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Melnikov Functions and Bautin Ideal

Robert Roussarie

Laboratoire de Topologie, U.M.R. 5584 du C.N.R.S., Université de Bourgogne, 9 avenue Alain Savary, B.P. 47 870, 21078 Dijon Cedex, France
E-mail: roussari@u-bourgogne.fr

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The computation of the number of limit cycles which appear in an analytic unfolding of planar vector fields is related to the decomposition of the displacement function of this unfolding in an ideal of functions in the parameter space, called the Ideal of Bautin. On the other hand, the asymptotic of the displacement function, for 1-parameter unfoldings of hamiltonian vector fields is given by Melnikov functions which are defined as the coefficients of Taylor expansion in the parameter. It is interesting to compare these two notions and to study if the general estimations of the number of limit cycles in terms of the Bautin ideal could be reduced to the computations of Melnikov functions for some 1-parameter subfamilies.

Key Words: Cyclicity, Melnikov function, desingularization

1. INTRODUCTION

Let $X = X_H$ be a real analytic Hamiltonian vector field with a closed orbit γ and let (X, γ) be the germ of X near γ. We want to consider analytic unfoldings $(X_\lambda, (\gamma, 0))$ i.e., germs of analytic families near γ with $X_0 = X$ and a parameter $\lambda \in (\mathbb{R}^k, 0)$. Recall that the cyclicity of $(X_\lambda, (\gamma, 0))$, noted $\mathrm{Cycl}(X_\lambda, (\gamma, 0))$ is the upper bound of the number of limit cycles which bifurcate from γ at $\lambda = 0$. In the present case of a closed orbit, the cyclicity is just the upper bound of the number of zeros which bifurcate in a function unfolding. To see this, one considers a transversal arc σ to γ, parameterized by $x \in]-\eta, \eta[$ with $\eta > 0$. For any η', $0 < \eta' < \eta$, there exists a neighborhood of $0 \in \mathbb{R}^k$ such that one can define a return map for X_λ from $]-\eta', \eta'[$ to $]-\eta, \eta[$:

$$P(x, \lambda) :]-\eta', \eta'[\times W \to]-\eta, \eta[.$$

The map $P(x, \lambda)$ is analytic. One defines the displacement function :

$$\delta(x, \lambda) = P(x, \lambda) - x$$

The cyclicity $\mathrm{Cycl}(X_\lambda, (\gamma, 0))$ is defined by :

$$\mathrm{Cycl}(X_\lambda, (\gamma, 0)) = Inf\{N(\varepsilon_1, \varepsilon_2) \mid \varepsilon_1 \to 0, \varepsilon_2 \to 0\}$$

where $N(\varepsilon_1, \varepsilon_2) = Sup\{N(\varepsilon_1, \lambda) \mid \|\lambda\| \le \varepsilon_2\}$ and $N(\varepsilon_1, \lambda)$ is the number of isolated roots of $\{\delta(x, \lambda) = 0\}$ on $]-\varepsilon_1, \varepsilon_1[$. Here we choose $0 < \varepsilon_1 \le \eta'$ and $\lambda \in W$.

It follows for instance from the theorem of Gabrielov [9] that the cyclicity is always finite (see also [6], [14]). An important question is to compute it explicitely for a given unfolding. This is easy for 1-parameter unfoldings. Suppose given such an unfolding, parameterized by $\varepsilon \in (\mathbb{R}, 0)$. One can expand the displacement function $\delta(x, \varepsilon)$ in series of ε. If the unfolding is not trivial, i.e. if X_ε is not an Hamiltonian vector field for each ε, then there is a number $\nu \neq 0$ and an analytic function $M_\nu(x)$, not identical to zero, such that :

$$\delta(x, \varepsilon) = \varepsilon^\nu M_\nu(x) + o(\varepsilon^\nu)$$

The function M_ν is called the (first) Melnikov function of the unfolding. Its order or multiplicity at $\{x = 0\}$ is a majorant for the cyclicity.

It is easy to compute explicitely this order. One associates to X_ε its dual 1-form unfolding $\omega_\varepsilon = dH + \varepsilon\tilde{\omega} + o(\varepsilon)$. If $\nu = 1$ the Melnikov function is given by the integral $M_1(x) = \int_{\gamma_x} \tilde{\omega}$ where γ_x is the level of H by $x \in \sigma$ and $\gamma = \gamma_0$.

If $\nu > 1$, similar integral formulas to compute $M_\nu(x)$ were obtained by J.P.Françoise. ([4], [5], [7]; see also [15]. It is easy to deduce from them the order of $M_\nu(x)$ at 0.

These observations about the Melnikov functions of 1- parameter unfoldings, arise the following question : *is it possible to reduce the computation of the cyclicity of a general unfolding to the computation of Melnikov functions?.*(see [8] for instance)

Let us consider again a general analytic unfolding $(X_\lambda, (\gamma, 0))$. It seems natural to consider analytic arcs $\xi(\varepsilon) : (\mathbb{R}, 0) \to (\mathbb{R}^k, 0)$ through the origin in the parameter space, and to look at the 1-parameter unfoldings $X_{\xi(\varepsilon)}$ obtained by restriction. Let M_ξ be the order of the first Melnikov function of $X_{\xi(\varepsilon)}$ (we take $M_\xi = 0$ if $X_{\xi(\varepsilon)}$ is a trivial unfolding). We will call M_ξ, the *Melnikov multiplicity* of the arc ξ. The principal result of this paper is to prove that the Melnikov functions indeed give a majorant for the cyclicity :

Theorem 1. $\mathrm{Cycl}(X_\lambda, (\gamma, 0)) \le SupM_\xi$, *where the upper bound is taken above all the analytic arc germs at* $0 \in R^k$.

It would seem reasonable to prove the above theorem, using the selecting lemma of Milnor [12]. A version of the selecting lemma for sub-analytic sets appeared in [2], in the line of the theory of semi-analytic and sub-analytic sets developped in [11] and [3]. (a more recent introduction to this theory can be found in [1]) The subset in the parameter space in which the number of limit cycles near γ is equal to $\mathrm{Cycl}(X_\lambda, (\gamma, 0))$ is a sub-analytic subset which contains 0 in its closure. Then, one can select inside, an analytic arc $\xi_0(\varepsilon)$ such that for each ε, one has exactly $n = Cycl(X_\lambda, (\gamma, 0))$ zeros $x_1^\varepsilon, \ldots, x_n^\varepsilon$, for the equation $\{\delta(x, \xi_0(\varepsilon)) = 0\}$ and $x_i^\varepsilon \to 0$ when $\varepsilon \to 0$. As a consequence, $\mathrm{Cycl}(X_\lambda, (\gamma, 0)) \le M_{\xi_O}$.

Here, we will give a different proof, based on the desingularization of the Bautin Ideal. The definition of this ideal, associated to the unfolding X_λ, will be recalled below. The advantage of this proof is to give an algorithm to find an explicit arc $\xi_0(\varepsilon)$ as above. This allows a computation of an index s_M defined below in paragraph 4. This index is an upper bound for $\mathrm{Cycl}(X_\lambda, (\gamma, 0))$.

2. THE BAUTIN IDEAL OF AN ANALYTIC UNFOLDING

Let us consider an analytic unfolding X_λ as above, with $\lambda = (\lambda_1, \ldots, \lambda_k) \in (\mathbb{R}^k, 0)$. Expanding the displacement map at $\{x = 0\}$,

$$\delta(x, \lambda) = \sum_{i=0}^{\infty} a_i(\lambda) x^i$$

we define the Bautin ideal $\mathcal{I}$ of X_λ to be the ideal generated by the germs $(a_i, 0)$ in the ring $\mathcal{O} = \mathcal{O}_0(\mathbb{R}^k)$ of analytic germs of functions at $0 \in \mathbb{R}^k$. This ideal is Noetherian and then generated by a finite number of germs : $\mathcal{I} = \mathcal{O}\{\varphi_1, \ldots, \varphi_l\}$. We recall now some definitions and simple basic results from [13],[14]. First, one can decompose locally the function δ in the system of generators $\{\varphi_1, \ldots, \varphi_l\}$:

$$\delta(x, \lambda) = \sum_{i=1}^{l} \varphi_i(\lambda) h_i(x, \lambda)$$

where the factors h_i are analytic in a neighborhood of $(0, 0) \in \mathbb{R} \times \mathbb{R}^k$.

For a 1-parameter nontrivial unfolding X_ε, one has

$$\delta(x, \varepsilon) = \varepsilon^\nu M_\nu(x) + o(\varepsilon^\nu)$$

for some ν and a Melnikov function $M_\nu(x)$ which is non identical to zero. In this case, the Bautin Ideal is the ideal generated by ε^ν, at $0 \in \mathbb{R}$, and all the following considerations are trivial.

Next, one can choose the system of generators to be minimal in the sense that classes of the φ_i generate a vector space basis of $\mathcal{I}/\mathcal{M}\mathcal{I}$ ($\mathcal{M}$ is the maximal ideal of the ring $\mathcal{O}$). This follows of the Nakayama's lemma and the dimension $l = dim_{\mathbb{R}}\mathcal{I}/\mathcal{M}\mathcal{I}$ is a well-defined invariant for $\mathcal{I}$. Let us suppose chosen such a minimal system of generators. The factor functions $H_i(x) = h_i(x, 0)$, where the h_i come from a decomposition of δ as above, are also well-defined (independent on the choice of the h_i) and moreover, the system of analytic functions $\{H_1, \ldots, H_l\}$ is independent over $\mathbb{R}$.

It follows from this independence that we can define an index s_R (see [14]), by :

$$s_R = \text{Inf}\{n \mid \dim \mathbb{R}\{j^n H_1(0), \ldots, j^n H_l(0)\} = l\}$$

Another way to define the index s_R is the following. There exist adapted minimal systems of generators for $\mathcal{I}$ such that :

$$\text{ord}(H_1) < \text{ord}(H_2) < \ldots < \text{ord}(H_l)$$

(if $g(x)$ is any analytic germ at $0 \in \mathbb{R}$, $\text{ord}(g)$ is the order of g at $\{x = 0\}$).

It is easy to see that $s_R = \text{ord}(H_l)$, for any adapted minimal system of generators. This index s_R does not depend on the choice of a minimal system of generators. Moreover, it is proved in [14] that one have the bound :

$$\text{Cycl}(X_\lambda, (\gamma, 0)) \leq s_R$$

To prove this inequality, one considers a covering of a small neighborhood W of $0 \in \mathbb{R}^k$ by the sets :

$$V_i = \{\lambda \in W \mid\mid \varphi_i(\lambda) \mid \geq \mid \varphi_j(\lambda) \mid, \quad j = 1, \ldots, l\}$$

and proves that on each V_i the behavior of δ is dominated by the one of φ_i. This implies that on V_i, the number of roots of $\{\delta = 0\}$ is less than $\text{ord}(H_i)$.

It is possible to obtain a better result by restricting the choice of i to well-chosen subsets $I \subset \{1, \ldots, l\}$. Taking any r, $0 < r \leq 1$, one can define larger sets :

$$V_i^r = \{\lambda \in W \mid\mid \varphi_i(\lambda) \mid \geq r \mid \varphi_j(\lambda) \mid, j = 1, \ldots, l\}$$

We have now the following :

Proposition 2. *Suppose that there exists some* r, $0 < r \leq 1$, *and some* $I \subset \{1, \ldots, l\}$, *such that* $\cup_{i \in I} V_i^r$ *is a neighborhood of* $0 \in W$, *then*

$$\mathrm{Cycl}(X_\lambda, (\gamma, 0)) \leq \mathrm{Sup}\{\mathrm{ord}(H_i) \mid i \in I\}$$

The proof of this proposition is exactly the same as the proof for s_R given in [14], which corresponds to $r = 1$ and $I = \{1, \ldots, l\}$.

Let us consider now any analytic arc $\xi(\varepsilon)$ through the origin in the parameter space. We can restrict the unfolding to this arc. The displacement function of this restriction is just :

$$\delta(x, \xi(\varepsilon)) = \sum \varphi(\xi(\varepsilon)) h_i(x, \xi(\varepsilon))$$

It follows from this the inequality $M_\xi \leq s_R$ and then $s_A = \mathrm{Sup}_\xi M_\xi \leq s_R$.
In the following paragraphs, one will use the above proposition to prove that s_A is an upper bound for the cyclicity.

3. THE CASE OF A BAUTIN IDEAL GENERATED BY MONOMIALS

In this paragraph we consider the very special following case : *the Bautin Ideal is assumed to have an adapted minimal set of generators which are monomials in* $(\lambda_1, \ldots, \lambda_k)$.

Let us introduce the following notations :

$$\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{N}^k, \quad \lambda = (\lambda_1, \ldots, \lambda_k) \in (\mathbb{R}^k, 0), \quad \lambda^\beta = \lambda_1^{\beta_1} \ldots \lambda_k^{\beta_k}$$

We suppose that the Bautin ideal $\mathcal{I}$ is generated by $S = \{\lambda^{\beta^1}, \ldots, \lambda^{\beta^l}\}$ for some finite set $\{\beta^1, \ldots, \beta^l\} \subset \mathbb{N}^k$. The set

$$\Delta = S + \mathbb{N}^k$$

is the Newton's diagram of the ideal. It is generated by the subset of its minimal points M in the sense that $\Delta = M + \mathbb{N}^k$, and the elements of M are minimal in Δ for the partial order defined by the norm $\mid \beta \mid = \beta_1 + \ldots + \beta_k$. In fact , as $\{\lambda^{\beta^1}, \ldots, \lambda^{\beta^l}\}$ is supposed to be a minimal set of generators, we have that $M = \{\lambda^{\beta^1}, \ldots, \lambda^{\beta^l}\}$.

Let us consider now the convex hull Δ^c of Δ in $\mathbb{N}^k$ and the subset E of extremal points of Δ^c. One have $E \subset d\Delta^c = \Delta \cap \partial\Delta^c \subset M$. We will call E, the set of convex extremal points. This set E is the 0-skeleton of a simplicial decomposition of $\partial\Delta^c$.

EXAMPLE 3. For $k = 2$ and $S = M = \{\lambda_1^8, \lambda_1^6\lambda_2, \lambda_1^4\lambda_2^2, \lambda_1^2\lambda_2^4, \lambda_1\lambda_2^7, \lambda_2^8\}$, one has $E = \{\lambda_1^8, \lambda_1^4\lambda_2^2, \lambda_1^2\lambda_2^4, \lambda_2^8\}$ since $\lambda_1^6\lambda_2 \in d\Delta^c \backslash E$ and $\lambda_1\lambda_2^7 \in M \backslash d\Delta^c$.

From now on, we change somewhat the indexation, writing $\{\varphi_\beta = \lambda^\beta \mid \beta \in M\}$ for the given set of generators. In this set of generators, the decomposition of the displacement function, is written :

$$\delta(x, \lambda) = \sum_{\beta \in M} \varphi_\beta(\lambda) h_\beta(x, \lambda)$$

Let $H_\beta(x) = h_\beta(x, 0)$ for any $\beta \in M$, and $O(\beta) = \mathrm{ord}(H_\beta)$. By assumption, the map $\beta \to O(\beta)$ is injective from M to $\mathbb{N}$ (but have nothing to do with the lexicographic order among the coefficients β).

We introduce now the index $s_E = \mathrm{Sup}\{O(\beta) \mid \beta \in E\}$. Clearly, one has $s_E \leq s_R$ where s_R is the index defined in the last paragraph, for the general unfolding. One wants to establish the following :

THEOREM 4. *(a)* $\mathrm{Cycl}(X_\lambda, (\gamma, 0)) \leq s_E$,

(b) $s_E \leq \mathrm{Sup}_\xi M_\xi$ *where the upper bound is taken over all the analytic arcs* $\xi(\varepsilon)$ *through* $0 \in \mathbb{R}^k$.

Remark 5. In the proof we will give an explicit arc $\xi_0(\varepsilon) = \varepsilon^\alpha = (\varepsilon^{\alpha_1}, \ldots, \varepsilon^{\alpha_k})$ for some $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ such that $s_E = M_{\xi_0}$.

Proof. Let W be a sufficiently small neighborhood of $0 \in \mathbb{R}^k$. Using the indexation by the $\beta \in M$, we introduce as above the sets :

$$V_\beta^r = \{\lambda \in W \mid\mid \varphi_\beta(\lambda) \mid \geq r \mid \varphi'_\beta(\lambda) \mid, \forall \beta' \in M\}$$

The point (a) will be a consequence of the proposition 2 if we prove that for some r, $0 < r \leq 1$, the union of the V_β^r, for $\beta \in E$, is a neighborhood of the origin in W. This property is equivalent to the following one :

There exists a constant $C > 0$ *such that, on some compact neighborhood* W *of* $O \in \mathbb{R}^k$, *one has that* $Sup\{\mid \varphi_\beta(\lambda) \mid\mid \beta \in E\} \geq C \mid \varphi_{\beta'}(\lambda) \mid$ *for any* $\beta' \in M$.

Observe first, that for any $\beta' \in M$, there exists some $\beta'' \in \partial\Delta^c$ such that $\beta' \in \beta'' + \mathbb{N}^k$. Then it suffices to prove the above claim for the $\beta' \in \partial\Delta^c$. Let β' be any coefficient in $\partial\Delta^c$. It belongs to a simplex of the simplicial decomposition of $\partial\Delta^c$, with the set E as 0-skeleton. Then, there exist coefficients $\mu_\beta \geq 0$ for $\beta \in E$, such that $\sum_{\beta \in E} \mu_\beta = 1$ and $\beta' = \sum_{\beta \in E} \mu_\beta \beta$. This implies the following formula :

$$\varphi_{\beta'} = \lambda^{\beta'} = \prod_{\beta \in E} \varphi_\beta^{\mu_\beta}$$

The desired inequality follows from it, if we take $C = D^{|E|-1}$, where $D = \text{Sup}\{| \varphi_\beta(\lambda) |\}$ for $\beta \in E$ and $\lambda \in W$.

To prove the point (b), we show that for any $\beta_0 \in E$, we can find a monomial arc :

$$\xi_0(\varepsilon) = \varepsilon^\alpha = (\varepsilon^{\alpha_1}, \ldots, \varepsilon^{\alpha_k})$$

such that $M_\xi = O(\beta_0)$. To this end, one chooses α such that the hyperplane direction $\{u \to \alpha \cdot u = \sum \alpha_i u_i\}$ has a strict contact with Δ^c : this means that there exists an affine hyperplane $P \subset \mathbb{N}^k$, parallel to this direction, with the property that $P \cap \Delta^c = \{\beta_0\}$.

For any monomial β one has $\varphi_\beta(\xi(\varepsilon)) = \varepsilon^{\beta \cdot \alpha}$ and, as the contact is strict, one has that $\beta \cdot \alpha > \beta_0 \cdot \alpha$ for all $\beta \in M \backslash \{\beta_0\}$.

It follows from this the expansion that one has

$$\delta(x, \xi(\varepsilon)) = \varepsilon^{\beta_0 \cdot \alpha} H_{\beta_0}(x) + o(\varepsilon^{\beta_0 \cdot \alpha})$$

As a consequence it follows the identity $M_\xi = \text{ord} H_{\beta_0}$. This means that

$$s_E = O(\beta_0) = M_\xi$$

for the given arc ξ, if we choose β_0 such that $O(\beta_0) = s_E$. ▮

Remark 6. In the next paragraph, we will need a slightly more general version of the theorem 4 : The parameter will be equal to $\tilde{\Lambda} = (\Lambda, \Lambda')$, and the Bautin ideal will have an adapted minimal system of generators Λ^β, depending only on the variable Λ. It is trivial to extend the theorem to this case.

4. THE CASE OF A GENERAL BAUTIN IDEAL

To reduce the case of a general ideal to the particular of an ideal generated by monomials, we will use the desingularization theorem of Hironaka [10] for real analytic functions.

Let us recall it briefly. One considers a germ of an analytic function φ at the origin of $\mathbb{R}^k$. Then, there exists a pair of an algebraic manifold of dimension k and an algebraic compact subset (A, Σ), a surjective proper analytic map Π from this pair to $(\mathbb{R}^k, 0)$, such that $\Pi^{-1}(0) = \Sigma$ which gives to $\varphi \circ \Pi$ the property of normal crossing. This means the following. Each point $\tilde{\Lambda}_0 \in \Sigma$ has a chart W with analytic coordinates $\tilde{\Lambda} = (\Lambda, \Lambda')$, with $\tilde{\Lambda}_0 = 0 \in \mathbb{R}^k$ and $\Lambda = (\Lambda_1, \ldots, \Lambda_c), \Lambda' = (\Lambda'_1, \ldots, \Lambda'_{k-c})$, for some $c, 1 \le c \le k$. On this chart, the composition map $\varphi \circ \Pi : W \to \mathbb{R}^k$ is monomial :

$$\varphi \circ \Pi(\tilde{\Lambda}) = u(\tilde{\Lambda}) \Lambda^\beta$$

where $u(0) \neq 0$ and $\beta \in \mathbb{N}^c$

We want to apply this theorem to the Bautin Ideal $\mathcal{I}$ of an unfolding $(X_\lambda, (\gamma, 0))$. We suppose that we have chosen an adapted minimal set of generators $\{\varphi_1, \ldots, \varphi_l\}$ for $\mathcal{I}$. We apply the above Hironaka's desingularization theorem to the product function $\varphi = \prod_{i=1}^{l} \varphi_i$. Then, for each $\tilde{\Lambda}_0 \in \Sigma$ and in the coordinates of normal crossing for φ, we can write for each $i = 1, \ldots, l$:

$$\Phi_i(\tilde{\Lambda}) = \varphi_i \circ \Pi(\tilde{\Lambda}) = u_i(\tilde{\Lambda}) \Lambda^{\beta^i}$$

where $u_i(0) \neq 0$ and $\beta^i \in \mathbb{N}^c$

The existence of a system of normal crossing coordinates implies that the set Σ is a union of a finite number of strata $\sigma_1, \ldots, \sigma_d$, the boundary of each one being a union of strata of strictly smaller dimension.

Let σ be one of these strata and $c(\sigma)$ its codimension, $1 \leq c(\sigma) \leq k$. For each function Φ_i, the associate coefficient β^i is constant, all along σ. We call it $\Delta_\sigma(i)$.

Let us look more closely to the *induced unfolding* $(X_{\Pi(\tilde{\Lambda})}, (\gamma, \tilde{\Lambda}_0))$. Its displacement function is equal to :

$$\tilde{\delta}(x, \tilde{\Lambda}) = \delta(x, \Pi(\tilde{\Lambda})) = \sum_{i=1}^{l} \Phi_i(\tilde{\Lambda}) \tilde{h}_i(x, \tilde{\Lambda})$$

where $\tilde{h}_i(x, \tilde{\Lambda}) = h_i(x, \Pi(\tilde{\Lambda}))$.

We call $S(\sigma) = \Delta_\sigma(\{1, \ldots, l\})$ the image of $\{1, \ldots, l\}$ in $\mathbb{N}^c$ and $M(\sigma)$ the subset of minimal points of $S(\sigma)$, as in the last paragraph. Each $\beta \in S(\sigma)$ is contained in a set $\{\beta'\} + \mathbb{N}^k$, for some $\beta' \in M(\sigma)$, not necessarily unique. Let us call $M_\sigma(i)$ a map which associates an element $\beta' \in M(\sigma)$ to each $i \in \{1, \ldots, l\}$, when $\beta = \Delta_\sigma(i)$ and $\beta \in \{\beta'\} + \mathbb{N}^k$. Taking together the factors $\tilde{h}_i$ we can rewrite $\tilde{\delta}$:

$$\tilde{\delta}(x, \tilde{\Lambda}) = \sum_{\beta \in M(\sigma)} \Lambda^\beta \bar{h}_\beta(x, \tilde{\Lambda})$$

where $\bar{h}_\beta(x, \tilde{\Lambda}) = \sum_{\{i | M_\sigma(i) = \beta\}} u_i(\tilde{\Lambda}) \Lambda^{\Delta_\sigma(i) - \beta} \tilde{h}_i(x, \tilde{\Lambda})$.

It follows from this formula that the Bautin's Ideal $\tilde{\mathcal{I}}$ of the induced unfolding is generated by the Λ^β for $\beta \in M(\sigma)$. This set of generators is clearly minimal. As above we consider the factor functions of the decomposition, $\bar{H}_\beta(x) = \bar{h}_\beta(x, 0)$. As the numbers $\mathrm{ord} H_i$ for the ideal $\mathcal{I}$, are two by two distinct, we have, for all β in $M(\sigma)$:

$$\mathrm{ord} \bar{H}_\beta = \mathrm{Inf}\{\mathrm{ord} H_i \mid \Delta_\sigma(i) = M_\sigma(i) = \beta\}$$

As a consequence, the numbers ord$\bar{H}_\beta$ are also two by two distinct and the set $\{\Lambda^\beta \mid \beta \in M(\sigma)\}$ is an adapted minimal system of generators for the induced unfolding. Moreover, it does not depend on the choice of the point $\tilde{\Lambda}_0$ on σ. Let us call $E(\sigma) \subset M(\sigma)$, the set of convex extremal points and $s_E(\sigma)$ the index defined in the last paragraph.We can apply the results of the last paragraph for the induced unfolding at any point $\tilde{\Lambda}_0$. In particular its cyclicity is bounded by $s_E(\sigma)$.

Let us consider now the given initial unfolding $(X_\lambda, (\gamma, 0))$. We define for it the following index :

$$s_M = \mathrm{Sup}_\sigma s_E(\sigma)$$

Now, as the image by the map Π of a neighborhood of Σ in A covers a neighborhood of 0 in $\mathbb{R}^k$ and as Σ is compact, it is easy to show that $\mathrm{Cycl}(X_\lambda, (\gamma, 0))$ is equal to the upper bound of the cyclicities of the induced unfoldings at the different points $\tilde{\Lambda}_0$ of Σ. In fact, just a finite number of such points is sufficient. Also, the image by the map Π of an analytic arc at a point $\tilde{\Lambda}_0$ of Σ is an analytic arc at 0 in $\mathbb{R}^k$. Then, as a consequence of theorem 4 we obtain, for a general unfolding $(X_\lambda, (\gamma, 0))$:

THEOREM 7. *(a)* $\mathrm{Cycl}(X_\lambda, (\gamma, 0)) \leq s_M$,
(b) $s_M \leq \mathrm{Sup}_\xi M_\xi$ *where the upper bound is taken over all the analytic arcs* $\xi(\varepsilon)$ *through* $0 \in \mathbb{R}^k$.

Remark 8. We can interpret the index s_M as the multiplicity of the Melnikov function associated to a particular analytic arc $\Pi \circ \tilde{\xi}$ where $\tilde{\xi}$ is a monomial arc at a point $\tilde{\Lambda}_0$ in a stratum σ with a maximal index $s_E(\sigma)$, chosen as in the last paragraph.

5. FINAL REMARKS

The index s_M which is an upper bound for the cyclicity, may be different from it. For instance, if $\delta(x, \varepsilon) = \varepsilon x^2$, one has $s_M = 2$, but the cyclicity is equal to one. In general the cyclicity does not just depend on the values of the functions $H_i(x) = h_i(x, O)$. For instance, if $\delta(x, \varepsilon) = \varepsilon x^2 + \varepsilon^2$, $s_M = 2$ and the cyclicity is also equal to two, because one has the term ε^2.

The question to see if the index s_M is equal to the cyclicity depends on the properties of the ideal. Let us say that the ideal $\mathcal{I}$ is *regular* if it has a minimal set of independent generators $\{\varphi_1, \ldots, \varphi_l\}$ (this means that $\{d\varphi_1 \wedge \ldots \wedge d\varphi_l(0) \neq 0\}$. We can also suppose that this minimal set of generators is adapted. Then if the sequence of orders is the sequence $0, \ldots, l-1$, one find that the cyclicity is equal to $s_M = l - 1$. Moreover in this case, the bifurcation diagram of δ is analytically diffeomorphic to the product of $\mathbb{R}^{k-l}$ by the bifurcation diagram of the universal unfolding of codimension $l-1$: $\mu_0 + \mu_1 x + \ldots + \mu_{l-1}x^{l-1}$. If the sequence of orders has some gap, i.e. if $s_M > l-1$, it is possible to prove that the cyclicity is always greater or equal to $l-1$ ([14]).

The bound s_M is attained in general, by the Melnikov multiplicity M_ξ of a non-linear arc ξ. It is easy to construct examples where the maximum of the Melnikov multiplicity over all the linear arcs through the origin for instance, is strictly smaller than s_M.

As it is mentioned in the introduction, it is possible to work directly with the cyclicity. If $\lambda = \Phi(\mu)$ is an analytic map, such that $\Phi(0) = 0$, one can look at the induced unfolding $\tilde{X}_\mu = X_{\Phi(\mu)}$. Obviously, $\mathrm{Cycl}(\tilde{X}_\mu, (\gamma, O)) \leq Cycl(X_\lambda, (\gamma, O))$. Conversely, as indicated in the introduction, it is possible to obtain $\mathrm{Cycl}(X_\lambda, (\gamma, O))$ by considering the unfolding induced along a single well-chosen analytic arc. Then, one has the equality :

$$\mathrm{Cycl}(X_\lambda, (\gamma, O)) = \mathrm{Sup}\{Cycl(X_{\xi(\varepsilon)}, (\gamma, O)) \mid \xi\}$$

where the supremum is taken over all the analytic arcs ξ through the origin in the parameter space. Nevertheless, it is not possible in general to compute easily the cyclicity along an arc, but just to bound it by the Melnikov multiplicity and then, the above formula reduces to the inequality proved in theorem 1.

The index s_M may be strictly smaller than the index s_R. For instance, if we consider the displacement function $\delta(x, \lambda) = \lambda_1^2 + \lambda_1\lambda_2 x^2 + \lambda_2^2 x$, one has $s_R = 2$ and $s_M = 1$.

The definition of the index s_M seems to depend on the choice of an adapted minimal system of generators and also of the desingularization. It is possible to obtain an index depending just on the unfolding by considering the lower bound of the values of the index s_M for the different choices. In fact, one may conjecture that s_M is indeed independent of these choices. It would be the case if it was true that s_M coincides with the upper bound $\mathrm{Sup}_\xi M_\xi$.

Let us give an example where it is the case. One supposes that $k = 2$. One supposes also there exists a desingularization such that for each stratum σ of dimension 0, the set of convex extremal points $E(\sigma)$ is limited to $\{\beta^1(\sigma), \beta^2(\sigma)\} \subset \mathbb{N}^2$, where $\beta_1(\sigma), \beta_2(\sigma) \in \mathbb{N}^*$ and $\beta^1(\sigma) = (\beta_1(\sigma), 0)$, $\beta^2(\sigma) = (0, \beta_2(\sigma))$. Let $\xi(\varepsilon)$ be any analytic arc through the origin in the parameter space. One can lift it to the desingularization domain into an analytic arc $\tilde{\xi}(\varepsilon)$ through a point $\tilde{\Lambda}_0$ of some strata σ. If σ is of dimension one, $E(\sigma)$ reduces to a point $\beta \in \mathbb{N}$ and $M_{\tilde{\xi}} = O(\beta)$. If σ is of dimension zero, we will apply the hypothesis. We call $F(\sigma)$ the side

$[\beta^1(\sigma), \beta^2(\sigma)]$ of $\partial\Delta^c$. We have $\tilde{\xi}(\varepsilon) = (v_1\varepsilon^{\alpha_1}(1+o(\varepsilon)), v_2\varepsilon^{\alpha_2}(1+o(\varepsilon))$ for $v_1, v_2 \neq 0$, and $\alpha_1, \alpha_2 \in \mathbb{N}^*$.

If the direction $\beta \to \beta \cdot \alpha$ is parallel to $F(\sigma)$, i.e. if $\alpha_1\beta_1(\sigma) = \alpha_2\beta_2(\sigma) = \nu$, the displacement function is equal to :

$$\delta(x, \tilde{\xi}(\varepsilon)) = \varepsilon^\nu \tilde{M}(x) + o(\varepsilon^\nu)$$

where $\tilde{M}(x) = \sum_{\beta\in F(\sigma)\cap M(\sigma)} u_\beta v^\beta x^{O(\beta)}$. Here, the u_β are some non zero constants and $v^\beta = v_1^{\beta_1} v_2^{\beta_2}$ for $\beta = (\beta_1, \beta_2)$.

Since the orders $O(\beta)$, for $\beta \in F(\sigma) \cap M(\sigma)$ are two by two distinct, $M_{\tilde{\xi}} = \mathrm{ord}(\tilde{M})$ is equal to $\mathrm{Inf}\{O(\beta) \mid \beta \in F(\sigma) \cap M(\sigma)\}$, which is less $\mathrm{Sup}\{O(\beta^1(\sigma)), O(\beta^2(\sigma))\}$.

If the direction $\beta \to \beta \cdot \alpha$ is not parallel to $F(\sigma)$, for instance if α_1 or $\alpha_2 = 0$, there exists a line parallel to it, which contains just one point of $E(\sigma)$: $\beta = \beta^1$ or β^2. In this case, $M_{\tilde{\xi}} = O(\beta)$.

In all the case we obtain that $M_\xi = M_{\tilde{\xi}} \leq \mathrm{Sup}\{O(\beta) \mid \beta \in E(\sigma)\} \leq s_M$. This gives that, $\mathrm{Sup}_\xi M_\xi \leq s_M$ and finally, using theorem 7, the equality $s_M = \mathrm{Sup}_\xi M_\xi$.

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