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On the Geometry in the Neighborhood of Infinity of Quadratic Differential Systems with a Weak Focus

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In this work we use the algebro-geometric concept of divisor on a projective curve in the study of planar quadratic vector fields. We introduce here specific divisors to encode global information about the geometry at infinity of quadratic systems with a weak focus which is not a center and we show that these concepts organise and unify in an intrinsic way this information. This geometric approach forms a link between chart-dependent classification studies of quadratic (or cubic) differential systems and affine-invariant results based on the algebraic theory of invariants of differential systems.

1. INTRODUCTION

In this work we consider planar vector fields

$$V(x,y) = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y \tag{1}$$

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and their associated differential systems

$$\begin{aligned} dx/dt &= P(x,y), \\ dy/dt &= Q(x,y) \end{aligned} \tag{2}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials with real coefficients. We consider systems (2) which are quadratic, i.e. such that $\max(\deg(P), \deg(Q)) = 2$.

The study of the class of planar quadratic vector fields is very far from being completed. For the moment only some special classes of quadratic systems were studied but even these works are not all entirely satisfactory. Indeed, a part of these studies was done using chart dependent normal forms and lead to results which are stated in terms of the coefficients of the systems written with respect to these normal forms and in terms of inequalities involving these coefficients.

To apply these results to specific systems one needs to perform a reduction to a canonical form, a non constructive procedure. Naturally one aims at obtaining affine-invariant results, applicable to systems, independent of their canonical forms. Works in which results are formulated in an affine-invariant way were realized by K.S. Sibirski and his school. These results are formulated in terms of the theory of algebraic invariants of differential equations developed by K.S. Sibirsky and his school (cf. [16]). This is a computationally powerful tool. However, the impact of this theory in the works of the specialists in the west has been rather unsubstantial.

Not enough effort was put into giving a geometric meaning to most of these purely algebraic invariants. There is thus a need to infuse this theory with more geometric meaning.

The work in this article is a bridge between the two approaches : it represent an effort to move from a non-invariant, chart-dependent approach, towards an invariant and at the same time geometrically meaningful approach. To do this we needed to introduce new global concepts which could encode non local information in compact ways. The initial step in this direction was taken in [8] where a complete study of quadratic Hamiltonian systems with a center was done in terms of the algebro-geometric notions of multiplicity of intersection of algebraic curves and of singular projective curves.

Here we use another algebro-geometric concept, namely the notion of divisor on a projective curve to capture the geometry at infinity of quadratic systems with a weak focus.

In this article we use term *weak focus* to mean a singularity with pure imaginary eigenvalues, which is not a center.

The purely geometric concept of divisor was used (in an implicit way) in the study of quadratic systems in [13]. Here and in [10] we make explicit use of this concept. We define specific divisors for the study of quadratic systems with a weak focus and show to which extent they determine the topological classification of these systems in the neighborhood of the line at infinity.

While earlier studies of the infinite singular points of quadratic systems were done using an (ε, δ)-definition for the multiplicity of such points (see for example Definition 3 on page 475 of [17]), we define this multiplicity by making use of the algebro-geometric notion of intersection multiplicity of projective curves and we assemble these multiplicities in a unique divisor on the line at infinity. In doing this we connect the basic theory of planar quadratic systems with some basic concepts of algebraic geometry.

The work proceeds as follows : In section 2 we briefly discuss the foliation associated to these differential systems. In section 3 we describe the basic framework for encoding the global information about the multiplicities of singularities at infinity, i.e. we define the two divisors at infinity which encode these multiplicities. In section 4 we compute the two divisors for the class of quadratic vector fields with a weak focus. In section 5 we apply the Newton Polyhedra construction to obtain information about the topology of phase curves around a singular point at infinity. In section 6 we give the results for the topological phase portraits and in section 7 we draw some conclusions.

2. THE FOLIATION WITH SINGULARITIES ON THE PROJECTIVE PLANE ASSOCIATED TO A PLANAR POLYNOMIAL SYSTEM

We shall suppose that the polynomials P and Q with real coefficients have no non-constant common factor. For the vector field (1) or equivalently for the differential system (2), we consider the associated differential 1-form $\omega_1 = Q(x,y)dx - P(x,y)dy$ and the differential equation $\omega_1 = 0$. Clearly the equation $\omega_1 = 0$ defines a foliation with singularities on $\mathbb{C}^2$. The affine plane $\mathbb{C}^2$ is compactified to the complex projective space $\mathbb{C}P(2) = (\mathbb{C}^3 - \{0\})/\sim$, where $(X,Y,Z) \sim (X',Y',Z')$ if and only if $(X,Y,Z) = t(X',Y',Z')$ for some $t \neq 0$, $t \in \mathbb{C}$. The equivalence class of (X,Y,Z) will be denoted by $[X:T:Z]$. The foliation with singularities defined by the equation $\omega_1 = 0$ on $\mathbb{C}^2$ can be extended to a foliation with singularities on $\mathbb{C}P(2)$ and the one-form ω_1 can be extended to a meromorphic one-form on $\mathbb{C}P(2)$ [4]. This meromorphic one-form on $\mathbb{C}P(2)$ can be described by a single one form on $\mathbb{C}^3$ given by $A^*(X,Y,Z)dX + B^*(X,Y,Z)dY + C^*(X,Y,Z)dz = 0$, where A^*, B^* and C^* are homogeneous polynomials. Indeed, consider the application $i : \mathbb{C}^3 - \{(X,Y,Z) \mid Z \neq 0\} \to \mathbb{C}^2$ given by $i(X,Y,Z) = (X/Z, Y/Z)$ and suppose that $\max(\deg(P), \deg(Q)) = \ell > 0$. From the relations $x = X/Z$, $y = Y/Z$ we have $dx = (ZdX - XdZ)/Z^2$,

$dy = (ZdY - YdZ)/Z^2$, the pull back $i^*(\omega_1)$ has poles at $Z = 0$ and the equation $\omega_1 = 0$ can be written in coordinates X, Y, Z as follows : $i^*(\omega_1) = Q(X/Z, Y/Z)(ZdX - XdZ)/Z^2 - P(X/Z, Y/Z)(ZdY - YdZ)/Z^2 = 0$. Then $\omega = Z^{\ell+2} i^*(\omega_1)$ has polynomial coefficients of degree $\ell + 1$ and for $Z \neq 0$, the equation $\omega = 0$ and $i^*(\omega_1) = 0$ have the same solutions. The differential equation $\omega = 0$ can be written as

$$\omega = A^*(X, Y, Z)dX + B^*(X, Y, Z)dY + C^*(X, Y, Z)dZ = 0 \tag{3}$$

where

$$\begin{aligned} A^*(X, Y, Z) &= ZQ^*(X, Y, Z) \\ B^*(X, Y, Z) &= -ZP^*(X, Y, Z) \\ C^*(X, Y, Z) &= YP^*(X, Y, Z) - XQ^*(X, Y, Z) \end{aligned} \tag{4}$$

and

$$\begin{aligned} P^*(X, Y, Z) &= Z^\ell P(X/Z, Y/Z) \\ Q^*(X, Y, Z) &= Z^\ell Q(Z/Z, Y/Z) \end{aligned} \tag{5}$$

are the homogeneous polynomials in X, Y, Z obtained from P and Q. Clearly A^*, B^* and C^* are homogeneous polynomials of degree $\ell + 1$ in the variables X, Y, Z and $XA^* + YB^* + ZC^* = 0$ which implies $\omega \wedge d\omega = 0$, i.e. the complete integrabibity condition for the equation $\omega = 0$. We will denote by F_C the foliation with singularities on $\mathbb{C}P(2)$ defined by the equation $\omega = 0$. Since P and Q are polynomials with real coefficients, in the same way as above, a real foliation with singularities F_R on the real projective plane $\mathbb{R}P(2)$ is induced by the equation $\omega_1 = Q(x, y)dx - P(x, y)dy = 0$.

To study the foliation with singularities defined by the equation $\omega = 0$ in the neighborhood of the line at infinity $Z = 0$, we consider the charts $(u, z) = (Y/X, Z/X)$, $X \neq 0$, and $(v, w) = (X/Y, Z/Y)$, $Y \neq 0$. Note that in the intersection of the charts (x, y) and (u, z) (respectively (v, w)) we have the well-known change of coordinates $x = 1/z$, $y = u/z$ (respectively $x = v/w$, $y = 1/w$). Except for the point $[0 : 1 : 0]$, the foliation defined by the equation $\omega = 0$ in the neighborhood of the line at infinity is topologically equivalent to the foliation (with singularities) defined by the integral curves of the vector field associated with the system of equations:

$$\begin{aligned} du/dt &= uP^*(1, u, z) - Q^*(1, u, z) = C^*(1, u, z), \\ dz/dt &= zP^*(1, u, z). \end{aligned} \tag{6}$$

Similarly, the foliation with singularities defined by the equation $\omega = 0$ in the neighbourhood of the line at infinity, without the point $[1 : 0 : 0]$, is topologically equivalent to the foliation (with singularities) defined by the integral curves of the vector field associated with the system of equations:

$$\begin{aligned} dv/dt &= vQ^*(v,1,w) - P^*(v,1,w) = -C^*(v,1,w), \\ dw/dt &= wQ^*(v,1,w). \end{aligned} \tag{7}$$

In the next chapters we will be using the systems of equations (6) and (7) for the topological study of the phase portraits in the neighbourhood of the line at infinity.

3. MULTIPLICITY OF SINGULARITIES AT INFINITY OF PLANAR VECTOR FIELDS. THE GENERAL FRAMEWORK

When considering the problem of orbital topological classification of quadratic differential systems, the very first step is the study of their singular points. These are of two kinds : finite and infinite. We are interested here in determining the local phase portraits around each one of the singularities at infinity and how these combine to yield the phase portrait around the line at infinity. At the same time we want a classification which is as transparent as possible, i.e. one whose meaning is easily understood without relying on complicated formulas or computer calculations but rather on geometrical concepts involved. For these reasons we shall use for this classification the algebro-geometric concept of multiplicity of intersection of projective curves and the notion of divisor on a curve.

In Section 2 we described the foliations with singularities F_C and F_R on the complex projective space $\mathbb{C}P(2)$, respectively the real projective space $\mathbb{R}P(2)$, associated with the differential system (2). In homogeneous coordinates these foliations are described by the polynomial differential equation (3) with $A^*X + B^*Y + C^*Z = 0$. The singularities of the foliation defined by the equation (3) are the intersection points of the curves $A^* = 0$, $B^* = 0$, $C^* = 0$ and these points need to be counted with multiplicities of intersections over $\mathbf{C}$. We have several ways of defining the concept of multiplicity of intersection of algebraic curves. A definition which generalizes easily for intersection multiplicity at a point, of several curves is via local rings [6].

Definition 1. The intersection multiplicity $I_p(R_1, R_2)$ at a point p, of two algebraic curves $C_1 : R_1(x,y) = 0$ and $C_2 : R_2(x,y) = 0$ in $\mathbb{C}^2$ is defined as being zero if the curves do not intersect at p, infinity if the curves have a common component and otherwise

$$I_p(R_1, R_2) = \dim_{\mathbb{C}} \mathbf{O}_p/(R_1, R_2) \tag{8}$$

where $\mathbf{O}_p$ is the local ring of the affine complex plane $\mathbf{A}^2(\mathbb{C}) = \mathbb{C}^2$ at p, i.e. $\mathbf{O}_p$ is the ring of rational functions $q(x,y)/r(x,y)$ which are defined at p, i.e. with $r(p) \neq 0$.

We can define the multiplicity of intersection $I_p(R,S)$ at a point $p = [X_0 : Y_0 : Z_0]$ of two projective curves given by the equations $R(X,Y,Z) = 0$, $S(X,Y,Z) = 0$ by using an appropriate chart and Definition 1. 1. (cf.[5]). The value of $I_p(R,S)$ is independent of the particular chart.

The concept of intersection multiplicity defined above easily extends to that of intersection multiplicity of several curves at a point of the projective plane (cf. [5]). Indeed, this is done by taking an affine chart and by using the ideal (R_1, R_2, R_3) instead of (R_1, R_2) in (8). In particular we will be interested in the way the curves $A^* = 0$, $B^* = 0$ and $C^* = 0$ intersect and hence in the values of $I_p(A^*, B^*, C^*)$ for p in the projective plane.

Notation 2.

$$m(p) = I_p(A^*, B^*, C^*) \tag{9}$$

From Section 2 in [10] we have the following:

Proposition 3. *Let p be a finite or infinite singular point of the differential equation (3) induced by the planar polynomial vector field (1) and A^*, B^*, C^* are as defined in (4) and (5). Then $I_p(P^*, Q^*)$, $I_p(C^*, Z)$ and $m(p) = I_p(A^*, B^*, C^*)$ are invariant with respect to the affine transformations and*

$$I_p(A^*, B^*, C^*) = I_p(C^*, Z^*) + I_p(P^*, Q^*) \tag{10}$$

$$m(p) = I_p(P, Q) \Leftrightarrow p \quad \text{is finite.}$$

The infinite singularities are given by intersecting the line at infinity $Z = 0$ with the curve $C^* = 0$. If p is a point on $C^* = 0$ and $Z = 0$, then $I_p(C^*, Z)$, tells us how many of the points of $C^* = 0$ which are also on $Z = 0$ coalesced at p. We note that among the singular points at infinity we may have points of intersection of the projective curves $P^* = 0$ with $Q^* = 0$ lying on $Z = 0$. These are points p where $I_p(A^*, B^*, C^*)$ is greater than or equal to two because $I_p(P^*, Q^*)$ and $I_p(C^*, Z)$ are each greater than or equal to one. The points p on $Z = 0$ and on both $P^* = 0$ and on $Q^* = 0$ are in some sense more complicated than the finite intersection points of $P = 0$ and $Q = 0$ and also more complicated than the other points on $Z = 0$ lying on $C^* = 0$ but which are not common to $P^* = 0$ and $Q^* = 0$. From the above we have the following.

PROPOSITION 4. *Let p be a singular point of (3), finite or infinite.*

i) p is a finite singular point if and only if $I_p(P^*,Q^*) \neq 0 = I_p(C^*,Z)$.

ii) p is an infinite singular point if and only if $I_p(C^*,Z) \neq 0$. *In this case we have two possibilities:*

iia) $I_p(P^*,Q^*) = 0$. *In this case, when the system is perturbed within the quadratic family, no finite singular point could arise in the perturbed system from the singularity p.*

iib) $I_p(P^*,Q^*) \neq 0$. *In this case when the system is perturbed within the quadratic family both finite and infinite singularities arise, in the perturbed system in the vicinity of p.*

DEFINITION 5. A singular point p is called of mixed type if and only if

$$I(C^*,Z) \cdot I(P^*,Q^*) \neq 0.$$

Remark 6. Due to the above proposition, if p is a singular point of mixed type, a small perturbation of the system will have m singularities in a small neighborhood of p with $m > 1$. A singular point at infinity which is of mixed type is of a more degenerate nature. To discuss multiplicity of a singular point we need both $I_p(P^*,Q^*)$ and $I_p(C^*,Z)$.

DEFINITION 7. A singular point p, finite or infinite, is said to be of multiplicity $M_{r,s}$ or simply (r,s) if and only if

$$(r,s) = \big(I_p(P^*,Q^*), I_p(C^*,Z)\big)$$

NOTATION 8. *We shall write* $M_{r,s}(p)$ *if p is of multiplicity* $M_{r,s}$, *(or of multiplicity* (r,s)*).*

In this article we are interested in the phase portraits in the neighborhood of the line at infinity (i.e. the line $Z = 0$) of the planar system (2) of a special class of quadratic systems, i.e. those with a weak focus. The phase portraits is a neighborhood of the line at infinity are determined by the local phase portraits around the singular points at infinity of the systems and we have at most three such singular points in the real projective plane. The two multiplicities : $I_p(P^*,Q^*)$ and $I_p(C^*,Z)$ have implications for the phase portraits. We need a more global notion to sum up the information about multiplicities at infinity and for this we make the following definition:

DEFINITION 9. Les S be a system (2). A *divisor at infinity* of the system S is a formal expression $D = \sum_{p\in\{Z=0\}} n_p p$ where p are points in $\mathbb{C}P^2$ on

$Z=0$, n_p's are integers and only a finite number of n_p's are non-zero. The *degree* of a divisor at infinity D is $\deg(D)=\sum n_p$.

We note that divisors could be added and subtracted. Let $G(D)=\{D : D$ is a divisor on $Z=0\}$. Then $\big(G(Z),+\big)$ is a group. We define the following two divisors at infinity for systems S :

$$\begin{aligned}&D_S(C^*,Z)=\textstyle\sum_{p\in\{Z=0\}} I_p(C^*,Z)p\\ &\text{and}\\ &D_S(P^*,Q^*;Z)=\textstyle\sum_{p\in\{Z=0\}} I_p(P^*,Q^*)p.\end{aligned} \tag{11}$$

As will be shown in the next chapter, the above divisors are well defined for planar quadratic systems S with a weak focus. In this work we describe the relationship between the two divisors $D_S(C^*,Z)$ and $D_S(P^*,Q^*;Z)$ and we show their role in determining the phase portraits in the neighborhood of the line at infinity.

4. SINGULAR POINTS AND DIVISORS AT INFINITY OF REAL PLANAR VECTOR FIELDS WITH A WEAK FOCUS

In this section we shall use the divisors (11) to encode the global information concerning the multiplicities corresponding to all the singularities at infinity of the quadratic systems with weak foci. We shall assume that the systems have a singularity which is a weak focus, i.e. a singularity with pure imaginary eigenvalues but which is not a center. On the family of such systems acts the group of affine transformation and positive time rescaling. We may therefore assume that the weak focus is placed at the origin. Due to the group action we have the following normal form:

$$\begin{aligned}dx/dt&=-y+kx^2+mxy+ny^2=P(x,y),\\ dy/dt&=x+ax^2+bxy=Q(x,y).\end{aligned} \tag{12}$$

The systems of equation (12) depend on the parameter $\lambda=(k,m,n,a,b)\in\mathbb{R}^5$. We consider systems which are nonlinear i.e. $\lambda=(k,m,n,a,b)\neq 0$. In this case the system can be rescaled, hence the parameter space needed is actually the real projective space $\mathbb{R}P(4)$ and not $\mathbb{R}^5$.

We shall show that the divisors (11) are well defined for quadratic systems with weak foci. We first perform the calculations, then sum up the results in Table 1. Let us recall some basic facts on the systems with weak focus. First we recall the following:

Lemma 10 (Shi's Lemma [14]). *Consider the polynomial system:*

$$\begin{aligned}\tfrac{dx}{dt}&=-y+P_2(x,y)+P_3(x,y)+\ldots+P_\ell(x,y),\\ \tfrac{dy}{dt}&=x+Q_2(x,y)+Q_3(x,y)+\ldots+Q_\ell(x,y)\end{aligned} \tag{13}$$

where

$$P_i = \sum_{j=0}^{i} a_{ij} x^{i-j} y^j, \qquad Q_i = \sum_{j=0}^{i} b_{ij} x^{i-j} y^j.$$

Then there exists a formal power series $F \in \mathbb{Q}[a_{20}, \dots, b_{0\ell}][[x, y]]$,

$$F = \frac{1}{2}(x^2 + y^2) + F_3(x, y) + F_4(x, y) + \dots,$$

and there exist $V_1, \dots, V_i, \dots \in \mathbb{Q}[a_{20}, \dots, b_{0\ell}]$ *such that*

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} P + \frac{\partial F}{\partial y} Q = \sum_{i=1}^{\infty} V_i (x^2 + y^2)^{I+1}.$$

The quantities V_i are not uniquely determined. For each i there is a infinite number of possibilities for a V_i. But according to a result also proved by Shi, all such V_i's are in the same coset modulo the ideal generated by $V_1, V_2, \dots, V_{i-1}$ in the ring $\mathbb{Q}[a_{20}, \dots b_{0\ell}]$. From the work of Poincaré 1985 [11] it follows that the system (13) has a center at the origin if and only if $V_i = 0$ for all i. By Hilbert's basis theorem, the ideal $I = \langle V_1, V_2, \dots, V_i, \dots \rangle$ has a finite basis. It follows from the work of Bautin [2] that for quadratic systems ($\ell = 2$) this ideal is determined by the values of V_i with $i \leq 3$. This implies that for the system (12) which is quadratic we have $V_1 = V_2 = V_3 = 0$ if and only if $V_i = 0$ for all i and the origin is a center. Calculations (cf. [18]) yield that $V_1 = W_1$, $V_2 = W_2 \pmod{V_1}$, $V_3 = W_3 \pmod{V_1, V_2}$ where we put

$$\begin{aligned} W_1 &= m(k+n) - a(b+2k), \\ W_2 &= (m-5a)[am(2k+b+n) - (b+2k)(k+n)(b+n)], \\ W_3 &= -a(2a^2+2n^2+kn)[am(2k+b+n) - (b+2k)(k+n)(b+n)]. \end{aligned} \tag{14}$$

DEFINITION 11. Consider a quadratic system with weak focus (12). The origin is a weak focus of order j with $j \leq 3$, if and only if we have $W_i = 0$ for $i < j$ and $W_j \neq 0$.

It is known (for example cf. Ye [19], Theorem 12.3) that we have :

PROPOSITION 12. *Consider the quadratic system (12). Then*

1) The origin is a center if and only if $W_i = 0$ *for* $i \leq 3$.

2) If the origin is a weak focus of order j *with* $j \leq 3$, *the stability of the weak focus is determined by the sign of* W_j *: the weak focus is stable if* $W_j < 0$ *and unstable if* $W_j > 0$.

The differential equation (3) corresponding to the planar quadratic system with a weak focus in normal form (12) can be written :

$$\omega = A^*(X,Y,Z)dX + B^*(X,Y,Z)dY + C^*(X,Y,Z)dZ = 0 \qquad (15)$$

where

$$\begin{aligned} A^*(X,Y,Z) &= ZQ^*(X,Y,Z) = Z(XZ + aX^2 + bXY), \\ B^*(X,Y,Z) &= -ZP^*(X,Y,Z) \\ &= -Z(-YZ + kX^2 + mXY + nY^2), \\ C^*(X,Y,Z) &= YP^*(X,Y,Z) - XQ^*(X,Y,Z) \\ &= nY^3 + mXY^2 + (k-b)X^2Y - aX^3 - Y^2Z - X^2Z. \end{aligned} \qquad (16)$$

Singularities at infinity of a system 12 correspond to solutions of $C^*(X,Y,Z) = 0$ situated on $Z = 0$, i.e. to solutions of $C^*(X,Y,0) = 0$.

If $q_0 = [1 : u_0 : 0]$ and $r_0 = [v_0 : 1 : 0]$ are singular points for the differential equation (15) we have $C^*(1,u_0,0) = 0$ and $C^*(v_0,1,0) = 0$. Then $(u_0,0)$ is a singular point for the differential system (6) given in chart (u,z) and $(v_0,0)$ is a singular point for the differential system (7) given in chart (v,w). It is convenient to translate the singular points $(u_0,0)$ and $(v_0,0)$ to the origin. Using the Taylor expansion and the translations $u' = u - u_0$, $z' = z$ and $v' = v - v_0$, $w' = w$, after renoting u' as u, v' as v, the differential systems (6) and (7) can be written respectively:

$$U \begin{cases} \frac{du}{dt} = (\partial C^*/\partial Y)(q_0)u - (1+u_0^2)z \\ \qquad +\frac{1}{2}(\partial^2 C^*/\partial Y^2)(q_0)u^2 - 2u_0 uz \\ \qquad +\frac{1}{6}(\partial^3 C^*/\partial Y^3)(q_0)u^3 - u^2 z, \\ \frac{dz}{dt} = P^*(q_0)z + (\partial P^*/\partial Y)(q_0)uz - u_0 z^2 \\ \qquad +\frac{1}{2}(\partial^2 P^*/\partial Y^2)(q_0)u^2 z - uz^2. \end{cases} \qquad (17)$$

and

$$V \begin{cases} \frac{dv}{dt} = -(\partial C^*/\partial X)(r_0)v + (1+v_0^2)w \\ \qquad -\frac{1}{2}(\partial^2 C^*/\partial X^2)(r_0)v^2 + 2v_0 vw \\ \qquad -\frac{1}{6}(\partial^3 C^*/\partial X^3)(r_0)v^3 + v^2 w, \\ \frac{dw}{dt} = Q^*(r_0)w + (\partial Q^*/\partial X)(r_0)vw + v_0 w^2 \\ \qquad +\frac{1}{2}(\partial^2 Q^*/\partial X^2)(r_0)v^2 w + vw^2. \end{cases} \qquad (18)$$

Let $P_2(x,y)$ and $Q_2(x,y)$ denote the zero or homogeneous polynomials of degree two such that $P(x,y) = -y + P_2(x,y)$ and $Q(x,y) = x + P_2(x,y)$. We introduce the following.

NOTATION 13. *Let us denote by*

$$\begin{aligned} G(u) &= C^*(1,u,0) = uP_2(1,u) - Q_2(1,u) \quad \text{and} \\ H(v) &= -C^*(v,1,0) = vQ_2(v,1) - P_2(v,1). \end{aligned} \tag{19}$$

A simple calculation yields:

$$\begin{aligned} G'(u_0) &= \tfrac{\partial G}{\partial u}(u_0) = \tfrac{\partial C^*}{\partial Y}(q_0), \\ G''(u_0) &= \tfrac{\partial^2 G}{\partial u^2}(u_0) = \tfrac{\partial^2 C^*}{\partial Y^2}(q_0), \\ G'''(u_0) &= \tfrac{\partial^3 G}{\partial u^3}(u_0) = \tfrac{\partial^3 C^*}{\partial Y^3}(q_0). \end{aligned} \tag{20}$$

and analogously

$$\begin{aligned} H'(v_0) &= -(\partial C^*/\partial X)(r_0), \\ H''(v_0) &= -(\partial^2 C^*/\partial X^2)(r_0), \\ H'''(v_0) &= -(\partial^3 C^*/\partial X^3)(r_0). \end{aligned} \tag{21}$$

In order to write the system of equations (17) and (18) in a simpler form we also introduce the following notations:

NOTATION 14.

$$\begin{aligned} P_2'(u_0) &= \tfrac{d}{du} P_2(1,u)|_{u=u_0} = (\partial P^*/\partial Y)(q_0), \\ P_2''(u_0) &= \tfrac{d^2}{du^2} P_2(1,u)|_{u=u_0} = (\partial^2 P^*/\partial Y^2)(q_0), \\ Q_2'(v_0) &= \tfrac{d}{dv} Q_2(v,1)|_{v=v_0} = (\partial Q^*/\partial X)(r_0) \quad \text{and} \\ Q_2''(v_0) &= \tfrac{d^2}{dv^2} Q_2(v,1)|_{v=v_0} = (\partial^2 Q^*/\partial X^2)(r_0). \end{aligned} \tag{22}$$

Using the above notations the system of equations (17) and (18) can be written respectively:

$$(U)\ \begin{cases} \frac{du}{dt} = G'(u_0)u - (1+u_0^2)z + \frac{G''(u_0)}{2}u^2 - 2u_0uz + \frac{G'''(u_0)}{6}u^3 - u^2z, \\ \frac{dz}{dt} = P_2(1,u_0)z + P_2'(u_0)uz - u_0z^2 + \frac{P_2''(u_0)}{2}u^2z + uz^2. \end{cases} \tag{23}$$

and

$$(V)\ \begin{cases} \frac{dv}{dt} = H'(v_0)v + (1+v_0^2)w + \frac{H''(v_0)}{2} + 2v_0vw + \frac{H'''(v_0)}{6}v^3 + v^2w, \\ \frac{dw}{dt} = Q_2(v_0,1)w + Q_2'(v_0)vw + v_0w^2 + \frac{Q_2''(v_0)}{2}v^2w + vw^2. \end{cases} \tag{24}$$

The determination of the local phase portraits around the each one of the singularities at infinity is reduced now to the study of the local phase portraits around the origin for the system of equation (23) and (24). The following terminology will be used.

DEFINITION 15. A singular point p of the system of equation (2) is

1) a *degenerate* (respectively *non degenerate)* singular point if and only if the determinant DetL(p) of the matrix of the linearization at p is zero (respectively DetL(p) $\neq 0$ or equivalently both eigenvalues at p are not zero).

2) an *elementary* singular point if and only if at least one of the eigenvalues at p is not zero and a *degenerate elementary* singular point if it is *elementary* and it has one zero eigenvalue at p.

3) a *nilpotent* singular point if both eigenvalues at p are zero but the matrix of the linearized system at p is not zero.

PROPOSITION 16. *Let $(u_0, 0)$ be a singular point of the system (17). Then*

1) $(u_0, 0)$ is a non degenerate point of the system (17) if $q_0 = [1 : u_0 : 0]$ is not a point of either one of the curves $P^* = 0$ or $\partial C^*/\partial Y = 0$.

2) $(u_0, 0)$ is a degenerate elementary singular point of the system (17) if $q_0 = [1 : u_0 : 0]$ is a point of only one of the curves $P^* = 0$, $\partial C^*/\partial Y = 0$.

3) $(u_0, 0)$ is a nilpotent singular point of the system (17) if $q_0 = [1 : u_0 : 0]$ is a common point of the curves $P^* = 0$, $\partial C^*/\partial Y = 0$.

Proof. The matrix of the linearized system at the point $(u_0, 0)$ for the system (17) is:

$$\begin{bmatrix} (\partial C^*/\partial Y)(1, u_0, 0) & -(1+u_0^2) \\ 0 & P^*(1, u_0, 0) \end{bmatrix}.$$

Therefore the eigenvalues are $\lambda_1 = (\partial C^*/\partial Y)(1, u_0, 0)$ and $\lambda_2 = P^*(1, u_0, 0)$ and using Definition 15 the proposition is proven. ▌

PROPOSITION 17. *Let $(v_0, 0)$ be a singular point of the system (18). Then*

1) $(v_0, 0)$ is a non degenerate singular point of the system (18) if $r_0 = [v_0 : 1 : 0]$ is not a point of either one of the curves $Q^* = 0$ and $\partial C^*/\partial X = 0$.

2) $(v_0, 0)$ is a degenerate elementary singular point of the system (18) if $r_0 = [v_0 : 1 : 0]$ is a point of only on of the curves $Q^* = 0$ or $\partial C^*/\partial X = 0$.

3) $(v_0, 0)$ is a nilpotent singular point of the system (18) if $r_0 = [v_0 : 1 : 0]$ is common point of the curves $Q^* = 0$ and $\partial C^*/\partial X = 0$.

We observe that from (6) it follows that if $q_0 = [1 : u_0 : 0]$ is a singular point at infinity for the differential system (12) and if $P^*(1, u_0, 0) = 0$, then q_0 is a common point of the projective curves $P^* = 0$ and $Q^* = 0$.

In classifying the phase portraits of quadratic systems with a weak focus via the topological equivalence our first step is to consider the number of distinct points of intersection of the curves $Z = 0$ with $C^* = 0$ and respectively with $P^* = 0$ and $Q^* = 0$. The singular points at infinity for the differential equation (15) lying on the curve $P^* = 0$, $Q^* = 0$ are the solution of the following system of equations:

$$\begin{aligned} &Z = 0, \\ &P^*(X,Y,Z) = -YZ + kX^2 + mXY + nY^2, \\ &Q^*(X,Y,Z) = XZ + aX^2 + bXY. \end{aligned} \tag{25}$$

We introduce the following:

Notation 18. *Let $V(R_1, \ldots, R_s)$ be the set of all common solutions in $\mathbb{C}P(2)$ of the homogeneous polynomials R_i in $\mathbb{R}[X, Y, Z]$.*

We have:

$$\begin{aligned} V(P^*, Q^*, Z) &= V(kX^2{+}mXY{+}nY^2, X, Z) \cup V(kX^2{+}mXY{+}nY^2, aX{+}bY, Z) \\ &= V(nY, X, Z) \cup V(kX^2 + mXY + nY^2, aX + bY, Z) \end{aligned}$$

Notation 19. *We shall denote by p_+, p_- be the intersecting points, real or complex and in the finite plane or at infinity, of the projective curves $P^*(X,Y,Z) = -YZ + kX^2 + mXY + nY^2 = 0$ and $Z + aX + bY = 0$. Any point located on $Z = 0$ could be written in the form $p_{c,d} = [-d : c : 0]$ with $c^2 + d^2 \neq 0$. We shall use the notation $p_0 = [0 : 1 : 0]$, $q_0 = [1 : 0 : 0]$.*

Notation 20. *For our equations (12) let $\Delta_\lambda = (m+a)^2 - 4k(n+b)^2$, let $\Delta_{k,m,n} = m^2 - 4kn$ and let $K_\lambda = kb^2 - mab + na^2 = P_2(-b, a) = P^*(p_{a,b})$.*

Δ_λ is the discriminant of the quadratic form $\mathrm{Res_Z(Z + aX + bY, P^*)}$ obtained by eliminating Z from P^* and $Z + aX + bY$.

For $a = b = 0$, Δ_λ becomes $\Delta_{k,m,n}$ which is the discriminant of the quadratic from $P_2(X,Y) = kX^2 + mXY + nY^2$.

$K_\lambda = kb^2 - amb + a^2n = P_2(-b, a) = P^*(p_{a,b})$. We observe that $K_\lambda = \mathrm{Res_v\big(P_2(1,v), a + bv\big)}$, i.e. the result of the elimination of v from $P_2(1,v)$, $a + bv$, if $b \neq 0$ and analogously $K_\lambda = \mathrm{Res_u(P_2(1,u,0), a + bu)}$ when $b \neq 0$.

Clearly we have:

$$V(nY, X, Z) = \begin{cases} \phi & \Leftrightarrow \quad n \neq 0 \\ \{p_0\} & \Leftrightarrow \quad n = 0. \end{cases}$$

We observe that $n = \mathrm{Res_v}\big(\mathrm{P_2(v,1),v}\big)$ (the resultant of the elimination of from $P(v,t)$ and v).

$$V(kX^2+mXY+nY^2, aX+bY, Z) = \begin{cases} V(P_2,Z) & \text{if} \quad a=b=0 \\ & \quad\ \text{and } (k,m,n)\neq 0 \\ \{p_{a,b}\} & \text{if} \quad (a,b)\neq(0,0) \\ & \quad\ \text{and } K_\lambda \neq 0 \\ \phi & \text{if} \quad (a,b)\neq(0,0) \\ & \quad\ \text{and } K_\lambda \neq 0. \end{cases}$$

A system (12) depends on the parameter $\lambda = (k,m,n,a,b) \in \mathbb{R}P(4)$ and hence the set $V(P^*,Q^*,Z)$ above will be denoted by $V_\lambda(P^*,Q^*,Z)$.

Proposition 21. *For any $\lambda = (k,m,n,a,b) \in \mathbb{R}P(4)$ the curves $P^*=0$ and $Q^*=0$ have at most two common points at infinity ($\#V_\lambda(P^*,Q^*,Z)$ ≤ 2).*

We have:

$$V(kX^2+mXY+nY^2, AX+bY, Z) = \begin{cases} V(P^*,Z) & \text{if } a=b=0 \\ \{p_{a,b}\} & \text{if } b\neq 0 = K_\lambda \\ \{p_0\} & \text{if } b=0=n\neq a \\ \phi & \text{if } bK_\lambda \neq 0 \text{ or} \\ & \qquad b=0\neq an. \end{cases}$$

Combining the above with the values of $V(nY,X,Z)$ we have:

$$V_\lambda(P^*,Q^*,Z) = \begin{cases} \phi & \text{if } nbK_\lambda \neq 0 \text{ or } b=0\neq an \\ V(P^*,Z) & \text{if } a=b=0\neq n \\ V(P^*,Z)\cup\{p_0\} & \text{if } a=b=0=n \\ \{p_{a,b}\} & \text{if } bn\neq 0=K_\lambda \\ \{p_0\} & \text{if } b=0=n\neq a \\ \{p_{a,b},p_0\} & \text{if } b\neq 0=K_\lambda=n. \end{cases}$$

The following proposition follows easily from the above evaluations:

PROPOSITION 22. *For a quadratic system (12) we have:*

$$V_\lambda(P^*,Q^*,Z)=\begin{cases}\{p_+,p_-\} & \text{if } a=b=0\neq n\Delta_{k,m,n}\\ \{p_{m,2n}\} & \text{if } a=b=\Delta_{k,m,n}=0\neq n\\ \{p_0,p_{m,k}\} & \text{if } a=b=0=n\neq\Delta_{k,m,n}\\ \phi & \text{if } \begin{cases} bnK_\lambda\neq 0\\ \text{or}\\ b=0\neq nK_\lambda a\end{cases}\\ \{p_{a,b}\} & \text{if } bn\neq 0=K\lambda\\ \{p_0\} & \text{if } \begin{cases} bK_\lambda\neq 0=n\\ \text{or}\\ n=0=b=K_\lambda\neq a\end{cases}\\ \{p_0,p_{a,b}\} & \text{if } b\neq 0=K_\lambda=n.\end{cases}$$

We would like to compute the divisors defined in the preceding section for the systems (12) for the various values of the parameter λ. For this we first introduce the:

NOTATION 23. *1) The two divisors at infinity (Definition 9) for the differential equations (12) are:*

$$D_\lambda(P^*,Q^*;Z)=\sum_{p\in\{z=0\}} I_p(P^*,Q^*)p \qquad \text{and}$$

$$D_\lambda(C^*,Z)=\sum_{p\in\{Z=0\}} I_p(C^*,Z)p$$

where $\lambda=[k:m:n:a:b]\in\mathbb{R}P(4)$.

From the above expression for $V_\lambda(P^*,Q^*;Z)$ we see that the values of $D_\lambda(P^*,Q^*,Z)$ must be among the following types : $0, p, 2p$, $p+q$, $2p+q$, $3p$.

To compute the above mentioned two divisors we need to use the value for $V_\lambda(P^*,Q^*;Z)$ in Proposition 17 and for each p on $Z=0$, we need to compute $I_p(P^*,Q^*)$ and $I_p(C^*,Z)$.

THEOREM 24. *For the family of systems(12) depending on the parameter* $\lambda\in\mathbb{R}P(4)$ *we have:* $0\leq\deg D_\lambda(P^*,Q^*;Z)\leq 3$ *and the values of* $D_\lambda(P^*,Q^*;Z)$ *are given by the Table 1.*

Proof. Since the origin is a weak focus and the system is quadratic clearly $\deg D_\lambda(P^*,Q^*;Z)\leq 3$. We then use the values of $V_\lambda(P^*,Q^*,Z)$ in Proposition 22. To make sure we cover all possible cases we carry out the computation depending on whether or not $(a,b)=0$ or $(a,b)\neq 0$. Assume first that $(a,b)=0$. Then we have one and only one of the possibilities : $n\Delta_{k,m,n}\neq 0$ or $n=0\neq\Delta_{k,m,n}$ or $n\neq 0=\Delta_{k,m,n}$ or $n=0=\Delta_{k,m,n}$.

TABLE 1.

$\deg D_\lambda(P^*, Q^*; Z)$	value of $D_\lambda(P^*, Q^*; Z)$	conditions in the space of parameters
0	0	$bnK_\lambda \neq 0$ or $anK_\lambda \neq 0 = b$
1	$p_{a,b}$ or p_0	$\Delta_\lambda bn \neq 0 = K_\lambda$ or $bK_\lambda \neq 0 = n$
2	$p_+ + p_-$ or $p_0 + p_{a,b}$ or $2p_{m,2n}$ or $2p_{a,b}$ or $2p_0$	$n\Delta_{k,m,n} \neq 0 = a = b$ or $b\Delta_\lambda \neq 0 = n = K_\lambda$ or $n \neq 0 = a = b = \Delta_{k,m,n}$ or $bn \neq 0 = K_\lambda = \Delta_\lambda$ or $a\Delta_\lambda \neq 0 = n = b (\Rightarrow K_\lambda = 0)$
3	$2p_0 + p_{k,m}$ or $p_0 + 2p_{a,b}$ or $3p_0$	$\Delta_{k,m,n} \neq 0 = a = b = n$ or $b \neq 0 = n = K_\lambda = \Delta_\lambda$ or $a \neq 0 = n = b = \Delta_\lambda$

However the last case is eliminated since it yields $W_1 = W_2 = W_3 = 0$ which is true for a center at the origin while we have a weak focus. We consider the three cases for $(a, b) = 0$:

Case 1. Assume $a = b = 0 \neq n\Delta_{k,m,n}$. In this case $V_\lambda(P^*, Q^*; Z) = \{p_+, p_-\}$ where $p_\pm = [2n : -m \pm \sqrt{\Delta_{k,m,n}} : 0]$. Since $n \neq 0$, $I_{p_+}(P^*, X) = 0$. Hence $I_{p_+}(P^*, Q^*) = I_{p_+}(P^*, Z) = 1$. Since $\Delta_{k,m,n} \neq 0$. Analogously $I_{p_-}(P^*, Q^*) = 1$ which yields $D_\lambda(P^*, Q^*; Z) = p_+ + p_-$.

Case 2. Assume $a = b = \Delta_{k,m,n} = 0 \neq n$. Then $V_\lambda(P^*, Q^*, Z) = \{p_{m,2n}\}$, $p_{m,2n} = [-2n : m : 0]$. $\Delta_{k,m,n} = m^2 - 4kn = 0$. Since $n \neq 0$, $m \neq 0$ since otherwise $k = 0$ and hence $W_1 = W_2 = W_3 = 0$ which contradicts having the origin as a weak focus. $I_{p_{m,2n}}(P^*, X) = 0$ but $I_{p_{m,2n}}(P^*, Z) = 2$ since $m \neq 0$ implies that $Z = 0$ is tangent to $P^* = 0$ at $p_{m,2n}$. So $D_\lambda(P^*, Q^*; Z) = 2p_{m,2n}$.

Case 3 Assume $a = b = n = 0 \neq \Delta_{k,m,n} = m^2$. In this case $V(P^*, Q^*, Z) = \{p_0, p_{k,m}\}$ where $p_0 = [0 : 1 : 0]$ and $p_{k,m} = [-m : k : 0] \neq p_0$.

$I_{p_0}(P^*,X) = I_{(0,0)}(-Z+mX+kX^2,X) = 1$ and $I_{p_0}(P^*,Z) = I_{(0,0)}(-Z+mX+kX^2,Z) = 1$. We also have $I_{p_{k,m}}(P^*,X) = 0$ and $I_{p_{k,m}}(P^*,Z) = I_{(-\frac{k}{m},0)}(k+mY-YZ,Z)$. Since $\frac{\partial(k+mY-YZ)}{\partial Y}|_{(-\frac{k}{m},0)} = m \neq 0$, $Z=0$ is not tangent to $k+mY-YZ=0$ at $(-\frac{k}{m},0)$ and hence $I_{p_{k,m}}(P^*,Z) = 1$. So $D_\lambda(P^*,Q^*;Z) = 2p_0 + p_{k,m}$.

Assume now $(a,b) \neq 0$. Then either $b \neq 0$ or $a \neq 0$. Assume first $b \neq 0$. Then either $nK_\lambda \neq 0$ or $n \neq 0 = K_\lambda$ or $n = 0 \neq K_\lambda$ or $n = 0 = K_\lambda$.

$\mathbf{b} \neq 0$ Case 1: If $bnK_\lambda \neq 0$, $V_\lambda(P^*,Q^*,Z) = \phi$ and hence $D_\lambda(P^*,Q^*;Z) = 0$.

$\mathbf{b} \neq 0$ Case 2: $n \neq 0 = K_\lambda$. In this case $V_\lambda(P^*,Q^*,Z) = \{p_{a,b}\}$ and $p_{a,b} = [-b:a:0]$. Clearly $I_{p_{a,b}}(P^*,X) = 0$ and hence $I_{p_{a,b}}(P^*,Z+aX+bY)$. We have $I_{p_{a,b}}(P^*,Z+aX+bY) = 2$ if $\Delta_\lambda = 0$ and it is 1 if $\Delta_\lambda = 0$. This yields

$$D_\lambda(P^*,Q^*;Z) = \begin{cases} p_{a,b} & \text{if } \Delta_\lambda \neq 0 \\ 2_{p_{a,b}} & \text{if } \Delta_\lambda = 0. \end{cases}$$

$\mathbf{b} \neq 0$ Case 3: $n = 0 \neq K_\lambda$. In this case $V_\lambda(P^*,Q^*,Z) = \{p_0\}$, $I_{p_0}(P^*,X) = I_{(0,0)}(-Z+mX+kX^2,X) = 1$ and $I_{p_0}(P^*,Z+aX+bY) = 0$. So $D_\lambda(P^*,Q^*;Z) = p_0$.

$\mathbf{b} \neq 0$ Case 4: Assume now $n = 0 = K_\lambda$. In this case $V_\lambda(P^*,Q^*,Z) = \{p_0,p_{a,b}\}$ and we have $I_{p_0}(P^*,X) = 1$. We also have $I_{p_0}(P^*,Z+aX+bY) = 0$ and hence $I_{p_0}(P^*,Q^*) = 1$. Clearly we have $I_{p_{a,b}}(P^*,X) = 0$ and $I_{p_{a,b}}(P^*,Z+aX+bY) = I_{(-\frac{a}{b},0)}(-yz+k+my,z+a+by)$. The line $z+a+by=0$ is tangent to $-yz+k+my=0$ at $(-\frac{a}{b},0)$ if and only if $\Delta_\lambda = 0$. So we have $I_p(P^*,Z+aX+bY) = 1$ if $\Delta_\lambda \neq 0$ and $I_p(P^*,Z+aX+bY) = 2$ if $\Delta_\lambda = 0$. This yields

$$D_\lambda(P^*,Q^*;Z) = \begin{cases} p_0 + p_{a,b} & \text{if } \Delta_\lambda \neq 0 \\ p_0 + 2p_{a,b} & \text{if } \Delta_\lambda = 0. \end{cases}$$

∎

Assume now that $b = 0 \neq a$. Then we have either $nK_\lambda \neq 0$ or $n = 0$ (in which case also $K_\lambda = 0$) or $n \neq 0 = K$. We consider now separately these three cases.

$\mathbf{b} = 0 \neq a$ Case 1: $nK_\lambda \neq 0$. In this case $V(P^*,Q^*,Z) = \phi$ and hence $D_\lambda(P^*,Q^*;Z) = 0$.

$\mathbf{b} = 0 \neq a$ Case 2: $n = 0$ ($\Rightarrow K_\lambda = 0$). We have two possibilities : $\Delta_\lambda \neq 0$ and $\Delta_\lambda = 0$. Assume first $\Delta_\lambda \neq 0$. In this case $V(P^*,Q^*,Z) = \{p_0\}$. $I_{p_0}(P^*,X) = I_{(0,0)}(-Z+kX^2+mX,X) = 1$ and $I_{p_0}(P^*,Z+aX) =$

$I_{(0,0)}(-Z+mX+kX^2, Z+aX)$. By hypothesis $\Delta_\lambda \neq 0$ and $\Delta_\lambda = (m+a)^2$. So $m \neq -a$ and hence $I_{(0,0)}(-Z+mX+kX^2, Z+aX) = 1$. This yields $I_{p_0}(P^*, Q^*) = 2$ and $D_\lambda(P^*, Q^*; Z) = 2p_0$. Assume now $\Delta_\lambda = 0$. In this case $V_\lambda(P^*, Q^*, Z) = \{p_0\}$, $I_{p_0}(P^*, X) = 1$ and $I_{p_0}(P^*, Z+aX) = 2$. Hence $D_\lambda(P^*, Q^*; Z) = 3p_0$.

We now need to compute $D_\lambda(C^*, Z)$ for all λ in $\mathbb{R}P(4)$. For this we need the values of $I_p(C^*, Z)$ for all $p = [X_0 : Y_0 : 0]$ on $C^* = 0$.

Proposition 25. *Let $p_0 = [X_0 : Y_0 : 0]$ be a singular point at infinity of a quadratic differential system (2). Then $X_0 Y - Y_0 X$ is a factor of multiplicity $I_{p_0}(C^*, Z)$ of the form $YP_2 - XQ_2$ where P_2 (resp. Q_2) contains only quadratic terms or is zero.*

Proof. The result follows from the fact that $I_p(C^*, Z) = I_p(YP_2 - XQ_2, Z)$ and the properties of intersection multiplicity (cf. [5], p. 74–75). ▮

For the system (12) the cubic form $YP_2 - XQ_2$ is

$$F(X,Y) = YP_2 - XQ_2 = nY^3 + mXY^2 + (k-b)X^2Y - aX^3 \quad (26)$$

Notation 26. *Let us denote by Δ_G the discriminant of a binary form G and let Hess(G) be the Hessien of F, i.e.*

$$\text{Hess(G)} = \begin{vmatrix} \frac{\partial^2 G}{\partial X^2} & \frac{\partial^2 G}{\partial Y \partial X} \\ \frac{\partial^2 G}{\partial X \partial Y} & \frac{\partial^2 G}{\partial Y^2} \end{vmatrix}$$

For our form $F(X,Y)$ (26) we have:

$$\text{Hess(F)} = -(3am + (k-b)^2)X^2 - (9an + (k-b)m)XY + (3n(k-b) - m^2)Y^2.$$

$$\text{Hess(F)} = 0 \Leftrightarrow 3n(k-b) - m^2 = 9an + m(k-b) = 3am + (k-b)^2 = 0.$$

The following is a well known result form the theory of binary forms and it can be easily proved:

Proposition 27. *A binary cubic form G over $\mathbb{C}$ has a double linear factor if and only if $\Delta_G = 0$ and Hess$(G) \neq 0$ and it has a linear factor of multiplicity three if and only if Hess$(G) = 0$.*

Proposition 28. *For the systems (12) with a weak focus at the origin and their associated differential equation (3), the divisors $D_\lambda(C^*, Z)$ are* *well defined for all values of* λ *in* $RP(4)$ *and we have:*

$$D_\lambda(C^*, Z) = \begin{cases} p+q+s & \text{if} \quad \Delta_F \neq 0 \\ 2p+q & \text{if} \quad \Delta_F = 0 \text{ and } \mathrm{Hess}(F) \neq 0 \\ 3p & \text{if} \quad \mathrm{Hess}(F) = 0 \end{cases}$$

where p, q, s *are distinct points satisfying* $C^* = 0 = Z$.

Proof. The form F is not zero since for $n = m = k - b = a = 0$ the origin is a center. Hence Z does not divide C^* and the above follows from the preceeding Proposition. ∎

We are now interested in assembling simultaneously the information we have about the two families of divisors $D_\lambda(P^*, Q^*, Z)$ and $D_\lambda(C^*, Z)$. If $\deg D_\lambda(P^*, Q^*; Z) = 0$, the above Proposition is all we need to have. However in the case $\deg D_\lambda(P^*, Q^*; Z) \geq 1$ we need to see how the conditions in Table 1 combine with those in the above Proposition. For a simultaneous consideration of the two families of divisors we introduce the following:

Definition 29. A divisor on the line $Z = 0$ with coefficients in $\mathbb{Z}^s$, $s \geq 1$ is a formal sum

$$\sum_{p \in \{Z=0\}} n_p p$$

where $n_p \in \mathbb{Z}^s$ and only a finite number of n_p's are not zero.

Notation 30. *For the family (12) with a weak focus at the origin we consider the following family of divisors with coefficients in* $\mathbb{Z}^2$

$$D_\lambda = \sum_{p \in \{Z=0\}} \begin{pmatrix} I_p(P^*_\lambda, Q^*_\lambda \\ I_p(C^*_\lambda, Z) \end{pmatrix}$$

where $\lambda \in \mathbb{R}P(4)$.

Since this is the first time these divisors at infinity are used for polynomial differential systems, we want to give here explicit calculations for the family (12) with a weak focus at the origin.

To give explicit results for D_λ when, $\deg D_\lambda(P^* Q^*; Z) \geq 1$ we need to consider common points $p = [X_0 : Y_0 : 0]$ of $P^* = 0$ and $Q^* = 0$. If $X_0 \neq 0$ then $F(X,Y) = \frac{1}{X_0^3}(X_0 Y - Y_0 X) F_1(X,Y)$ where

$$F_1(X,Y) = nX_0^2Y^2 + X_0(mX_0 + nY_0)XY + [(k-b)X_0^2 + mX_0Y_0 + nY_0^2X^2]. \tag{27}$$

If $Y_0 \neq 0$ then $F(X,Y) = \frac{1}{Y_0^3}(X_0Y - Y_0X)F_2(X,Y)$ where $F_2(X,Y) = aY_0^2X^2 - Y_0(mY_0 - aX_0)XY + [(k-b)Y_0^2 + mX_0Y_0 - aX_0^2]Y^2$.

NOTATION 31. *Let* $m(C^*, Z) = \max\{I_p(C^*, Z) \mid p = [X:Y:0] \in \mathbb{C}P(2)\}$, *sometimes we just write* m_{C^*} *for* $m(C^*, Z)$,

$$d = \deg(\mathrm{D}_\lambda(\mathrm{P}^*, \mathrm{Q}^*; \mathrm{Z}))$$
$$m = \max_{z\in\{Z=0\}}(m(p)) = \max_{p\in\{Z=0\}}(I_p(P^*, Q^*) + I_p(C^*, Z))$$
$$m_{T\infty} = \sum_{p\in\{Z=0\}} m(p).$$

THEOREM 32. *For the family (12) of systems with weak focus at the origin, depending on the parameter* $\lambda \in \mathbb{R}P(4)$, *the values of the family of divisors* D_λ *are given in Table 2.*

Proof. Assume first $\mathbf{degD}_\lambda(\mathbf{P}^*, \mathbf{Q}^*; \mathbf{Z}) = \mathbf{1}$. According to Table 1 then we either have $p_{a,b}$ or p_0 as the only solution $P_2 = Q_2 = 0$. Consider the first case i.e. of the point $p_{a,b}$ which occurs only when $\Delta_\lambda bn \neq 0 = K_\lambda$. Then $F(X,Y)$ has : only simple linear factors if and only if $\Delta_F = 0$ in which case $D_\lambda = \begin{pmatrix}1\\1\end{pmatrix} p_{a,b} + \begin{pmatrix}1\\1\end{pmatrix} 1 + \begin{pmatrix}1\\1\end{pmatrix} q$; a double linear factor if and only if $\Delta_F = 0$ and $\mathrm{Hess}(F) \neq 0$ and it has a triple linear factor if and only if $\mathrm{Hess}(F) = 0$. In the second case if $aX + bY$ is the double linear factor (i.e. when $F_1(-b,a) = 0$) then $D_\lambda = \begin{pmatrix}1\\2\end{pmatrix} p_{a,b} + \begin{pmatrix}0\\1\end{pmatrix} q$ and if $aX + bY$ is not the double factor of F (i.e. when $F_1(-b,a) \neq 0$), then $D_\lambda = \begin{pmatrix}1\\1\end{pmatrix} p_{a,b} + \begin{pmatrix}0\\2\end{pmatrix} q$. In the third case, from the first one of the condition for $\mathrm{Hess}(F) = 0$ we obtain $k - b = m/3n$. Replacing this in the third and second condition for $\mathrm{Hess}(F) = 0$ we obtain $m(27n^2 + m^3) = 0 = 27an^2 + m^3$. Since $m = 0 = 27an^2 + m^3$ implies $m = a = 0$, case where the origin is a center, we must have $m \neq 0 = 27an^2 + m^3 = K_\lambda$. In this case $D_\lambda = \begin{pmatrix}1\\3\end{pmatrix} p_{a,b}$.

Assume now the point is p_0, i.e. $bK_\lambda \neq 0 = n$. In this case $F(X,Y) = mXY^2 + (k-b)X^2Y - aX^3 = X(mY^2 + (k-b)XY - aX^2)$ and $\Delta_F = am^2[(k-b)^2 + 4am]$. F has only simple factors if $\Delta_F \neq 0$. Then $D_\lambda = \begin{pmatrix}1\\1\end{pmatrix} p_0 + \begin{pmatrix}1\\1\end{pmatrix} l + \begin{pmatrix}1\\1\end{pmatrix} q$. In the case $\Delta_F = 0$, X is a double factor of F iff $m = 0 \neq k - b$ in which case $D_\lambda = \begin{pmatrix}1\\2\end{pmatrix} p_0 + \begin{pmatrix}0\\1\end{pmatrix} q$, and it is a simple factor of F whenever $am \neq 0 = (k-b)^2 + 4am$. In this case

$D_\lambda = \begin{pmatrix}1\\1\end{pmatrix} p_0 + \begin{pmatrix}0\\2\end{pmatrix} q$. X is a triple factor iff $m = 0 = k - b$. Since $K_\lambda \neq 0$ we have $m = k - b \neq k$ and in this case $D_\lambda = \begin{pmatrix}1\\3\end{pmatrix} p_0$.

Assume now that $\mathbf{deg D_\lambda(P^*, Q^*; Z) = 3}$. According to the Table 1 we need to consider three cases. We consider the first case : $D_\lambda(P^*, Q^*; Z) = 2p_0 + p_{k,m}$, occurring whenever $\Delta_{k,m,n} \neq 0 = a = b = n$. In this case we have $m \neq 0$ and $F(X, Y) = XY(mY + kX)$. $k \neq 0$ since otherwise the origin in a center. Since also $m \neq 0$ we have three simple factors of F and hence $D_\lambda = \begin{pmatrix}1\\1\end{pmatrix} p_0 + \begin{pmatrix}1\\1\end{pmatrix} p_{k,m} + \begin{pmatrix}1\\1\end{pmatrix} q_0$ where q_0 is here $[1 : 0 : 0]$.

Consider now the case $D_\lambda(P^*, Q^*; Z) = p_0 + 2p_{a,b}$ occurring whenever $b \neq 0 = n = K + \lambda = \Delta_\lambda$. In this case $K_\lambda = b(kb - am) = 0$ and since $b \neq 0$ we have $k = am/b$. Replacing this in $\Delta_\lambda = (a + m)^2 - 4k(b + n) = 0$, we obtain $(a - m)^2 = 0$ and hence $a = m \neq 0$ since otherwise the origin is a center. Replacing the data in $F(X, Y)$ we obtain $F(X, Y) = aXY^2 + (a^2 - b^2)X^2Y/b - aX^3 = X(abY^2 + (a^2 - b^2)XY - abX^2)/b$. Since $ab \neq 0$, X is not a multiple factor of F. The discriminant of the quadratic factor of F is $(a^2 - b^2)^2 + 4a^2b^2 = (a^2 + b^2)^2 \neq 0$ and hence $D_\lambda = \begin{pmatrix}1\\1\end{pmatrix} p_0 + \begin{pmatrix}2\\1\end{pmatrix} p_{a,b} + \begin{pmatrix}0\\1\end{pmatrix} p_{b,-a}$.

Consider now the case $D_\lambda(P^*, Q^*; Z) = 3p_0$ occurring whenever $a \neq 0 = n = b = \Delta_\lambda$. Since $\Delta_\lambda = (a + m)^2 = 0$, $m = -a$ and hence $F(X, Y) = -aXY^2 + kX^2Y - aX^3 = X(-aY^2 + kXY - aX^2)$. So X cannot be a double factor of F. F contains a double linear factor iff $k^2 - 4a^2 = 0$ and otherwise three distinct factors. So we have $D_\lambda = \begin{pmatrix}3\\1\end{pmatrix} p_0 + \begin{pmatrix}0\\1\end{pmatrix} l + \begin{pmatrix}0\\1\end{pmatrix} s$ whenever $k^2 - 4a^2 \neq 0$ and $D_\lambda = \begin{pmatrix}3\\1\end{pmatrix} p_0 + \begin{pmatrix}0\\2\end{pmatrix} q$ whenever $k^2 - 4a^2 = 0$.

The calculations for the case $\mathbf{deg D_\lambda(P^*, Q^*; Z) = 2}$ are analogously treated. We shall only consider here the case when $D_\lambda(P^*, Q^{*}`Z) = 2p_{a,b}$ occurring whenever $bn \neq 0 = K_\lambda = \Delta_\lambda$. Since $b \neq 0$ and $K_\lambda = kb^2 - abm + na^2 = 0$, then if $a = 0$ we get $k = 0$ and hence from $\Delta_\lambda = 0$ we get $m = 0$, a system for which the origin is a center. So $a \neq 0$ and hence from $K_\lambda = 0$ we obtain $m = (kb^2 + na^2)/ab$. After replacing this value in Δ_λ we obtain $\Delta_\lambda = (a^2b - kb^2 + na^2)^2/(ab) = 0$ which yields $k = a^2(b + n)/b^2$ and hence $m = a(b + 2n)/b$. Replacing the values of m and k in F and knowing that $aX + bY$ divides F we obtain $F(X, Y) = (aX + bY)F_1(X, Y)/b^3$ where $F_1(X, Y) = b(bnY^2 + a(b+n)XY - b^2X^2)$. $F_1(-b, a) = b(bna^2 - a(b+n)ba - b^4) = b^2(na^2 - a(b + n)a - b^3) = -b^3(a^2 + b^2) \neq 0$. So $I_{p_{a,b}}(C^*, Z) = 1$.

Hence $D_\lambda = \begin{pmatrix} 2 \\ 1 \end{pmatrix} p_{a,b} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} q$ if $\Delta_{F_1} = 0$ (i.e. when $a^2(b+n)^2 + 4b^3 n = 0$)
or $D_\lambda = \begin{pmatrix} 2 \\ 1 \end{pmatrix} p_{a,b} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} q + \begin{pmatrix} 0 \\ 1 \end{pmatrix} s$ if $\Delta_{F_1} = a^2(b+n)^2 + 4b^3 n \neq 0$. ▮

TABLE 2.

d	m_{C^*}	m	D_λ	Conditions on the parameters	$m_{T,\infty}$
0	1	1	$\binom{0}{1}p + \binom{0}{1}q + \binom{0}{1}\ell$	$bnK_\lambda\Delta_F \neq 0$ or $anK_\lambda\Delta_F \neq 0 = b$	3
	2	2	$\binom{0}{1}p + \binom{0}{2}q$	($bnK_\lambda \neq 0 = \Delta_\lambda$ or $anK_\lambda \neq 0 = \Delta_F$) and Hess$(F) \neq 0$	
	3	3	$\binom{0}{3}p$	($bnK_\lambda \neq 0$ or $anK_\lambda \neq 0 = b$) and Hess$(F) = 0$	
1	1	2	$\binom{1}{1}p_{a,b} + \binom{0}{1}\ell + \binom{0}{1}q$	$\Delta_\lambda bn\Delta_F \neq 0 = K_\lambda$	4
			$\binom{1}{1}p_0 + \binom{0}{l}\ell + \binom{0}{1}q$	$abK_\lambda m[(k-b)^2 + 4am] \neq 0 = n$	
	2		$\binom{1}{1}p_{a,b} + \binom{0}{2}\ell$	$\Delta_\lambda bn \neq 0 = K_\lambda = \Delta_F$ and Hess$(F) \neq 0$ (i.e. $F_1(-b,a) \neq 0$)	
			$\binom{1}{1}p_0 + \binom{0}{2}\ell$	$abK_\lambda m \neq 0 = n = (k-b)^2 + 4am$	
		3	$\binom{1}{2}p_{a,b} + \binom{0}{1}\ell$	$\Delta_\lambda bn \neq 0 = K_\lambda = F_1(-b,a) \neq \Delta_{F_1}$	
			$\binom{1}{2}p_0 + \binom{0}{1}\ell$	$bK_\lambda(k-b) \neq 0 = m = n$	
	3	4	$\binom{1}{3}p_{a,b}$	$\Delta_\lambda bnm \neq 0 = K_\lambda = 27an^2 + m^3$	
			$\binom{1}{3}p_0$	$bK_\lambda \neq 0 = n = m = k-b$	
2	1	2	$\binom{1}{1}p_+ + \binom{1}{1}p_- + \binom{0}{1}\ell$	$kn\Delta_{k,m,n} \neq 0 = a = b$	5
			$\binom{1}{1}p_0 + \binom{1}{1}p_{a,b} + \binom{0}{1}\ell$	$b\Delta_\lambda \neq 0 = n = K_\lambda$	
		3	$\binom{2}{1}p_{a,b} + \binom{0}{1}\ell + \binom{0}{1}q$	$[a^2(b+n)^2+4b^3n]bn \neq 0 = K_\lambda = \Delta_\lambda$	
	2		$\binom{2}{1}p_{a,b} + \binom{0}{2}\ell$	$bn \neq 0 = K_\lambda = \Delta_\lambda = a^2(b+n)^2 + 4b^3n$	
			$\binom{1}{2}p_{a,b} + \binom{1}{1}p_0$	$b \neq 0 = n = K_\lambda = \Delta_\lambda$	
			$\binom{1}{2}p_0 + \binom{1}{1}p_{a,b}$	$ab \neq 0 = n = m = k$	
		4	$\binom{2}{2}p_0 + \binom{0}{1}p_{a,-k}$	$ka\Delta_\lambda \neq 0 = n = b (\Rightarrow K_\lambda = 0)$	
			$\binom{0}{1}q_0 + \binom{2}{2}p_{m,2n}$	$n \neq 0 = a = b = \Delta_{k,m,n}$	
3	1	3	$\binom{2}{1}p_0 + \binom{1}{1}p_{k,m} + \binom{0}{1}q_0$	$k.\Delta_{k,m,n} \neq 0 = a = b = n$	6
			$\binom{1}{1}p_0 + \binom{2}{1}p_{a,b} + \binom{0}{1}p_{b,-a}$	$b \neq 0 = n = K_\lambda = \Delta_\lambda$	
		4	$\binom{3}{1}p_0 + \binom{0}{1}\ell + \binom{0}{1}q$	$a(k^2 - 4a^2) \neq 0 = n = b = \Delta_\lambda$	
	2		$\binom{3}{1}p_0 + \binom{0}{2}\ell$	$a \neq 0 = n = b = \Delta_\lambda = k^2 - 4a^2$	

THEOREM 33. *For any quadratic system (2) with a singularity which is a weak focus, for any singular point p finite or infinity we have:*

$$m(p) \leq 4.$$

Proof. Since the multiplicities of singularities, finite or infinite are not changed when we change coordinates, we may assume the system to be in the normal form (12) and the result follows from $m(p) = I_p(P^*, Q^*)$ if p is finite and from the Table 2 if p is infinite. ▮

COROLLARY 34. *If a quadratic system S has a singular point p_0 which is a center for the linear part at p_0 and if for S we have an infinite point p of multiplicity $M_{3,2}$ (resp $M_{3,3}$ or M_{23}) then necessarily p is a center.*

Proof. We may assume $p_0 = (0,0)$ and the system in normal form (12). Since a linear center is either a weak focus or a center the Corollary follows form the Theorem 33 and the fact that if $M_{r,s}$ is either $M_{3,2}$ or $M_{3,3}$ or $M_{2,3}$ then $m(p) = r + s \geq 5$. ▮

5. THE NEWTON POLYHEDRA CONSTRUCTION FOR SINGULARITIES AT INFINITY OF TYPE $M_{1,2}$

To determine the phase portrait in a neighborhood of a real singular point at infinity $p = [X_0 : Y_0 : 0]$ we shall use truncations of the vector fields defined by the system of equation (23), $u_0 = Y_0/X_0$ and (24), $v_0 = X_0/TY_0$. Our aim is to find a truncation which is simpler and belongs to the same topological equivalence class as the initial field. We will follow [3] and use the notion of the principal part of a vector field defined by Newton Polyhedra. We are interested here in the case when the origin is a nilpotent singularity for the system of equations (23) (respectively for (24)). The origin is a nilpotent singularity for (23) (respectively for (24)) whenever

$$P_2(1, u_0) = 0 = G'(u_0) \text{ (respectively } \mathrm{Q}_2(\mathrm{v}_0, 1) = 0 = \mathrm{H}'(\mathrm{u}_0)) \qquad (28)$$

Remark 35. We observe that the conditions above are equivalent to : $I_p(P^*, Z) \geq 1$ and $I_p(C^*, Z) \geq 2$ (respectively $I^p(Q^*, Z) \geq 1$ and $I_p(C^*, Z) \geq 2$). We shall be interested in the case when the nilpotent singularity at the origin for (23) (respectively for (24)) satisfies a nondegeneracy condition:

$$P_2'(u_0) \cdot G''(u_0) \neq 0 \text{ (respectively } \mathrm{Q}_2'(\mathrm{v}_0)\mathrm{H}_2''(\mathrm{v}_0) \neq 0) \qquad (29)$$

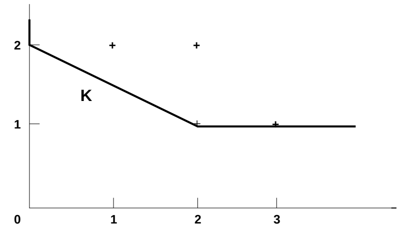

FIG. 1.

Remark 36. We observe that the conditions (28) and (29) are equivalent to: $I_p(P^*, Z) = 1$ and $I_p(C^*, Z) = 2$ (respectively $I_p(Q^*, Z) = 1$ and $I_p(C^*, Z) = 2$.

For the system of equation (23) we denote also by U the planar vector field associated and consider the formal expansion of U around the origin which we write as follows (cf. [3]):

$$\begin{aligned} U &= \left[-(1+u_0^2)u^{-1}z + \tfrac{G''(u)}{2}u - 2u_0 z + \tfrac{G'''(u_0)}{6}u^2 - uz\right]u\tfrac{\partial}{\partial x} + \\ &\quad \left[P_2'(u_0)u - u_0 z + \tfrac{P_2''(u_0)}{2}u^2 - uz\right]z\tfrac{\partial}{\partial z}. \end{aligned}$$

The *support* S of the field U is defined as a subset of $\mathbb{R}^2$.

$$\begin{aligned} S &= \{(m+1, n+1) \mid \exists j \text{ such that } \mathrm{a_{m,n,j}} \neq 0\} \text{ where} \\ U &= \sum a_{m,n,1} u^m z^n u \frac{\partial}{\partial u} + \sum a_{m,n,2} u^m z^n z \frac{\partial}{\partial z}. \end{aligned}$$

It is easy to see that in our case $S = \{(0,2),(2,1),(1,2),(3,1),(2,2)\}$. The *Newton Polyhedron* of U is the convex envelope $co(T)$ of the set

$$T = \cup_{(m,n)\in S}\{(m,n) + \mathbb{R}_+^2\}$$

and the *Newton Diagram* is the union K of the compact faces of the Newton Polyhedron $co(T)$. Again it is clear that in our case K is the segment determined in the plane by the points $(0,2)$ and $(2,1)$ as shown in Figure 1.

The *Principal Part* U_P of the vector field U is the vector field obtained from U considering only the coefficients which correspond to support points lying on the Newton Diagram K. Therefore

$$U_P = \left(-(1+u_0^2)z + \frac{G''(u_0)}{2}u^2\right)\frac{\partial}{\partial u} + P_2'(u_0)uz\frac{\partial}{\partial z} \qquad (30)$$

and similar considerations show that the field V defined by the system of equations (24) has exactly the same support S, hence

$$V_P = \left((1+v_0^2)w + \frac{H''(v_0)}{2}v^2\right)\frac{\partial}{\partial v} + Q_2'(v_0)vw\frac{\partial}{\partial w}. \tag{31}$$

In both cases the Newton Diagram has only one compact face. The vector field that is associated with a compact face in the Newton Diagram is called a *quasi-homogeneous component* of the principal part relative to that compact part. In our situation the principal part is the only quasi-homogeneous component. We note also that the origin is the only singular point for both U_P and V_P (therefore for all quasi-homogeneous components) which means that the principal parts are non degenerate (cf. Brunella and Miari [3]).

DEFINITION 37. (cf.[3]) Two smooth planar vector fields X and Y, $X(0,0) = Y(0,0) = (0,0)$, are locally topologically equivalent around $(0,0)$ modulo center-focus when either

i) X and Y have characteristic orbits at $(0,0)$ and are topologically equivalent in a neighborhood of the isolated singularity $(0,0) \in \mathbb{R}^2$, or

ii) X and Y have each $(0,0)$ as an isolated singularity which is eigher a center of focus.

We shall use the following theorem from [3]:

THEOREM A. *Let X, $X(0,0) = (0,0)$ be a smooth planar vector field with non degenerate principal part X_P such that the origin is an isolated singularity of X_P. Then X is locally topologically equivalent to X_P modulo center-focus.*

We have:

PROPOSITION 38. *Consider the real foliation with singularities in the two dimensional real projective space defined by the differential equation (15). Let $p = [X_0 : Y_0 : 0]$ be a singular point at infinity of type $M_{1,2}$ for (12). Then the foliation in the neighborhood of p is topologically equivalent to the foliation (with singularities) in the neighborhood of the origin defined by U_P, $u_0 = Y_0/X_0$, $X_0 \neq 0$ or V_P, $v_0 = X_0/Y_0$, $Y_0 \neq 0$.*

Proof. Assume that the singular point at infinity $p = [X_0 : Y_0 : 0]$ is of type $M_{1,2}$. It has been shown in Chapter 4 that for p in the domain of chart (u, z) the foliation in the neighborhood of p is topologically equivalent to the foliation (with singularities) in the neighborhood of the origin defined by the phase curves of the vector field associated with the system of equations (23). Similarly for p in the domain of chart (v, w), the foliation in the neighborhood of p is topologically equivalent to the foliation (with singularities) in the neighborhood of the origin defined by the phase curves of the vector field associated with the system of equations (24). Since the singular point at infinity p is of type $M_{1,2}$ the conditions (28) and (29) are satisfied for the system (23) or (24). They follow from the Remarks 35 and 36 and the conditions in Table 2. The statement then follows from Theorem A and the fact that the line at infinity is invariant for the differential system (23), (24), the center-focus case being excluded. ∎

The reduced fields U_P and V_P will be used in the qualitative study for a singular point of type $M_{1,2}$. For a complete description of this type of singularity the blowing up method is required. (For a nilpotent singularity, the theorems stated as in Andronov [1] do not give a complete description of the topological nature of the singular point.) We will present in what follows the blowing up of the isolated degenerate singularity located at the origin of the vector field U_P. To keep the notations simple we will note $X = U_P$.

We first perform the affine blow up in one of the two possible directions. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be, defined by $f(u, \ell) = (u, \ell u)$. The pull back $X^1 = f^*X$ of the vector field X, restricted to $\mathbb{R}^2 - \{x{-}\text{axix}\}$, extends to the whole plane and we shall still call it X^1

$$X^1(u,\ell) = \big[-(1+u_0^2)u\ell + \frac{G''(u_0)}{2}u^2\big]\partial/\partial u + \big[\big(P_2'(u_0) - \frac{G''(u_0)}{2}\big)u\ell + (1+u_0^2)\ell^2\big]\partial/\partial\ell.$$

The only singularity of the field X^1 located on the ℓ-axis is the origin which is also an isolated degenerate singularity, therefore the procedure needs to be repeated. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $g(u,t) = (u, tu)$ and let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $h(s,\ell) = (s\ell, \ell)$. The pull pack $X^{11} = g^*X^1$ of the vector field X^1, restricted to $\mathbb{R}^2$-$\{\ell\text{-axis}\}$, after division by u can be extended to the whole plane. We shall still call it X^{11}

$$X^{11}(u,t) = \big(\frac{G''(u_0)}{2}u - (1+u_0^2)ut\big)\partial/\partial u + \big[(P_2'(u_0) - G''(u_0))t + 2(1+u_0^2)t^2\big]\partial/\partial t.$$

Clearly X^{11} gives a complete picture of the solution curves of X^1 in a neighborhood of $(0,.0)$ with the ℓ-axis removed. To give a complete picture of the integral curves for X^1 in a neighborhood of the ℓ-axis we have to use the map $h(s,\ell) = (s\ell,\ell)$. The pull back $X^{12} = h^*X^1$ of the vector field X^1, restricted to $\mathbb{R}^2 - \{u{-}\text{axis}\}$, after division by ℓ can be extended the whole plane. We shall still call it X^{12}

$$X^{12}(s,\ell) = \big[-2(1+u_0^2)s - \big(P_2'(u_0) - G''(u_0)\big)s^2\big]\partial/\partial s + \big[(1+u_0^2)\ell + \big(P_2'(u_0) - \frac{G''(u_0)}{2}\big)\ell s\big]\partial/\partial \ell.$$

It is clear that the origin $(0,0)$ is a saddle for the vector field X^{12}. Since we are using the map $h(s,\ell) = (s\ell,\ell)$ to describe the picture of the solution curves for X^1 only in a small neighborhood of the ℓ-axis we can consider the behaviour for the vector field X^{12} only around the origin $(0,0)$. The singularities of X^{11} lying on the t axis and the linear part of X^{11} are:

$$(u,t) = (0,0), \quad DX^{11}(0,0) = \begin{bmatrix} G''(u_0)/2 & 0 \\ 0 & P_2'(u_0) - G''(u_0) \end{bmatrix}$$

and

$$(u,t) = (0,t_1), \quad t_1 = \frac{P_2'(u_0) - G''(u_0)}{-2(1+u_0^2)},$$
$$DX^{11}(0,t_1) = \begin{bmatrix} P_2'(u_0)/2 & 0 \\ 0 & -P_2'(u_0) + G'')u_0) \end{bmatrix}.$$

We distinguish seven cases according to all the combinations that are possible for the eigenvalues of $X^{11}(0,0)$ and $X^{11}(0,t_1)$. We use the figures included in Appendix A to describe all possible cases for the vector field X^1. In figures A.1 – A.8 in Appendix A are shown the phase curves for X^1 in the middle, in the left for X^{11} and in the right for X^{12}. We have:

$$\begin{array}{lll} \text{Case 1.1} & 0 < P_2^1(u_0) < G''(u_0) & \text{in Figure A.1} \\ \text{Case 1.2} & P_2'(u_0) < 0 < G''(u_0) & \text{in Figure A.2} \\ \text{Case 1.3} & P_2'(u_0) < G''(u_0) < 0 & \text{in Figure A.3} \\ \text{Case 1.4} & 0 < G''(u_0) < P_2'(u_0) & \text{in Figure A.4} \\ \text{Case 1.5} & G''(u_0) < 0 < P_2'(u_0) & \text{in Figure A.5} \\ \text{Case 1.6} & G''(u_0) < P_2'(u_0) < 0 & \text{in Figure A.6} \\ \text{Case 1.7} & P_2'(u_0) = G''(u_0) > 0 & \text{in Figure A.7} \\ & \text{and} & \\ & P_2'(u_0) = G''(u_0) < 0 & \text{in Figure A.8.} \end{array} \tag{32}$$

The vector field X^1 gives a complete picture of the solution curves of X in a neighborhood of $(0,0)$ with the z-axis removed. To give a complete picture of the integral curves for X in a neighborhood of the z-axis we have to use the map $i(s,z) = (sz,z)$. the pull back $X^2 = i^*X$ of the vector field X, restricted to $\mathbb{R}^2 - \{u\text{−axis}\}$, after division by z can be extended to the whole plane. We shall still call it X^2

$$X^2(s,z) = \Big[-(1+u_0^2) - \Big(P_2'(u_0) - \frac{G''(u_0)}{2}\Big)s^2 z\Big]\partial/\partial s + P_2'(u_0)sz^2\ \partial/\partial z$$

with

$$X^2(0,0) = \big(-(1+u_0^2), 0\big).$$

Clearly the vector field X^2 has no singular points on the s-axis and the phase curves are approximately parallel with the s-axis in the neighborhood of the origin. For each case in the list (32) the corresponding blowing up at the origin for the vector field $X = U_P$ is presented in Appendix A, Figure A.9–A.15 respectively.

Before we can summarize the results obtained for points $M_{1,2}$ by the blowing up procedure we introduce some notations.

NOTATION 39. *In the topological description of a singular point p at infinity of type $M_{r,s}$ we shall use the notation $T^i_{r,s}$ where i is the index of the singular point. For an elementary singularity we shall write $T = N$ if the singular point p is a node, $T = S$ if it is a saddle and $T = SN$ if it is a saddle-node. For more complicated singular points a listing of sectors around the singular point will be given, starting from the positive u axis and going counterlockwise. An elliptic sector will be indicated by E, a hyperbolic sector will be indicated by H and a parabolic sector by P. We have*

PROPOSITION 40. *For the vector field U_P the singular point located at the origin is:*

1) a saddle $S^{-1}_{1,2}$ with 4 hyperbolic sectors if $G''(u_0)\cdot P_2'(u_0) < 0$. Three of the hyperbolic sectors are in the half plane $z > 0$ if $G''(u_0) - P_2'(u_0) > 0$ and in the half plane $z < 0$ if $G''(u_0) - P`'_2(u_0) < 0$.

2) an $EH^1_{1,2}$ if $G''(u_0) = P_2'(u_0) > 0$ or a $HE^1_{1,2}$ if $G''(u_0) = P_2'(u_0) < 0$.

3) a

$$\begin{array}{lll} PEPH^1_{1,2} & if & 0 < P_2'(u_0) < G''(u_0) \quad or \\ HPEP^1_{1,2} & if & P_2^1(u_0) < G''(u_0) < 0 \quad or \\ PHPE^1_{1,2} & if & P_2'(u_0)) < G''(u_0) < 0 \quad or \\ EPHP^1_{1,2} & if & 0 < G''(u_0) < P_2'(u_0). \end{array}$$

Clearly we have a very similar situation for the field V_P.From Proposition 38 and Proposition 40 we have a complete description for a singular point at infinity of type $M_{1,2}$. The goal for the next chapter is to give the description for the topological nature of other mixed type singularities at infinity.

6. THE TOPOLOGICAL NATURE OF SINGULAR POINTS AT INFINITY FOR A PLANAR QUADRATIC SYSTEM WITH A WEAK FOCUS

In section 2 we described the foliation with singularities F_R on the real projective space $\mathbb{R}P(2)$, associated with the differential system (2). In homogeneous coordinates the foliation F_R is defined by the polynomial differential equation $ZQ^*dX - ZP^*dY + (YP^* - XQ^*)dZ = 0$. In coordiantes (u, z) and (v, w) the foliation is described at infinity by the planar polynomial differential systems $du/dt = C^*(1, u, z) = R(u, z)$, $dz/dt = zP^*(1, u, z) = S(u, z)$ and $dv/dt = -C^*(v, 1, w) = M(v, w)$, $dw/dt = wQ^*(v, 1, w) = L(v, w)$ respectively, given by (6) and (7). We give below the geometric interpretation of the multiplicity of a point p at infinity in terms of the multiplicity (r, s) where p is of type $M_{r,s}$.

PROPOSITION 41. *For the foliation F_R consider the singular point at infinity $p = [X_0 : Y_0 : 0]$. If X_0 is not zero, $u_0 = Y_0/X_0$ (respectively Y_0 is not zero, $v_0 = Y_0/Y_0$) then*

$$I_{(u_0,0)}(R, S) = m(p) = I_p(P^*, Q^*) + I_p(C^*, Z). \qquad (33)$$

(respectively $I_{(v_0,0)}(M, L) = I_p(P^*, Q^*) + I_p(C^*, Z)$).

Proof. From relations (4) and (5) it is clear that $X^{\ell+1}R(Y/X, Z/X) = YP^* - XQ^*$, $X^{\ell+1}S(Y/X, Z/X) = ZP^*$ and $Y^{\ell+1}(X/Y, Z/Y) = YP^* - XQ^*$, $Y^{\ell+1}L(X/Y, Z/Y) = ZQ^*$ where $\ell = \max(\mathrm{degP}, \mathrm{degQ})$. Then

$$\begin{aligned}
&I_{(u_0,0)}(R, S) = I_p(C^*, ZP^*) = I_p(C^*, Z) + I_p(YP^* - XQ^*, P^*) = \\
&I_p(C^*, Z) + I_p(XQ^*, P^*) = I_p(C^*, Z) + I_p(Q^*, P^*) + I_p(X, P^*) = \\
&I_p(C^*, Z) + I_p(Q^*, P^*) \quad \text{and} \\
&I_{(v_0,0)}(M, L) = I_p(C^*, Z) + I_p(YP^* - XQ^*, Q^*) = \\
&I_p(C^*, Z) + I_p(YP^*, Q^*) = I_p(C^*, Z) + I_p(Q^*, P^*) + I_p(Y, Q^*) = \\
&I_p(C^*, Z) + I_p(Q^*, P^*).
\end{aligned}$$

∎

Now we consider the topological nature of the singular points at infinity of type other than $M_{1,2}$ for the planar quadratic system with a weak focus. The local phase portrait and the type of multiplicity $M_{r,s}(p)$ for a singular point p is not normal form dependent. Therefore we may consider the topological nature for singular points at infinity for the normal form (12) which is equivalent to the study of the local phase portraits around the origin for the system of equations (23) and (24). the local phase portraits around these singularities are illustrated in Appendix B, Figure B.1–B.6.

PROPOSITION 42. *Assume that $p = [X_0 : Y_0 : 0]$ is a singular point at infinity of type $M_{r,1}$ for the differential equation (15), $r = 1, 2$ or 3. We have:*

1) The singular point p of type $M_{1,2}$ is a saddle node $SN^0_{1,2}$ of total multiplicity $m(p) = 2$ as shown in Figure B.4.

2) The singular point p of type $M_{2,1}$ is a saddle node $N^1_{2,1}$ or a saddle $S^{-1}_{2,1}$ of total multiplicity $m(p) = 3$ as shown in Figure B.5 and Figure B.3 respectively.

3) The singular point p of type $M_{3,1}$ is saddle node $SN^0_{3,1}$ of total multiplicity $m(p) = 4$ as shown in Figure B.6.

Proof. First we consider the singular point at infinity $p = [X_0 : Y_0 : 0]$ in the domain of the chart (u, z). Since $I_p(C^*, Z) = 1$ we have the eigenvalue $\alpha = (dG/du)(u_0) \neq 0$ and the point $(u_0, 0)$, $u_0 = Y_0/X_0$ is an elementary singular point for the equation (23) with a center manifold transversal to the line at infinity. Without loss of generality we may assume that the system (23) has been transformed via a non singular linear transformation into $du/dt = \alpha u + R_2(u, z)$, $dz/dt = S_2(u, z)$ where by R_2 and S_2 we denote the parts not containing the constant or linear terms. Since $(\partial/\partial u)(\alpha u + R_2(u, v))(0, 0) = \alpha \neq 0$, the equation $\alpha u + R_2(u, z) = 0$ has a unique solution $u = h(z)$ in the neighborhood of the origin, $\alpha h(z) + R_2(h(z), z) = 0$ and $dh/dz(0) = 0$. Then $I_{(0,0)}(\alpha u + R_2(u, z), S_2(u, z)) = k \geq 2$, where $S_2(h(z), z) = a_k z^k + O(z^{k+1})$, $a_k \neq 0$. Using Proposition 41 we have $k = r + 1$ and the statement simply follow from the classification Theorem 65 given in Andronov [1]. In chart (u, w) we use equation (24), transformed into $dv/dt = \beta v + M_2(v, w)$, $dw/dt = L_2(v, w)$, $\beta = dH/dv(v_0) \neq 0$, and the proof follows exactly the same way as above, the coefficient a_k being determined by $L_2(g(w), w) = a_k w^k + Q(w^{k+1})$, $v = g(w)$, $\beta g(w) + M_2(g(w), w) = 0$, $g(0) = 0$, $dg/dw(0) = 0$.

There are no simple formulas for a_2 and a_3. We give their expression, since p is a node if $a_2 = 0$ and $\alpha a_3 > 0$ or saddle if $a_2 = 0$ and $\alpha a_3 < 0$. We have in chart (u, z):

$$\begin{aligned}
&a_2 = P_2'(u_0) + u_0 Q_2'(u_0)\\
&a_3 = \theta P_2'(u_0) + \tfrac{P''(u_0)}{2}(1 + u_0^2)^2 - \alpha(1 + u_0^2) \text{ where}\\
&\alpha = G'(u_0) = u_0 P_2'(u_0) - Q_2'(u_0) \neq 0 \text{ is the eigenvalue and}\\
&\theta = (1 + u_0^2)u_0 + \alpha^{-1}(1 + u_0^2)^2\big(P_2'(u_0) - \tfrac{G''(u_0)}{2}\big).
\end{aligned} \tag{34}$$

In chart (v, w) it is sufficient to give the values of a_2 and a_3 only when $p = [0 : 1 : 0]$. Using equation (24) we have by an easy calculation $\beta = dH/dv(0) = -m \neq 0$, $a_2 = b$ and $a_3 = -bk/m + m + a$. ∎

For singular points at infinity of type $M_{0,s}$ we have:

PROPOSITION 43. *Assume that* $p = [X_0 : Y_0 : 0]$ *is a singular point at infinity of type* $M_{0,s}$ *for the differential equation (15),* $s = 2$ *or* 3. *We have:*

1) The singular point p *of type* $M_{0,2}$ *is a saddle node* $SN^0_{0,2}$ *of total multiplicity* $m(p) = 2$ *in Figure B.1.*

2) The singular point p *of type* $M_{0,3}$ *or saddle* $S^{-1}_{0,3}$ *of total multiplicity* $m(p) = 3$ *as shown in Figure B.2.*

Proof. Assume that the singular point $p = [X_0 : Y_0 : 0]$ is the domain of chart (u, z) and $u_0 = Y_0/X_o$. In equation (23) we have $P_2(1, u_0) \neq 0$ and $dG/du(u_0) = 0$ since $s = I_p(C^*, Z) \geq 2$. Then the point $(u_0, 0)$ is an elementary singular point for the equation (23) with a center manifold tangent to the line at infinity. Using the linear transformation $u = u_1 - (1 + (u_0)^2)z_1$, $z = \nu z_1$ and renoting for simplicity, we may assume that the system (23) has been transformed into $du/dt = R_2(u, z)$, $dz/dt = \nu z + S_2(u, z)$. Since $\partial/\partial z(\nu z + S_2(u, z))(0, 0) = \nu \neq 0$ and $S_2(u, 0) = 0$ the equation $\nu z + S_2(\nu z) = 0$ admits $z = h(u) = 0$ as the unique solution in the neighborhood of the origin. It is clear that $R_2(u, h(u)) = (d^2G/du^2)(u_0)u^2/2 + (d^3G/du^3)(u_0)u^3/6$. We note that $I_p(C^*, Z) = 2$ if $(dG/du)(u_0) = 0$, $(d^2G/du^2)(u_0) \neq 0$ and $I_p(C^*, Z) = 3$ if $(dG/du)(u_0) = 0$, $(d^2G/du^2)(u_0) = 0$ and $(d^3G/du^3)(u_0) \neq 0$. Now we can apply directly the classification Theorem 65 in [1]. Note also that p is a node when $(d^2G/du^2)(u_0) = 0$ and $\nu(d^3G/du^3)(u_0) > 0$, and it is a saddle when $(d^2G/du^2)(u_0) = 0$ and $\nu(d^3G/du^3)(u_0) < 0$. In chart (u, v) the proof is similar since equations (23) and (24) have the same form. ▮

Remark 44. From Proposition 16 and Proposition 17 it follows immediately that a singular point p of type $M_{0,1}$ is a first-order elementary node $N^{-1}_{0,1}$ or saddle $S^{-1}_{0,1}$.

PROPOSITION 45. *Let* $p = [X_0 : Y_0 : 0]$ *be a singular point at infinity for the differential equation (15) of type* $M_{2,2}$ *or* $M_{1,3}$. *Then the point* p *of total multiplicity* $m(p) = 4$ *is a saddle node* $SN^0_{2,2}$ *as shown in Figure B.7 or* $SN^0_{1,3}$ *as shown in Figure B.8.*

Proof. First we consider the case $M_{2,2}$. We can assume that in chart (u, z) the singular point has coordinates $(u_0, 0)$, $u_0 = Y_0/X_0$. In equation (23) we have $P_2(1, u_0) = 0$, $dG/du(u_0) = 0$, and $(d^2G/du^2)(u_0) \neq 0$ since $I_p(C^*, Z) = 2$ and $I_p(P^*, Q^*) = 2$. Using the linear transformation $u_1 = u$, $z_1 = -(1+(u_0)^2)z$ and changing the notations for simplicity we may assume that the system (23) has been transformed into $du/dt = z + R_2(u, z)$, $dz/dt = S_2(u, z)$. We have the approximate near $(0, 0)$ the implicit function $h(u) = cu^2 + O(u^3)$, $h(u) + R_2(u, h(u)) = 0$.

Since $h(u) = -\frac{G''(u_0)}{2}u^2 + O(u^3)$ it follows that $P_2'(u_0) = 0$, i.e. $I_p(P^*, Z) \geq 2$.

Otherwise $\Psi(u) = S_2(u(h(u)) = \frac{P_2'(u_0)G''(u_0)}{2}u^3 + O(u^4)$ which is in contradiction with the fact that p is of type $M_{2,2}$.

Hence we may consider that

$$G(u_0) = G'(u_0) = P_2(1, u_0) = P_2'(u_0) = 0, \text{ and } \mathrm{G''(u_0)} \neq 0 \tag{35}$$

Doing the calculations we have:

$$\begin{aligned} &h(u) = -\tfrac{G''(u_0)}{2}u^2 + O(u^3) \\ &\Psi(u) = S_2(u, h(u)) = \left[\tfrac{u_0}{4(1+u_0^2)}G''(u_0)^2 - \tfrac{P''(u_0)G''(u_0)}{4}\right] u^4 + O(u^5) \\ &\Phi(u) = \tfrac{\partial}{\partial u}R_2(u, h(u)) + \tfrac{\partial}{\partial z}S_2(u, h(u)) = G''(u_0)u + O(u^2). \end{aligned} \tag{36}$$

According to Theorem 32 in chart (u, z) this situation occurs only when

$$\begin{aligned} &a = b = 0, \ \Delta_{k,m,n} = m^2 - 4kn = 0, \ n \neq 0 \\ &u_0 = -\tfrac{m}{2n}, \ P_2''(u_0) = 2n \text{ and } \mathrm{G''(u_0) = -m} \neq 0 \end{aligned}$$

therefore

$$\frac{u_0}{4(1+u_0^2)}G''(u_0)^2 + \frac{P''(u_0)G''(u_0}{4} = \frac{4n^3m}{4n^2+m^2} \neq 0.$$

Using Theorem 67 given in [1] it follows that p is a saddle node.

Now we consider the case $M_{1,3}$. Again we can assume that in chart (u, z) the singular point p has coordinates $(u_0, 0)$. In equation (23) we gave $P_2(1, u_0) = 0$, $(dG/du)(u_0) = 0$, $(d^2G/du^2)(u_0) = 0$ and $(d^3G/du^3)(u_0) \neq 0$ since $I_p(C^*, Z) = 3$ and $I_p(P^*, Q^*) = 1$. We have to approximate near $(0, 0)$ the implicit function $h(u) = cu^2 + O(u^3)$, $h(u) + R_2(u, h(u)) = 0$. By simple calculations we have:

$$\begin{aligned} &h(u) = -\tfrac{G''(u_0)}{6}u^3 + O(u^4) \\ &\Psi(u) = S_2(u, h(u)) = -\tfrac{P_2'(u_0)G''(u_0)}{6}u^4 + O(u^5) \\ &\Phi(u) = \tfrac{\partial}{\partial u}R_2(u, h(u)) + \tfrac{\partial}{\partial z}S_2(u, h(u)) = P_2'(u_0)u + O(u^2). \end{aligned} \tag{37}$$

Since the singular point p is of type $M_{1.3}$ we have $I_p(P^*, Q^*) + I_p(C^*, Z) = 4$. From Proposition 41 follows that necessarily $P_2'(u_0) \neq 0$ i.e. $I_p(P^*, Z) \neq 0$, otherwise it would result that $I_p(P^*, Q^*) + I_p(C^*, Z) > 4$. Applying again Theorem 67 [1] it follows that p is a saddle node.

Points at infinity of type $M_{2,2}$ and $M_{1,3}$ occur also when $p = [0 : 1 : 0]$. From Theorem 32 it follows that $p = [0 : 1 : 0]$ is a singular point of type $M_{2,2}$ only in the case when $H(0) = n = 0$, $(dH/dv)(0) = -m = 0$, $(d^2H/dv^2)(0) = -2k \neq 0$, $Q_2'(0) = b = 0$, $Q_2''(0) = 2a \neq 0$. Also the point $p = [0 : 1 : 0]$ is a singular point of type $M_{1,3}$ in the case $bK_\lambda \neq 0 = n = m = k - b$ when $I_p(C^*, Z) = 3$ $I_p(Q^*, Z) = 1$, therefore $(dH/dv)(0) = -m = 0$, $(d^2Hdv^2)(0) = 2(b-k) = 0$, $(d^3H/dv^3)(0) = 6a \neq 0$ and $Q_2'(0) = b \neq 0$. Since the equations (23) and (24) are very similar we can write immediately the corresponding formulas with (36) and (37) for equation (24). We may assume that the system (24) has been transformed into $dv/dt = z + M_2(v,w)$, $dw/dt = L_2(v,w)$. If $p = [0 : 1 : 0]$ is of type $M_{2,2}$ we have:

$$\begin{aligned} h(v) &= -\tfrac{H''(0)}{2}v^2 + O(v^3) = kv^2 + O(v^3) \\ \Psi(v) &= L_2(v, h(v)) = -\tfrac{Q''(0)H''(0)}{4}v^4 + O(v^5) = akv^4 + O(v^5) \\ \Phi(v) &= \tfrac{\partial}{\partial v}M_2(v,h(v)) + \tfrac{\partial}{\partial z}L_2(v,h(v)) = H''(0)v + O(v^2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad = -2kv + O(v^2). \end{aligned} \tag{38}$$

For $p = [0 : 1 : 0]$ of type $M_{1,3}$ we have:

$$\begin{aligned} h(v) &= -\tfrac{H'''(0)}{6}v^3 + O(v^4) = -av^3 + O(v^4) \\ \Psi(v) &= L_2(v, w(v)) = -\tfrac{Q_2'(0)H'''(0)}{6}v^4 + O(v^5) = -abv^3 + O(v^5) \\ \Phi(v) &= \tfrac{\partial}{\partial v}M_2(v,w(v)) + \tfrac{\partial}{\partial w}L_2(v,w(v)) = Q_2'(0)v + O(v^2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad = bv + O(v^2). \end{aligned} \tag{39}$$

As above, according to Theorem 67 in [1] the statement in this proposition is proven. ▮

7. CONCLUSION

In this work we were interested in the behaviour of phase curves of planar vector fields in the neighbourhood of infinity. We have two ways of regarding this infinity:

1) The Poincaré compactification of the vector field on the sphere, i.e. we place the plane (x, y) on the plane $Z = 1$ in the three-dimensional space (X, Y, Z) and we consider the central projection of the vector field on the upper hemisphere H_+ of $S^2 = \{(X, Y, Z) \mid X + Y + Z = 1\}$, and also on the lower hemisphere H_-. It is well known that there is an analytic vector field on the whole sphere whose oriented phase curves coincide with those on H_+ obtained above. We can view the equator of the sphere as the infinity in this compactification.

2) We may also regard as infinity for our planar vector field, the line $Z = 0$ in the real projective plane $\mathbb{R}P(2)$ on which we compactify the foliation with singularities associated to the planar vector field on $\mathbb{R}^2$.

On the other hand, to any real planar vector field, we may associate its complex foliation $F_{\mathbb{C}}$ on the complex projective plante $\mathbb{C}P(2)$. This point of view is very advantageous and it is the one we take in order to introduce our divisors. Indeed, it is known that Bezout's theorem on the number of intersection points of two algebraic curves (cf.[5]) is obtained over the complex field and in the complex projective plane. For this theorem the notion of intersection number of two curves at a point is used.

In this work we made use of this concepts to introduce two specific divisors on the line $Z = 0$, purely geometric objects, globally encoding the multiplicities of all the singularities at infinity of a given vector field. To our knowledge, these divisors were introduced for the first time in our article [9] (a preliminary version of the present article). We are interested in global problems on planar vector fields and these divisors turned out to be very useful for this purpose because they organise the global information about multiplicities of singularities at infinity in a clear, coherent way.

We computed here these divisors for the whole class of quadratic systems with a weak focus with respect to the normal form (12). However, the types of these divisors are independent of any specifically chosen normal form. Thus we find in our Table 2 the divisors : $\begin{pmatrix} 2 \\ 2 \end{pmatrix} p_{a,b} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} p_{a,-k}$ and $\begin{pmatrix} 2 \\ 2 \end{pmatrix} p_{m,n} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} q_0$. These divisors are of the same type, i.e. they are both of the form $\begin{pmatrix} 2 \\ 2 \end{pmatrix} p + \begin{pmatrix} 0 \\ 1 \end{pmatrix} q$. If a system has the divisor of this form, then any affine coordinate change will not change the form of this divisor which is thus an affine invariant. These invariants have a clear global geometric content as we have shown in Section 4. We also showed the extent of their role in the determination of the local phase portraits around singularities (cf. Sections 5 and 6).

From the calculation of these divisors, some global results were obtained for the whole class of quadratic systems with a weak focus, (for example cf. Theorem 33 or Corollary 34).

The divisors will be used in the study of the classes of systems with weak foci of specific order. (Results on the class of quadratic systems with a weak focus of third order, using the notion of divisor on the line at infinity, has recently been obtained by J. Llibre and the first author).

This work forms a bridge between classifications results (which are chart dependent) about low degree polynomial differential systems and works of the school of Sibirsky which are based on the theory of algebraic invariants.

Although we used the normal form (12), the types of the divisors we obtained are affine invariants of the systems, which have geometric content. While researchers from Sibirsky's school produced results which are chart independent, these results are formulated in terms of algebraic invariants and commitants and these are in most cases without a clear geometric meaning. This work is an attempt to bridge this gap. We fully realise the value of the work of Sibirsky and of his school: they built a powerful computational apparatus, applicable to any particular normal form. But this machinery needs an infusion of geometrical concepts which will make it geometrically more meaningful and also more user friendly. Our hope is that this work forms a basis for future contributions, linking three distinct currents in classification problem: the chart dependent studies of quadratic and cubic vector fields, our geometric approach and the computationally powerful methods of the school of Sibirsky. Unifying these currents is one of our future goals.

The authors wish to thank J.C. Artés and N. Vulpe for reading the first part of this manuscript and for observations.

APPENDIX A

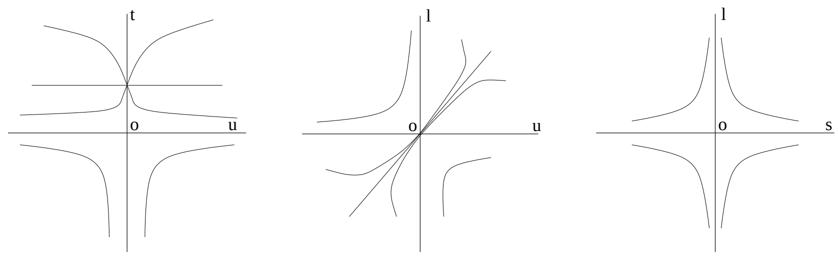

FIG. A.2.

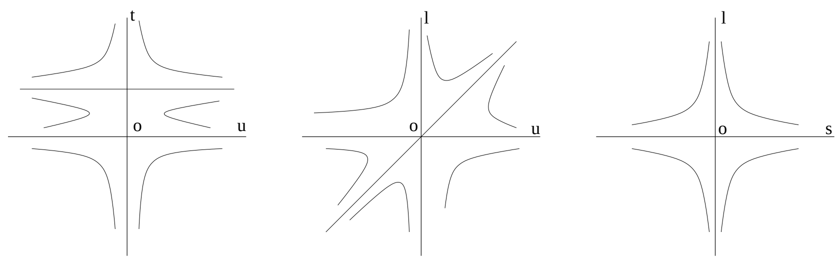

FIG. A.3.

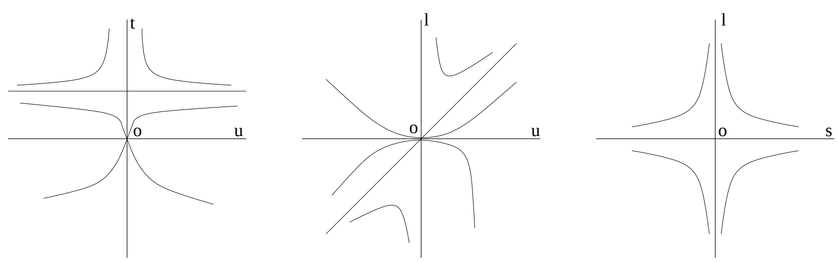

FIG. A.4.

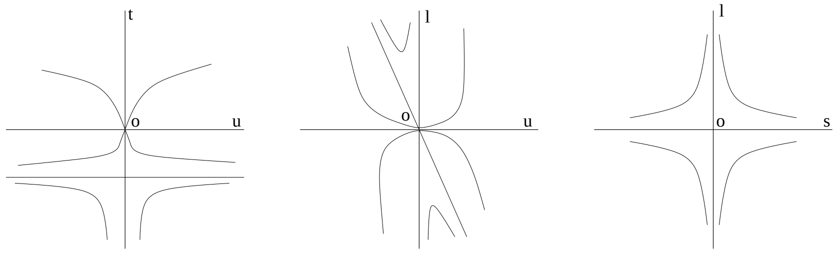

FIG. A.5.

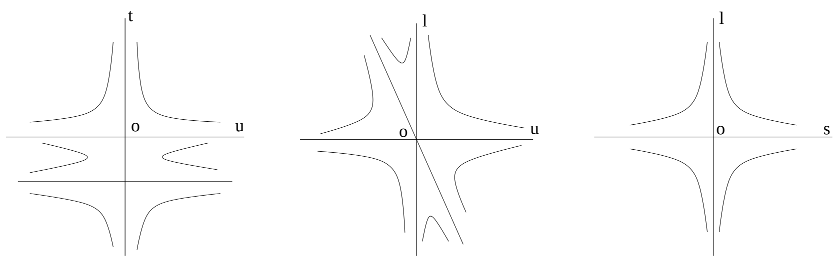

FIG. A.6.

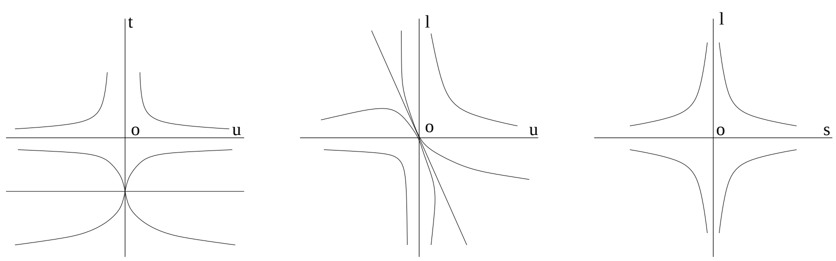

FIG. A.7.

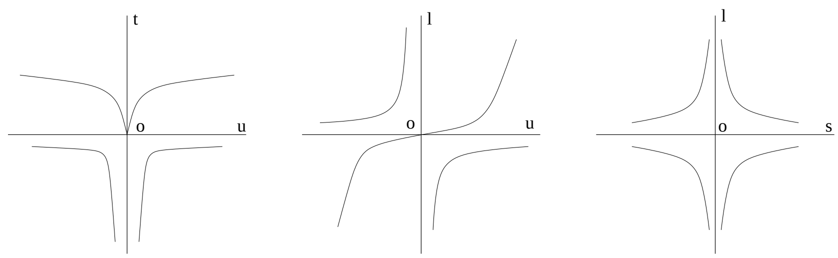

FIG. A.8.

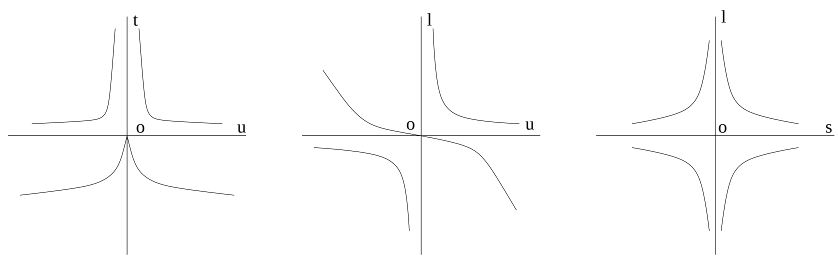

FIG. A.9.

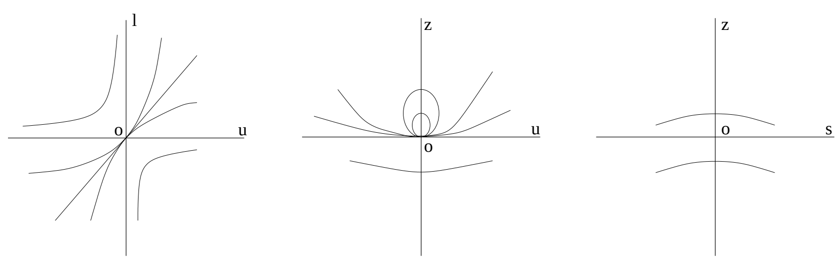

FIG. A.10.

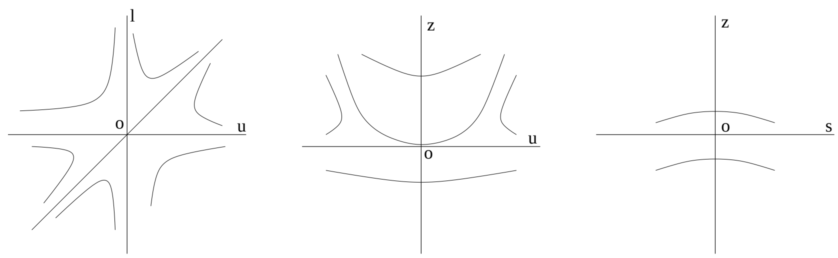

FIG. A.11.

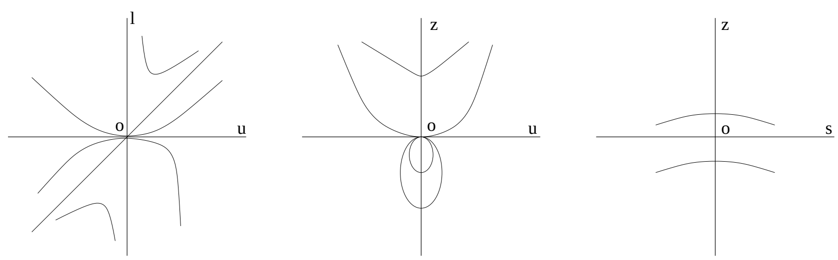

FIG. A.12.

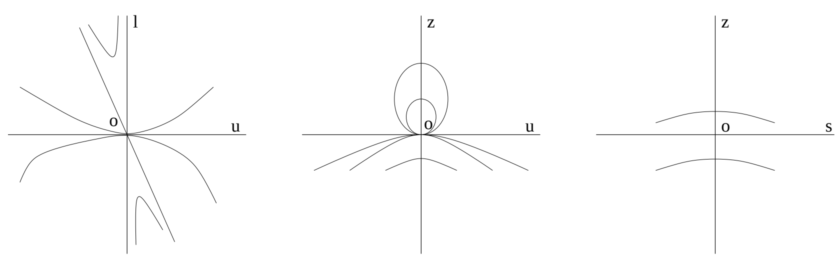

FIG. A.13.

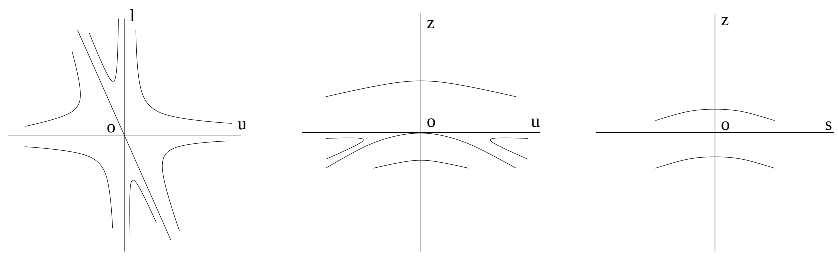

FIG. A.14.

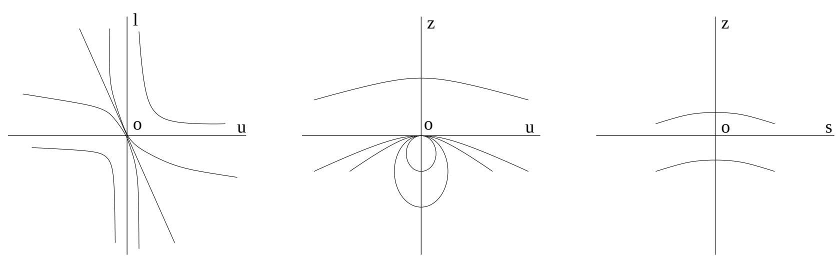

FIG. A.15.

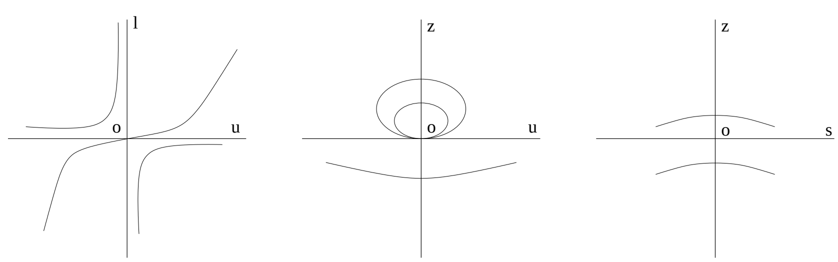

FIG. A.16.

APPENDIX B

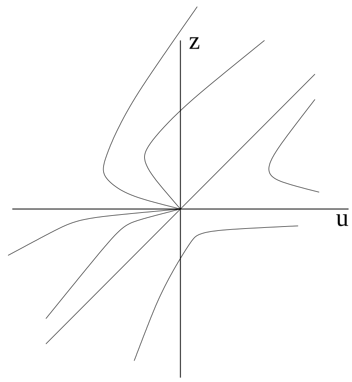

FIG. B.1.

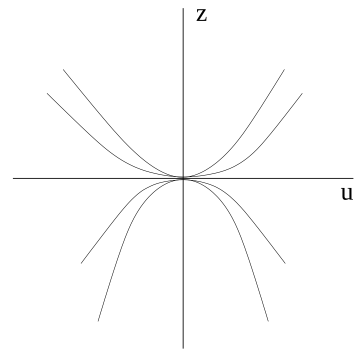

FIG. B.2.

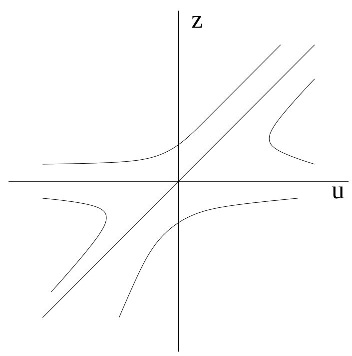

FIG. B.3.

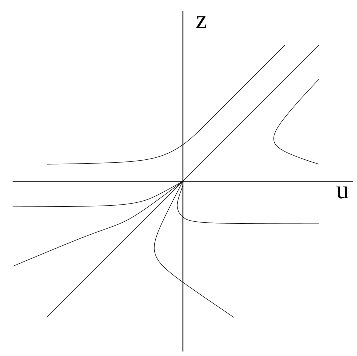

FIG. B.4.

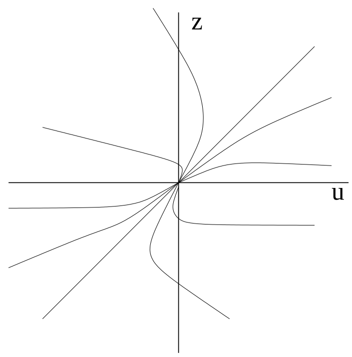

FIG. B.5.

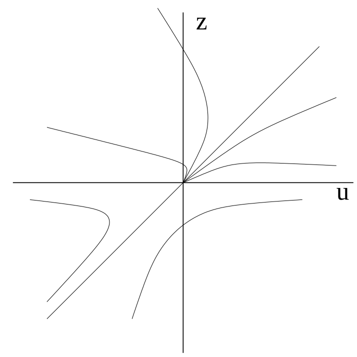

FIG. B.6.

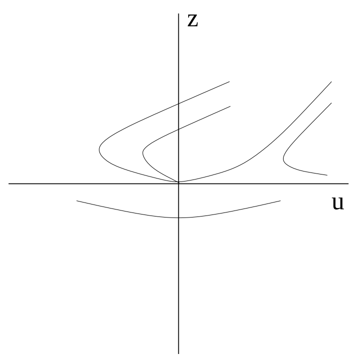

FIG. B.7.

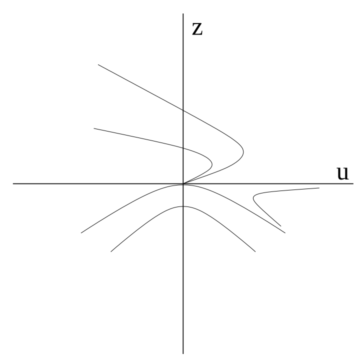

FIG. B.8.

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