\lambda}(s,t)}{C_0(\lambda)} = 1 &+ C_{10}(\lambda) C_0^{-\alpha_1(\lambda)}(\lambda) t^{1-\alpha_1(\lambda)} \\ &- \eta_{10}(\lambda) C_0^{-\overline{\alpha}_1(\lambda)}(\lambda)(1-t)^{1-\overline{\alpha}_1(\lambda)} + O(\lambda).\end{aligned}$$

Let $\mathcal{L}_{12}$ be the straight line in T_1 passing through (t_1, s_1) and (t_2, s_2). $\mathcal{L}_{12}$ can be parametrized by t. The first derivative of the restriction of $\delta_{1,\lambda}(s,t)$ to $\mathcal{L}_{12}$ is of the form

$$*C_0^{-\alpha_1(\lambda)}(\lambda) t^{-\alpha_1(\lambda)} + *C_0^{-\overline{\alpha}_1(\lambda)}(\lambda)(1-t)^{-\overline{\alpha}_1(\lambda)} + O(\lambda). \tag{27}$$

Since $\delta_{1,\lambda}(s,t)$ has at least two zeros in $\mathcal{L}_{12}$, the expression (27) has at least one zero, say t_3. Thus for any $\epsilon_1 > 0$ sufficiently small, both $C_0^{-\alpha_1(\lambda)}$ and $C_0^{-\overline{\alpha}_1(\lambda)}$ must be bounded, i.e. we are interested in the region Λ_1 of the parameter space Λ where there exist $m, M > 0$ such that

$$0 < m < C_0^{-\alpha_1(\lambda)}(\lambda) < M. \tag{28}$$

Indeed, when $t_3 \in [\epsilon_1, 1-\epsilon_1]$, condition (28) follows directly from the vanishing of (27). If $t_3 \in [0, \epsilon_1)$ or $t_3 \in (1-\epsilon_1, 1]$, we need to split the discussion in the two cases $\alpha_1 < 0$ and $\alpha_1 > 0$. In the case $\alpha_1 > 0$ and for sufficiently small λ, $C_0^{-\alpha_1(\lambda)}(\lambda) \gg 0$ and $C_0^{-\overline{\alpha}_1(\lambda)}(\lambda)$ is small. Moreover, we have that for all $t \in (0,1)$

$$\begin{aligned} 0 \leq (1-t)^{-\overline{\alpha}_1(\lambda)} &\leq 1; \\ t^{-\alpha_1(\lambda)} &\geq 1. \end{aligned} \tag{29}$$

From Equation (29), $C_0^{-\alpha_1(\lambda)}(\lambda)t^{-\alpha_1(\lambda)} \gg 0$. The vanishing of equation (27) at t_3 excludes t_3 small and t_3 large.

In the case $\alpha_1 < 0$ we use the same argument as in the case $\alpha_1 > 0$ where we interchange t and $(1-t)$, and also $\alpha_1(\lambda)$ and $\overline{\alpha}_1(\lambda)$. ▮

3. THE FINITE CYCLICITY PROPERTY

3.1. Reduction to a 1−Variable Problem

This reduction is made possible by the existence of the invariant Möbius band. In practice it is achieved via the implicit function theorem to solve $\delta_{2,\lambda}(s,t) = \tilde{\delta}_{2,\lambda} \circ \Phi(s,t) = 0$, yielding s as a function of t.

Let us introduce the two following variables:

$$\nu_1 = X(t)\omega\big(X(t),\lambda\big) \qquad \text{and} \qquad \nu_2 = U(t)\overline{\omega}\big(U(t),\lambda\big) \tag{30}$$

where $X(t)$ and $U(t)$ are defined in (22).

Since $Y = U(t) + f_{1,\lambda}\big(C_0^{\mu}s\big)$, using Lemma 17, we can consider the function $\delta_{2,\lambda}(s,t)$ (the second line of (18)) as a C^K function of the variables s, t, t^{μ}, ν_1 and ν_2. We use the notation:

$$F_{\lambda}\left(s,t,t^{\mu},\nu_1,\nu_2\right) = \delta_{2,\lambda}(s,t), \tag{31}$$

i.e. F_{λ} is C^K in its variables. For all points of the curve

$$s_1(t) = D_0\eta_{10}^{\mu}(0)(1-t)^{\mu}, \tag{32}$$

we have:

$$\begin{cases} F_0\left(s_1(t),t,0,0\right) = 0 \\ \partial_s F_0\left(s_1(t),t,0,0\right) = -\big(-C_{10}^{-1}(0)(1-t)\big)^{-\mu} < 0. \end{cases}$$

We can apply the implicit function theorem to Equation (31) to solve for s around any solution of Equation (32) in a small neighborhood of $\lambda = 0$.

Moreover, for a sufficiently small neigbourhood Λ' of $\lambda = 0$ we can write s explicitly in terms of (t, t^μ, ν_1, ν_2) which are functions of t only. From Lemmae 18 and 19, the equation $F_\lambda = 0$ is equivalent to (after substitution of the ν_i using Equation (30)):

$$
\begin{aligned}
&s \cdot \bigl(1 + \mathcal{O}_{13,\lambda}(s,t)\bigr) = \eta_{10}^{\mu}(1-t)^{\mu} \\
&\times \left[\sum_{i=0}^{K} D_{i0} \left(C_0 t + \sum_{j=1}^{K} \alpha_j (C_0 t)^j \omega(C_0 t, \lambda)\bigl(1 + \mathcal{O}_{14,j}(t)\bigr) \right)^{i} + D_{01}(C_0 t)^{\mu} \right. \\
&\left. \times \left(\sum_{i=0}^{K} D_{i0} \eta_{10}^{i} \bigl(C_0(1-t)\bigr)^{i} \bigl(1 + \mathcal{O}_{15,i,\lambda}(s,t)\bigr) + D_{01} C_0^{\mu} s \bigl(1 + \mathcal{O}_{16,\lambda}(s,t)\bigr) \right) \right].
\end{aligned} \tag{33}
$$

Lemma 21. *The zeros of $\delta_{2,\lambda}(s,t)$ (the second line of (18)) in the neighbourhood of a solution of Equation (32) are of the form $\bigl(s(t), t\bigr)$ where*

$$
\begin{aligned}
&s(t) = \eta_{10}^{\mu}(1-t)^{\mu} \\
&\quad \times \sum_{i=0}^{K} D_{i0} \Biggl[\Bigl(C_0 t \bigl(1 + \mathcal{O}_{17}(t)\bigr) \quad + \alpha_1 (C_0 t) \omega(C_0 t, \lambda) \bigl(1 + \mathcal{O}_{18}(t)\bigr) \Bigr)^{i} \\
&\qquad\qquad + D_{01} \eta_{10}^{i} C_0^{i+\mu} t^{\mu} (1-t)^{i} \bigl(1 + \mathcal{O}_{19,i}(t)\bigr) \Biggr]
\end{aligned} \tag{34}
$$

Proof. Equation (34) is obtained directly from Equation (33) using the fact that, since $D_{10} \neq 0$ or $D_{01} \neq 0$ (because $\mathcal{R}_\lambda$ is a diffeomorphism), we can group all terms either in a term with a coefficient $D_{i0} \neq 0$ or with coefficient D_{01}. ∎

We use the notation

$$
V_\lambda(t) = \delta_{1,\lambda}\bigl(s(t), t\bigr).
$$

Proposition 22. *The fixed points of the 2–return map are in one to one correspondance with the zeros of the map $V_\lambda(t)$, where*

$$
\begin{aligned}
V_\lambda(t) &= c + \sum_{i=1}^{K} C_0^i \Bigg(\alpha_i t^i \omega(C_0 t, \lambda)\big(1 + \mathcal{O}_{20,i}(t)\big) \\
&- \overline{\alpha}_i (1-t)^i \overline{\omega}(C_0(1-t), \lambda)\big(1 + \mathcal{O}_{21,i}(t)\big)\Bigg) + \sum_{i+j=1}^{K} C_0^{i+j\mu} \sum_{\substack{||M||=j\\ M=(m_\ell)\\ 0\le \ell\le k}} \binom{j}{M} \\
&\qquad \times \prod_{\ell=0}^{K} \big(D_{\ell 0} C_0^\ell\big)^{m_\ell} \Bigg(C_{ij} t^{i+j\mu}(1-t)^{\sum \ell m_\ell}\big(1 + \mathcal{O}_{22,M,i,j}(t)\big) \\
&\qquad\qquad - (1-\delta_{i0})\eta_{ij}(1-t)^{i+j\mu} t^{\sum \ell m_\ell}\big(1 + \mathcal{O}_{23,M,i,j}(t)\big)\Bigg) \qquad (35)
\end{aligned}
$$

where $c = c(\lambda)$ is some constant, δ_{i0} is the Kronecker delta, and $\binom{j}{M}$ is the multinomial coefficient

$$
\binom{j}{M} = \frac{j!}{m_1! m_2! \cdots m_K!}.
$$

Remark: Note that Γ_0 is of finite codimension if and only if at least one of the coefficients in $V_\lambda(t)$ is nonvanishing, up to an adequate power of C_0.

Proof (Proof of Proposition 22). We need to apply Lemmae 18 and 19 in which we replace s by its value $s(t)$ given in Lemma 21. To substitute it in equations (25) and (26) we first need to calculate s^j and $\big(D_0 + f_{2,\lambda}(C_0^\mu s) + \tilde{r}_{2,\lambda} \circ \Phi(s,t)\big)^j$ (see (14) and (17)).

$$
\begin{aligned}
s^j(t) = \eta_{10}^{j\mu}(1-t)^{j\mu} \Bigg(&\sum_{\substack{||M||=j\\ M=(m_\ell)}} \binom{j}{M} \prod_{\ell=0}^{K} \big(D_{\ell 0} C_0^\ell t^\ell\big)^{m_\ell}\big(1 + \mathcal{O}_{24,M}(t)\big) \\
&+ \alpha_1 (C_0 t)\omega(C_0 t, \lambda) \cdot F_{1,j,\lambda}(t)\Bigg).
\end{aligned}
$$

Also we have

$$\begin{aligned}
&D_0 + f_{2,\lambda}(C_0^\mu s) + \tilde{r}_{2,\lambda} \circ \Phi(s,t)\\
&= D_0\big(1+\mathcal{O}_{25}(t)\big) + D_{01}C_0^\mu s\big(1+\mathcal{O}_{26}(t)\big)\\
&\quad + \sum_{i=1}^{K} D_{i0}\eta_{10}^i C_0^i (1-t)^i \big(1+\mathcal{O}_{27,i}(t)\big)\\
&= D_0\big(1+\mathcal{O}_{28}(t)\big) + \sum_{i=1}^{K} D_{i0}\eta_{10}^i C_0^i (1-t)^i \big(1+\mathcal{O}_{29,i}(t)\big)\\
&\quad + D_{01}C_0^\mu \eta_{10}^\mu (1-t)^\mu \sum_{i=1}^{K} D_{i0}\Bigg(\Big(C_0 t\big(1+\mathcal{O}_{30,i}(t)\big)\\
&\quad + \alpha_1 C_0 t\omega(C_0 t,\lambda)\big(1+\mathcal{O}_{31,i}(t)\big)\Big)^i + D_{01}C_0^{i+\mu}t^\mu(1-t)^i\big(1+\mathcal{O}_{32,i}(t)\big)\Bigg)\\
&= D_0\big(1+\mathcal{O}_{28,i}(t)\big) + \sum_{i=1}^{K} D_{i0}C_0^i(1-t)^i\Big(\eta_{10}^i\big(1+\mathcal{O}_{29,i}(t)\big) + *C_0^\mu t^\mu \mathcal{O}_{33,i}(t)\Big)\\
&\quad + D_{01}C_0^\mu \eta_{10}^\mu (1-t)^\mu\\
&\quad \times \sum_{i=1}^{K} D_{i0}\Big(C_0 t\big(1+\mathcal{O}_{30,i}(t)\big) + \alpha_1 C_0 t\omega(C_0 t,\lambda)\big(1+\mathcal{O}_{31,i}(t)\big)\Big)^i\\
&= D_0\big(1+\mathcal{O}_{28}(t)\big) + \sum_{i=1}^{K} D_{i0}\eta_{10}^i C_0^i (1-t)^i \big(1+\mathcal{O}_{34,i}(t)\big)\\
&\quad + D_{01}C_0^\mu \eta_{10}^\mu (1-t)^\mu \sum_{i=1}^{K} D_{i0}\alpha_1 C_0 t\omega(C_0 t,\lambda)\big(1+\mathcal{O}_{31,i}(t)\big)F_{2,i,\lambda}(t),
\end{aligned}$$

where $F_{2,1,\lambda}(t) \equiv 1$ and for $i > 1$, the $F_{2,i,\lambda}(t)$ are $I_0^K(t)$. Therefore

$$\begin{aligned}
&\big(D_0 + f_{2,\lambda}(C_0^\mu s) + \tilde{r}_{2,\lambda} \circ \Phi(s,t)\big)^j\\
&\qquad = \sum_{\substack{||M||=j\\ M=(m_\ell)}} \binom{j}{M} \prod_{\ell=0}^{K} \big(D_{\ell 0}\eta_{10}^\ell C_0^\ell (1-t)^\ell\big)^{m_\ell}\big(1+\mathcal{O}_{35,M}(t)\big)\\
&\qquad\qquad + \alpha_1 C_0^{1+\mu} t\omega(C_0 t,\lambda)F_{2,\lambda}(t),
\end{aligned}$$

where, for terms in ω, all the D_{i0} are included in $F_{2,\lambda}(t)$ which is $I_0^K(t)$. The result follows from Lemmae 18 and 19. We have used the hypothesis that Γ_0 is of finite codimension to get rid of the higher order terms in the expansion. Indeed there exists at least one nonvanishing term of the expansion in which we can include the higher order terms. ∎

Corollary 23. *For codimensions* 1 *and* 2, $V_\lambda(t)$ *is of the same form as studied in [5], i.e.*

$$V_\lambda(t) = c + C_0\Big(\alpha_1 t\omega(C_0t,\lambda)\big(1+\mathcal{O}_{20,1}(t)\big)$$
$$-\overline{\alpha}_1(1-t)\overline{\omega}(C_0(1-t),\lambda))\big(1+\mathcal{O}_{21,1}(t)\big)\Big) + C_0t(C_{10}+\eta_{10})\big(1+\mathcal{O}_{36}(t)\big).$$

We will limit our study to codimensions $k > 2$ $\big($i.e. $\alpha_1(0) = 0$ and $C_{10}(0) = -1 = -\eta_{10}(0)\big)$.

3.2. The differentiability properties of the generalized monomials

In the region $t \in [0,\epsilon]$, we use a derivation-division algorithm on $V_\lambda(t)$ which is a generalization of Rolle's theorem. Each derivation must kill one term. In between the derivations we multiply the function by functions which are positive for t in the whole region $(0, 1-\epsilon)$. The details of the algorithm are long to write and lead to an explicit bound which is a function of $\mu(\lambda)$, see Propositions 26 and 27.

We recall the nice differential properties of the generalized monomials (which can be found in [17] for instance).

1. Everywhere in the sequel, $*$ denotes a nonvanishing constant (which may be a differentiable function of λ).
2.

$$\frac{d\omega(x,\lambda)}{dx} = -x^{-1-\alpha_1(\lambda)} = x^{-1}\big(\alpha_1(\lambda)\omega(x,\lambda)+1\big).$$

3. The derivative of a monomial $g = x^\beta\omega^\ell(x,\lambda)$ is

$$\frac{dg}{dx} = *x^{\beta-1}\omega^\ell(x,\lambda)[1+g_1(x,\lambda)],$$

where $g_1(x,\lambda)$ is $I_0^K(x)$.

4. More generally, if i and ℓ are integers such that $\ell \le i \le h$, then

$$\frac{d^h\big(x^i\omega^\ell(x,\lambda)\big)}{dx^h} = \begin{cases} x^{i-h-\alpha_1(\lambda)}\displaystyle\sum_{j=0}^{\ell-1} *\omega^j(x,\lambda) & \text{if } i<h \\ \displaystyle\sum_{j=0}^{\ell} *\omega^j(x,\lambda) & \text{if } i=h. \end{cases}$$

5. If $h < \beta$,

$$\frac{d^h\big(x^\beta \omega^\ell(x,\lambda)\big)}{dx^h} = *x^{\beta-h}\omega^\ell(x,\lambda)[1+f_{\beta h\ell}(x,\lambda)],$$

where $f_{\beta h\ell}(x,\lambda)$ is $I_0^K(x)$.

The n–th derivative of a generalized monomial $f_1 = x^{i+j\mu}\omega^\ell(x,\lambda)$ is thus given by

$$\frac{\partial^n f_1}{\partial x^n} = \begin{cases} *x^{i-n+j\mu(\lambda)}\omega^\ell\big(x,\lambda\big)[1+f_{ij\ell n}(x,\lambda)] & \text{if } j\geq 1 \text{ or} \\ & j=0 \text{ and } n<i \\ \displaystyle\sum_{k=0}^{\ell} *\omega^k\big(x,\lambda\big) & \text{if } j=0 \text{ and } n=i \\ *x^{i-n-\alpha_1(\lambda)}\omega^{\ell-1}\big(x,\lambda\big)[1+f_{ij\ell n}(x,\lambda)] & \text{if } j=0 \text{ and } n>i, \end{cases}$$

where $f_{ijln}(x,\lambda)$ are $I_0^{K-n}(x)$.

Lemma 24.

1.Let $f_i(X,\lambda)$ *be* $I_0^{K-n}\big(\rho(X,\lambda)\big)$, *and let* $F_i(t) \overset{\text{def}}{=} f_i\big(tC_0(\lambda)\big)$. *Then* $F_i(t)$ *is* $J_0^{K-n}(t)$.

2.Let $\overline{f}_i(Y,\lambda)$ *be* $I_0^{K-n}\big(\overline{\rho}(Y,\lambda)\big)$, *and let* $\overline{F}_i(t) \overset{\text{def}}{=} \overline{f}_i\big(C_0(\lambda)(1-t+\tilde{\overline{f}}_{1,\lambda}(t))\big)$. *Then on* $[0,1-\epsilon]$, $\overline{F}_i(t,\lambda)$ *is analytic in* t *and* $\lim_{\lambda\to 0}\overline{F}_i(t,\lambda)=0$ *uniformly.*

Proof. We have that for all $0\leq n\leq K-(k+1)$

$$\lim_{X\to 0}\big(X^{1+\alpha_1(\lambda)}\omega\big)^n\frac{\partial^n f_i(X,\lambda)}{\partial X^n} = 0 = \lim_{Y\to 0}\big(Y^{1+\overline{\alpha}_1(\lambda)}\overline{\omega}\big)^n\frac{\partial^n\overline{f}_i(Y,\lambda)}{\partial Y^n},$$

uniformly for $\lambda\in\Lambda$. Since $\dfrac{X}{X^{1+\alpha_1(\lambda)}\omega(X,\lambda)}$ is bounded, we then have the following limit:

$$\lim_{X\to 0}X^n\frac{\partial^n f_i(X,\lambda)}{\partial X^n} = 0 = \lim_{Y\to 0}Y^n\frac{\partial^n\overline{f}_i(Y,\lambda)}{\partial Y^n}.$$

We easily obtain that for all $0\leq n\leq K-(k+1)$

$$\lim_{C_0(\lambda)\to 0}\frac{\partial^n\overline{F}_i(t,\lambda)}{\partial t^n} = \lim_{C_0(\lambda)\to 0}\frac{\partial^n F_i(t,\lambda)}{\partial t^n} = 0 = \lim_{(t,\lambda)\to(0,0)}t^n\frac{\partial^n F_i(t,\lambda)}{\partial t^n},$$

the first limits being uniform in λ. ▮

3.3. Algorithm for $t \in [0,\epsilon]$ with Γ_0 of codimension k

The idea here is to write an asymptotic expansion for $V_\lambda(t)$ in terms of well-ordered monomials such as those in (9)–(12). A bound for the number of small zeros is then found by an iteration of Rolle's theorem in the form of a succession of derivations and divisions. The exact treatment depends on the type of the bifurcation.

In this section, the notation $\mathcal{O}_\lambda(t)$ is used to note a function such that if we note $\mathcal{O}_\lambda(0) = f(\lambda)$, then $f(\lambda) = O(\lambda)$ and $\mathcal{O}_\lambda(t) - f(\lambda)$ is at least $J_0^{K-2(I_1+JI_2+[J\mu]+1)}(t)$. Thus

$$\lim_{(t,\lambda)\to(0,0)} \partial_t^j \mathcal{O}_\lambda(t) = 0,$$

for all $0 \le j \le K - 2(I_1 + JI_2 + [J\mu] + 1)$.

3.3.1. Case 1: Γ_0 of type $(\boldsymbol{I}_1, \boldsymbol{I}_2, \boldsymbol{J}, \boldsymbol{L})$ with $(\boldsymbol{J}, \boldsymbol{L}) \neq (0,0)$.

This is the case where $\alpha_{I_1}(0) \neq 0$ or $C_{I_1 J} D_{I_2 0} \neq 0$. Let $I = I_1 + JI_2$ and

$$I_3 = \begin{cases} I_2 & \text{if } I_2 \neq 0 \\ I = I_1 & \text{otherwise.} \end{cases}$$

The introduction of I_3 is motivated by the fact that when $D_{I_2 0} \neq 0$, then terms D_{i0} with $i > I_2$ can be grouped with the $D_{I_2 0}$ term.

Lemma 25. *For $t \in (0,\epsilon]$, the vanishing of the $(I+[J\mu]+1)^{th}$ derivative of Equation (35) is equivalent to the vanishing of*

$$\begin{aligned} \overline{T}_{I+[J\mu]+1,\lambda}(t) = &\sum_{i=1}^{I+[J\mu]} *C_0^i \alpha_i t^i \big(1 + \mathcal{O}_{37,i,\lambda}(t)\big) \\ &+ \sum_{\substack{1 \le i+j \le I+J\mu \\ j \neq 0}} C_0^{i+j\mu} p_{ij}(\lambda) t^{i+j\mu+\alpha_1(\lambda)} \big(1 + \mathcal{O}_{38,i,j,\lambda}(t)\big), \end{aligned} \tag{36}$$

where

$$p_{ij}(\lambda) \overset{\mathrm{def}}{=} \sum_{\substack{||M||=j \\ M=(m_\ell) \\ 0\le \ell \le I_3}} \sum_{\substack{i \le i_1 + \sum m_\ell \ell \le I+[(J-j)\mu] \\ 0 \le i_1 \le i}} *C_0^{i_1+\sum m_\ell \ell - i} C_{i_1 j} \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{m_\ell} \right), \tag{37}$$

and $*$ are nonvanishing functions of λ.

Proof. For $t \in (0, \epsilon]$, $V_\lambda(t)$ $\big($Equation (35)$\big)$ is of the following form

$$V_\lambda(t) = c + \sum_{i=1}^{I+[J\mu]} C_0^i \Bigg(\alpha_i t^i \omega(C_0 t, \lambda)\big(1 + \mathcal{O}_{38,i,\lambda}(t)\big) - \overline{\alpha}_i (1-t)^i \overline{\omega}(C_0(1-t), \lambda))\big(1 + \mathcal{O}_{21,i,\lambda}(t)\big)\Bigg) + \mathcal{V}_\lambda(t),$$

where, using the relation $(1-t)^A = \sum_{i'=0}^{A} * t^{i'}$, the rest function $\mathcal{V}_\lambda(t)$ is of the form

$$\begin{aligned}
\mathcal{V}_\lambda(t) = &\sum_{i+j\mu=1}^{I+J\mu} C_0^{i+j\mu} \sum_{\substack{||M||=j \\ M=(m_\ell) \\ 0\le \ell \le k}} \binom{j}{M} \prod_{\ell=0}^{I_3} \big(D_{\ell 0} C_0^\ell\big)^{m_\ell} \Bigg(C_{ij} t^{i+j\mu} (1-t)^{\sum \ell m_\ell} \\
&\times \big(1 + \mathcal{O}_{30,M,i,j}(t)\big) - (1-\delta_{i0})\eta_{ij}(1-t)^{i+j\mu} t^{\sum \ell m_\ell}\big(1 + \mathcal{O}_{31,M,i,j}(t)\big)\Bigg) \\
= &\sum_{i+j\mu=1}^{I+J\mu} C_0^{i+j\mu} t^i \Bigg[\sum_{\substack{||M||=j \\ M=(m_\ell) \\ 0\le \ell \le I_3}} \sum_{\substack{i\le i_1+\sum m_\ell \ell \le I+[(J-j)\mu] \\ 0\le i_1\le i}} * C_0^{i_1+\sum m_\ell \ell - i} C_{i_1 j} \\
&\times \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{m_\ell}\right) t^{j\mu}\big(1 + \mathcal{O}_{39,M,i_1,j,\lambda}(t)\big) \\
&- \sum_{\substack{||N||=j \\ N=(n_\ell) \\ 0\le \ell\le I_3 \\ 0\le \sum n_\ell \ell \le i}} \sum_{i\le i_2+\sum n_\ell \ell \le I+[(J-j)\mu]} *(1-\delta_{i_2 0}) C_0^{i_2+\sum n_\ell \ell - i} \eta_{i_2 j} \\
&\times \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell}\right) (1-t)^{j\mu}\big(1 + \mathcal{O}_{40,N,i_2,j,\lambda}(t)\big)\Bigg] + C_0^{I+J\mu}\mathcal{O}_{41,k,\lambda}(t).
\end{aligned}$$

The rest function $C_0^{I+J\mu}\mathcal{O}_{41,k,\lambda}(t)$ can be included in the term with coefficient $*C_0^I \alpha_I$ or $*C_0^{I+J\mu} C_{I_1 J} D_{I_2 0}^J$. The $(I+[J\mu]+1)^{\text{th}}$ derivative of equation $V_\lambda(t)$ is then of the form

$$\sum_{i=1}^{I+[J\mu]} * C_0^{i-\alpha_1(\lambda)} \alpha_i \Big(t^{i-(I+[J\mu]+1+\alpha_1(\lambda))}\big(1+\mathcal{O}_{42,i,\lambda}(t)\big) + *\big(1+\mathcal{O}_{43,i,\lambda}(t)\big)\Big) + \mathcal{V}_{I+[J\mu]+1,\lambda}(t), \quad (38)$$

where $\mathcal{V}_{I+[J\mu]+1,\lambda}(t)$ is of the following form:

$$\begin{aligned}
&\mathcal{V}_{I+[J\mu]+1,\lambda}(t)\\
&\quad= \sum_{\substack{1\le i+j\mu\le I+J\mu\\ j\neq 0}} C_0^{i+j\mu}\Bigg[\sum_{\substack{||M||=j\\ M=(m_\ell)\\ 0\le \ell\le I_3}}\ \sum_{\substack{i\le i_1+\sum m_\ell \ell\le I+[(J-j)\mu]\\ 0\le i_1\le i}} *C_0^{i_1+\sum m_\ell\ell-i}C_{i_1 j}\\
&\qquad\qquad \times\left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{m_\ell}\right) t^{i-(I+[J\mu]+1)+j\mu}\big(1+\mathcal{O}_{44,M,i_1,j,\lambda}(t)\big)\\
&\quad - \sum_{\substack{||N||=j\\ N=(n_\ell)\\ 0\le\ell\le I_3\\ 0\le\sum n_\ell\ell\le i}}\ \sum_{i\le i_2+\sum n_\ell\ell\le I+[(J-j)\mu]} *C_0^{i_2+\sum n_\ell\ell-i}(1-\delta_{i_2 0})\eta_{i_2 j}\left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell}\right)\\
&\qquad\qquad\qquad\qquad \times\big(1+\mathcal{O}_{45,N,i,j,\lambda}(t)\big)\Bigg).
\end{aligned}\tag{39}$$

Indeed for all $i+j\mu < I+[J\mu]+1$,

$$\frac{d^{I+[J\mu]+1}t^i(1-t)^{j\mu}}{dt^{I+[J\mu]+1}} = *\big(1+\mathcal{O}_{46,i,j,\lambda}(t)\big),$$

with $*$ a nonvanishing function of λ.

We multiply Equation (38) by $t^{I+[J\mu]+1+\alpha_1(\lambda)}$ and in the first summation we include $C_0^{-\alpha_1(\lambda)}$ in $*$ using Equation (28). We can then factor t^i in the term with coefficient $*C_0^i\alpha_i$,

$$\Big(t^i\big(1+\mathcal{O}_{42,i,\lambda}(t)\big) + *t^{I+[J\mu]+1+\alpha_1(\lambda)}\big(1+\mathcal{O}_{43,i,\lambda}(t)\big)\Big) = t^i\big(1+\mathcal{O}_{37,i,\lambda}(t)\big).$$

Moreover, if $j\neq 0$,

$$\eta_{i_2 j}(\lambda) = C_{i_2 j}(\lambda) + \sum_{0<j'<j,\, i'<i_2} C_{i'j'}(\lambda)\cdot O(\lambda). \tag{40}$$

Hence all terms in the second summation of $\mathcal{V}_{I+[J\mu]+1,\lambda}(t)$ $\big($ Equation (39)$\big)$ have the form $*C_{i'j'}\left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{m_\ell}\right)$ multiplied by at least the same power of C_0 and a greater power of t than the corresponding term in the first summation. Thus

$$\mathcal{V}_{I+[J\mu]+1,\lambda}(t) = \sum_{i+j\mu=1}^{I+J\mu} C_0^{i+j\mu}p_{ij}(\lambda)t^{i+j\mu-\alpha_1(\lambda)}\big(1+\mathcal{O}_{38,i,j,\lambda}(t)\big)$$

where $p_{ij}(\lambda)$ is defined in (37). Indeed, fix (i, i_2, j, N) with

$$\begin{cases} ||N|| = j \\ 0 \le \sum_{\ell=0}^{I_3} n_\ell \ell \le i \\ i \le i_2 + \sum_{\ell=0}^{I_3} n_\ell \ell \le I + [(J-j)\mu]. \end{cases}$$

Then from Equation (40)

$$\eta_{i_2 j} \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell} \right) t^{I+[J\mu]+1} = \\ \left(C_{i_2 j}(\lambda) + \sum_{0<j'<j,\, i'<i_2} C_{i'j'}(\lambda) \cdot O(\lambda) \right) \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell} \right) t^{I+[J\mu]+1}. \quad (41)$$

Moreover, if $j > j'$ and $N = N' + N''$ with any $N' = (n'_\ell)$ such that $||N'|| = j'$, then

$$\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell} = \prod_{\ell=0}^{I_3} D_{\ell 0}^{n'_\ell} \times \prod_{\ell=0}^{I_3} D_{\ell 0}^{n''_\ell}.$$

Let Equation (41), and for each $j' < j$ choose such a N'. Then

$$\eta_{i_2 j} \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell} \right) t^{I+[J\mu]+1} = t^{I+[J\mu]+1} \\ \times \left(C_{i_2 j}(\lambda) \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n_\ell} \right) + \sum_{\substack{1<i'+j'\mu<i_2+j\mu \\ (j'\neq 0)}} C_{i'j'}(\lambda) \left(\prod_{\ell=0}^{I_3} D_{\ell 0}^{n'_\ell} \right) \cdot O(\lambda) \right), \quad (42)$$

i.e. the term in Equation (42) with coefficient $C_{i'j'}(\lambda)$ can be included in the rest function of the corresponding term in the first summation of Equation (38) with $M = N'$. ▮

To simplify Equation (36), we homogenize the coefficients. Let the following homogenization.

$$\begin{cases} *\alpha_i(\lambda) = C_0^{(I-i)+J\mu}\overline{\tau}_i(\lambda) \\ *p_{ij}(\lambda) = *C_0(\lambda)^{(I-i)+(J-j)\mu}\overline{\rho}_{ij}(\lambda) \quad (j \neq 0). \end{cases} \tag{43}$$

Then, coefficients $\overline{\tau}_i(\lambda)$ and $\overline{\rho}_{ij}(\lambda)$ may not be bounded at 0. To eliminate this problem, let

$$L(\lambda) = \left(\sum_{\substack{(i,j)\leq(I,J)\\ j\neq 0}} \left(\overline{\tau}_i^2(\lambda) + \overline{\rho}_{ij}^2(\lambda)\right) \right)^{1/2} > \delta > 0,$$

where the first inequality comes from the finite codimension hypothesis. Indeed

$$\begin{cases} \overline{\tau}_I(0) = \alpha_I(0) \neq 0 & \text{if } L = 1 \\ \overline{\rho}_{IJ}(0) = p_{IJ}(0) = C_{I_1J}(0)D_{I_20}^J(0) \neq 0 & \text{if } L = 0 \neq J. \end{cases}$$

Let

$$\begin{cases} \tau_i(\lambda) = \dfrac{\overline{\tau}_i(\lambda)}{L(\lambda)} \\ \rho_{ij}(\lambda) = \dfrac{\overline{\rho}_{ij}(\lambda)}{L(\lambda)}. \end{cases} \tag{44}$$

Remark: There are $N = k(I_1, I_2, J, L) - (I + [J\mu])$ equations in system (44).

Hence, even if $L(\lambda)$ is not bounded at $\lambda = 0$, at least one of the inequalities $\tau_i(\lambda) \geq 1/N$ or $\rho_{ij}(\lambda) \geq 1/N$ is satisfied.

We divide $\overline{T}_{I+[J\mu]+1,\lambda}(t)$ $\big($Equation (36)$\big)$ by $C_0^{I+J\mu}L(\lambda)$.

$$T_{I+[J\mu]+1,\lambda}(t) = \sum_{i=1}^{I+[J\mu]} \tau_i t^i \big(1 + \mathcal{O}_{47,i,\lambda}(t)\big) + \sum_{\substack{1<i+j\mu\leq I+J\mu\\ j\neq 0}} \rho_{ij} t^{i+j\mu+\alpha_1}\big(1 + \mathcal{O}_{48,i,j,\lambda}(t)\big). \tag{45}$$

PROPOSITION 26. *For sufficiently small $\lambda \in \Lambda$, $V_\lambda(t)$ has at most $k(I_1, I_2, J, L) + 1$ zeros in $[0, \epsilon]$. ($k(I_1, I_2, J, L)$ is defined in Definition* 10.)

Proof. All terms corresponding to polynomial terms in Equation (35) have been killed by derivation, thus there are at most $k(I_1, I_2, J, L) - (I +$

$[J\mu]$) terms in Equation (45). Moreover, monomials t^i and $t^{i+j\mu+\alpha_1(\lambda)}$ with $j \neq 0$ are well ordered and form a Chebyshev system (cf. [17]). Using a derivation-division algorithm in each cone where either τ_i or ρ_{ij} is the largest coefficient, we thus obtain that for sufficiently small $\lambda \in \Lambda$, $T_{I+[J\mu]+1,\lambda}(t)$ has at most $k(I_1, I_2, J, L) - (I + [J\mu])$ zeros in $[0, \epsilon]$. The result follows from Rolle's theorem. ▮

Remark: As stated in the previous proof, monomials t^i and $t^{i+j\mu+\alpha_1(\lambda)}$ with $j \neq 0$ are well ordered and form a Chebyshev system. If a function has an expansion in these monomials and if at least one of the coefficients is nonvanishing, then a derivation-division algorithm yields that the number of its small zeros is at most the order of the nonvanishing coefficient minus one.

3.3.2. *Case 2:* Γ_0 *of type* $(2\boldsymbol{I}_1 + 1, 0, 0, 0)$.

When Γ_0 is of type $(I, 0, 0, 0)$, with $I = 2I_1 + 1$, we must be careful in the algorithm not to kill the leading term t^I with coefficient $*C_0^I C_{I0}$. Indeed, following the proof of Lemma 25, the I^{th} derivative of Equation (35) is of the form

$$\begin{aligned}
&\overline{\overline{T}}_{I,\lambda}(t) = \sum_{i=1}^{I-1} *C_0^i \alpha_i t^{i-I-\alpha_1(\lambda)}\big(1 + \mathcal{O}_{49,i,\lambda}(t)\big) \\
&+ *C_0^I \alpha_I \Big(\omega\big(C_0 t, \lambda\big)\big(1 + \mathcal{O}_{50,I_1,\lambda}(t)\big) + *\overline{\omega}\big(C_0(1-t), \lambda\big)\big(1 + \mathcal{O}_{51,I_1,\lambda}(t)\big)\Big) \\
&+ \Bigg(\sum_{\substack{1 \le i+j\mu \le I \\ j \neq 0}} *C_0^{i+j\mu} p_{ij}(\lambda) t^{i-I+j\mu}\big(1+\mathcal{O}_{52,M,i,j}(t)\big)\Bigg) + *C_0^I p_{I0}\big(1+\mathcal{O}_{53,I,\lambda}(t)\big)
\end{aligned} \tag{46}$$

where, up to multiplication by a nonvanishing function of λ, the $p_{ij}(\lambda)$ are the ones given in Equation (37), and

$$p_{I0}(\lambda) = *\big(C_{I0}(\lambda) + (-1)^{I+1}\eta_{I0}(\lambda)\big).$$

Let the homogenization given in Equation (43). We subdivide the parameter space in the following cones:

$$E_\ell(\Lambda_1) \stackrel{\mathrm{d}ef}{=} \left\{\lambda \in \Lambda_1 \;\middle|\; |\tau_\ell|(\lambda) = \max_{\substack{k\leq I\\(m,n)\leq(I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|\big)\right\}$$

$$E_{ij}(\Lambda_1) \stackrel{\mathrm{d}ef}{=} \left\{\lambda \in \Lambda_1 \;\middle|\; |\rho_{ij}(\lambda)| = \max_{\substack{k\leq I\\(m,n)\leq(I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|\big)\right\},$$

with $0 \in \Lambda_1 \subseteq \Lambda$.

The only cone which requires a discussion different from Proposition 26 is the cone $E_{I0}(\Lambda_1)$. We need to subdivide the cone $E_{I0}(\Lambda_1)$ in the following ones:

$$E^1_{I0}(\Lambda_1) \stackrel{\mathrm{d}ef}{=} \left\{\lambda \in E_{I0}(\Lambda_1) \;\middle|\; |\tau_I| \leq |\tau_1|\right\}$$
$$E^2_{I0}(\Lambda_1) = \Lambda_1 \backslash E^1_{I0}(\Lambda_1).$$

Since $\big(\tau_i(\lambda), \rho_{ij}(\lambda)\big) \in \mathbb{S}^k$,

$$\Lambda_1 = \left(\bigcup_{\ell=0}^{I} E_\ell(\Lambda_1)\right) \bigcup \left(\bigcup_{\substack{0<i+j\mu<I\\(j\neq 0)}} E_{ij}(\Lambda_1)\right) \bigcup E_{I0}(\Lambda_1).$$

Notice that if

$$E'_\ell(\Lambda_1) \stackrel{\mathrm{d}ef}{=} \left\{\lambda \in E^2_{I0}(\Lambda_1) \;\middle|\; |\tau_\ell(\lambda)| = \max_{\substack{k\leq I\\(m,n)<(I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|\big)\right\}$$

$$E'_{ij}(\Lambda_1) \stackrel{\mathrm{d}ef}{=} \left\{\lambda \in E^2_{I0}(\Lambda_1) \;\middle|\; |\rho_{ij}(\lambda)| = \max_{\substack{k\leq I\\(m,n)<(I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|\big)|\right\},$$

then

$$E^2_{I0}(\Lambda_1) = \left(\bigcup_{\ell=0}^{I} E'_\ell(\Lambda_1)\right) \bigcup \left(\bigcup_{\substack{0<i+j\mu<I\\(j\neq 0)}} E'_{ij}(\Lambda_1)\right). \tag{47}$$

Proposition 27. *For sufficiently small $\lambda \in \Lambda$, $V_\lambda(t)$ has at most $k(I,0,0,0)$ zeros in $[0,\epsilon]$. ($k(I_1,I_2,J,L)$ is defined in Definition* 10.)

Proof. We first divide Equation (46) by $C^I_0 L(\lambda)$ and denote by $\tilde{T}_{I,\lambda}(t)$ the resulting equation.

1. Let $\lambda \in E^1_{I0}(\Lambda_)$. In $\tilde{T}_{I,\lambda}(t)$, we group the terms with coefficient in τ_I with the terms with coefficient in τ_1 ($|\tau_I/\tau_1| < 1$ if $\tau_1 \neq 0$ or both

terms vanish). The monomials in $\tilde{T}_{I,\lambda}$ are then well ordered and form a Chebyshev system. The result follows from the remark above.

2. Let $\lambda \in E^2_{I0}(\Lambda_1)$. We first divide $\tilde{T}_{I,\lambda}(t)$ by $\big(1+\mathcal{O}_{53,I,\lambda}(t)\big)$ and then differentiate once with respect to t. We obtain a function whose vanishing is equivalent to the vanishing of $T_{I+1,\lambda}(t)$, see Equation (45). The result follows from Proposition 26 and Equation (47). ▮

▮

3.4. Algorithm for $t \in [\epsilon, 1-\epsilon]$ with Γ_0 of codimension k

Here we need to bound the number of zeros of an analytic function (V_λ) on a global domain. The idea to compute this bound is that a certain derivative of $V_\lambda(t)$ is close, uniformly in λ, to a Pfaffian function with only isolated zeros, the number of zeros of which can be explicitly estimated. The explicit estimation is a lovely application of Khovanskiĭ's fewnomial theory.

3.4.1. Case 1: Γ_0 *of type* $(\boldsymbol{I_1}, \boldsymbol{I_2}, \boldsymbol{J}, \boldsymbol{L})$ *with* $(\boldsymbol{J}, \boldsymbol{L}) \neq (0,0)$.

Let $I = I_1 + JI_2$ and, as in the previous section,

$$I_3 = \begin{cases} I_2 & \text{if } I_2 \neq 0 \\ I = I_1 & \text{otherwise.} \end{cases}$$

Lemma 28. *For $t \in [\epsilon, 1-\epsilon]$ the vanishing of the $(I+[J\mu]+1)^{th}$ derivative of Equation (35) is equivalent to the vanishing of*

$$\begin{aligned} \overline{\overline{\overline{T}}}_{I+[J\mu]+1,\lambda}(t) = \sum_{i=1}^{I+[J\mu]} *C_0^i \alpha_i \Big(& t^i (1-t)^{I+[J\mu]+1} \big(1+\mathcal{O}_{54,i,\lambda}(t)\big) \\ & + (-1)^{I+[J\mu]} t^{I+[J\mu]+1} (1-t)^i \big(1+\mathcal{O}_{55,i,\lambda}(t)\big) \Big) \\ + \sum_{\substack{1 \le i+j\mu \le I+J\mu \\ j \neq 0}} C_0^{i+j\mu} t^i \Big(& \overline{q}_{ij} t^{j\mu} (1-t)^{I+[J\mu]+1} \big(1+\mathcal{O}_{56,i,j,\lambda}(t)\big) \\ & - \check{\overline{q}}_{ij} t^{I+[J\mu]+1} (1-t)^{j\mu} \big(1+\mathcal{O}_{57,i,j,\lambda}(t)\big) \Big), \end{aligned} \tag{48}$$

where

$$\overline{q}_{ij}(\lambda) \stackrel{\mathrm{def}}{=} \sum_{\substack{||M||=j\\ M=(m_\ell)\\ 0\le \ell\le I_3}} \sum_{\substack{0\le i_1\le i\\ i\le i_1+\sum m_\ell \ell\le I+(J-j)\mu}} *C_0^{i_1+\sum m_\ell \ell - i}\left(\prod_{\ell=0}^{I_3}\big(D_{\ell 0}(\lambda)\big)^{m_\ell}\right) C_{i_1 j}(\lambda),$$

and

$$\overline{\check{q}}_{ij}(\lambda) \stackrel{\mathrm{def}}{=} \sum_{\substack{||N||=j\\ N=(n_\ell)\\ 0\le \ell\le I_3}} \sum_{\substack{0\le \sum n_\ell \ell\le i\\ i\le i_2+\sum n_\ell \ell\le I+(J-j)\mu}} *C_0^{i_2+\sum n_\ell \ell - i}\left(\prod_{\ell=0}^{I_3}\big(D_{\ell 0}(\lambda)\big)^{n_\ell}\right)(1-\delta_{i_2 0})\eta_{i_2 j}(\lambda).$$

Remark 29. All coefficients $\overline{q}_{ij}(\lambda)$ in the summation are equal, up to multiplication by a nonvanishing function of λ, to the corresponding coefficient $p_{ij}(\lambda)$ defined in Equation (37).

Proof (Proof of Lemma 28). The $(I+[J\mu]+1)^{\mathrm{th}}$ derivative of Equation (35) has the form

$$\sum_{i=1}^{I+[J\mu]} *C_0^{i-\alpha_1(\lambda)}\bigg(\alpha_i t^{i-(I+[J\mu]+1+\alpha_1(\lambda))}\big(1+\mathcal{O}_{58,i,\lambda}(t)\big)$$
$$+(-1)^{I+[J\mu]+1}\overline{\alpha}_i(1-t)^{i-(I+[J\mu]+1+\overline{\alpha}_1(\lambda))}\big(1+\mathcal{O}_{59,i,\lambda}(t)\big)\bigg)+\overline{\mathcal{V}}_{I+[J\mu]+1,\lambda}(t), \tag{49}$$

where

$$\overline{\mathcal{V}}_{I+[J\mu]+1,\lambda}(t) = \sum_{\substack{1\le i+j\mu\le I+J\mu\\ j\neq 0\\ ||M||=j\\ A+B=I+[J\mu]+1\\ 0\le B\le \sum m_\ell \ell\\ i_1+\sum m_\ell \ell = i}} C_0^{i+j\mu}$$
$$\times\bigg(p_{i_1,j,A,B,M}t^{i_1+j\mu-A}(1-t)^{\sum m_\ell \ell - B}\big(1+\mathcal{O}_{60,M,i_1,j,A,B,\lambda}(t)\big)$$
$$-\check{p}_{i_1,j,A,B,M}t^{\sum m_\ell \ell - B}(1-t)^{i_1+j\mu-A}\big(1+\mathcal{O}_{61,M,i_1,j,A,B,\lambda}(t)\big)\bigg),$$

with

$$p_{i_1,j,A,B,M}(\lambda) \stackrel{\mathrm{def}}{=} f_{1,i_1,j,A,B,M}(\lambda)\left(\prod_{l=0}^{I_3}\big(D_{l0}(\lambda)\big)^{m_l}\right)C_{i_1 j}(\lambda),$$

and

$$\check{p}_{i_1,j,A,B,M}(\lambda) \stackrel{\mathrm{def}}{=} f_{2,i_1,j,A,B,M}(\lambda)\left(\prod_{l=0}^{I_3}\big(D_{l0}(\lambda)\big)^{m_l}\right)(1-\delta_{i_1 0})\eta_{i_1 j}(\lambda),$$

where coefficients $f_{i,i_1,j,A,B,M}(\lambda)$ are nonvanishing functions appearing as a result of the derivations which, in the sequel, we simply write as $*$.

We multiply Equation (49) by $t^{I+[J\mu]+1}(1-t)^{I+[J\mu]+1}$ and, in the first summation, we include $C_0^{-\alpha_1(\lambda)}$ in $*$ using Equation (28). We can then factor t^i in the term with coefficient $*C_0^i\alpha_i$. Indeed, using the identity $t^{i-\alpha_1(\lambda)} = t^i\big(1+(t^{-\alpha_1(\lambda)}-1)\big) = t^i\big(1+\mathcal{O}_{62,i,\lambda}(t)\big)$,

$$\begin{aligned} & t^{i-\alpha_1(\lambda)}(1-t)^{I+[J\mu]+1}\big(1+\mathcal{O}_{58,i,\lambda}(t)\big) \\ &\qquad + (-1)^{I+[J\mu]}t^{I+[J\mu]+1}(1-t)^{i-\overline{\alpha}_1(\lambda)}\big(1+\mathcal{O}_{59,i,\lambda}(t)\big) \\ &= t^i(1-t)^{I+[J\mu]+1}\big(1+\mathcal{O}_{54,i,\lambda}(t)\big)+(-1)^{I+[J\mu]}t^{I+[J\mu]+1}(1-t)^i\big(1+\mathcal{O}_{55,i,\lambda}(t)\big). \end{aligned}$$

From Equation (49), we then obtain a function of the form

$$\begin{aligned} \sum_{i=1}^{I+[J\mu]} & *C_0^i\alpha_i\Big(t^i(1-t)^{I+[J\mu]+1}\big(1+\mathcal{O}_{54,i,\lambda}(t)\big) \\ & + (-1)^{I+[J\mu]+1}t^{I+[J\mu]+1}(1-t)^i\big(1+\mathcal{O}_{55,i,\lambda}(t)\big)\Big) + \overline{\overline{\mathcal{V}}}_{I+[J\mu]+1,\lambda}(t), \end{aligned}$$

where

$$\begin{aligned}\overline{\overline{\mathcal{V}}}_{I+[J\mu]+1,\lambda}(t) = &\sum_{\substack{1\le i+j\mu\le I+J\mu\\ j\neq 0}} C_0^{i+j\mu}\\ &\times \sum_{\substack{||M||=j\\ i_1+\sum m_\ell \ell = i}} \sum_{\substack{A+B=I+[J\mu]+1\\ 0\le B\le \sum m_\ell \ell}} \Big[(1-t)^{I+[J\mu]+1}\\ \times \Big(p_{i_1,j,A,B,M}t^{i_1+I+[J\mu]+1+j\mu-A}&(1-t)^{\sum m_\ell\ell-B}\big(1+\mathcal{O}_{60,M,i_1,j,A,B,\lambda}(t)\big)\Big)\\ &- t^{I+[J\mu]+1}\\ \times\Big(\check{p}_{i_1,j,A,B,M}t^{\sum m_\ell\ell-B}(1-t)^{i_1+I+[J\mu]+1+j\mu-A}&\big(1+\mathcal{O}_{61,M,i_1,j,A,B,\lambda}(t)\big)\Big)\Big].\end{aligned} \tag{50}$$

Consider, for fixed (i,j,M), the polynomial

$$\begin{aligned}\sum_{i\le i_1+\sum m_\ell\ell\le I+(J-j)\mu} \sum_{\substack{A+B=I+[J\mu]+1\\ 0\le B\le \sum m_\ell\ell}} C_0^{i_1+\sum m_\ell\ell-i} p_{i_1,j,A,B,M}t^{i_1+I+[J\mu]+1-A}&\\ \times (1-t)^{\sum m_\ell\ell-B},&\end{aligned} \tag{51}$$

and, for fixed (i,j,N), the polynomial

$$\begin{aligned}\sum_{i\le i_2+\sum n_\ell\ell\le I+(J-j)\mu} \sum_{\substack{A+B=I+[J\mu]+1\\ 0\le B\le \sum n_\ell\ell}} C_0^{i_2+\sum n_\ell\ell-i} \check{p}_{i_1,j,A,B,N}&\\ \times t^{\sum n_\ell\ell-B}(1-t)^{i_2+I+[J\mu]+1-A}.&\end{aligned} \tag{52}$$

Let $\overline{p}_{i,j,M}(\lambda)$ $\big($resp. $\overline{\check{p}}_{i,j,N}(\lambda)\big)$ be the coefficient of the monomial t^i in Equation (51) $\big($resp. (52)$\big)$ after expansion of terms of the form $(1-t)^a$. Then from Equation (50),

$$\begin{aligned}\overline{\overline{\mathcal{V}}}_{I+[J\mu]+1,\lambda}(t) = &\sum_{\substack{1\le i+j\mu\le I+J\mu\\ j\neq 0}} C_0^{i+j\mu}\Big[t^{j\mu}(1-t)^{I+[J\mu]+1}\\ \times\Big(&\sum_{\substack{||M||=j\\ i_1+\sum m_\ell\ell=i}} \sum_{\substack{A+B=I+[J\mu]+1\\ 0\le B\le\sum m_\ell\ell}} p_{i_1,j,A,B,M}t^{i_1+I+[J\mu]+1-A}(1-t)^{\sum m_\ell\ell-B}\end{aligned}$$

$$\times\big(1+\mathcal{O}_{60,M,i_1,j,A,B,\lambda}(t)\big)\Bigg) - t^{I+[J\mu]+1}(1-t)^{j\mu}$$

$$\times\Bigg(\sum_{\substack{||N||=j\\ i_2+\sum n_\ell \ell=i}}\ \sum_{\substack{A+B=I+[J\mu]+1\\ 0\le B\le \sum n_\ell \ell}} \check{p}_{i_2,j,A,B,N}t^{\sum n_\ell \ell-B}(1-t)^{i_2+I+[J\mu]+1-A}$$

$$\times\big(1+\mathcal{O}_{61,N,i_2,j,A,B,\lambda}(t)\big)\Bigg)\Bigg]$$

$$= \sum_{\substack{1\le i+j\mu\le I+J\mu\\ j\neq 0}} C_0^{i+j\mu}t^i\Bigg[t^{j\mu}(1-t)^{I+[J\mu]+1}\Bigg(\sum_{\substack{||M||=j\\ M=(m_\ell)\\ 0\le \ell\le I_3}}\overline{p}_{i,j,M}\big(1+\mathcal{O}_{63,M,i,j,\lambda}(t)\big)\Bigg)$$

$$- t^{I+[J\mu]+1}(1-t)^{j\mu}\Bigg(\sum_{\substack{||N||=j\\ N=(n_\ell)\\ 0\le \ell\le I_3}}\overline{\check{p}}_{i,j,N}\big(1+\mathcal{O}_{64,i,j,\lambda}(t)\big)\Bigg)\Bigg].$$

∎

Let the following homogenization of the coefficients:

$$\begin{cases} *\alpha_i(\lambda) = C_0^{(I-i)+J\mu}\overline{\tau}_i(\lambda)\\ *\overline{q}_{ij}(\lambda) = C_0(\lambda)^{I-i+(J-j)\mu}\overline{\rho}_{ij}(\lambda),\\ *\overline{\check{q}}_{ij}(\lambda) = C_0(\lambda)^{I-i+(J-j)\mu}\overline{\check{\rho}}_{ij}(\lambda).\end{cases}$$

Once again coefficients $\overline{\tau}_i(\lambda)$, $\overline{\rho}_{ij}(\lambda)$ and $\overline{\check{\rho}}_{ij}(\lambda)$ may not be bounded at 0. However, since

$$\overline{\rho}_{I,J}(\lambda) = C_{I_1J}(\lambda)\Bigg(D^J_{I_2,0}(\lambda) + \sum_{i_2=0}^{I_2} D_{i_2,0}h_{I_1,i_2,J}(\lambda)\Bigg) + \sum_{i_1+ji_2+j\mu=1}^{I_1+JI_2+J\mu} C_{i_1,j}D_{i_2,0}h_{i_1,i_2,j}(\lambda)$$

where $h_{i_1,i_2,j}(\lambda)$ are polynomials in $C_{ij}(\lambda)$ and $D_{\ell 0}(\lambda)$, either $\overline{\tau}_I(0) = \alpha_I(0)\neq 0$ or $\overline{\rho}_{I,J}(0) = C_{I_1J}(0)D^J_{I_2,0}(0)\neq 0$. We can thus compactify the coefficient space as we did in section 3.3.1.

Lemma 30. *For $t\in[\epsilon,1-\epsilon]$ the vanishing of the $(I+[J\mu]+1)^{th}$ derivative of Equation (35) is equivalent to the vanishing of $\mathcal{G}_{I+[J\mu]+1,\lambda}\big(t,(1-t)^\mu,t^\mu\big)$,* *where*

$$\mathcal{G}_{I+[J\mu]+1,\lambda}(t,y,z)=\sum_{i+(j+l)\mu=1}^{2I+[J\mu]+J\mu+1}\xi_{ijl}t^iy^jz^l+\mathcal{O}_{65,\lambda,k}(t,y,z),$$

with $\xi_{ijl}(\lambda)$ *polynomials in* $\tau_{i'}(\lambda)$, $\rho_{i'j'}(\lambda)$ *and* $\check{\rho}_{i'j'}(\lambda)$.

Proof. We divide $\overline{\overline{\overline{T}}}_{I+[J\mu]+1,\lambda}(t)$ $\big($Equation (48)$\big)$ by $C_0^{I+J\mu}L(\lambda)$ and obtain

$$\tilde{\tilde{\tilde{T}}}_{I+[J\mu]+1,\lambda}(t)=\sum_{i=1}^{I+[J\mu]}\tau_i\big(t^i(1-t)^{I+[J\mu]+1}+(-1)^{I+[J\mu]}(1-t)^it^{I+[J\mu]+1}\big)$$
$$+\sum_{\substack{1\leq i+j\mu\leq I+J\mu\\ j\neq 0}}t^i\Big(\rho_{ij}t^{j\mu}(1-t)^{I+[J\mu]+1}-\check{\rho}_{ij}t^{I+[J\mu]+1}(1-t)^{j\mu}\Big)+\mathcal{O}_{66,k}(t).\tag{53}$$

The result follows by setting $\xi_{ijl}(\lambda)$ such that

$$\mathcal{G}\big(t,(1-t)^\mu,t^\mu\big)=\tilde{\tilde{\tilde{T}}}_{I+[J\mu]+1,\lambda}(t).$$

∎

PROPOSITION 31. *Let* $n=2(I+[J\mu])+1$. *For sufficiently small* $\lambda\in\Lambda$, $V_\lambda(t)$ *has at most* $\frac{1}{2}\big(n(4n^2+16n+37)+1\big)$ *zeros in* $[\epsilon,1-\epsilon]$.

To prove this proposition, we will need the following lemma.

LEMMA 32. *Let* $0<t_2<t_3$. *If* $T(t,\lambda)\overset{\mathrm{def}}{=}P(t,\lambda)+f(t,\lambda)$ *where* $P(t,\lambda)$ *and* $f(t,\lambda)$ *are some analytic functions depending on* λ, *and* $f(t,\lambda)$ *is such that for all* $n\leq k$ *we have on* $[t_2,t_3]$

$$\lim_{\lambda\to 0}\frac{\partial^n f(t,\lambda)}{\partial t^n}=0.$$

We suppose $P(t,0)\not\equiv 0$. *Let* N *be a bound for the number of zeros of* $P(t,\lambda)$ *on* $[t_2,t_3]$, *for* λ *in a neighborhood of* Λ_0. *Then there exists a neighborhood* $\Lambda_P\subseteq\Lambda_0$ *of* $\lambda=0$ *such that* $T(t,\lambda)$ *has at most* N *zeros on* $[t_2,t_3]$.

Proof. This result is stated in [9] for P a polynomial. The proof is similar.

Let $N\in\mathbb{N}$ such that for all sufficiently small λ, $P(t,\lambda)$ has at most N zeros counted with multiplicities on $[t_2,t_3]$. Moreover, assume there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ converging to 0 and such that $T(t,\lambda_n)$ has at least M zeros counted with multiplicities in $[t_2,t_3]$:

$$t_n^{(1)} \le t_n^{(2)} \le \cdots \le t_n^{(M)}.$$

We can take a subsequence $(\lambda_{n_k})_{n_k\in\mathbb{N}}$ such that the $t_{n_k}^{(i)}$ converge on $[t_2,t_3]$ to $t^{(i)}$ with

$$t^{(1)} \le t^{(2)} \le \cdots \le t^{(M)}.$$

Since $\lim_{\lambda\to 0} f(t,\lambda) = 0$ uniformly on $[t_2,t_3]$, we have that $P\big(t^{(i)},0\big) = 0$ for all $1 \le i \le M$.

We now show that the zeros $t^{(i)}$ of $P(t,0)$ are counted with multiplicities: this is done using Rolle's theorem. Let $t^{(j)} = t^{(j+1)} = \cdots = t^{(j+s)}$. Using Rolle's theorem for the derivatives of $T(t,\lambda_{n_k})$, we can find convergent sequences $(t_{n_k,\ell})$ with $\lim_{n_k\to\infty} t_{n_k,\ell} = t^{(j)}$ such that $t_{n_k}^{(j)} \le t_{n_k,\ell} \le t_{n_k}^{(j+s)}$ and

$$\frac{\partial^\ell T}{\partial t^\ell}\big(t_{n_k,\ell},\lambda_{n_k}\big) = 0 \qquad \text{for } \ell \le j-1.$$

Since

$$\lim_{\lambda\to 0} \frac{\partial^n f(t,\lambda)}{\partial t^n} = 0,$$

uniformly in $t\in[t_2,t_3]$, we have

$$\frac{\partial^\ell P}{\partial t^\ell}\big(t^{(j)},0\big) = 0 \qquad \text{for } \ell \le j-1.$$

Therefore $T(t,\lambda)$ has at most N zeros counted with multiplicities, i.e. $M \le N$. ▮

Proposition 31 can be proven using Lemma 32 and the following theorem.

THEOREM 33. *Let $f_1(t,t^\mu,(1-t)^\mu)$ be a polynomial of degree at most n in the variables t, t^μ and $(1-t)^\mu$, $f\not\equiv 0$, and let $P_n(t) = f_1(t,t^\mu,(1-t)^\mu)$. Then for all $\epsilon>0$ and μ irrational, the number of zeros of $P_n(t)$ on $[\epsilon,1-\epsilon]$ counted with multiplicity is bounded from above. Moreover*

$$\#_0\big(P_n(t)\big) \overset{\text{def}}{=} \#\big\{t\in[\epsilon,1-\epsilon] \mid P_n(t)=0\big\} \le n(2n^2+8n+18),$$

where the solutions t are counted with multiplicities.

Proof. The proof is delayed until section 3.6. ∎

Proof (Proof of Proposition 31). As shown in the proof of Lemma 30, the vanishing of the $(I+[J\mu]+1)^{\text{th}}$ derivative of Equation (35) is equivalent to the vanishing of $\mathcal{G}_{I+[J\mu]+1,\lambda}\big(t, t^\mu, (1-t)^\mu\big)$ which is of the form stated in Lemma 32. Let

$$P_\lambda(t,y,z) = \sum_{i+(j+l)\mu=1}^{2I+[J\mu]+J\mu+1} \xi_{ijl} t^i y^j z^l,$$

and

$$f_\lambda(t,y,z) = \mathcal{O}_{65,\lambda,k}(t,y,z).$$

Then from Equation (53),

$$\mathcal{G}_{I+[J\mu]+1,\lambda}\big(t,t^\mu,(1-t)^\mu\big) = P_\lambda\big(t,t^\mu,(1-t)^\mu\big) + f_\lambda\big(t,t^\mu,(1-t)^\mu\big).$$

To conclude, we use Theorem 33. To apply the theorem, we simply have to show that $P_\lambda\big(t,t^\mu,(1-t)^\mu\big)$ is not trivial. We compactify the coefficient space as we did in Section 3.3.2.

1. Let $\lambda \in E_{i_1}(\Lambda_1)$, then

$$\mathcal{G}_\lambda(t,y,z) = t(1-t)P_{1,\lambda}(t) + \sum_{\substack{1 \le i+(j+l)\mu \le 2I+[J\mu]+J\mu \\ j+l>0}} \xi_{ijl} t^i y^j z^l + \mathcal{O}_{67,k}(t,x,z),$$

where $P_{1,\lambda}(t)$ is the following (nontrivial) polynomial

$$\begin{aligned} P_{1,\lambda}(t) &= \sum_{i=0}^{I+[J\mu]-1} \tau_{i+1}\Big(t^i(1-t)^{I+[J\mu]} + (-1)^{I+[J\mu]}(1-t)^i t^{I+[J\mu]}\Big) \\ &= \sum_{i=0}^{I+[J\mu]-1} c_i(\lambda)t^i + o(t^{I+[J\mu]-1}), \end{aligned}$$

where the $c_i(\lambda)$ are obtained by expanding all terms $(1-t)^A$. $P_{1,\lambda}(t)$ is nontrivial. Indeed, let $V_1(t) = (c_0, c_1 t, \dots, c_{I+[J\mu]-1}t^{I+[J\mu]-1})$, $V_2 = (\tau_1, \tau_2, \dots, \tau_{I+[J\mu]})$ and let $\mathcal{M}_\lambda(t)$ be the lower triangular $(I+[J\mu]\times I+[J\mu])$ matrix with $m_{ij,\lambda}(t) = *t^{i-1}$ such that $V_1^T(t) = \mathcal{M}_\lambda(t)\cdot V_2^T$. Then $P_{1,\lambda}(t)$ is not identically zero since $V_2 \neq 0$ $(\tau_{i_1} \neq 0)$ and $\mathcal{M}_\lambda(t)$ is inversible for all $(t,\lambda) \in [\epsilon, 1-\epsilon] \times \Lambda_1$.

2. Let $\lambda \in E_{i_1 j_1}(\Lambda_1)$, then

$$\mathcal{G}_\lambda(t,y,z) = (1-t)^{I+[J\mu]} z^{j_1} P_{2,\lambda}(t) + \sum_{j \neq j_1} \xi_{ijl} t^i y^j z^l + \mathcal{O}_{68,k}(t,x,z),$$

where $P_{2,\lambda}(t,z)$ is the following (nontrivial) polynomial:

$$P_{2,\lambda}(t) = \sum_{i=0}^{I+[(J-j)\mu]} \rho_{ij_1} t^i.$$

We have thus shown that for sufficiently small $\lambda \in \Lambda$, $\tilde{\tilde{\tilde{T}}}_{I+[J\mu]+1,\lambda}(t)$ has at most $n(2n^2+8n+18)$ zeros and the result follows yielding at most $n(2n^2+8n+18) + \frac{1}{2}(n+1)$ zeros for $V_\lambda(t)$. ▮

3.4.2. Case 2: Γ_0 *be of type* $(2\boldsymbol{I}+\mathbf{1},0,0,0)$

Proposition 34. *Let* $n = 4I+3$. *For sufficiently small* $\lambda \in \Lambda$, $V_\lambda(t)$ *has at most* $\frac{1}{2}\big(n(4n^2+18n+37)+1\big)$ *zeros in* $[\epsilon, 1-\epsilon]$.

Proof. We proceed as in the proof of Proposition 27, but we subdivide the cone $E_{I0}(\Lambda_1)$ in a different way.

We have that the vanishing of the I^{th} derivative of Equation (35) is equivalent to the vanishing of

$$\begin{aligned}\check{T}_{I,\lambda}(t) = \sum_{i=1}^{I-1} \tau_i \Big(& t^{i-I-\alpha_1}\big(1+\mathcal{O}_{69,i,\lambda}(t)\big) + *(1-t)^{i-I-\overline{\alpha}_1}\big(1+\mathcal{O}_{70,i,\lambda}(t)\big)\Big) \\ & + \tau_I \Big(\omega\big(C_0 t,\lambda\big)\big(1+\mathcal{O}_{71,I,\lambda}(t)\big) + *\overline{\omega}\big(C_0(1-t),\lambda\big)\big(1+\mathcal{O}_{72,I,\lambda}(t)\big)\Big) \\ + \sum_{\substack{1\le i+j\mu \le I+J\mu \\ j\neq 0}} & \Big(\rho_{ij} t^{i+j\mu-I}\big(1+\mathcal{O}_{73,i,j,\lambda}(t)\big) - *\check{\rho}_{ij} t^i (1-t)^{j\mu-I}\big(1+\mathcal{O}_{74,i,j,\lambda}(t)\big)\Big) \\ & + \rho_{I0}\big(1+\mathcal{O}_{75,I,\lambda}(t)\big).\end{aligned}$$

Let

$$\begin{aligned} E_{I0}^{1''}(\Lambda_1) &\stackrel{\text{def}}{=} \big\{\lambda \in \Lambda_1 \bigm| |\tau_I| \le |\lambda| \cdot |\tau_1|\big\} \\ E_{I0}^{2''}(\Lambda_1) &= \Lambda_1 \backslash E_{I0}^{1''}(\Lambda_1). \end{aligned}$$

As before, if

$$E'''_\ell(\Lambda_1) \overset{\text{def}}{=} \left\{\lambda \in E^{2''}_{I0}(\Lambda_1) \;\middle|\; |\tau_\ell(\lambda)| = \max_{\substack{k<I\\(m,n)<(I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|, |\tau_I(\lambda)|\big)\right\}$$

$$E'''_{ij}(\Lambda_1) \overset{\text{def}}{=} \left\{\lambda \in E^{2''}_{I0}(\Lambda_1) \;\middle|\; |\rho_{ij}(\lambda)| = \max_{\substack{k<I\\(m,n)<(I,0)}} \big(|\tau_k(\lambda)|, |\rho_{mn}(\lambda)|, |\tau_I(\lambda)|\big)\right\},$$

then

$$E^{2''}_{I0}(\Lambda_1) = \left(\bigcup_{\ell=0}^{I} E'''_\ell(\Lambda_1)\right) \bigcup \left(\bigcup_{\substack{0<i+j\mu<I\\(j\neq 0)}} E'''_{ij}(\Lambda_1)\right).$$

1. Let $\lambda \in E^{1''}_{I0}(\Lambda_1)$. In $\check{T}_{I,\lambda}(t)$ we group the term with coefficient in τ_I with the term with coefficient in τ_1 ($|\tau_I/\tau_1| \le |\lambda|$ if $\tau_i \neq 0$ or both terms vanish). We obtain a function of the form of Equation (53). Note that in Proposition 27, the term with coefficient in τ_I is added as $O(t)$ whereas here it is added as $O(\lambda)$.

2. Let $\lambda \in E^{2''}_{I0}(\Lambda_1)$. We divide $\check{T}_{I,\lambda}(t)$ by $\big(1 + \mathcal{O}_{75,I,\lambda}(t)\big)$ which is nonzero on $[\epsilon, 1-\epsilon]$ for λ in a sufficiently small neighborhood and differentiate once more with respect to t. We obtain a function of the form of Equation (53) (but in which all coefficients may be small).

The result follows using the same argumentation as in the proof of Proposition 31. ∎

3.5. General conclusion for $t \in [0,1]$ with Γ_0 of codimension k

PROPOSITION 35. *Let Γ_0 be of type (I_1, I_2, J, L), $I = I_1 + JI_2$, and $n = 2(I + [J\mu]) + 1$. There exists a neighborhood Λ_0 of $\lambda = 0$ such $V_\lambda(t)$ has at most $N = \frac{1}{2}\big(n(4n^2 + 16n + 37) + 1\big)$ roots on $[0,1]$.*

Proof. As we saw in the previous sections, we can divide the coefficient space in several cones noted $E_\ell(\Lambda_1)$ and $E_{ij}(\Lambda_1)$. We prove the result on each cone.

Let us restrict the parameter space to any of the cones. Moreover, assume the n^{th}-derivative $V_\lambda^{(n)}(t)$ of $V_\lambda(t)$ has a maximum of d zeros on this cone. Choose a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ converging to 0 such that $V_{\lambda_i}^{(n)}(t)$ has d zeros. Of those d zeros, assume m_0 go to 0 and m_1 go to 1 (the m_ℓ can of course be 0). Let $1 - t_1$ be the lower bound of the set of roots that go to 1, and t_2 the upper bound of the set of roots that go to 0. Note $\epsilon_2 = \min\{t_1, t_2\}$, the minimum of the two.

We then have that $V_0^{(n)}(t)$ is of the following form:

$$
\begin{aligned}
V_0^{(n)}(t) = &\sum_{i=1}^{I+[J\mu]} \tau_i\Big(t^i(1-t)^{I+[J\mu]+1} + (-1)^{I+[J\mu]}(1-t)^i t^{I+[J\mu]+1}\Big)\\
+ &\sum_{\substack{1\le i+j\mu\le I+J\mu\\ j\neq 0}} t^i\Big(\rho_{ij}t^{j\mu}(1-t)^{I+[J\mu]+1} - \check{\rho}_{ij}t^{I+[J\mu]+1}(1-t)^{j\mu}\Big) + V_{k,0}^{(n)}(t),
\end{aligned}
\tag{54}
$$

in which we can factor $t^{m_0}(1-t)^{m_1}$ and where

$$
V_{k,0}^{(n)}(t) = \begin{cases} \tau_{I+[J\mu]}(1-t)^{I+[J\mu]}t^{I+[J\mu]}\times \\ \qquad\qquad \Big((1-t)+(-1)^{I+[J\mu]}t\Big) & \begin{cases}\text{if } (J,L)\neq(0,0)\\ \text{or if the cone is}\\ \qquad\qquad E_{I0}^2(\Lambda_1)\end{cases} \\ \rho_{I0}(1-t)^I t^I & \text{otherwise.}\end{cases}
$$

Let $\tilde{\mathcal{G}}_\lambda\big(t,(1-t)^\mu,t^\mu\big) = V_0^{(n)}(t)$. As in the proof of Proposition 31, the result follows from Khovanskiĭ's fewnomials theory if Equation (54) is nontrivial, which was proven either in Proposition 31 or in Proposition 34, since $\tilde{\mathcal{G}}_\lambda(t,y,z)$ is of degree at most $2(I+[J\mu])+1-m_0-m_1$. ∎

Corollary 36. *Let Γ_0 be of type (I_1,I_2,J,L), $I=I_1+JI_2$, and $n=2(I+[J\mu])+1$. Then $Cycl(\Gamma_0)\le \frac{1}{4}\big(n(4n^2+10n+37)+3\big)$.*

Proof. The result follows from Proposition 35 and Facts 8. ∎

3.6. Khovanskiĭ's fewnomial theory and proof of Theorem 33.

In this section we prove Theorem 33. The result is obtained using Khovanskiĭ's method of reducing a transcendental system to nondegenerate polynomial ones; our setting is one of the simplest nontrivial cases of the theory. The theory in its full generality can be found in [11]. In their article [8], Il'yashenko and Yakovenko used the theory to bound the cyclicity of elementary polycycles on $\mathbb{R}^2$ in generic families. Section 2 of their paper is certainly a good introduction to the subject. We illustrate the theory for the simplest case, when $P_n(t)$ has nondegenerate zeros.

3.6.1. The zeros of $\boldsymbol{P_n(t)}$ are solutions of a system of transcendental equations on $\mathbb{R}^3$.

We first transform the problem of bounding the number of zeros of $P_n(t)$ to bounding the number of solutions of a transcendental system on $\mathbb{R}^3$.

Define the following two functions on $\mathbb{R}^3$:

$$\begin{cases} f_{2,A}(t,y,z) \stackrel{\text{def}}{=} y - At^{\mu} \\ f_{3,B}(t,y,z) \stackrel{\text{def}}{=} z - B(1-t)^{\mu}, \end{cases} \tag{55}$$

where $(A,B) \in {\mathbb{R}^+}^2$ and consider the system of transcendental equations

$$\mathsf{S}_0 = \begin{cases} f_1(t,y,z) = 0 \\ f_{2,1}(t,y,z) = 0 \\ f_{3,1}(t,y,z) = 0, \end{cases} \tag{56}$$

defined on $\mathcal{D}_\epsilon$, where

$$\mathcal{D}_\epsilon \stackrel{\text{def}}{=} [\epsilon, 1-\epsilon] \times \left[\epsilon^{\mu}, (1-\epsilon)^{\mu}\right]^2 \subseteq \mathbb{R}^3.$$

Lemma 37. *Solving $P_n(t) = 0$ on $[\epsilon, 1-\epsilon]$ is equivalent to solving system S_0 on $\mathcal{D}_\epsilon$.*

We use Khovanskiĭ's method to compute an explicit upper bound for the number of isolated zeros of the transcendental system S_0, system (56). The method consists in transforming the transcendental problem in algebraic ones, allowing to use Bezout's theorem. This is done in four main steps:

1. We verify that the system has a finite number of solutions which are then isolated.

2. We unfold the transcendental systems in a family of systems where all degeneracies have been eliminated in the generic systems.

3. Using the fact that the manifolds $\{f_{2,A} = 0\}$ and $\{f_{3,B} = 0\}$ are integral separating solutions of polynomial Pfaff equations (to be defined below), we embed the system in a nondegenerate system S of Pfaff forms and polynomials. Indeed the transcendental functions $f_{2,1}$ and $f_{3,1}$ in S_0 are separating solutions of polynomial Pfaff 1-forms. For instance the function $f_{2,1}$ is an integral solution of the polynomial Pfaff 1-form

$$w_2 \stackrel{\text{def}}{=} t dy - \mu y dt,$$

and the function $f_{3,1}$ is an integral solution of the polynomial Pfaff 1-form

$$w_3 \stackrel{\text{def}}{=} (1-t)dz + \mu z dt.$$

The two solutions in $\mathcal{D}_\epsilon$ are given in Figure 6.

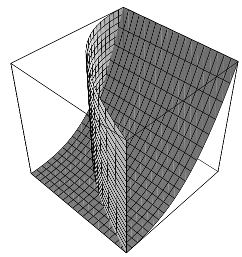


Figure 6. The manifolds $\{f_{2,1} = 0\}$ and $\{f_{3,1} = 0\}$ in $\mathcal{D}_\epsilon$.

4. Finally we iterate Khovanskiĭ's reduction method to bound the number of zeros of S by the sum of the number of zeros of polynomial systems having nondegenerate roots and to which we can apply Bezout's theorem.

3.6.2. The smooth manifold with boudary $\boldsymbol{M_\epsilon} \supseteq \boldsymbol{\mathcal{D}_\epsilon}$.

The theory applies to systems defined on smooth manifolds. We must thus define a smooth manifold with boundary M_ϵ which contains $\mathcal{D}_\epsilon$ and on which the system (55) is smooth. Let

$$M_\epsilon \stackrel{def}{=} \left\{(t,y,z) \in \mathbb{R}^3 | F(t,y,z) \stackrel{def}{=} t(1-t)y(1-y)z(1-z) \geq \epsilon^{3\mu}(1-\epsilon)^3\right\},$$

and denote by M_ϵ^0 its interior. We then have that $M_\epsilon \supseteq \mathcal{D}_\epsilon$. We can also choose ϵ small and such that the algebraic surface $f_1 = 0$ is in general position with respect to the boundary $F = \epsilon^{3\mu}(1-\epsilon)^3$.

3.6.3. Bounding the number of solutions of $\mathbf{S}_0$.

Lemma 38. *For all* $(A, B) \in {\mathbb{R}^+}^2$, *the system*

$$\mathcal{S}_{0,A,B} = \begin{cases} f_1(t,y,z) = 0 \\ f_{2,A}(t,y,z) = 0 \\ f_{3,B}(t,y,z) = 0 \end{cases}$$

has a finite number of solutions.

Proof. By hypothesis the polynomial $f_1(t,y,z)$ has at least one nonzero coefficient.

1. If f_1 is a polynomial in only one of the variables t, y, or z, the result follows (e.g. from Rolle's theorem).

2. If f_1 is a polynomial in at least the y and t variables, we can write f_1 as a function of y of the following form (in the neighbourhood of 0):

$$f_1(t, y, z) = \sum_{i \geq 0} \tilde{f}_i(t, z) y^i.$$

Zeros of f_1 are thus solutions of the following equation:

$$-\tilde{f}_0(t, z) = \sum_{i \geq 1} \tilde{f}_i(t, z) y^i. \tag{57}$$

There exists $i \in \mathbb{N}^+$ such that after expanding $z = B(1-t)^\mu$ (if it occurs in the $\tilde{f}_i$) as a function of t in the neighbourhood of 0 and substituting $y = At^\mu$, Equation (57) can be written in the following form:

$$a_{k_1} t^{k_1} \big(1 + O(t)\big) = b_{k_2} t^{k_2 + i\mu} \big(1 + O(t)\big), \tag{58}$$

where $b_{k_2} \neq 0$. Let $k_3 = \min\{k_1, k_2 + i\mu\}$, and let c_{k_3} be the nonzero coefficient corresponding to k_3. Dividing Equation (58) by t^{k_3} and taking $t = 0$, we get that there exists $\epsilon > 0$ sufficiently small such that the system has no zeros for $t \in (0, \epsilon)$. From the analycity of the functions on $(0, 1)^3$, we have that on any M_ϵ with $\epsilon > 0$ the system has a finite number of solutions.

3. If f_1 is a polynomial in only the z and t variables, we use the same argument as in the previous case where we interchange y and z, and expand around $z = 0$ and $t = 1$.

∎

3.6.4. Khovanskiĭ's reduction procedure

In this section, we will only consider the case where $f_1 = 0$ is a nondegenerate algebraic surface (a regular surface), i.e. $f_1 = 0$ has no singular points in M_ϵ, and $f_1 = f_{3,1} = 0$ is a nondegenerate curve in $f_{3,1} = 0$. This simple case illustrates all important geometric ideas of the method. The result is true for a general algebraic surface $f_1 = 0$, but the generalization of the method is much more technical since we need to control all possible pathologies (cf. [11], Chapter 3).

Definition 39. A **contact point** of a curve and a vector field in the plane is a point of the curve in which the tangent vector to the curve and the vector of the vector field are collinear.

It is easily seen that between two points of intersection of a connected component of $f_1 = f_{3,1} = 0$ with $f_{2,1} = 0$ there exists a contact point of

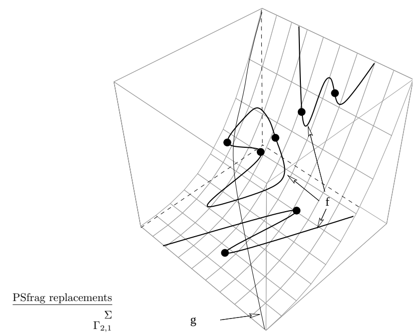


Figure 7. Example of contact points on $\{f_{3,1}=0\}$: $\Sigma=\{f_1=f_{3,1}=0\}$ and $\Gamma_{2,1}=\{f_{2,1}=f_{3,1}=0\}$

$f_1=f_{3,1}=0$ and w_2 (Figure 7) . Hence

$$\begin{aligned}\#_0\big(P_n(t)\big) \le{}& \text{number of contact points of } f_1=f_{2,1}=0 \text{ and } w_2=0\\ &+\text{number of noncompact connected components of } f_1=f_{3,1}=0.\end{aligned}$$

Define the following map $*$ mapping 3-forms to functions.

Definition 40. Let $\alpha = f\,dx\wedge dy\wedge dz$ be a 3-form on M_ϵ. Then

$$*(\alpha) \overset{def}{=} f.$$

The equation of the contact points on $f_1=f_{3,1}=0$ is given by

$$\begin{aligned}&f_1=f_{3,1}=0\\ &W_1 \overset{def}{=} *(w_3\wedge w_2\wedge df_1) \qquad\qquad (\deg W_1 = n+1),\end{aligned}$$

which we can again consider as a Pfaffian system:

$$\mathsf{S}_1 = \begin{cases} f_1 = 0\\ W_1=0\\ w_3=0.\end{cases} \tag{59}$$

Each noncompact connected component of $f_1=f_{3,1}=0$ intersects ∂M_ϵ in at least two points. Hence the number of noncompact components is bounded by

$$\frac{1}{2}\#\{f_1 = f_{3,1} = F = 0\}, \tag{60}$$

where $\partial M_\epsilon = \{F = 0\}$. We can also consider (60) as a Pfaffian system

$$\mathsf{S}_2 = \begin{cases} f_1 = 0 \\ F = 0 \\ w_3 = 0. \end{cases} \tag{61}$$

The elimination of w_3 in systems (59) and (61) is similar although it is simpler in system (61). We now consider the curve $f_1 = W_1 = 0$ which, for the moment, we suppose regular.

Between two intersection points of $f_1 = W_1 = 0$ with $f_{3,1} = 0$ there is at least one contact point with w_3. Hence

$$\begin{aligned} \#_0\{f_1 = W_1 = f_{3,1} = 0\} &\le \#\big\{f_1 = W_1 = *(df_1 \wedge dW_1 \wedge w_3)\big\} \\ &\quad + \frac{1}{2}\#\{f_1 = W_1 = F = 0\} \\ &= 2n^2(n+1) + 3n(n+1). \end{aligned}$$

Let

$$\begin{aligned} W_2 &\stackrel{\text{def}}{=} *(df_1 \wedge dW_1 \wedge w_3) \qquad &(\deg = 2n) \\ W_3 &\stackrel{\text{def}}{=} *(df_1 \wedge dF \wedge w_3) \qquad &(\deg = n+5). \end{aligned}$$

In the case of system (61), $F = 0$ is a compact manifold without boundary. Hence

$$\#_0\{f_1 = f_{3,1} = F = 0\} \le \#\big\{f_1 = F = W_3\big\} = 6n(n+5).$$

Therefore

$$\begin{aligned} \#_0\big(P_n(t)\big) \le{} & \#\big\{f_1 = W_1 = W_2\big\} \\ & + \frac{1}{2}\Big(\#\{f_1 = W_1 = F = 0\} + \#\big\{f_1 = F = W_3\big\}\Big), \end{aligned}$$

i.e.

$$\#_0\big(P_n(t)\big) \le 2n^2(n+1) + 3n(n+1) + 3n(n+5) = n(2n^2 + 8n + 18).$$

3.6.5. The case of degenerate systems

As we have seen, the case of degenerate systems can be of different nature:

1. the intersections are not transversal: the remedy is to count points with multiplicity;
2. the surface $f_1 = 0$ is not regular;
3. the intersection of the surface $f_1 = 0$ with $f_{3,1} = 0$ is not a regular curve;
4. the curve $f_1 = W_1 = 0$ is not regular.

The solution exhibited by Khovanskiĭ is to introduce an unfolding of the Pfaffian system

$$\mathsf{S}_{\lambda,6} = \begin{cases} f_{1,\lambda} \overset{\mathrm{def}}{=} f_1(t) + \displaystyle\sum_{i+j\mu=0}^{k} a_{ijl} t^i y^j z^l = 0 \\ w_{2,\lambda} \overset{\mathrm{def}}{=} w_2 + \displaystyle\sum_{i=1}^{3} (\xi_{2i0} + \xi_{2i1} t + \xi_{2i2} y + \xi_{2i3} z) dx_i = 0 \\ w_{3,\lambda} \overset{\mathrm{def}}{=} w_3 + \displaystyle\sum_{i=1}^{3} (\xi_{3i0} + \xi_{3i1} t + \xi_{3i2} y + \xi_{3i3} z) dx_i = 0 \end{cases}$$

with $x_1 = t$, $x_2 = y$, $x_3 = z$ and $\lambda = (a_{ijl}, \xi_{2ij}, \xi_{3ij})$.

We repeat the previous argument (Section 3.6.4) for all systems $\mathsf{S}_{\lambda,6}$ where λ is a regular value of the parameter (a value for which none of the previous pathologies occur) in a small neighborhood Λ of 0. Let $\Lambda_0 \subseteq \Lambda$ be the set of regular values of the parameter and $B(\lambda)$ the bound obtained by the method. (This set Λ_0 is of full mesure, cf. [11, Prop. 3, Section 3.9].) Then

$$\#_0\big(P_n(t)\big) \le \max_{\lambda \in \Lambda_0} B(\lambda).$$

This ends the proof of Theorem 33.

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