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Rotation Rate of a Trajectory of an Algebraic Vector Field Around an Algebraic Curve

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For a Lipschitzian vector field in $\mathbb{R}^n$, angular velocity of its trajectories with respect to any stationary point is bounded by the Lipschitz constant. The same is true for a rotation speed around any integral submanifold of the field. However, easy examples show that a trajectory of a C^∞-vector field in $\mathbb{R}^3$ can make in finite time an infinite number of turns around a straight line. We show that for a trajectory of a polynomial vector field in $\mathbb{R}^3$, its rotation rate around any algebraic curve is bounded in terms of the degree of the curve and the degree and size of the vector field. As a consequence, we obtain a linear in time bound on the number of intersections of the trajectory with any algebraic surface.

1. INTRODUCTION

Let $v(x)$, $x \in \mathbb{R}^n$, be a Lipschitzian vector field, defined in a certain domain of $\mathbb{R}^n$. Let x_0 be a stationary point of v, $v(x_0) = 0$. Then for any $x \neq x_0$, the angular velocity of the trajectory of v, passing through x, with

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respect to x_0, does not exceed

$$\frac{\|v(x)\|}{\|x-x_0\|} = \frac{\|v(x)-v(x_0)\|}{\|x-x_0\|} \le K$$

where K is the Lipschitz constant of v. In particular, for any trajectory $w(t)$ of the field v, the length of the spherical curve $s(t) = \frac{w(t)-x_0}{\|w(t)-x_0\|}$ between t_1 and t_2 does not exceed $K(t_2-t_1)$.

Exactly in the same way, one can prove that for any linear subspace $L \subseteq \mathbb{R}^n$, which is invariant for v (i.e. for any $x \in L$, $v(x)$ is tangent to L), the rotation speed of v in the orthogonal direction is bounded by K.

On the other hand, the rotation of W around a straight line, which is not invariant under v, can be unbounded, even for v – a C^∞-vector field.

Consider the following example: Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism, defined by

$$\Phi(x_1,x_2,x_3) = (x_1,x_2,x_3)$$

for $x_1 \le 0$,

$$\Phi(x_1,x_2,x_3) = (x_1, x_2+\omega_1(x_1), x_3+\omega_2(x_1))$$

for $x_1 \ge 0$, where

$$\omega_1(x_1) = e^{-1/x_1^2}\cos(\frac{1}{x_1}) \ ,$$

$$\omega_2(x_1) = e^{-1/x_1^2}\sin(\frac{1}{x_1}) \ .$$

One can easily check that Φ is a C^∞-diffeomorphism of a neighborhood of $0 \in \mathbb{R}^3$. Now the image of the positive x_1-semiaxis under Φ is a line w, which makes an infinite number of turns around $0x_1$ in any neighborhood of the origin.

Consider a vector field v in $\mathbb{R}^3$, which is an image under Φ of the constant vector field $(1,0,0)$. Clearly, w is a trajectory of the C^∞-vector field v, and it makes an infinite number of turns around $0x_1$ in finite time. In coordinates,

$$v(x_1,x_2,x_3) = (1, \omega_1'(x_1), \omega_2'(x_1)) \ .$$

Notice that in this example, the orthogonal components of v on the line $0x_1$ itself have an infinite number of sign changes, accumulating to the origin.

The main result of this paper is that the kind of a degeneracy, represented by the above example, is impossible for polynomial (in fact, for analytic) vector fields and submanifolds. The main point of the proof is that too fast rotation of a trajectory around a submanifold forces too many sign changes of (some of) orthogonal components of the vector field on this manifold.

There exists a rich theory of nonoscillation of trajectories of algebraic vector fields (see [1],[2],[4]-[7]). We believe that in this paper we provide a simple, but adequate additional reason for such a nonoscillation.

For an algebraic vector field v in $\mathbb{R}^3$ define its norm $\|v\|$ as the sum of the absolute values of the coefficients of the polynomials, defining this field.

Below we always assume that all the objects considered (trajectories of the vectorfields, algebraic curves) are contained in the unit ball B in $\mathbb{R}^3$.

Theorem 1. *Rotation of any trajectory $w(t)$ of an algebraic vector field v in $\mathbb{R}^3$ around an algebraic curve V, between the time moments t_1 and t_2, is bounded by*

$$C_1(d_1, d_2) + C_2(d_1, d_2)\|v\|(t_2 - t_1)$$

Here the constants $C_1(d_1, d_2)$ and $C_2(d_1, d_2)$ depend only on the degrees d_1, d_2 of the field v and the curve V, respectively. We postpone definition of a "rotation around an algebraic subvariety" till Section 2 below.

Theorem 2. *For any trajectory $w(t)$ of an algebraic vector field v, and for any algebraic surface W in $\mathbb{R}^3$, the number of intersection points of $w(t)$ with W between the time moments t_1 and t_2 is bounded by*

$$C_3(d_1, d_2) + C_4(d_1, d_2)\|v\|(t_2 - t_1)\ .$$

As far as the dependence on the norm $\|v\|$ and on the time interval is concerned, these bounds are obviously sharp. Taking in the example above ω_1 and ω_2 to be polynomials in x_1 of degree d with d real roots on $[0, 1]$ (shifted one with respect to another), we get d turns around the x_1-axis in time one. Hence the dependence of the constants C_i on d, obtained in Proposition 3 below, is also sharp. Finally, rescaling the time in this example, we get d turns in an arbitrarily small time. Taking then the polynomials ω_1 and ω_2 to be sufficiently small, we see that the first term in the inequality of Theorem 1 cannot be omitted. The same vector field and the plane $x_3 = 0$ provide examples for Theorem 2.

2. PROOFS

In fact, we give a detailed proof only of a special case of Theorem 1 (Proposition 3 below), and an outline of the proofs of Theorems 1 and 2. We plan to present the details in [3].

Proposition 3. *Rotation of any trajectory $w(t)$ of an algebraic vector field v in $\mathbb{R}^3$ around any straight line l between the time moments t_1 and t_2, is bounded by*

$$C_1(d) + C_2(d)\|v\|(t_2 - t_1) \ ,$$

assuming that the projection of $w(t)$ on l is monotone in this time interval.

Here d is the degree of v. The rotation of $w(t)$ around l is defined as a rotation around zero of the projection of $w(t)$ onto the orthogonal plane to l.

We can assume l to be the axis $0x_1$. Let M be the maximum of the partial derivatives $\frac{\partial v_i}{\partial x_2}, \frac{\partial v_i}{\partial x_3}$, $i = 2, 3$ in B. Clearly, M is bounded by twice the degree of v, multiplied by the norm of v. M is also the the Lipschitz constant of v in the orthogonal direction $0x_2x_3$.

Assume that from t_1 to t_2 the trajectory $w(t)$ made N turns around $0x_1$. Then the rotation velocity of w around $0x_1$ "mostly" exceeds $\frac{2\pi N}{t_2 - t_1}$. More accurately, one can show by an easy integral-geometric argument, that for "many" directions q in the plane $0x_2x_3$, the projection of the orthogonal velocity vector $(0, v_2, v_3)$ of w onto q takes at least $[C_1N]$ times both the values $\frac{C_2N}{t_2-t_1}$ and $-\frac{C_2N}{t_2-t_1}$, in alternating order (where C_1 and C_2 are certain absolute constants). Assume now that $\frac{N}{t_2-t_1} \geq (\frac{2}{C_2})M$, where M is,

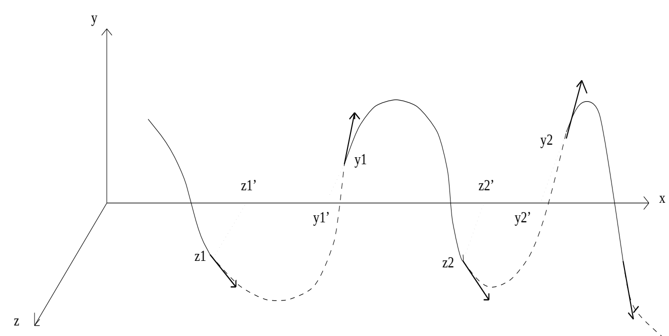

FIG. 1.

as above, the Lipschitz constant of v in the orthogonal direction $0x_2x_3$. Assuming also that the direction q coincides with $0x_2$, we obtain that at least at $\kappa = [C_1N]$ points y_j on the trajectory w, $v_2(x)/(x_2^2 + x_3^2)^{1/2}$ takes the value $2M$, and at least at other κ points z_j it takes the value $-2M$.

Now consider the projections y_j' and z_j' of y_j and z_j onto the line $0x_1$, and denote by δ_j (ρ_j) the distance between y_j and y_j' ($z_j$and z_j' respectively). Since the Lipschitz constant of v_2, v_3 in the orthogonal direction $0x_2x_3$

does not exceed M, and since $v_2(y_j) \geq 2M\delta_j$, $v_2(z_j) \leq -2M\rho_j$, we get $v_2(y'_j) \geq M\delta_j > 0$, $v_2(z'_j) \leq -M\delta_j < 0$ (see Figure 1).

Since, by assumptions, the projection of the trajectory on the line is monotone, the points y'_j, z'_j alternate. Hence between each couple of them there is a zero of v_2. But v_2, restricted to $0x_1$, is a polynomial of degree at most $d = \deg v$. Hence $\kappa \leq d$, or $N \leq \frac{1}{C_1}d$.

What we have shown is that if the "average rotation speed" $\frac{N}{t_2-t_1}$ of the trajectory around the line l is big ($\frac{N}{t_2-t_1} \geq (\frac{2}{C_2})M$), then it makes at most $\frac{1}{C_1}d$ turns around l. In other words, the trajectory can keep a high rotation velocity only for a short time interval.

Easy examples show that if the rotation is slow enough, all the orthogonal components of v may preserve the sign on the axis $0x_1$. Consider, for instance, the field

$$v(x_1, x_2, x_3) = (1, 1-\rho+\rho x_3, 1-\rho-\rho x_2) \ ,$$

with $\rho = x_2^2 + x_3^2$. Then on the axis $0x_1$, $v(x_1, 0, 0) = (1, 1, 1)$, but any trajectory of v, starting on the cylinder $\rho = 1$, remains on this cylinder and rotates around $0x_1$ with the angular velocity $\frac{1}{2\pi}$.

Now we have two possibilities: either $\frac{N}{t_2-t_1} \leq (\frac{2}{C_2})M$, and hence $N \leq (\frac{2}{C_2})(t_2-t_1)M$, or $N \leq \frac{1}{C_1}d$. Taking into account that M does not exceed twice the degree d of v, multiplied by the norm of v, we conclude that in both cases the rotation N does not exceed $C_1(d) + C_2(d)\|v\|(t_2-t_1)$, with $C_1(d) = \frac{1}{C_1}d$ and $C_2(d) = \frac{4d}{C_2}$. This completes the proof of Proposition 3.

Let us now give a short outline of the proof of Theorems 1 and 2.

First of all, the assumption of the monotonicity of the projection of the trajectory on the line, can be reduced as follows. If the number of the "monotonicity intervals" of the projection is small, one can use Proposition 3. If, in contrary, the projection of our trajectory on the line l oscillates, one can show that there is a point on l, such that the rotation around this point is of the same order as the rotation around l. But one can show that the rotation rate around a point is bounded for any Lipschitzian vector field.

Extension to any algebraic curve (instead of the straight line) is rather straightforward. However, we are not aware of any "invariant" definition of a "metric" rotation around a curve. Consequently, we use a noninvariant definition, where the orthogonal plane to the straight line is replaced by a family of parallel planes, transversal to the curve.

To complete the proof of Theorem 1, we have to take into account a possible oscillation of the curve itself, but for algebraic curves this oscillation can be bounded in terms of the degree.

Theorem 2 is implied by Theorem 1, as follows. If the trajectory of a polynomial vector field v crosses an algebraic surface W, the sign of the normal (to W) component of v corresponds to the direction of the crossing. Hence multiple crossings happen alternatively at the parts of W, where the normal component of v is positive or negative, respectively. But then one can show that our trajectory necessarily rotates around the algebraic curve, defined in W by vanishing of the normal component of v.

Both the statements of the results above and the proofs can be extended to higher dimensions. However the technical difficulties seem to be rather serious. We plan to present some of these extensions separately.

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