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A New Classification of Planar Homogeneous Quadratic Systems

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Using Invariant Theory, we give a new classification of the planar homogeneous quadratic systems with respect to the general linear group. This classification involves less canonical forms than those known before. It takes into account some algebraic properties like the existence of the common factor of the right hand. Thereby, it is more adapted to the phase portraits described in [7]. The computations are made with Maple.

Key Words: classification, invariant theory, orbits of general linear groups, qualitative theory of ordinary differential equations, quadratic systems.

1. INTRODUCTION AND FIRST NOTATIONS

Let $\mathcal{A}$ be the linear space of planar homogeneous quadratic systems

$$\begin{aligned} \frac{dx^1}{dt} &= a^1_{11}(x^1)^2 + 2a^1_{12}x^1x^2 + a^1_{22}y(x^2)^2 = P^1(x), \\ \frac{dx^2}{dt} &= a^2_{11}(x^1)^2 + 2a^2_{12}x^1x^2 + a^2_{22}y(x^2)^2 = P^2(x) \end{aligned} \qquad (1)$$

and G, the group of invertible 2×2 matrices, $p = (p^i_\alpha)_{i,\alpha=1,2}$. Throughout this work the coefficients $a^i_{\alpha\beta}$ and p^i_α belong to the real field $\mathbb{R}$. Let $GL(\mathcal{A})$ be the group of the linear automorphisms of $\mathcal{A}$. The group G acting over $\mathbb{R}^2$ ($(p, x) \mapsto p^{-1}x$) induces a linear rational representation

$$\rho : G \to GL(\mathcal{A})$$

where $\rho(p)(a) = b$ is defined by

$$b^i_{\alpha\beta} = \sum_{j=1}^{2}\sum_{\gamma=1}^{2}\sum_{\delta=1}^{2} q^i_j p^\gamma_\alpha p^\delta_\beta a^j_{\gamma\delta}, \quad q = p^{-1}, \quad \forall\, i, \alpha, \beta = 1, 2. \tag{2}$$

Considering the transformation laws (2), the $\mathbb{R}$-linear space $\mathcal{A}$ is nothing that the tensorial product $(\mathbb{R}^2)^* \bigotimes \mathcal{S}_2$ where $(\mathbb{R}^2)^*$ is the dual of $\mathbb{R}^2$ and $\mathcal{S}_2$ the space of the quadratic forms . The tensorial interpretation of the vectors $a \in \mathcal{A}$ (i.e. once contravariant and twice covariant) explains why we have adopted the notation as in (1) .

The transformation laws (2) are the same as those of the two-dimensional nonassociative algebras with the structure constants $a^i_{\alpha\beta}$ (see [5]) and the homogeneous quadratic polynomial mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ (see [1]). So, the present classification may be applied to these objects.

Two systems s and s' of $\mathcal{A}$ are G- equivalent if there exists $p \in G$ such that $\rho(p)s = s'$. The equivalence classes are called the G-orbits. The problem of the linear classification of the systems (1) consists in describing the G-orbits with the help of a finite number of algebraic functions which are constant over orbits.

Many works ([4, 5, 1, 6], ...) are devoted to this question. Some of them are completed by other type of classification like the geometric and/or the topological ones ([4, 5, 1, 6], ...).
The present paper is inspired by the monograph ([6]). We remark that the linear and topological classifications given there do not reflect one another. The aim of this paper is to present a new linear classification which is closer to the topological one. The tool used is Invariant Theory.

This paper is composed of three sections. In the first one, we recall the main results about the algebra of covariants of the systems (1). In the second section we analyze geometric and algebraic properties of particular covariants and in the third section we give the expected linear classification of the systems (1) .

2. AROUND THE COVARIANTS OF PLANAR HOMOGENEOUS QUADRATIC SYSTEMS

2.1. The algebra of covariants

Throughout this part, instead of the real field $\mathbb{R}$, we could take any field of characteristic zero.
In this work the notion of a covariant is understood in the following sense :

Definition 1. A polynomial function $Q : \mathcal{A} \times \mathbb{R}^2 \to \mathbb{R}$ is a covariant of $\mathcal{A}$ with respect to the group G (or a G-covariant) if there exists a function (group character) $\lambda : G \to \mathbb{R}$ such that

$$\forall p \in G,\, \forall a \in \mathcal{A}, \quad Q(\rho(p)a, p^{-1}x) = \lambda(p)Q(a, x).$$

If the polynomial function Q does not depend on x, it is called an *invariant* of $\mathcal{A}$.
From [7, 2] the character group λ is a power of the determinant of $p \in G$:

$$\lambda(p) = det(p)^{-\kappa}$$

where the integer κ is the weight of the covariant. The covariant Q is absolute if $\kappa = 0$. Otherwise, it is relative.
The above definition works for any subgroup of G and in particular, for the subgroup SL_2 of matrices of determinant 1. It is obvious that the homogeneous G-covariants are SL_2-covariants . The converse is also true. This confers to the set of the SL_2-covariants of $\mathcal{A}$ a structure of bi-gradate algebra with respect to the pair $(a, x) \in \mathcal{A} \times \mathbb{R}^2$. One knows (by the famous Hilbert basis theorem) that this algebra has a finite system of generators. Let us denote it $\mathcal{K}$. It contains the subalgebra of invariants $\mathcal{I}$.
In carrying out the study of these algebras, two classical problems arise : construct concretely minimal systems of generators and find the ideal of syzygies (polynomial identities between the covariants, including the invariants). Both problems are solved for $\mathcal{K}$ (and thereby, for $\mathcal{I}$) [7]. Using the Einstein notation (condensed form), we give here the expressions of the 13 generators of $\mathcal{K}$:

$$A = a^{\alpha}_{pr}a^{\beta}_{\alpha q}a^{\gamma}_{\beta s}a^{\delta}_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs}, \quad B = a^{\alpha}_{pr}a^{\beta}_{\alpha q}a^{\gamma}_{\delta s}a^{\delta}_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},$$

$$C = a^{\alpha}_{pr}a^{\beta}_{\beta q}a^{\gamma}_{\gamma s}a^{\delta}_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs}, \quad D = a^{\alpha}_{pr}a^{\beta}_{qk}a^{\gamma}_{\alpha s}a^{\delta}_{\delta l}a^{\mu}_{\beta\gamma}a^{\nu}_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl},$$

$$K = a^{\alpha}_{\alpha\beta}x^{\beta}, \quad L = a^{p}_{\alpha\beta}x^{\alpha}x^{\beta}x^{q}\varepsilon_{pq}, \quad M = a^{\alpha}_{\alpha\beta}a^{\beta}_{\gamma\delta}x^{\gamma}x^{\delta},$$

$$N = a^{\alpha}_{\beta\gamma}a^{\beta}_{\alpha\delta}x^{\gamma}x^{\delta} \quad R_1 = a^{\alpha}_{\alpha p}a^{\beta}_{\gamma q}a^{\gamma}_{\beta\delta}x^{\delta}\varepsilon^{pq}, \quad R_2 = a^{\alpha}_{\alpha p}a^{\beta}_{\delta q}a^{\gamma}_{\beta\gamma}x^{\delta}\varepsilon^{pq},$$

$$R_3 = a^{\alpha}_{\beta\nu}a^{\beta}_{\alpha\gamma}a^{\gamma}_{\delta\mu}x^{\delta}x^{\mu}x^{\nu}, \quad R_4 = a^{\alpha}_{\mu p}a^{\beta}_{\alpha q}a^{\gamma}_{\beta\nu}a^{\delta}_{\gamma\delta}x^{\mu}x^{\nu}\varepsilon^{pq},$$

$$R_5 = a^{\alpha}_{pr}a^{\beta}_{\nu q}a^{\gamma}_{\alpha s}a^{\delta}_{\beta\gamma}a^{\mu}_{\delta\mu}x^{\nu}\varepsilon^{pq}\varepsilon^{rs}.$$

where $\varepsilon^{12} = \varepsilon_{12} = -\varepsilon^{21} = -\varepsilon_{21} = 1$ et $\varepsilon^{11} = \varepsilon_{11} = -\varepsilon^{22} = -\varepsilon_{22} = 0$.
In this work we need only the following invariants and covariants :

$$A := (a^1_{11})^3a^1_{22} - (a^1_{11})^2(a^1_{12})^2 + (a^1_{11})^2a^1_{12}a^2_{22} + (a^1_{11})^2a^1_{22}a^2_{12}$$

$$- 3a^1_{11}(a^1_{12})^2a^2_{12} + 2a^1_{11}\, a^1_{12}a^1_{22}a^2_{11} + 2a^1_{11}a^1_{12}a^2_{12}a^2_{22} + 2\, a^1_{11}a^1_{22}a^2_{11}a^2_{22}$$
$$- a^1_{11}\, a^1_{22}\, (a^2_{12})^2 + a^1_{11}\, a^2_{12}\, (a^2_{22})^2 - (a^1_{12})^3\, a^2_{11} - (a^1_{12})^2\, a^2_{11}\, a^2_{22}$$
$$- 4\, (a^1_{12})^2\, (a^2_{12})^2 + 2\, a^1_{12}\, a^1_{22}\, a^2_{11}\, a^2_{12} + a^1_{12}\, a^2_{11}\, (a^2_{22})^2 - 3\, a^1_{12}\, (a^2_{12})^2\, a^2_{22}$$
$$+ 2\, a^1_{22}\, a^2_{11}\, a^2_{12}\, a^2_{22} - a^1_{22}\, (a^2_{12})^3 + a^2_{11}\, (a^2_{22})^3 - (a^2_{12})^2\, (a^2_{22})^2,$$

$$B := (a^1_{11})^3\, a^1_{22} - (a^1_{11})^2\, (a^1_{12})^2 + \; a^1_{12}\, a^2_{22} - (a^1_{11})^2\, a^1_{22}\, a^2_{12}$$
$$- a^1_{11}\, (a^1_{12})^2\, a^2_{12} + 4\, a^1_{11}\, a^1_{12}\, a^1_{22}\, a^2_{11} + 3\, a^1_{11}\, a^1_{22}\, (a^2_{12})^2 + a^1_{11}\, a^2_{12}\, (a^2_{22})^2$$
$$- 3\, (a^1_{12})^3\, a^2_{11} + 3\, (a^1_{12})^2\, a^2_{11}\, a^2_{22} - 4\, (a^1_{12})^2\, {a^2_{12}}^2 - 4\, a^1_{12}\, a^1_{22}\, a^2_{11}\, a^2_{12}$$
$$- a^1_{12}\, a^2_{11}\, (a^2_{22})^2 - a^1_{12}\, (a^2_{12})^2\, a^2_{22} + 2\, {a^1_{22}}^2\, (a^2_{11})^2 + 4\, a^1_{22}\, a^2_{11}\, a^2_{12}\, a^2_{22}$$
$$- 3\, a^1_{22}\, (a^2_{12})^3 + a^2_{11}\, (a^2_{22})^3 - (a^2_{12})^2\, (a^2_{22})^2,$$

$$C := (a^1_{11})^3\, a^1_{22} - (a^1_{11})^2\, (a^1_{12})^2 + (a^1_{11})^2\, a^1_{12}\, a^2_{22} + 3\, (a^1_{11})^2\, a^1_{22}\, a^2_{12}$$
$$+ 2(a^1_{11})^2(a^2_{22})^2 - 5a^1_{11}(a^1_{12})^2a^2_{12} - 4a^1_{11}a^1_{12}a^2_{12}a^2_{22} + 3a^1_{11}a^1_{22}(a^2_{12})^2$$
$$+ a^1_{11}\, a^2_{12}\, (a^2_{22})^2 + (a^1_{12})^3\, a^2_{11} + 3\, (a^1_{12})^2\, a^2_{11}\, a^2_{22} - 4\, (a^1_{12})^2\, (a^2_{12})^2$$
$$+ 3\, a^1_{12}\, a^2_{11}\, (a^2_{22})^2 - 5\, a^1_{12}\, (a^2_{12})^2\, a^2_{22} + a^1_{22}\, (a^2_{12})^3 + a^2_{11}\, {a^2_{22}}^3$$
$$-(a^2_{12})^2\, (a^2_{22})^2,$$

$$D := -4\, (a^1_{12})^4 a^2_{11} a^2_{12} - 3\, (a^1_{11})^2\, (a^1_{12})^2\, a^2_{12}\, a^2_{22} + 3\, (a^1_{11})^2 a^1_{12} a^1_{22} a^2_{11} a^2_{22}$$
$$- 3(a^1_{11})^2(a^1_{22})^2a^2_{11}a^2_{12} - (a^1_{11})^2a^2_{12}(a^2_{22})^3 - (a^1_{22})^2a^2_{11}(a^2_{12})^3$$
$$- 3a^1_{11}(a^1_{22})^2a^2_{11}(a^2_{12})^2 + a^1_{11}a^1_{22}{a^2_{12}}^3a^2_{22} - 6(a^1_{12})^3a^2_{11}a^2_{12}a^2_{22}$$
$$+ 3(a^1_{12})^2a^1_{22}{a^2_{11}}^2a^2_{22} - 4(a^1_{12})^2(a^2_{12})^3a^2_{22} + 4a^1_{12}a^1_{22}(a^2_{12})^4$$
$$- 2a^1_{12}(a^2_{12})^3{a^2_{22}}^2 + a^1_{22}(a^2_{11})^2(a^2_{22})^3 + a^1_{22}(2(a^2_{12})^4a^2_{22} + (a^1_{12})^3(a^2_{11})^2)$$
$$+ (a^1_{11})^4a^1_{22}a^2_{22} - (a^1_{11})^3(a^1_{22})^2a^2_{11} + 2(a^1_{12})^3((a^1_{11})^2a^2_{12} + 2a^1_{11}(a^2_{12})^2)$$
$$+ (a^1_{11})^3a^1_{12}(a^2_{22})^2 + a^1_{11}(a^2_{12})^2(a^2_{22})^3 + 3a^1_{12}a^1_{22}(a^2_{11})^2(a^2_{22})^2$$
$$- 3\, a^1_{12}a^1_{22}a^2_{11}(a^2_{12})^2a^2_{22} + 2a^1_{12}a^2_{11}\, a^2_{12}(a^2_{22})^3 - 3a^1_{22}a^2_{11}(a^2_{12})^2(a^2_{22})^2$$
$$+ 3a^1_{11}(a^1_{12})^2a^2_{11}\, (a^2_{22})^2 + 6a^1_{11}a^1_{12}a^1_{22}(a^2_{12})^3 + a^1_{11}a^1_{12}a^2_{11}(a^2_{22})^3$$
$$+ 3a^1_{11}a^1_{12}(a^2_{12})^2\, (a^2_{22})^2 - a^1_{11}a^2_{11}(a^2_{22})^4 - (a^1_{11})^3(a^1_{12})^2a^2_{22}$$
$$- 2(a^1_{11})^3a^1_{12}a^1_{22}a^2_{12} - (a^1_{11})^3a^1_{22}a^2_{12}a^2_{22} + 3(a^1_{11})^2(a^1_{12})^2a^1_{22}a^2_{11}$$
$$- 2a^1_{11}(a^1_{12})^4a^2_{11} - 3(a^1_{11})^2a^1_{22}(a^2_{12})^2a^2_{22} - a^1_{11}(a^1_{12})^3a^2_{11}a^2_{22}$$
$$+ 3a^1_{11}(a^1_{12})^2a^1_{22}\, a^2_{11}a^2_{12} - 3a^1_{11}a^1_{22}a^2_{11}\, a^2_{12}(a^2_{22})^2,$$

$$K := (a_{11}^1 + a_{12}^2)\, x^1 + (a_{12}^1 + a_{22}^2)\, x^2,$$

$$L := -a_{11}^2\, (x^1)^3 + (a_{11}^1 - 2\, a_{12}^2)\, x^2\, (x^1)^2 + (2\, a_{12}^1 - a_{22}^2)\, (x^2)^2\, x^1 + a_{22}^1\, (x^2)^3,$$

$$\begin{aligned} M := {} & ((a_{11}^1)^2 + a_{12}^1\, a_{11}^2 + a_{11}^1\, a_{12}^2 + a_{11}^2\, a_{22}^2)\, (x^1)^2 \\ & + (2\, a_{11}^1\, a_{12}^1 + 4\, a_{12}^1\, a_{12}^2 + 2\, a_{12}^2\, a_{22}^2)\, x^2\, x_1 \\ & + (a_{11}^1\, a_{22}^1 + a_{22}^1\, a_{12}^2 + a_{12}^1\, a_{22}^2 + (a_{22}^2)^2)\, (x^2)^2, \end{aligned}$$

$$\begin{aligned} N := {} & (2\, a_{12}^1\, a_{11}^2 + {a_{12}^2}^2 + (a_{11}^1)^2)\, (x^1)^2 \\ & + (2\, a_{12}^1\, a_{12}^2 + 2\, a_{12}^2\, a_{22}^2 + 2\, a_{11}^1\, a_{12}^1 + 2\, a_{22}^1\, a_{11}^2)\, x^2\, x^1 \\ & + ((a_{22}^2)^2 + (a_{12}^1)^2 + 2\, a_{22}^1\, a_{12}^2)\, (x^2)^2. \end{aligned}$$

Among them, we have four invariants A, B, C and D.

In [7], it was established that the ideal of syzygies of $\mathcal{I}$ is generated by the polynomial :

$$C^2B - A^3 - 2D^2 + A^2C - AC^2 \tag{3}$$

and that one of $\mathcal{K}$ by (3) and the polynomials :

$$\begin{aligned} CL - MR_2 + K^2R_1, \\ K^2(A + C) - 2CM + 2R_2^2, \\ C(K^2 - M) + 2R_1R_2 + AM - CN, \\ CR_1 + KD + (A - C)R_2, \\ KM(C - B) + 2(KR_1^2 + MR_5) - 2CR_3, \\ CK(R_1 - R_2) + AKR_1 + DM - CR_4, \\ A^2K - 2DR_2 + BCK - 2CR_5. \end{aligned}$$

Actually, the structure of the algebras $\mathcal{K}$ and $\mathcal{I}$ is more precise. Since the group $SL(2, \mathbb{R})$ is reductive these algebras are Cohen-Macaulay [3]. This means that they are finitely generated free modules over a submodule. For example, using the syzygy (3) between the elements of $\mathcal{I}$, we get :

$$\mathcal{I} = \mathbb{R}[A, B, C]D \oplus \mathbb{R}[A, B, C].$$

This decomposition is called the Hironaka decomposition of $\mathcal{I}$. Theoretically one could get the analog decomposition for the algebra $\mathcal{K}$. However, the computations are very long and useless for the present subject.

2.2. Algebraic and geometric properties of the covariants $\boldsymbol{K}, \boldsymbol{N}, \boldsymbol{L}$

In the following, all covariants will be regarded as elements of the ring $F[x^1, x^2]$ where F is the ring of polynomial functions over $\mathcal{A}$. A covariant is zero if and only if all its coefficients in x^1, x^2 vanish.

The jacobian of the vector field (1) is equal to $U = 2(K^2 - N)$ and its divergence to $2K$.

Remark 2. [and notation] The discriminant of the quadratic form U is equal to 8α where $\alpha = B + C - 2A$.

It is obvious that the study of the quadratic systems (1) of the form

$$\begin{pmatrix} P^1(x) \\ P^2(x) \end{pmatrix} = \begin{pmatrix} ax^1 + bx^2 \\ cx^1 + dx^2 \end{pmatrix} (ux^1 + vx^2)$$

can be reduced to the study of the linear or constant systems which are called "degenerated".

LEMMA 3 ([7],p.56). *A planar homogeneous quadratic system (1) is "degenerated" if, and only if,* $\alpha = 0$.

Proof. Note that the resultant of the polynomials $a^1_{11}(x^1)^2 + 2a^1_{12}x^1x^2 + a^1_{22}(x^2)^2$ and $a^2_{11}(x^1)^2 + 2a^2_{12}x^1x^2 + a^2_{22}(x^2)^2$ is equal to $\frac{1}{2}\alpha$. ∎

If the polynomials $P^1(x)$ and $P^2(x)$ are colinear the quadratic systems (1) are "degenerated" and

$$\frac{U}{8} = \begin{vmatrix} a^1_{11} & a^1_{12} \\ a^2_{11} & a^2_{12} \end{vmatrix} (x^1)^2 + \begin{vmatrix} a^1_{11} & a^1_{22} \\ a^2_{11} & a^2_{22} \end{vmatrix} x^1x^2 + \begin{vmatrix} a^1_{12} & a^1_{22} \\ a^2_{12} & a^2_{22} \end{vmatrix} (x^2)^2 = 0 \quad (4)$$

The converse is also true. Furthermore,

LEMMA 4. *The quadratic system (1) can be reduced (by a rotation) to the form*

$$\frac{dX^1}{dt} = 0, \quad \frac{dX^2}{dt} = b^2_{11}(x^1)^2 + 2b^2_{12}x^1x^2 + b^2_{22}y(x^2)^2$$

if, and only if, $U = 0$.

Proof. The necessary condition is trivial. Suppose that $U = 0$. From the relation (4) it follows that there exists two numbers $k_1, k_2 \in$ k such that $k_1^2 + k_2^2 = 1$ and $k_1 a^1_{\alpha\beta}x^\alpha x^\beta + k_2 a^2_{\alpha\beta}x^\alpha x^\beta = 0$.
The rotation $X^1 := k_1x^1 + k_2x^2$, $X^2 := -k_2x^1 + k_1x^2$ reduces the given system to the expected form. ∎

Up to now (see [4, 5, 1, 6, 7]), the linear classifications of the systems (1) with respect to the group G are based on the important property of the covariant L :

LEMMA 5 ([7],p.56). *The straight line* $\lambda_1 x^1 + \lambda_2 x^2 = 0$ *is an integral line of the system (1) if, and only if,* $L(-\lambda_2, \lambda_1) = 0$.

Taking into account the degree of the covariant L, the system (1) has at most three integral straight lines. Moreover, the x^2 - axis ($x^2 = 0$) is an integral straight line if, and only if, $a^1_{22} = 0$.
We complete this lemma by putting :

$$\text{discriminant }(L) = -\beta = -(27B - C - 18A).$$

3. THE LINEAR CLASSIFICATION OF THE PLANAR QUADRATIC HOMOGENEOUS SYSTEMS

In this section we substitute x^1 by x and x^2 by y.

3.1. The Sibirskii's classification

The classifications proposed in [4, 5, 1, 6, 7] were based on the number and the multiplicity of the integral straight lines of (1) which are defined, by the Lemma 5, with the covariant L. An account of these papers can be found in the monograph [7]. The first tableau (see Tab.1) we bring back here from [7] contains the different canonical forms which provide the parameterization of the G-orbits.
In the same monograph we also find topological phase portraits of the planar homogeneous quadratic systems with the condition $\alpha \neq 0$ (see Tab. 2). The case $\alpha = 0$ is analog (but not the same) to the linear or constant systems.
Curiously, the invariant α that plays a main role in the second tableau is absent in the first. The reason of this absence is that the covariant L does not describe the character of "degeneracy" of the systems (1). The different equations are, for $\beta < 0$:

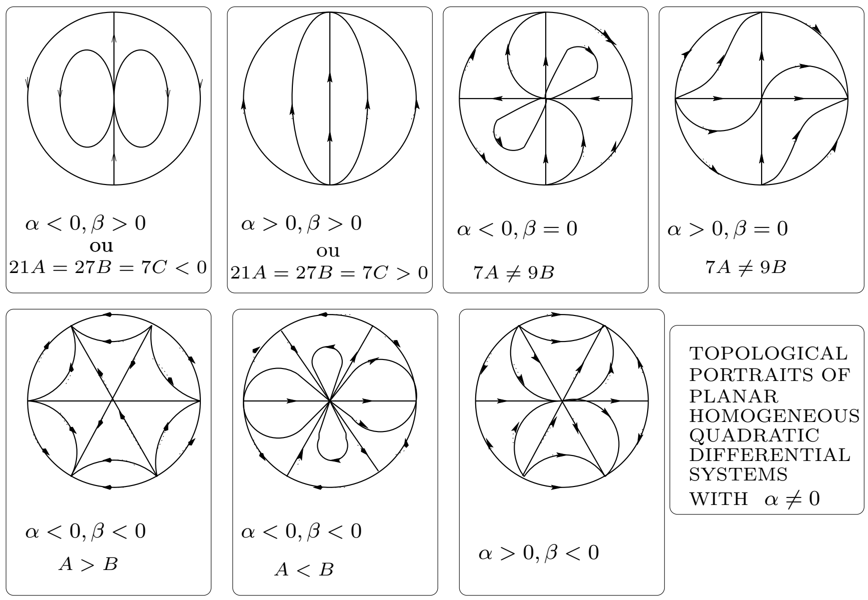
$$(c-b)^3 + \frac{4(C-3A)}{\beta}(c-b) + \frac{16\sqrt{2}|D|}{\sqrt{|\beta|^3}} = 0;$$
$$3(b+c) = 2 - \frac{8C\sqrt{|\beta|}(c-b)}{8\sqrt{2}D - [\sqrt{|\beta|}(c-b)]^3} \quad , \qquad (5)$$

INVARIANT CONDITIONS	CANONICAL FORMS	
$L = 0,\ K = 0$	$\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 0;$	
$L = 0,\ K \neq 0$	$\frac{dx}{dt} = x^2,\ \frac{dy}{dt} = xy$	
$\beta < 0$	$\begin{cases} \frac{dx}{dt} = bx^2 + (c-1)xy \\ \frac{dy}{dt} = (b-1)xy + cy^2; \end{cases}$	equations (5) $D > 0$
$\beta = 0,\ D \neq 0$	$\begin{cases} \frac{dx}{dt} = bx^2 + xy \\ \frac{dy}{dt} = (b-1)xy + y^2; \end{cases}$	$(9b-7)\sqrt[3]{D^2} = \sqrt[3]{32}A$
$B = K = 0, N \neq 0$	$\frac{dx}{dt} = x^2,\ \frac{dy}{dt} = -2xy;$	
$A = B = C = 0$ $KL \neq 0, 3M \neq 2K^2$	$\frac{dx}{dt} = bx^2,\ \frac{dy}{dt} = (b-1)xy;$	$\frac{2b}{3b-1} = \frac{M}{K^2}$
$A = B = C = 0$ $KL \neq 0, 3M = 2K^2$	$\frac{dx}{dt} = x^2,\ \frac{dy}{dt} = -x^2 + xy;$	-
$L \neq 0, K = N = 0$	$\frac{dx}{dt} = x^2,\ \frac{dy}{dt} = -x^2;$	-
$\beta = D = 0, C \neq 0$	$\begin{cases} \frac{dx}{dt} = xy \\ \frac{dy}{dt} = \mathrm{sgn}(C)x^2 + y^2; \end{cases}$	-
$\beta > 0$	$\begin{cases} \frac{dx}{dt} = bx^2 + (c+1)xy \\ \frac{dy}{dt} = -x^2 + bxy + cy^2; \end{cases}$	equations (6) $b \geq 0$

Tab. 1

and for $\beta > 0$:

$$\begin{aligned} 4(A-B) &= -(c+1)[b^2+(c-1)^2]; \\ b^3 - \frac{4(C-3A)}{\beta} b - \frac{16\sqrt{2}|D|}{\sqrt{|\beta|^3}} &= 0. \end{aligned} \tag{6}$$



Tab. 2

3.2. The new classification

We achieve the new classification in two steps, from the rougher to the finer. In the first step, we study the different reduced forms of the systems (1) related to the canonical forms of the covariant U.

3.2.1. Reduced forms

The reduced forms of the algebraic quadratic form U (which is an absolute covariant) are as follows :

$\alpha < 0$	$\alpha > 0$	$\alpha = 0$ and $U \neq 0$	$U = 0$
$2\varepsilon(x^2 + y^2)$	$2xy$	$2\varepsilon x^2$	0

Tab. 3

where $\varepsilon \in \{-1, 1\}$ following the sign (when it is defined) of U.

Theorem 6. *By a linear transformation, any system (1) can be reduced to one of the following forms :*

	invariant conditions	*reduced forms*	*additive conditions*
I	$\alpha < 0$	$\begin{cases} \frac{dx}{dt} = 2mxy \\ \frac{dy}{dt} = \frac{-\varepsilon}{m}x^2 + 2exy + \frac{\varepsilon}{m}y^2; \end{cases}$	$m \neq 0$
II	$\alpha > 0$	$\begin{cases} \frac{dx}{dt} = ax^2 + by^2 \\ \frac{dy}{dt} = cx^2 + dy^2; \end{cases}$	$ad - bc = 1$
III	$\alpha = 0$ $U \neq 0;$	$\begin{cases} \frac{dx}{dt} = ax^2 + 2bxy \\ \frac{dy}{dt} = cx^2 + 2dxy; \end{cases}$	$ad - bc = \varepsilon$
IV	$U = 0$	$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = ax^2 + 2bxy + cy^2; \end{cases}$	

Tab. 4

where ε is defined by the reduced form of the covariant U.

Proof. By the Lemma 3, if $\alpha = 0$ the system (1) is equivalent to a system of the form III or IV.
Assume that $\alpha > 0$. By the transformation that reduced U to the form II, the obtained system satisfies the relations

$$\begin{vmatrix} b_{11}^1 & b_{12}^1 \\ b_{11}^2 & b_{12}^2 \end{vmatrix} = 0, \begin{vmatrix} b_{11}^1 & b_{22}^1 \\ b_{11}^2 & b_{22}^2 \end{vmatrix} = 1, \begin{vmatrix} b_{12}^1 & b_{22}^1 \\ b_{12}^2 & b_{22}^2 \end{vmatrix} = 0.$$

Under the condition $\alpha \neq 0$, there is no common factor between the polynomials P^1 and P^2. Thus, the column vectors $(b_{11}^1, b_{11}^2)^T$ and $(b_{22}^1, b_{22}^2)^T$ do not vanish. That implies that the column vector $(b_{12}^1, b_{12}^1)^T$ is zero because, if not, the first and the third relations imply the colinearity of the right hand polynomials or in other words, by the Lemma 4, $U = 0$.

It remains the case $\alpha < 0$. The canonical form $2\varepsilon(x^2 + y^2)$ of U is invariant under the action of the group of the rotations. That corresponds to the relations :

$$\begin{vmatrix} b_{11}^1 & b_{12}^1 \\ b_{11}^2 & b_{12}^2 \end{vmatrix} = \varepsilon, \begin{vmatrix} b_{11}^1 & b_{22}^1 \\ b_{11}^2 & b_{22}^2 \end{vmatrix} = 0, \begin{vmatrix} b_{12}^1 & b_{22}^1 \\ b_{12}^2 & b_{22}^2 \end{vmatrix} = \varepsilon. \tag{7}$$

Taking into account the fact that any linear transformation $p = (p_j^i)_{i,j=1,2}$ transforms the coefficient a_{22}^1 into $\det(p)^{-1}L(p_2^1, p_2^2)$, there exits a rotation which makes $L(p_2^1, p_2^2)$ zero.
The Theorem 6 is proved. ▮

The obtained canonical forms in Tab. 1 achieve a first partition of the space of quadratic (homogeneous) differential systems $\mathcal{A}$ into G-invariant regions. Each region contains entire G-orbits that we have to identify with the help of absolute invariants. Except for the two first cases ($\alpha < 0$ and $\alpha > 0$), that needs a finer partition.

3.3. Characterization of the G-orbits

3.3.1. Case : $\boldsymbol{\alpha} < 0$

For the reduced form (I), we have

$$A := -\frac{4\,m^6\,e^2 - m^4 + m^2\,e^2 + 1 - m^6\,\varepsilon + 3\,e^2\,\varepsilon\,m^4 + m^2\,\varepsilon}{m^4},$$

$$B := -\frac{4\,m^6\,e^2 + 3\,m^4 + m^2\,e^2 + 1 - 3\,m^6\,\varepsilon + e^2\,\varepsilon\,m^4 - m^2\,\varepsilon}{m^4},$$

$$C := -\frac{m^6\,\varepsilon + 4\,m^6\,e^2 + 5\,e^2\,\varepsilon\,m^4 + 3\,m^2\,\varepsilon + 3\,m^4 + m^2\,e^2 + 1}{m^4},$$

$$\alpha := -8, \quad \beta = -8\frac{(2m^2 - \varepsilon)(1 + m^2e^2 - 2m\varepsilon)}{m^4},$$

$$L = \frac{x(\varepsilon(x^2 - 2emxy + (2m^2 - \varepsilon)y^2)}{m} \quad U := 2\,\varepsilon\,((y)^2 + (x)^2).$$

Combining the expressions of the relative invariants A, B et C we get a system of two equations verified by the parameters m and e :

$$m^2\,e^2 + 2\,\varepsilon\,m^2\,\frac{B - C}{\alpha} + (m^4 + 1) = 0 \tag{8}$$

$$m^6 + \varepsilon\,\frac{3B - C - 2A}{\alpha}\,m^4 + \frac{3B - A}{\alpha}\,m^2 + \varepsilon\,\frac{B - A}{\alpha} = 0. \tag{9}$$

Note that the coefficients of equations (8) and (9) are *rational absolute invariants.*
In view of their degrees, these equations may have twelve pairs of solutions (e, m) and one might think that they correspond to different G-orbits. However, this is is not the case : the systems I where the parameters e and m satisfy the relations (8) and (9) are G-equivalent.
Firstly, the substitution $x \leftrightarrow -x$, $y \leftrightarrow y$ transforms the pair (e, m) into $(-e, m)$. That explains the reason why the parameter e appears with the degree 2 in the equation (8).
Secondly, by the Lemma 5, the x^2-axis is an integral straight for the systems having the canonical form I. Since there may exist three integral straight lines, there are at most six possibilities (corresponding to six rotations) to get the canonical form I. Three of these rotations may be obtained from the other three by the symmetry $x \leftrightarrow -x$, $y \leftrightarrow -y$. That explains why we have got the biquadratic equation (9) of degree 6 on m.
Let us compute the discriminant of the equation (9), with respect to m^2 :

$$\Delta = -1024\,(-C + 27\,B - 18\,A)\,(-A^3 + A^2\,C - C^2\,A + B\,C^2).$$

Using the syzygy (3), we get :

$$\Delta = 2048\,\beta\,D^2.$$

There are more integral straight lines than solutions in m^2 of the equation (9).
Accordingly, the solutions (e, m) of the equations (8) and (9) characterize one and only one G-orbit of $\mathcal{A}$.

3.3.2. Case : $\boldsymbol{\alpha} > 0$

For the reduced form (II) we have :

$$A := a^3\,b + 2\,a\,b\,c\,d + c\,d^3,\; B := a^3\,b + 2\,b^2\,c^2 + c\,d^3,\; C := a^3\,b + 2\,a^2\,d^2 + c\,d^3,$$

$$U := 2\,x\,y\,(-b\,c + a\,d), \quad L := -c\,(x)^3 + a\,y\,(x)^2 - d\,{x_2}^2\,x + b\,(y)^3.$$

Thus,

$$\alpha = 2\,(-b\,c + a\,d)^2$$

$$A - B := 2\,b\,c\,(a\,d - b\,c), \quad C - A := 2\,a\,d\,(a\,d - b\,c)$$

If $A - C \neq 0$, then $ad \neq 0$ and by the transformation

$$x \leftrightarrow ax, \quad y \leftrightarrow dy,$$

the reduced form (II) becomes

$$\begin{cases} \dfrac{dx}{dt} &= x^2 + by^2 \\ \dfrac{dy}{dt} &= cx^2 + y^2 \end{cases}$$

where b and c verify the equations

$$\begin{aligned} 2\,A\,b^2\,c^2 - 2\,(C+B)\,b\,c + (2A-B-C)\,(b+c) + 2\,A &= 0 \\ 2(2\,A - C)\,b^2\,c^2 + -4Bbc + (2\,A - C - B)\,(b+c) + 2\,B &= 0 \\ 2\,C\,b^2\,c^2 + (2\,A - C - B)\,(b+c) - 4\,C\,b\,c + 2\,(2\,A - B) &= 0, \end{aligned}$$

or

$$\begin{aligned} 2\,A\,b^2\,c^2 - 2\,(C+B)\,b\,c + (2A-B-C)\,(b+c) + 2\,A &= 0, \\ B - A + (-B+C)\,b\,c + (-C+A)\,b^2\,c^2 &= 0. \end{aligned} \quad (10)$$

which are obtained from the expressions of $\dfrac{A}{\alpha}, \dfrac{B}{\alpha}, \dfrac{C}{\alpha}$.

The second equation provides two solutions $bc = 1$ and $bc = \dfrac{A-B}{C-A}$. The first one is incompatible with the condition $\alpha \neq 0$.

If $A - C = 0$, then $B - C \neq 0$ and, eventually, by the transposition $x \leftrightarrow y$, we get $a = 0$. With the help of the transformation

$$x \leftrightarrow \sqrt[3]{bc^2}x, \quad y \leftrightarrow \sqrt[3]{b^2c}y,$$

the systems II can be rewritten in the form

$$\begin{cases} \dfrac{dx}{dt} &= -y^2 \\ \dfrac{dy}{dt} &= -x^2 + dy^2; \end{cases}$$

where the parameter d satisfies the relation $\alpha\,d^3 + 2\,A = 0$.
Remark that when $A - C = 0$, the pair $(b, c) = (-1, -1)$ is a particular solution of the equations (10).
Consequently, when $\alpha > 0$, the systems II can be represented by

$$\begin{cases} \dfrac{dx}{dt} &= |sgn(A - C)|x^2 + by^2 \\ \dfrac{dy}{dt} &= cx^2 + dy^2; \end{cases}$$

where (b, c, d) is a solution of the following system of equations

$$\begin{aligned} 2\,A\,b^2\,c^2 - 2\,(C+B)\,b\,c + (2A - B - C)\,(b\,+\,c) + 2\,A \;&=\; 0,\\ B - A + (-B + C)\,b\,c + (-C + A)\,b^2\,c^2. \;&=\; 0,\\ (1\,-\,|sgn(A\,-\,C)|)(\alpha\,d^3\,+\,2\,A)\,+\,(d\,-\,1)|sgn(A\,-\,C)| \;&=\; 0. \end{aligned} \tag{11}$$

The parameter d is uniquely determined.
Concerning the parameters b and c, they have symmetric positions in these equations. Question : does a solution (v_1, v_2) correspond to (b, c) or (c, b)? In other words, what value v_1 or v_2 are we allowed to attribute to b (and another to c)? The answer to this question is given by the absolute invariant $\dfrac{A}{\alpha}$. Indeed, after substituting (b, c) by (v_1, v_2) and (v_2, v_1) we get the expressions : $\dfrac{a^3v_1 + 2adv_1v_2 + d^3v_2}{2(ad - v_1v_2)^2}$ $\dfrac{a^3v_2 + 2adv_1v_2 + d^3v_1}{2(ad - v_1v_2)^2}$. Their difference

$$\frac{(a-d)(v_1 - v_2)(a^2 - ad + d^2)}{2(ad - v_1v_2)^2}$$

is zero if, and only if, $a = d$ or $v_1 = v_2$. In both cases, the transposition $x \leftrightarrow y$ makes the corresponding systems equivalent. Otherwise, the value of the absolute invariant $\dfrac{A}{\alpha}$ identifies the appropriate pair.

3.3.3. Case : $\boldsymbol{\alpha} \,=\, 0$**,** $\quad \boldsymbol{U} \,\neq\, 0$

For the reduced form III we have :

$$A := -b^2\,(a^{2\ 2} + 4\,a\,d + 4\,d^2\,-\,\varepsilon),$$

$$B := -b^2\,(a^{2\ 2} + 4\,a\,d + 4\,d^2\,-\,3\varepsilon),$$

$$C := -b^2\,(a^{2\ 2} + 4\,a\,d + 4\,d^2\,+\,\varepsilon),$$

$$D := 2\,b^3\,\varepsilon\,(a\,+2\,d),$$

$$U := 2\,x^2\,\varepsilon.$$

Thus, $A\,-\,B \,=\, 2\varepsilon b^2$. It arises two possibilities : $b \neq 0$ and $b \,=\, 0$.

1. If $b \neq 0$, the mapping $x \leftrightarrow x, y \leftrightarrow \dfrac{ax + 2by}{2}$ transforms the reduced form III into :

$$\begin{cases} \dfrac{dx}{dt} \;=\; 2xy \\ \dfrac{dy}{dt} \;=\; \bar{c}x^2\,+\,2dxy. \end{cases} \tag{12}$$

The previous transformation preserves the *absolute* covariant U. So, $\overline{c} = -\varepsilon$ where ε is the sign of the definite quadratic form U. For the systems (12), we have also : $A - B = -2\varepsilon$ and $D = 4\varepsilon d$. Thus, d is equal to the positive or zero solution of the equation : $\varepsilon D^2 + d^2(A - B)^3 = 0$. We can take the positive value because the substitution $x \leftrightarrow -x$ transforms d into $-d$.

2. If $b = 0$ (that is to say $A - B = 0$), from $U \neq 0$, we get $ad \neq 0$. We can take $a = 1$. Then, by the transformation $x \leftrightarrow x$, $y \leftrightarrow ux + y$, the coefficient c becomes $u(a-2d)+c$. If $2M - 3U = 2(a-2d)x^2 \neq 0$, by an appropriate choice of u, we obtain the following system :

$$\frac{dx}{dt} = \varepsilon x^2 \quad \frac{dy}{dt} = 2dxy$$

where $d = \dfrac{U}{2M - U}$. Otherwise, it is equivalent to

$$\begin{cases} \dfrac{dx}{dt} = x^2 \\ \dfrac{dy}{dt} = \varepsilon x^2 + 2xy; \end{cases}$$

Consequently, when $\alpha = A - B = 0$, the initial system is equivalent to

$$\begin{cases} \dfrac{dx}{dt} = x^2 \\ \dfrac{dy}{dt} = \varepsilon x^2 + 2dxy; \end{cases}$$

where $d = \dfrac{U}{2M - U}$, $\varepsilon = 0$, if $2M - 3U \neq 0$ and $\varepsilon = 1$ if $2M - 3U = 0$.

3.3.4. Case : $\boldsymbol{U} = 0$

Following the Theorem 1, under the condition $U = 0$ the reduced form of the initial system is :

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = ax^2 + 2bxy + cy^2$$

for which $A = B = C = c^2(ac-b^2)$, $K = bx+cy$ et $M = c(ax^2+2bxy+cy^2)$. It is easy to verify that with the help of a linear triangular transformation (which preserves the reduced form (III)), the covariant $M \neq 0$ can be written in the form $c(ac-b^2)x^2+y^2$. Then, if $U = 0$ and $M \neq 0$, the initial system is equivalent to the system

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = sgn(A)x^2 + y^2$$

If $M = 0$ and $K \neq 0$, then $c = 0$ et $b \neq 0$ and the substitution $y \leftrightarrow \frac{a}{2} + by$ allows us to get the form

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 2xy.$$

If $K \neq 0$, the initial system is equivalent to

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = \mu x^2$$

where μ is 1 if $L \neq 0$ and 0, otherwise.

THEOREM 7. *Any differential system (1) can be reduced to one of the following where the parameters are uniquely determined by the indicated algebraic equations.*

Invariant Conditions	*Canonical Forms*	*Values of the parameters*
$\alpha < 0$	$\begin{cases} \frac{dx}{dt} = 2mxy \\ \frac{dy}{dt} = \frac{-\varepsilon}{m}x^2 + 2exy + \frac{\varepsilon}{m}y^2 \end{cases}$	*equations (8), (9)* $\varepsilon = sgn(U)$
$\alpha > 0$	$\begin{cases} \frac{dx}{dt} = \lvert sgn(A-C)\rvert x^2 + by^2 \\ \frac{dy}{dt} = cx^2 + dy^2; \end{cases}$	*equations (10)*
$\alpha = 0$ $A \neq B$	$\begin{cases} \frac{dx}{dt} = 2xy \\ \frac{dy}{dt} = \varepsilon x^2 + 2dxy; \end{cases}$	$d = \sqrt{\frac{D^2}{(A - B)^3}}$ $\varepsilon = sgn(U)$

Invariant Conditions	*Canonical Forms*	*Values of the parameters*
$\alpha = 0$, $A = B$ $U \neq 0$	$n \begin{cases} \frac{dx}{dt} = x^2 \\ \frac{dy}{dt} = cx^2 + 2dxy; \end{cases}$	$d = \frac{U}{2M - U}$, $c = \begin{cases} 0 \text{ if } 2M - 3U \neq 0, \\ 1 \text{ if } 2M - 3U = 0. \end{cases}$
$U = 0$, $M \neq 0$	$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = sgn\ (A)\, x^2 + y^2; \end{cases}$	
$U = 0$, $M = 0$, $K \neq 0$	$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = 2xy \end{cases}$	
$U = 0$, $K = 0$	$\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} = \mu\, x^2 \end{cases}$	$\mu = \begin{cases} 0 \text{ if } L = 0, \\ 1 \text{ if } L \neq 0. \end{cases}$

Tab. 5

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