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Extension of Bautin Theory to any Dimension

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Bautin made some years ago a decisive contribution to the algebraic approach of the perturbation theory of periodic orbits of plane polynomial vector fields. This article presents first steps of a general framework in which a generalization of Bautin's ideas to any dimension could be developed. The main result is the generalization of the algorithm of the successive derivatives of return mappings for 2-dimensional systems to any dimension in this framework.

Key Words: Algebraic Bifurcation Theory

1. INTRODUCTION

These last years, the dynamics of plane systems was extensively studied and several new techniques were developed. Some are specific to 2-dimensional systems but mostly often these methods can be appropriately extended to multi-dimensional systems. The algorithm of the successive derivatives was derived some years ago [1] to find the first non-vanishing derivative (relatively to the parameter ϵ) of the return mapping (near the origin) of a plane vector field $X_0 + \epsilon X_1$ of type:

$$\begin{aligned} X_0 + \epsilon X_1 \;=\; & x\partial/\partial y - y\partial/\partial x + \\ & \epsilon \textstyle\sum_{i,j/i+j=2}^{d} [a_{i,j} x^i y^j \partial/\partial x + b_{i,j} x^i y^j \partial/\partial y]. \end{aligned} \tag{1}$$

The algorithm was then used in the center-focus problem (cf. [1]), which directly relates to Hopf bifurcations of higher order and to several other problems on limit cycles of plane vector fields.
How to extend appropriately this situation in any dimension? We have

to perturb a dynamics which is integrable and displays only periodic orbits. Assuming that the perturbation depends of finitely many parameters (say for instance it is polynomial), we expect also that the perturbed system displays a first return-mapping which is analytic with a Taylor expansion with coefficients which depend polynomially of the parameters. This return-mapping should label all the periodic orbits (at first return)of the perturbed system by its fixed points. The principal aim of this article is to present a framework where such demands are realized.In this framework, a generalization of the algorithm of the successive derivatives is provided.

2. CONTROLLED NAMBU DYNAMICS AND (*)-PROPERTY

Let $f = (f_1, ..., f_{n-1}) : R^n \to R^{n-1}$ be a generic submersion (meaning that f is a submersion outside a critical set $f^{-1}(C)$, where C is a set of isolated points). Let $\Omega = dx_1 \wedge dx_2 \wedge ... dx_n$ be a volume form on R^n. Consider the vector field X_0 such that:

$$\iota_{X_0} dx_1 \wedge dx_2 \wedge ... \wedge dx_n = df_1 \wedge ... \wedge df_{n-1}. \tag{2}$$

The functions $f_i, (i = 1, ..., n-1)$ are first integrals of the vector field X_0:

$$\begin{aligned} df_i \wedge \iota_{X_0} dx_1 \wedge dx_2 \wedge ... \wedge dx_n &= (X_0.f_i) dx_1 \wedge dx_2 \wedge ... \wedge dx_n \\ &= df_i \wedge df_1 \wedge df_2 \wedge ... \wedge df_{n-1} = 0. \end{aligned} \tag{3}$$

This type of dynamics is well-known in Physics and named Nambu's dynamics.

For c varying in a neighborhood of 0, assume that the curves $f^{-1}(c)$ have a compact connected component γ_c. Let Σ be a small neighborhood of the zero-section of the normal bundle to γ_0. For c small enough, the curves γ_c are closed periodic orbits of X_0 and they cut transversely Σ. Choose c as a coordinate on the transverse section Σ to the flow of X_0.
Lastly, assume that there are 1-forms ω_i such that:

$$\iota_{X_0} \omega_i = df_i; i = 1, ..., n-1. \tag{4}$$

Depending of the type of regularity of the 1-form ω_i, this condition may be a consequence of the preceding assumptions. If the condition (4) is fulfilled, we will say that the singularity of the Nambu dynamics (3) is controlled (or alternatively that the Nambu dynamics itself is controlled).

The appropriated extension of the (*)-property first discussed in [1] is presented in the following.

DEFINITION 1. Let $f = (f_1, ..., f_{n-1}) : R^n \to R^{n-1}$ be a generic submersion. Assume that $f^{-1}(c)$ contains a compact curve γ_c. The application displays the (*)-property if for all polynomial 1-form ω such that

$$\int_{\gamma_c} \omega = 0, \tag{5}$$

for all c; there exist polynomial g_i, R such that:

$$\omega = g_1 df_1 + ... + g_{n-1} df_{n-1} + dR. \tag{6}$$

It was proved in [1] that the function $f_1 : R^2 \to R^1, f_1 : (x_1, x_2) \to (x_1^2 + x_2^2)$ displays the (*)-property. Several generalizations were proposed after but the core of the argument in the computation of the successive derivatives is captured in this notion.

The generalization proposed in this article provides a new presentation of the (*)-property which seems interesting as well for the 2-dimensional case. Indeed, the definition of the vector field X_0 given in the preceding introduction yields the:

PROPOSITION 2. *Let ω be a 1-form such that $\omega(X_0) = 0$, then there are functions $g_1, ..., g_{n-1}$ so that:*

$$\omega = g_1 df_1 + ... + g_{n-1} df_{n-1}. \tag{7}$$

Note that the condition $\omega(X_0) = 0$, equivalent to $\omega \wedge df_1 \wedge ... \wedge df_{n-1} = 0$, yields $\omega = g_1 df_1 + ... + g_{n-1} df_{n-1}$ where the coefficients g_k are obtained as ratio of minors of the Jacobian matrix of the f_j.

This displays an alternative to the (*)-property now presented as follows:

PROPOSITION 3. *A generic submersion $f : R^n \to R^{n-1}$ displays the (*)-property if for any polynomial 1-form ω such that*

$$\int_{\gamma_c} \omega = 0, \tag{8}$$

for all c; then there exists a polynomial R such that:

$$\omega(X_0) = X_0.R. \tag{9}$$

Such a function R can be (in principle) constructed with the following pattern. Choose R arbitrarily on the transverse section Σ, then extends R to the whole tubular neighborhood of γ_0 saturated by the orbits γ_c by integration of the 1-form ω along the orbits of X_0.

3. THE SUCCESSIVE DERIVATIVES OF THE FIRST RETURN MAPPING OF THE PERTURBED SYSTEM

Now perturb X_0 into $X_\epsilon = X_0 + \epsilon X_1$. Let M be a point of Σ close to 0 and let γ_ϵ be the trajectory of X_ϵ passing by the point M. The next first intersection point of γ_ϵ with Σ defines the so-called first return mapping of X_ϵ relatively to the transverse section Σ: $c \longmapsto L(c, \epsilon)$.
The mapping L is analytic and it displays a Taylor development (in ϵ):

$$L(c, \epsilon) = c + \epsilon L_1(c) + ... + \epsilon^k L_k(c) + O(\epsilon)^{k+1}. \tag{10}$$

The expression of the first coefficient $L_1(c)$ is classical and belongs to the lore of bifurcation theory. With the vector field X_ϵ and the 1-forms ω_i (cf. [1]), introduce the 1-forms:

$$\iota_{X_\epsilon}\omega_i = \iota_{X_0}\omega_i + \epsilon\iota_{X_1}\omega_i = df_i + \epsilon\iota_{X_1}\omega_i. \tag{11}$$

Definition 4. The perturbation X_ϵ of the controlled Nambu dynamics is said to be admissible if for all the 2-forms ω_i, the 1-forms $\iota_{X_1}\omega_i$ have polynomial coefficients.

Recall that the parameter c chosen as coordinates on the transverse section Σ is the restriction of the functions $f = (f_1, ..., f_{n-1})$ to the section.

Then the i^{th}-component of $L_1(c)$ is equal to:

$$L_{1,i}(c) = \int_{\gamma_0} \iota_{X_1}\omega_i. \tag{12}$$

Assume now that the first derivative $L_1(c)$ vanishes identically and that the submersion f displays the (*)-property then there exist g_{ij} and R_i such that:

$$\iota_{X_1}\omega_i = \sum_j g_{ij}df_j + dR_i. \tag{13}$$

Following the lines of the algorithm of the successive derivatives, the expression (13) yields:

$$L_{2,i}(c) = -\int_{\gamma_0} \sum_j g_{ij} \iota_{X_1} \omega_j. \tag{14}$$

This is indeed the second step of a general recursive scheme which displays as follows:

Assume that all the k^{th}-first derivatives of the return mapping of the perturbed vector field vanish identically. This yields:

$$L_{k,i}(c) = \int_{\gamma_0} \sum_j g_{ij}^{k-1} \iota_{X_1} \omega_j = 0. \tag{15}$$

The (*)-property yields new functions g_{ij}^k, R^k such that:

$$\sum_j g_{ij}^{k-1} \iota_{X_1} \omega_j = \sum_j g_{ij}^k df_j + dR^k. \tag{16}$$

This yields the following expression of the $(k+1)^{th}$-derivative of the return mapping of the perturbation:

$$L_{k+1,i}(c) = \int_{\gamma_0} \sum_j g_{ij}^k \iota_{X_1} \omega_j. \tag{17}$$

The algorithm implies of course the first

THEOREM 5. *Let X_0 be a controlled Nambu dynamics which displays the (*)-property and let X_1 be an admissible perturbation. Then the perturbed dynamics X_ϵ has an analytic first return. The coefficients of the Taylor expansion of this return mapping depend polynomially of the coefficients of the perturbation.*

From the general theory of projections of analytic sets (cf.[2],[3],[4]), it now follows:

THEOREM 6. *There exists a uniform bound to the number of isolated periodic orbits, which corresponds to fixed point of the first return mapping of $X_0 + \epsilon X_1$ which intersects the transverse section Σ in the neighborhood of* 0.

The general framework presented here should of course be illustrated with specific examples. Some have been worked out recently by Seok Hur (Paris VI) and will be matter to further publications.

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