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Solution of the Problem of the Centre for a Cubic Differential System with Three Invariant Straight Lines *

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For a cubic differential system $\dot{x} = y(1+x)(1-x+cx+fy)$, $\dot{y} = -(x+gx^2+dxy+by^2+sx^3+qx^2y+nxy^2+ly^3)$ we find coefficient conditions for the existence of three invariant straight lines. We resolve the problem of the centre in each of these conditions.

Key Words: Cubic systems of differential equations, center-focus problem, invariant algebraic curves, integrability.

1. INTRODUCTION

In this paper we consider the polynomial system of differential equations

$$\dot{x} = y + \sum_{j=2}^{n} P_j(x,y) \equiv P(x,y),\ \ \dot{y} = -\Big(x + \sum_{j=2}^{n} Q_j(x,y)\Big) \equiv -Q(x,y), \quad (1)$$

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where $P_j,\ Q_j,\ j=\overline{2,n}$ are homogeneous polynomials of degree j. Coefficients and variables are assumed to be real in (1). The origin $O(0,0)$ is a singular point of a centre or focus type for (1) ([13, 12]).

The origin is a center for system (1) if and only if in a neighborhood of $(0,0)$ system (1) has a C^ω nonconstant strong first integral ([12, 3]). Also, $O(0,0)$ is a centre iff (1) has in a neighborhood of $O(0,0)$ a holomorphic integrating factor of the form $\mu = 1+\sum \mu_j(x,y)$ ([1]).

There exists a formal power series $F(x,y)=\sum F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (1) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}:\ dF/dt=\sum\limits_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j$. The polynomials $\{(x^2+y^2)^j,\quad j=\overline{2,\infty}\}$ can be replaced with certain polynomials of even degree $\{\Psi_{2j}(x,y),\quad j=\overline{2,\infty}\}$ by condition $\int\limits_0^{2\pi}\Psi_{2j}(\cos\phi,\sin\phi)d\phi\neq 0,\ \forall j$. For these polynomials there exist a formal series $\Phi(x,y)=\sum\Phi_{2j}(x,y)$ such that $d\Phi/dt=\sum\limits_{j=2}^{\infty}\Lambda_{j-1}\Psi_{2j}(x,y)$ ([22]). Quantities $\Lambda_j(L_j),\ j=\overline{1,\infty}$ are polynomials in the coefficients of system (1). The quantities $L_j,\ j=\overline{1,\infty}$ are called the Liapunov quantities.

The origin is a centre for system (1) if and only if $\Lambda_j=0,\ j=\overline{1,\infty}$ (equivalent with $L_j=0,\ j=\overline{1,\infty}$). By the Hilbert's basis theorem there exists a natural number N such that the infinite system $\Lambda_j=0,\ j=\overline{1,\infty}$ is equivalent with a finite system $\Lambda_j=0,\ j=\overline{1,N}$. The number N is known only for quadratic systems $(n=2)\quad N=3$ ([7]). For cubic system $(n=3)$ the problem of the centre was found to be rather difficult. It is solved only in some particular cases. For example, if $n=3,\ P_2(x,y)\equiv 0,\ Q_2(x,y)\equiv 0$, then $N=5$ ([17, 23]). In this paper the cubic systems with invariant straight lines will be divided in five classes and the problem of the centre will be solved in one of these classes. We shall assume that the number of invariant straight lines is not less than three.

Various problems for polynomials systems of differential equations with invariant straight lines were investigated in ([2], [4]–[6], [8]–[11], [14]–[16], [18]–[20]).

2. INVARIANT ALGEBRAIC CURVES, LIAPUNOV QUANTITIES, CENTRE

An algebraic curve $f(x,y)=0$ (real or complex) is said to be an invariant curve of system (1) if there exists a polynomial $K(x,y)$ such that $P\cdot\partial f/\partial x - Q\cdot\partial f/\partial y = K\cdot f$. The polynomial K is called the cofactor of the invariant algebraic curve $f=0$. We shall consider only algebraic curves $f=0$ with f irreducible.

We shall say that ($f_j,\ j=\overline{1,M};\ L=N$) is *ILC* (I - invariant algebraic curves, L - Liapunov quantities, C - centre) for (1), if the existence of M algebraic curves $f_j(x,y)=0$ and the vanishing of the focal values $L_\nu,\ \nu=\overline{1,N}$ implies the origin $O(0,0)$ to be a centre for (1).

Denote by $[a]$ the integer part of a number a.

Theorem 1 ([21]). *$\left(f_j(x,y),\ f_j(0,0)\neq 0,\ j=\overline{1,\left[\frac{n(n+1)}{2}\right]-\left[\frac{n+1}{2}\right]};\ L=\left[\frac{n-1}{2}\right]\right)$ is ILC for system (1) ($n>2$).*

Corollary 2. *($f_j(x,y),\ f_j(0,0)\neq 0,\ j=\overline{1,4};\quad L=1$) is ILC for cubic system ($n=3$).*

In the case of invariant straight lines

$$1+Ax+By=0,\ A,B\in\mathbb{C},\ |A|+|B|\neq 0\ . \tag{2}$$

Corollary 2 can be formulated in the following way

Corollary 3. *($1+A_jx+B_jy,\ j=\overline{1,4};\quad L=1$) is ILC for cubic system ($n=3$).*

As homogeneous invariant straight lines $Ax+By=0$ the system (1) can have only the lines $x+iy=0$ and $x-iy=0$, $i^2=-1$.

Theorem 4 ([6, 20]). *($x\pm iy,\ 1+A_jx+B_jy,\ j=1,2;\quad L=2$) and ($x\pm iy,\ 1+Ax+By;\quad L=7$) are ILC for cubic system ($n=3$).*

Remark 5. From Corollary 2 and Theorem 4 follows that if a cubic system has four invariant straight lines (real, complex, real and complex) then the order of a weak focus is at most 2.

In the case of the cubic system with three invariant straight lines of the form (2) of which two are parallel, we have

Theorem 6. *($l_j=1+A_jx+B_jy,\ j=\overline{1,3},\ l_1||l_2;\quad L=5$) is ILC for cubic system ($n=3$).*

It should be noted that cubic system ($n=3$) cannot have more than two parallel invariant straight lines of the form (2), that is if $l_1||l_2||l_3$, then $l_3\equiv l_1$ or $l_3\equiv l_2$.

3. CLASSIFICATION OF CUBIC SYSTEMS WITH ONE REAL INVARIANT STRAIGHT LINE

For $n=3$, the system (1) can be written as follows

$$\begin{aligned}\dot{x}&=y+ax^2+cxy+fy^2+kx^3+mx^2y+pxy^2+ry^3\equiv P(x,y),\\ \dot{y}&=-(x+gx^2+dxy+by^2+sx^3+qx^2y+nxy^2+ly^3)\equiv -Q(x,y)\end{aligned} \tag{3}$$

in which all variables and coefficients are assumed to be real.

The straight line (2) is an invariant straight line for (3) if and only if there exist such numbers D, G, M, R, S, that

$$A \cdot P(x,y) - B \cdot Q(x,y) \equiv (1 + Ax + By)(Dx + Gy + Mx^2 + Rxy + Sy^2).$$

By equating the coefficients of monomials in x and y we reduce this identity to a system of nine equations for the unknowns A, B, D, G, M, R, S. We find that $D = -B,\ G = A,\ M = aA - gB + AB,\ R = cA - dB + B^2 - A^2,\ S = fA - bB - AB$, and A, B are the solutions of the system

$$\begin{aligned} &F_1(A,B) = A^2B + aA^2 - gAB - kA + sB = 0,\\ &F_2(A,B) = AB^2 - fAB + bB^2 + rA - lB = 0,\\ &F_3(A,B) = B^3 - 2A^2B + fA^2 + (c-b)AB - dB^2 - pA + nB = 0,\\ &F_4(A,B) = A^3 - 2AB^2 - cA^2 + (d-a)AB + gB^2 + mA - qB = 0. \end{aligned} \tag{4}$$

Let us assume that system (3) has at least one real invariant straight line. Via a rotation of axis about the origin we make this line parallel to the axis Oy. To find the equation of this line, we put in (4) $B = 0$. In this connection, taking account of $A \neq 0$, the system (4) becomes equivalent to the following series of equalities:

$$aA - k = A^2 - cA + m = fA - p = r = 0. \tag{5}$$

It is easily seen from (5) that there are five classes of systems of type (3) which have invariant straight lines parallel to the axis Oy:

I $a = f = k = p = r = m = 0,\ c \neq 0.$
The straight line is $x = -c^{-1}$;
II $a = f = k = p = r = 0,\ m = c^2/4,\ c \neq 0.$
The straight line is $x = -2c^{-1}$;
III $a = f = k = p = r = 0,\ m(c^2 - 4m) \neq 0.$
The straight lines are $x_{1,2} = (-c \pm \sqrt{c^2 - 4m})/(2m)$;
IV $a = k = r = 0,\ pf \neq 0,\ m = p(cf - p)f^{-2}.$
The straight line is $x = -fp^{-1}$;
V $r = 0,\ ak \neq 0,\ p = kfa^{-1},\ m = k(ac - k)a^{-2}.$
The straight line is $x = -ak^{-1}$.

In the class I $(x \pm iy,\ 1 + cx;\ L = 1)$ and $(1 + cx,\ 1 + A_jx + B_jy,\ j = 1,2;\ L = 1)$ are ILC ([5]);
- in the class II $(x \pm iy,\ 2 + cx;\ L = 1)$ and $(2 + cx,\ 1 + A_jx + B_jy,\ j = 1,2;\ L = 1)$ are ILC ([5]);
- in the class III $(x \pm iy,\ 1 + \frac{1}{2}(c \pm \sqrt{c^2 - 4m})x;\ L = 1)$ and $(1 + \frac{1}{2}(c \pm \sqrt{c^2 - 4m})x,\ 1 + Ax + By;\ L = 5)$ are ILC ([6]);

- in the class IV $(x \pm iy,\ 1+\frac{p}{f}x;\ L=5)$ is ILC ([20]);
- in the class V $(x \pm iy,\ 1+\frac{k}{a}x;\ L=7)$ is ILC ([20]).

In this paper we shall prove that $(1+\frac{p}{f}x,\ 1+A_jx+B_jy,\ j=1,2;\ L=5)$ is ILC in the class IV.

As appears from the above to solve completely the problem of the centre for cubic systems with at least three invariant straight lines, it remains to investigate the class V.

4. CONDITIONS FOR THE EXISTENCE OF THREE INVARIANT STRAIGHT LINES IN THE CLASS IV

Without loss of generality we can assume that $p=f$. In the class IV, the cubic system (3) is of the following form

$$\begin{aligned} &\dot{x} = y(1+x)(1-x+cx+fy), \quad f \neq 0,\\ &\dot{y} = -(x+gx^2+dxy+by^2+sx^3+qx^2y+nxy^2+ly^3), \end{aligned} \tag{6}$$

and the algebraic system (4) with condition $B \neq 0$ is equivalent to the following system

$$\begin{aligned} F_1(A) &= A^2 - gA + s = 0,\\ F_2(A,B) &= (A+b)B - fA - l = 0,\\ F_3(A,B) &= B^3 - dB^2 - (2A^2+(b-c)A+n)B + fA(A-1) = 0,\\ F_4(A,B) &= (g-2A)B^2 + (dA-q)B + A^3 - cA^2 + (c-1)A = 0. \end{aligned} \tag{7}$$

The first focal value for (6) looks

$$L_1 = q + 3l - d(b+g) + f(c-2b-1). \tag{8}$$

In order to simplify the formulas it is convenient to introduce the following notation: $\nu = b+g$, $\lambda = g+2b$, $\beta = s+b\nu$, $\gamma = f^2s+fgl+l^2$, $\delta = b+c-1$, $\tau = \nu - 1$.

4.1. Case $s + b\nu = 0$

Suppose first that $s=-b\nu$, then $F_1(A) = (A+b)(A-\nu)$. For $A=-b$ and $A=\nu$ we obtain respectively from (7) that

$$\begin{aligned} &l = bf,\ F_{31}(B) = B^3 - dB^2 + (n-b^2-bc)B + bf(b+1) = 0,\\ &F_{41}(B) = \lambda B^2 - (q+bd)B - b\delta(b+1) = 0. \end{aligned} \tag{9}$$

$$\begin{aligned} F_{22}(B) &= \lambda B - f\nu - l = 0, \\ F_{32}(B) &= B^3 - dB^2 + (n + \nu(c - b - 2\nu))B + f\nu\tau = 0, \\ F_{42}(B) &= \lambda B^2 + (q - d\nu)B + \nu(\nu(c - \nu) + 1 - c) = 0. \end{aligned} \tag{10}$$

If $\lambda = 0$, then systems (9) and (10) are equivalent. We have

$$\begin{aligned} F_{31}(B) = B^3 - dB^2 + (n - b^2 - bc)B + bf(b + 1) = 0, \\ F_{41}(B) = (q + bd)B + b\delta(b + 1) = 0. \end{aligned} \tag{11}$$

Assume that $F_{41}(B) \not\equiv 0$, if $F_{41}(B) \equiv 0$ the system (7) has at most two invariant straight lines. In order that (6) have three invariant straight lines, it is necessary and sufficient that a cubic equation $F_{31}(B) = 0$ have exactly two distinct roots. We shall distinguish here only the cubic systems (6) with three invariant straight lines which are not the limit cases of the systems (6) with four invariant straight lines. We have the following two cases:

1)

$$b = g = l = q = s = 0, \ c = 1. \tag{12}$$

The invariant straight lines are $1 + x = 0, \quad 1 + \frac{1}{2}(d \pm \sqrt{d^2 - 4n})y = 0.$

2)

$$b = -1, \ g = 2, \ l = -f, \ q = d, s = 1. \tag{13}$$

The invariant straight lines are $1 + x = 0, \quad 1 + x + \frac{1}{2}(d \pm \sqrt{d^2 + 4(1 - c - n)})y = 0.$

From (9) and (10) it is seen that the necessary condition for system (6) to have three invariant straight lines is $l = bf$.

Let $\lambda \neq 0$. The equation of the third degree with respect to $B : F_{31}(B) = 0$ can be reduced to an equation of the first degree by equation $F_{41}(B) = 0$

$$\begin{aligned} F^*_{31}(B) = \ & [n\lambda^2 - bc\lambda\tau + (q + bd)(q - d\nu) - b\lambda(1 + b\nu)]B + \\ & b(b + 1)[f\lambda^2 + \delta(q - d\nu)]. \end{aligned} \tag{14}$$

If $F^*_{31}(B) \equiv 0$ we obtain the following five series of conditions on the coefficient of system (6) to have three invariant straight lines:

3)

$$b = l = s = 0, \ c = g + 1. \ n = f(d - f), \ q = g(d - f). \tag{15}$$

The invariant straight lines are $1+x=0$, $1+(d-f)y=0$, $1+gx+fy=0$.

4)

$$\begin{aligned} b &= -1,\ c=g,\ l=-f,\ n=1-g+fd-f^2, \\ q &= 2f-d+g(d-f),\ s=g-1. \end{aligned} \tag{16}$$

The invariant straight lines are $1+x=0$, $1+x+(d-f)y=0$, $1+(g-1)x+fy=0$.

5)

$$\begin{aligned} l &= bf,\ s=-b\nu,\ q=d\nu-f\lambda^2\delta^{-1}, \\ n &= (b\lambda\delta^3-b^2\delta^3-b\delta^3+b\delta^2\lambda+df\delta\lambda^2-f^2\lambda^3)/(\delta^2\lambda). \end{aligned} \tag{17}$$

The invariant straight lines are $1+x=0$, $1-bx+B_jy=0$, where B_j, $j=1,2$ are found from the equation $\lambda B^2-(d\lambda-f^2\lambda^2\delta^{-1})B-b\delta(b+1)=0$.

6)

$$\begin{aligned} l &= bf,\ s=-b\nu, \\ d &= f+\tau(\delta-\nu)f^{-1}-f(b+1)\tau^{-1}, \\ n &= 2b\nu-fd\tau^{-1}-f^2(b\nu+1)\tau^{-2}, \\ q &= b\nu\tau f^{-1}-bf-f(b+1)\nu\tau^{-1}. \end{aligned} \tag{18}$$

The invariant straight lines are $1+x=0$, $1+\nu x+fy=0$, $1-bx-f(b+1)\tau^{-1}y=0$.

7)

$$\begin{aligned} d &= f+(b+c-\nu)\nu f^{-1}-bf\nu^{-1}, \\ l &= bf,\ s=-b\nu,\ n=(1+2b)\nu-bf^2\nu^{-1}, \\ q &= (b\nu+c-1)\nu f^{-1}-2bf. \end{aligned} \tag{19}$$

The invariant straight lines are $1+x=0$, $1+\nu x+fy=0$, $1-bx-bf\nu^{-1}y=0$.

4.2. Case $s+b\nu\neq 0$

Let us solve the system (7) assuming that $s+b\nu\neq 0$. From $F_1(A)=0$ follows that $A+b\neq 0$ and from $F_2(A,B)=0$ we find B: $B=(fA+l)/(A+b)$. Increasing by $F_1(A)=0$ degrees of equations $F_3(A,(fA+l)/(A+b))$ and $F_4(A,(fA+l)/(A+b))$ we obtain respectively equations of the first degree $F_3^*(A)=0$, $F_4^*(A)=0$. In order that cubic system (6) have three invariant straight lines, it is necessary that $F_3^*(A)\equiv 0$, $F_4^*(A)\equiv 0$. The solution of these identities leads to the following three series of conditions on the coefficients of system (6):

8)

$$\begin{aligned} d &= f - bf\nu^{-1} + (c-g)\nu f^{-1},\ l = -fs\nu^{-1}, \\ n &= \nu - 2s + f^2 s\nu^{-2},\ q = 2fs\nu^{-1} + (c-s-1)\nu f^{-1}. \end{aligned} \tag{20}$$

The invariant straight lines are $1+x=0,\ 1+A_jx+f(A_j-s\nu^{-1})(b+A_j)^{-1}y=0$, where $A_j,\ j=1,2$ obey the equation $A^2-gA+s=0$.

9)

$$\begin{aligned} d &= f(g-2)\tau^{-1} + (c-g-1)\tau f^{-1}, \\ l &= -f(b+s)\tau^{-1}, \\ n &= 1-c+g-2s+f^2(s-g+1)\tau^{-2}, \\ q &= f(2s-g)\tau^{-1} - s\tau f^{-1}. \end{aligned} \tag{21}$$

The invariant straight lines are $1+x=0,\ 1+A_jx+f(A_j-(b+s)\tau^{-1})(A_j+b)^{-1}y=0$, where $A_j,\ j=1,2$ are the solution of the equation $A^2-gA+s=0$.

10)

$$\begin{aligned} c &= \nu+1, \\ d &= (fs+gl-l)\beta\gamma^{-1} + (bfg+2bl+2fs+gl)\beta^{-1}, \\ q &= (fs\lambda+gl\nu-2ls)\beta^{-1} + s(f+l)\beta\gamma^{-1}, \\ n &= \gamma\beta^{-1} + (bf^2s-2l^2s+fls(\lambda-2)+l^2\nu(g-1))\gamma^{-1}. \end{aligned} \tag{22}$$

The invariant straight lines are $1+x=0,\ 1+A_jx+(l+fA_j)(b+A_j)^{-1}y=0$, where $A_j,\ j=1,2$ are the solution of the equation $A^2-gA+s=0$.

We note here that system (6) and conditions (22) can be obtained respectively from the system

$$\begin{aligned} \dot{x} &= y(cfx+f^2y+f-px)(f+px)/f^2, \\ \dot{y} &= -(x+gx^2+dxy+by^2+sx^3+qx^2y+nxy^2+ly^3). \end{aligned} \tag{23}$$

and the conditions

$$\begin{aligned} p =&\ f(c-\nu), \\ d =&\ (l\lambda+bfg+2fs)\beta^{-1} + \beta(bl-cl+fs+2gl)\gamma^{-1}, \\ q =&\ (fs\lambda+gl\nu-2ls)\beta^{-1} - s\beta(f\nu-cf-l)\gamma^{-1}, \\ n =&\ \gamma\beta^{-1} + (s(bcf^2+bfl-2cfl-2l^2)+ \\ &\ \nu(2l^2\nu-bf^2s-bl^2-cl^2+3fls))\gamma^{-1}, \end{aligned} \tag{24}$$

if we assume that $p=f$. The system (23) has the invariant straight lines $1+(c-b-g)x=0,\ 1+A_jx+(l+fA_j)(b+A_j)^{-1}y=0$, where $A_j,\ j=1,2$ are the solutions of the equation $A^2-gA+s=0$.

5. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A CENTRE

LEMMA 7. *The following are sufficient conditions for the origin to be a centre for system (6):*

1) $\lambda = \delta = 0,\ l = bf,\ q = -bd,\ s = -b\nu;$

2) $l = bf$, $q = d\nu - f\delta$, $s = -b\nu$, $n = fd - f^2 + \nu(1 + \lambda - c)$,
$\delta^2 - \lambda^2 = 0,\ (\delta - \lambda)(f^2 + \nu(1 - \nu)) = 0,$
$\lambda(\delta - \lambda)(\nu(b + \lambda) - b + fd)) = 0.$

3) $c = 1 - b$, $d = 3f$, $g = -2b$,
$l = -f(2b^2 + b + 2f^2)/(4f^2 + (2b + 1)^2),$
$n = (2b^2 + 5b + 2f^2 + 2)(2b^2 + b + 2f^2)/(4f^2 + (2b + 1)^2),$
$q = 3f(2b^2 + b + 2f^2)/(4f^2 + (2b + 1)^2),$
$s = (2b^2 + b + 2f^2)^2/(4f^2 + (2b + 1)^2).$

Proof. Suppose that for (6) conditions 1) hold, then $L_1 = 0$ (see (8)) and $F_{41}(B) \equiv 0$ (see (11)). The system (6) has the invariant straight lines $1 + x = 0$, $1 - bx + B_j y = 0$, where B_j are solutions of the equation $F_{31}(B) = 0$ (see (11)). If $b(b+1) \neq 0$ and the resultant $R(F_{31}(B), F'_{31}(B))$ of the polynomial $F_{31}(B)$ and its derivative $F'_{31}(B)$ is not equal to zero, then equation $F_{31}(B) = 0$ has three distinct roots ($B \neq 0$) and hence the system (6) has four invariant straight lines. By Corollary 3 the origin $(0, 0)$ is a centre for system (6). Since the centre variety is closed in the space of coefficients of the system (6), then $(0, 0)$ will be a centre and in the case when $R(F_{31}(B), F'_{31}(B)) = 0$.

If $b = 0$, we obtain conditions (12) and the system (6) has a Darboux integrating factor of the form

$$\mu(x, y) = l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3}, \tag{25}$$

where $l_1 = 1 + x$, $l_{2,3} = 1 + \frac{1}{2}(d \pm \sqrt{d^2 - 4n})y$ and $\alpha_1 = \alpha_2 = \alpha_3 = -1$.

If $b = -1$, we get conditions (13) (see (8)) and the system (6) has an integrating factor of the form (25) with $l_1 = 1 + x$, $l_{2,3} = 1 + x + \frac{1}{2}(d \pm \sqrt{d^2 - 4(n + 1)})y$ and $\alpha_1 = -2$, $\alpha_2 = \alpha_3 = -1$.

Under conditions 2) we have that $L_1 = 0$ (see (8)), $F_{32}(f) = F_{42}(f) = 0$ (see (10)) and $F^*_{31}(B) \equiv 0$ (see (14)). If $b\lambda(b + 1) \neq 0$ and the resultant $R(F_{41}(B), F'_{41}(B)) \neq 0$ (see (9)) then system (6) has four invariant straight lines $1 + x = 0$, $1 + \nu x + fy = 0$, $1 - bx + B_j y = 0$, where B_j, $j = 1, 2$ are solution of the equation $F_{41}(B) = \lambda B^2 - (f\delta - 2bd)B - b\delta(b + 1) = 0$. By Corollary 3 the system (6) has a centre at $(0, 0)$. The origin $(0, 0)$ will be a centre and in the cases when at least one of these two equalities $b\lambda(b + 1) = 0,\ R(F_{41}(B), F'_{41}(B)) = 0$ holds.

In the case 3) the system (6) has three invariant straight lines

$1+x=0,$
$1+A_{1,2}x+f(A_{1,2}-(2b^2+b+2f^2)((2b+1)^2+4f^2)^{-1}(b+A_{1,2})^{-1}y=0,$
where

$$A_{1,2}=-b\pm\sqrt{b^2-(2b^2+b+2f^2)^2((2b+1)^2+4f^2)^{-1}}$$

and a conic

$$1-2bx+2fy+((2b+1)^2+4f^2)^{-1}((2b^2+b+2f^2)x+fy)^2=0.$$

By Corollary 2 we obtain the desired result. ∎

LEMMA 8. *The following three series of conditions are sufficient conditions for the origin to be a centre for system (6):*

1) $c=1-b-\lambda,\ d=g-1,\ f=-\lambda,\ l=-b\lambda,\ s=-b\nu,$
$n=-5b^2-3bg-g,\ q=\nu(g-1)-\lambda^2.$

2) $c=1-b-\lambda,\ d=1-g,\ f=\lambda,\ l=b\lambda,\ s=-b\nu,$
$n=-5b^2-3bg-g,\ q=\lambda^2-\nu(g-1).$

3) $b=-1/2,\ c=3/2,\ d=(2f^2+s)/f,\ g=1,$
$l=-f/2,\ n=f^2,\ q=(2f^2+s)/(2f).$

Proof. Assume that the conditions 1), 2), 3) of lemma hold, then (6) is reversible ([24]). Indeed, in case the condition 1) of Lemma 8 holds the transformation

$$x=\frac{bX+2bY-1}{b(bX-1)},\quad y=\frac{bX-2bY+1}{b(bX-1)},$$

and in case the condition 2) of Lemma 8 holds the transformation

$$x=\frac{bX+2bY-1}{b(bX-1)},\quad y=\frac{bX-2bY+1}{b(1-bX)}$$

bring the system (6) to the form

$$\begin{aligned}\dot X=&\ (1+b-2b(b+2)Y-b^2(b-2\lambda+1)X^2+4b^2Y^2))(1-b^2X^2),\\ \dot Y=&\ X(\lambda(b+1)-b(b+b^2+4\lambda+2b\lambda)Y-b^2\lambda(1+b)X^2+\\ &4b^2(\lambda+b+b^2)Y^2+b^4(b-2\lambda+1)X^2Y-4b^4Y^3).\end{aligned}$$

In the cases 1) and 2) the singular point $O(0,0)$ is moved in $O_1(0,\frac{1}{2b})$ and $X=0$ is an axis of symmetry for the obtained system. In the case 3) the transformation

$$x=\frac{2X}{2-X},\qquad y=\frac{2Y}{2-X}$$

bring the system (6) to the form

$$
\begin{aligned}
\dot{X} =&\ fY(X^2-4)(fY+1),\\
\dot{Y} =&\ X(4f+f(4s-1)X^2+4(s+2f^2)Y+2f(1+2f^2)Y^2+f^2Y^3).
\end{aligned}
$$

$\blacksquare$

LEMMA 9. *The following four series of conditions are the sufficient conditions for the origin to be a centre for system (6):*

1) $c=2b\nu-b+1,\ d=\nu(2b\nu-3b-1)/f,\ l=bf,\ n=s=-b\nu,$
$q=-\nu^2(b+1)/f,\ f^2-\nu\tau=0.$

2) $c=-2b\nu-\nu-g+3,\ d=-\tau(2b\nu+2\nu+b)/f,\ l=bf,$
$s=-b\nu,\ n=b\nu+\lambda,\ q=-\nu\tau(3b+2)/f,\ f^2-\nu\tau=0.$

3)
$$
\begin{aligned}
&b=(cf^2-c\nu^2-f^2\nu-3f^2+\nu^3)/(f^2+\nu^2),\\
&d=f(3c\nu^2-cf^2+2f^2\nu+3f^2-2\nu^3-3\nu^2))/(\nu(f^2+\nu^2)),\\
&g=(c\nu^2-cf^2+2f^2\nu+3f^2)/(f^2+\nu^2),\\
&l=f(\nu-c\nu-2f^2)/(f^2+\nu^2),\\
&q=f(3c\nu+4f^2-2\nu^2-3\nu)/(f^2+\nu^2),\\
&s=\nu(c\nu+2f^2-\nu)/(f^2+\nu^2),\\
&n=(cf^2\nu-2c\nu^3+2f^4-3f^2\nu^2-f^2\nu+\nu^4+2\nu^3)/(\nu(f^2+\nu^2)).
\end{aligned}
\tag{26}
$$

4)
$$
\begin{aligned}
&b=(cf^2-c\tau^2-f^2\tau-f^2+\tau^3+2\tau^2)/(f^2+\tau^2),\\
&d=f(-cf^2+3c\tau^2+2f^2\tau-2\tau^3-6\tau^2)/(\tau(f^2+\tau^2)),\\
&g=(-cf^2+c\tau^2+2f^2\tau+2f^2-\tau^2)/(f^2+\tau^2),\\
&n=f^2(-c\tau-2f^2+2\tau^2+3\tau)/(\tau(f^2+\tau^2)),\\
&l=f(c\tau+f^2-\tau^2-2\tau)/(f^2+\tau^2),\\
&q=f^3(-c-2\tau)/(\tau(f^2+\tau^2)),\\
&s=f^2(-c+1)/(f^2+\tau^2).
\end{aligned}
\tag{27}
$$

Proof. In each of these cases the system (6) has an integrating factor of the form (25).

In the case 1): $l_1=1+x,\ l_2=1+\nu x+fy,\ l_3=1-bx-f\delta^{-1}(b+1)y,\ \alpha_1=(b-1)/(b+1),\ \alpha_2=-2b\nu/\lambda$ and $\alpha_3=2b^2\tau\lambda^{-1}(b+1)^{-1}-1$.

In the case 2): $l_1=1+x,\ l_2=1+\nu x+fy,\ l_3=1-bx-bf\nu^{-1}y,\ \alpha_1=-3,\ \alpha_2=2\tau(b+1)/\lambda$ and $\alpha_3=-(2b^2\nu+3b\lambda+2bg+2\nu)/(b\lambda)$.

In the case 3) the conditions (26) are contained in (20) and the invariant straight lines are preserved for (6). The exponents $\alpha_1,\ \alpha_2,\ \alpha_3$ in (25) can be found from the identity

$$
\alpha_1K_1(x,y)+\alpha_2K_2(x,y)+\alpha_3K_3(x,y)+\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\equiv 0,
$$

where $K_1(x,y)$, $K_2(x,y)$ and $K_3(x,y)$ are cofactors of the invariant straight lines. The expressions for α_1, α_2, α_3 were found to be large and we do not bring them here.

The conditions (27) are contained in (21). Next we proceed as in the case 3). ▮

6. THE PROBLEM OF THE CENTRE FOR CUBIC SYSTEM (6) WITH THREE INVARIANT STRAIGHT LINES

By "$\Longrightarrow$" denote "implies".

THEOREM 10. $(1 + A_j x + B_j y = 0,\ j = \overline{1,3};\ L = 5)$ *is ILC for system (6), i.e. if the cubic system (6) has three invariant straight lines (real, real and complex) then the order of a weak focus is at most 5.*

Proof. In order to solve the problem of the centre for cubic system (6) with three invariant straight lines which are limit systems of those with four invariant straight lines, it is enough that $L_1 = 0$ (8). Therefore, to prove theorem it is sufficient to show that in each series of conditions (12), (13), (15)–(22) the Liapunov quantities L_j, $j = \overline{1,\infty}$ vanish. For calculation of the quantities L_j we shall use the algorithms described in [22]. In the expressions for L_j we will neglect with denominators and non-zero factors.

In the case (12) $\Longrightarrow$ Lemma $7,1)\,(b=0)$.

In the case (13) the vanishing of the first Liapunov quantity gives $c = 2$ $\Longrightarrow$ Lemma $7,1)\,(b=-1)$.

In the cases (15) and (16) $\Longrightarrow$ Lemma $7,2)$.

In the case (17) the first Liapunov quantity is $L_1 = (c-b-g-1)(c+3b+g-1)$. If $c-b-g-1=0$ then $\Longrightarrow$ Lemma $7,2)$. Assume that $c+3b+g-1=0$, and let $c = 1-3b-g$. We have $L_2 = df + \lambda(\lambda+g-1) - f^2$. From $L_2 = 0$ we find d and substitute into the expression for L_3. We conclude that $L_3 = f_1 f_2 f_3$, where $f_1 = \nu^2 - \nu - f^2$, $f_2 = 2b + f + g$ and $f_3 = 2b - f + g$. If $f_1 = 0$ then $\Longrightarrow$ Lemma $7,2)$, if $f_2 = 0 \Longrightarrow$ Lemma $8,1)$ and if $f_3 = 0$ $\Longrightarrow$ Lemma $8,2)$.

In the case (18) we have $L_1 = f_1 f_2$, where $f_1 = \nu - c + 1$ and $f_2 = f^2 - \nu\tau$. If $f_1 = 0$ then $\Longrightarrow$ Lemma $7,2)$. Assume that $f_1 \neq 0$ and let $f_2 = 0$. The second Liapunov quantity with f^2 expressed from $f_2 = 0$ and canceled by non-zero factors looks as follows $L_2 = (\lambda+\delta)(2b\nu - \delta)$. If $\lambda + \delta = 0$ then $\Longrightarrow$ Lemma $7,2)$ and if $2b\nu - \delta = 0 \Longrightarrow$ Lemma $9,1)$.

In the case (19) the first Liapunov quantity looks $L_1 = (\nu-c+1)(f^2-\nu\tau)$. If $\nu - c + 1 = 0$ then $\Longrightarrow$ Lemma $7,2)$. Assume that $(\nu - c + 1) \neq 0$. The second Liapunov quantity is found to be $L_2 = f_1 f_2$, where $f_1 = \lambda + \nu$ and $f_2 = c + 2b\nu + \nu + g - 3$. If $f_1 = 0 \Longrightarrow$ Lemma $7,2)$ and if $f_2 = 0 \Longrightarrow$ Lemma $9,2)$.

In the case (20) the vanishing of the first Liapunov quantity gives $s = [\nu(c\nu-\nu-\sigma-f^2)]/(\nu^2+f^2)$, where $\sigma = (\nu^2-f^2)c+\lambda f^2-\nu^2 g$. The second one looks $L_2 = f_1 f_2 f_3$, where $f_1 = \delta-\lambda$, $f_2 = \sigma+3f^2$, $f_3 = c+3b+g-1$ (i.e. $f_3 = \delta+\lambda$). If $f_1 = 0$ then $\Longrightarrow$ Lemma $7, 2)$ and if $f_2 = 0$ then $\Longrightarrow$ Lemma $9, 3)$. Assume that $\nu f_1 f_2 \neq 0$ and let $f_3 = 0$. Then $c = 1-3b-g$ and $L_3 = f_{31} f_{32}$, where $f_{31} = f^2+\nu(1-\nu)$, $f_{32} = bf^2+6\nu^3+(3b-2)\nu^2+4f^2\nu$. If $f_{31} = 0$ then $\Longrightarrow$ Lemma $7, 2)$. From $f_{32} = 0$ we express b and calculate L_4 and L_5:

$$
\begin{aligned}
L_4 &= 5\nu^5-(5f^2+8)\nu^4+2f^2(2-5f^2)\nu^2-5f^4\nu-5f^6,\\
L_5 &= 10(8037f^2+29)\nu^{13}-2(40185f^4+85656f^2+232)\nu^{12}\\
&\quad+(266940f^2+74921)f^2\nu^{11}-(385250f^4+374583f^2+10632)f^2\nu^{10}\\
&\quad+5(57902f^2+17299)f^4\nu^9-(742360f^4+161425f^2+1148)f^4\nu^8\\
&\quad+2(30140f^2-12459)f^6\nu^7-2(361930f^4-59063f^2-3026)f^6\nu^6\\
&\quad-2(45945f^2+15512)f^8\nu^5-2(182185f^4-38265f^2+1122)f^8\nu^4\\
&\quad-(59060f^2-6413)f^{10}\nu^3-(82570f^4+55f^2-244)f^{10}\nu^2\\
&\quad-5(1966f^2+61)f^{12}\nu-5(916f^2+61)f^{14}.
\end{aligned}
$$

The resultant of the polynomials L_4 and L_5 by ν is not equal to zero and the obtained system of equations $L_4 = 0$ and $L_5 = 0$ has no real solutions, i.e. in this case the origin $O(0,0)$ is a focus.

In the case (21) from $L_1 = 0$ we find s: $s = (-bf^2\nu-2f^2\mu\tau+cf^2\tau+\nu^2\tau^2+g\nu\tau^2-c\nu\tau^2+f^2\tau^2-\nu\tau^3)(f^2+\tau^2)$ and substitute it into the expressions for L_2, L_3, L_4 and L_5. The second one looks $L_2 = f_1 f_2 f_3$, where $f_1 = \delta-\lambda$, $f_2 = (f^2+\tau^2)b-cf^2+c\tau^2+f^2\tau+f^2-\tau^3-2\tau^2$ and $f_3 = c+3b+g-1$.

If $f_1 = 0$ then $\Longrightarrow$ Lemma $7, 2)$ and if $f_2 = 0$ then $\Longrightarrow$ Lemma $9, 4)$. Assume that $\nu f_1 f_2 \neq 0$ and let $f_3 = 0$. Then $c = 1-3b-g$ and $L_3 = f_{31} f_{32}$, where $f_{31} = f^2-\nu\tau$, $f_{32} = (f^2+3\tau^2)b+4f^2\tau+f^2+6\tau^3+5\tau^2$.

If $f_{31} = 0$ then $\Longrightarrow$ Lemma $7, 2)$. From $f_{32} = 0$ we express b and calculate L_4 and L_5:

$$
\begin{aligned}
L_4 &= 5\tau^5+5f^2\tau^4+8\tau^4+10f^4\tau^2-4f^2\tau^2-5f^4\tau+5f^6,\\
L_5 &= 10(8037f^2+29)\tau^{13}+2(40185f^4+138306f^2+232)\tau^{12}\\
&\quad+(372240f^2+308471)f^2\tau^{11}+(385250f^4+664833f^2+114744)f^2\tau^{10}\\
&\quad+(725290f^2+454433)f^4\tau^9+(742360f^4+426295f^2+96116)f^4\tau^8\\
&\quad+2(382040f^2-2289)f^6\tau^7+2(361930f^4-18833f^2-16958)f^6\tau^6\\
&\quad+10(45783f^2-19450)f^8\tau^5+2(182185f^4-37005f^2-8922)f^8\tau^4\\
&\quad+25(5824f^2-1789)f^{10}\tau^3+(82570f^4+3745f^2-1612)f^{10}\tau^2\\
&\quad+5(3686f^2-403)f^{12}\tau+5(916f^2+403)f^{14}.
\end{aligned}
$$

The resultant of the polynomials L_4 and L_5 by τ is not equal to zero, therefore the origin $O(0,0)$ is a focus.

Suppose now that conditions (22) for the system (6) are satisfied. First we shall calculate and analyze the Liapunov quantities for the system (23) under conditions (24), then we shall apply the obtained results on the system (6). The first Liapunov quantity for (23), (24) looks as

$$L_1 = -\beta\sigma c + \gamma(bf - l) + \beta(2fs\nu + l(\nu(\nu + g) - s)),$$

where $\sigma = bl + fs + gl$. Now if $\gamma \neq 0$ and $\sigma = 0$ it follows that $L_1 \neq 0$. Therefore we shall assume that $\sigma \neq 0$. From $L_1 = 0$ we express c and substitute into the next two Liapunov quantities. The second quantity L_2 with non-zero factors removed looks $L_2 = g + 2b$. Let $g = -2b$. This implies the third Liapunov quantity to be $L_3 = f_1 f_2$, where $f_1 = l - bf$ and $f_2 = l^2 + s^2 + b^2 s + f^2 s$. If $f_1 = 0$, $p = f$, then $\Longrightarrow$ Lemma $8, 3)$ and if $f_2 = 0$, $p = f \Longrightarrow$ Lemma $7, 3)$. ▮

Note that $(1 + A_j x + B_j y,\, j = \overline{1,3};\; L = 5)$ is a minimal ILC, that is $(1 + A_j x + B_j y,\, j = \overline{1,3};\; L < 5)$ are not ILC for (6).

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