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Center and Composition Conditions for Abel Differential Equation, and Rational Curves

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We consider the Abel Equation $\rho' = p(\theta)\rho^2 + q(\theta)\rho^3$ (*) with $p(\theta)$, $q(\theta)$ - polynomials in $\sin\theta$, $\cos\theta$. The center problem for this equation (which is closely related to the classical center problem for polynomial vector fields on the plane) is to find conditions on p and q under which all the solutions $\rho(\theta)$ of this equation are periodic, i.e. $\rho(0) = \rho(2\pi)$ for all initial values $\rho(0)$. We consider the equation (*) as an equation on the complex plane $\frac{dy}{dz} = p(z)y^2 + q(z)y^3$ (**) with p, q – Laurent polynomials. Then the center condition is that its solution $y(z)$ is a univalued function along the circle $|z| = 1$. We study the behavior of the equation (**) under mappings of the complex plane onto Riemann Surfaces. This approach relates the center problem to the algebra of rational functions under composition and to the geometry of rational curves. We obtain the sufficient conditions for the center in the form $\int_{|z|=1} P^i Q^j dP = 0$ with $P = \int p$, $Q = \int q$.

Key Words: Abel equation, center conditions, composition, rational curves

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1. CENTER PROBLEM AND ABEL EQUATION

We consider the classical Center-Focus Problem for homogeneous polynomial vector fields on the plane (see e.g [16]):

Let $F(x,y)$, $G(x,y)$ be polynomials in x, y of degree d. Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + F(x,y) \\ \dot{y} = x + G(x,y) \end{cases} \tag{1}$$

A solution $x(t)$, $y(t)$ of (1) is closed if it is defined in the interval $[0, t_0]$ and $x(0) = x(t_0)$, $y(0) = y(t_0)$. The system has a center at the origin if all the solutions around zero are closed.

It was shown in [8] that one can reduce the system (1) with homogeneous F, G of degree d to the *trigonometric Abel equation*

$$\frac{d\rho}{d\theta} = p(\theta)\,\rho^2 + q(\theta)\,\rho^3,\ \ \theta \in [0, 2\pi], \tag{2}$$

where $p(\theta)$, $q(\theta)$ are polynomials in $\sin\theta$, $\cos\theta$ of degrees $d+1$, $2d+2$ respectively. Then (1) has a center if and only if (2) has all the solutions periodic on $[0, 2\pi]$, i.e. the solutions $\rho = \rho(\theta)$ satisfying $\rho(0) = \rho(2\pi)$. The classical center problem is to find conditions on p and q such that (2) has a center.

The following simple sufficient condition was introduced in [2]. Let $w(\theta) \in C^1[0, 2\pi]$ be a function such that $w(0) = w(2\pi)$. Let

$$\begin{cases} p(\theta) = \tilde{p}(w(\theta))w'(\theta) \\ q(\theta) = \tilde{q}(w(\theta))w'(\theta). \end{cases} \tag{3}$$

Then all the solutions of (2) have the form $\rho(\theta) = \tilde{\rho}(w(\theta))$, hence they satisfy the condition $\rho(0) = \rho(2\pi)$. Indeed, in this case the equation (2) takes the form

$$\frac{d\rho}{d\theta} = \tilde{p}(w(\theta))\frac{dw}{d\theta}\rho^2 + \tilde{q}(w(\theta))\frac{dw}{d\theta}\rho^3, \tag{4}$$

and after the change of variables $w(\theta) = u$ we obtain the equation in u:

$$\frac{d\tilde{\rho}}{du} = \tilde{p}(u)\,\tilde{\rho}^2 + \tilde{q}(u)\,\tilde{\rho}^3,$$

which for small initial values $\tilde{\rho}(0)$ has an analytic solution $\tilde{\rho}(u)$. By the uniqueness theorem for the solutions of the first order ODE the solution of

(2) through the point $\tilde{\rho}(0)$ is the same, i.e. $\rho(\theta) = \tilde{\rho}(u) = \tilde{\rho}(w(\theta))$.

We shall call the representation (3) *Composition Condition* on p, q (we shall explain it later in all the details). In this fashion we can get the so called Hamiltonian and symmetric components of center for the dynamical system (1) (see [16], [2]). The composition condition is obviously sufficient, but not necessary, as there are cases of center which are impossible to represent as a composition - see [1] for the detail.

Let us consider for a moment the *polynomial Abel equation*

$$y' = p(x)y^2 + q(x)y^3 \tag{5}$$

with $p(x)$, $q(x)$ – usual *algebraic* polynomials in x. Fix two points $a, b \in \mathbb{C}$. The problem is to find conditions on p and q under which $p(a) = p(b)$ for all the solutions $p(x)$. This problem does not correspond exactly to the center problem on the plane, but is important by itself and was studied intensively in [12], [13], [3], [4], [5], [6], [7]. Here we believe that the composition condition

$$\begin{cases} p(x) = \tilde{p}(w(x))w'(x) \\ q(x) = \tilde{q}(w(x))w'(x), \end{cases}$$

for a polynomial $w(x)$ s.t. $w(a) = w(b)$ is not only sufficient, but also necessary. In [7] we checked it for $p(x)$, $q(x)$ of small degrees. This fact was verified also for some related problems in [4], [5].

In the present paper we study composition conditions for polynomial and trigonometric Abel equation, and we relate them to

- composition of rational functions
- algebraic geometry of rational curves
- differential equations on Riemann Surfaces
- generalized moments $\int_0^{2\pi} P^i(x)Q^j(x)p(x)dx = 0$.

The main result of the paper is the following simple sufficient condition for Abel equation (2) to have a center: $\int_0^{2\pi} P^i(x)Q^j(x)p(x)dx = 0$ for all $i, j \geq 0$, where $P(x) = \int_0^x p(\theta)d\theta$, $Q(x) = \int_0^x q(\theta)d\theta$.

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2. ABEL EQUATION ON RIEMANN SURFACES

We may rewrite the differential equation (2) in an invariant form

$$d\rho = dP(\theta)\,\rho^2 + dQ(\theta)\,\rho^3, \tag{6}$$

then expressing $\sin\theta$ and $\cos\theta$ through $x = e^{i\theta}$, i.e.

$$\begin{cases} \cos\theta = \frac{x+x^{-1}}{2}, \\ \sin\theta = \frac{x-x^{-1}}{2i}, \\ \rho = y, \end{cases}$$

we obtain P and Q in the form of *Laurent polynomials* in x, and the differential equation is

$$dy = y^2\,dP + y^3\,dQ$$

considered on the circle $x([0, 2\pi]) = S^1$.

In general let X be a domain on a connected Riemann Surface and let P and Q be two analytic functions on X. This is the case above, where X is a neighborhood of the unit circle S^1 on $\mathbb{C}$, Laurent polynomials P and Q are analytic on X. We consider the following *Abel differential equation* on X:

$$dy = y^2\,dP + y^3\,dQ \tag{A}$$

A (local) solution of (A) is an analytic function y on an open set Ω in X, such that the differential forms dy and $y^2\,dP + y^3\,dQ$ coincide in Ω.

If x is a local coordinate in Ω, (A) takes the usual form

$$\frac{dy}{dx} = y^2\,p(x) + y^3\,q(x), \tag{7}$$

where

$$p(x) = \frac{d}{dx}P(x),\ Q(x) = \frac{d}{dx}Q(x).$$

Let $Y \to X$ be the universal covering of X. The equation (A) can be lifted onto Y. One can easily show, that for any $a \in Y$ and for any $c \in \mathbb{C}$, there is a unique solution y_c of (A) on Y, satisfying $y_c(a) = c$, whose singularities tend to infinity as c tends to zero. In what follows we always assume that c is sufficiently small, so y_c is regular and univalued on any compact part of Y, but in general is multivalued on X.

Definition 1. Let γ be a closed curve in X. We say that the Abel equation (A) has *a center along the curve* γ, if for any small $c \in \mathbb{C}$ y_c is univalued along γ.

Notice that in this definition it is sufficient to assume that for any sufficiently small c and some $a \in \gamma$, $y_c(a) = y_{c,\gamma}(a)$, where $y_{c,\gamma}$ is a result of an analytic continuation of y_c along γ. Indeed, by uniqueness of a solution of the first order differential equation (A), $y_c(a) = y_{c,\gamma}(a)$ implies that y_c and $y_{c,\gamma}$ coincide in a neighborhood of a. Then by analytic continuation y_c coincide with $y_{c,\gamma}$ along the the whole γ, i.e. y_c is univalued.

Definition 2. We say that (A) has a *total center on* X, if it has a center along any closed curve in X.

In particular, this is always the case for X – simply-connected. Indeed, in this case any closed curve is homotopic to a point, so after analytic continuation along any closed curve we will obtain the initial value at this point.

Definition 3. Let $\tilde{X}$ be a domain on another Riemann Surface, $\tilde{P}$ and $\tilde{Q}$ be analytic functions on $\tilde{X}$. Assume there is an analytic mapping $w : X \to \tilde{X}$, such that $P(x) = \tilde{P}(w(x))$, $Q(x) = \tilde{Q}(w(x))$. We say that the Abel equation (A) on X *is induced from* the Abel equation

$$dy = y^2\, d\tilde{P} + y^3\, d\tilde{Q} \tag{$\tilde{A}$}$$

on $\tilde{X}$ by the mapping w. We also say that (A) *is factorized through* $\tilde{X}$.

Notice, that the words "factorized through $w : X \to \tilde{X}$" are equivalent to the words "composition representation $P(x) = \tilde{P}(w(x))$, $Q(x) = \tilde{Q}(w(x))$.". When we deal with Abel differential equation, we shall use the word "factorization", and when we deal with P and Q, we shall use the word "composition".

Lemma 4. *Let (A) be induced from ($\tilde{A}$) by w. Then any solution y of (A) is induced from a corresponding solution $\tilde{y}$ of ($\tilde{A}$), i.e. $y = \tilde{y} \circ w$.*

Proof. Let $y = y_c$ take a value c at some point $a \in X$. Consider the solution $\tilde{y}_c$ of (3), taking the value c at $w(a) \in \tilde{X}$. By the "invariance of the first differential", $\tilde{y}_c \circ w$ is a solution of (1), and it satisfies $\tilde{y}_c \circ w(a) = c$. Hence locally, $\tilde{y}_c \circ w \equiv y_c$, and analytic continuation completes the proof ∎

Corollary 5. *If (A) is induced from ($\tilde{A}$) by w, and if ($\tilde{A}$) has a center along $w(\gamma)$, then (A) has a center along γ.*

Definition 6. Let a and b be two different points in X. We say that a and b are *conjugate with respect to the equation (A) and with respect to a certain homotopy class of curves γ, joining a and b in X*, if for any solution

y_c of (A) with sufficiently small c ($y_c(a) = c$), its continuation $y_{c,\gamma}$ along γ satisfies $y_{c,\gamma}(b) = c$. In other words, any solution of (A) takes equal values at a and b after analytic continuation along γ.

A priori it is not evident that these definitions are natural and that conjugate points can appear at all. However, the following proposition gives a basic reason for their appearance:

PROPOSITION 7. *Let X, $\tilde{X}$, P, Q, $\tilde{P}$, $\tilde{Q}$, w be as above. Consider two different points a, $b \in X$ and a path γ joining them. If $w(a) = w(b)$ and $(\tilde{A})$ has a center along the closed curve $w(\gamma)$, then a and b are conjugate along γ for the Abel equation (A). In particular, if $\tilde{X}$ is simply connected, any two points a, b with $w(a) = w(b)$ are conjugate along any γ joining them.*

Proof. Follows immediately from Lemma 4 ▮

2.1. Example: Polynomial Composition Conjecture in the case $X = \mathbb{C}$, P and Q – polynomials in x

Here X is simply-connected, P and Q are analytic on X and hence the Abel equation

$$\frac{dy}{dx} = p(x)\, y^2 + q(x)\, y^3, \tag{8}$$

with $p(x) = \dfrac{d}{dx}P(x)$, $q(x) = \dfrac{d}{dx}Q(x)$, has a center along any closed curve γ.

As far as a factorization of (8) is concerned, assume that there exist polynomials w, $\tilde{P}$, $\tilde{Q}$, such that $P(x) = \tilde{P}(w(x))$, $Q(x) = \tilde{Q}(w(x))$. Then for $\tilde{X} = \mathbb{C}$ (8) is induced by w from

$$\frac{dy}{dw} = \tilde{P}'(w)\, y^2 + \tilde{Q}'(w)\, y^3. \tag{9}$$

By proposition 7, any two points a and b such that $w(a) = w(b)$ are conjugate along any path γ.

In [3], [4], [5], [6] and [7] this example was investigated in some details. In particular, for small degrees of P and Q it was shown, that conjugate points can appear only in this way.

The following conjecture was proposed in [3]:

Polynomial Composition Conjecture: *Two different points a and b in $\mathbb{C}$ are conjugate for the Abel equation (8) with P, Q - polynomials in x, if and only if the following* **Polynomial Composition Condition** *is* *satisfied: there exists a factorization* $P(x) = \tilde{P}(w(x))$, $Q(x) = \tilde{Q}(w(x))$ *with polynomial mapping* w*, such that* $w(a) = w(b)$.

Notice, that although the equation (8) is non-symmetric with respect to p and q, this conjecture proposes the symmetric condition: if a and b are conjugate for (8), then they are conjugate also for the equation

$$\frac{dy}{dx} = q(x)\,y^2 + p(x)\,y^3. \tag{10}$$

This situation is not unique in the center problem. In [9] it was shown that the Lienard system

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x) = 0$$

has a center at the origin if and only if the functions $F(x) = \int_0^x f(t)dt$, $G(x) = \int_0^x g(t)dt$ can be represented as a composition $F(x) = \tilde{F}(z(x))$, $G(x) = \tilde{G}(z(x))$ for an analytic function $z(x)$, with $z'(0) < 0$.

2.2. Example: Laurent Composition Condition in the case X – a neighborhood of the unit circle $S^1 = \{|x| = 1\} \subseteq \mathbb{C}$, P and Q – Laurent series, convergent on X

We shall discuss this case in much more detail below, because it corresponds directly to the classical Center-Focus Problem for homogeneous polynomial vector fields on the plane.

The one of possible factorizations in this case, which we shall call *Laurent Composition Condition*, takes the form $P = \tilde{P}(w)$, $Q = \tilde{Q}(w)$, where w is a Laurent series, and $\tilde{P}$, $\tilde{Q}$ are regular analytic functions on the disk D in $\mathbb{C}$, containing the image $w(S^1)$.

Lemma 8. *Laurent Composition Condition implies center along* S^1.

Proof. We have a factorization $w : X \to D$, and since D is simply connected, the Abel equation $dy = y^2 d\tilde{P} + y^3 d\tilde{Q}$ on D has a total center, then by Corollary 5 it implies center along S^1 ∎

In this case the conjecture that this is the only reason for center is not true: there are known cases of center along S^1 for the equation (8), when p and q can not be represented as a Laurent composition (the example was considered by Alwash in [1]).

Nevertheless, for $p(z)$, $q(z)$ – Laurent polynomials of small degrees up to (4,4), i.e. of the form $z^{-2}(a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0)$, the composition representation is the only possible reason for center. These computations will appear separately.

3. FACTORIZATION (COMPOSITION) FOR THE CASE $X \subseteq \mathbb{C}$, P AND Q – RATIONAL FUNCTIONS ON X

3.1. Composition and rational curves

It turns out that the assumption that P and Q are rational functions puts the composition (factorization) question completely in the framework of algebraic geometry of rational curves. We shall show that the above examples of Polynomial composition and Laurent composition are in some sense "generic", and any analytic factorization of rational function can be reduced to composition of rational functions.

The following facts are very basic in algebraic geometry. We restate them for convenience of our presentation. For details we address the reader to any classical algebraic geometry text (e.g. [15]).

Definition 9. Two rational functions P and Q define a curve $Y = \{(P(t), Q(t)), t \in \mathbb{C}\}$ in $\mathbb{C}^2$.

Lemma 10. *The curve Y for* rational *P and Q is an algebraic curve.*

Proof. We need to prove that for sufficiently large d there exists a polynomial $F(x, y)$ of the degree d, such that $F(P(t), Q(t)) \equiv 0$. Without loss of generality we may assume that P and Q have the same denominator R. Consider all the products $P(t)^i Q(t)^j R^d$, $i+j \le d$. These are polynomials of t of the degrees less than or equal to $i \deg P + j \deg Q + d \deg R \le d(\deg P + \deg Q + \deg R)$. But there exist $\frac{d(d-1)}{2}$ such products $P^i Q^j$, therefore for $\frac{d(d-1)}{2} > d(\deg P + \deg Q + \deg R)$, i.e. for $d > 2(\deg P + \deg Q + \deg R) + 1$ there exists a linear dependence over $\mathbb{C}$:

$$\sum_{i,j} \beta_{ij} P(t)^i Q(t)^j R(t)^d \equiv 0, \text{hence} \sum_{i,j} \beta_{ij} P(t)^i Q(t)^j \equiv 0.$$

This polynomial $F(x, y) = \sum \beta_{ij} x^i y^j$ vanishes on Y. Such polynomial of minimal degree defines an algebraic curve Y. ▌

Lüroth theorem *Any subfield of a field of rational functions is generated by a rational function.*

Corollary 11. *There exist rational functions $r(t)$, $\bar{P}$, $\bar{Q}$, s.t. $P(t) = \bar{P}(r(t))$, $Q(t) = \bar{Q}(r(t))$ and s.t. the map*

$$\bar{\gamma} : \mathbb{C} \to Y, \quad z \mapsto (\bar{P}(z), \bar{Q}(z))$$

defines a birational isomorphism between $\mathbb{C}$ and Y. In particular, Y is a rational curve.

Proof. **1.** Notice that $K = \mathbb{C}(P(t), Q(t))$ is is a subfield of the field of rational functions $\mathbb{C}(t)$. By Lüroth theorem $K = \mathbb{C}(r(t))$ for some rational

function $r(t)$. In particular, P and Q belong to K, hence there exist rational functions $\bar{P}(t)$, $\bar{Q}(t)$ such that $P(t) = \bar{P}(r(t))$, $Q(t) = \bar{Q}(r(t))$
2. It is obvious that $\bar{\gamma}$ is surjective and rational. Let's prove that there exists an inverse map $\bar{\gamma}^{-1} : Y \to \mathbb{C}$. Since $r(t) \in \mathbb{C}(P(t), Q(t))$, there exist a rational function $R(x, y)$ s.t. $R(P(t), Q(t)) \equiv r(t)$, i.e. $R(\bar{P}(r(t)), \bar{Q}(r(t))) \equiv r(t)$. Obviously it is a rational function from Y to $\mathbb{C}$. Let us prove that $R = \bar{\gamma}^{-1}$. Indeed, $R \circ \bar{\gamma} \equiv id : \mathbb{C} \to \mathbb{C}$: $R(\bar{\gamma}(z)) = R(\bar{P}(z), \bar{Q}(z)) = R(P(r(t)), Q(r(t))) = r(t) = z$, since for any z there exists t: $z = r(t)$. Vice versa: $\bar{\gamma} \circ R \equiv id : Y \to Y$, since $\bar{\gamma}(R(P(t), Q(t))) = \bar{\gamma}(r(t)) = (\bar{P}(r(t)), \bar{Q}(r(t))) = (P(t), Q(t))$ ∎

Definition 12. The *degree of a map* $\gamma = (P, Q) : \mathbb{C} \to Y$ is the degree of the algebraic extension $[\mathbb{C}(t) : \mathbb{C}(P, Q)]$.

Definition 13. The parameterization of a rational curve Y, $\gamma : \mathbb{C} \to Y$ $z \mapsto (P(z), Q(z))$ is called *minimal* if $\deg \gamma = 1$.

From corollary 11 it follows that a minimal parameterization defines a birational isomorphism between $\mathbb{C}$ and Y.

Definition 14. The mapping $f : X \to Y$ is called *"not 1-1"*, if there exists an open set $\Omega \subseteq Y$, s.t. each point of Ω has more than one preimage under f.

Definition 15. A rational function r is called *common divisor under composition* of rational functions P and Q, if $P = \tilde{P}(r)$, $Q = \tilde{Q}(r)$. The common divisor r is called *nontrivial*, if r is not 1-1.

Definition 16. A rational function r is called *Composition Greatest Common Divisor (CGCD)* of rational functions P, Q, if r is a common divisor under composition of P and Q, and if $\tilde{r}$ is another common divisor of P and Q under composition, then $r = R(\tilde{r})$ for a rational function R.

Definition 17. The *degree of a rational function* is a maximum of degrees of numerator and denominator.

Let's notice that among rational functions only linear functions (functions of degree 1) are 1-1. Respectively, if we are looking for nontrivial CGCD in the class of rational functions, it must have degree greater than 1. It is easy to show that CGCD exists and satisfies all the properties of usual greatest common divisor. In particular,

Proposition 18. *For any rational functions $P(t)$, $Q(t)$ their CGCD $r(t)$ exists and is given by corollary 11. CGCD is unique in the algebra of rational functions under compositions up to composition with an invertible rational function (i.e. a function of degree 1), i.e. two CGCD of a* *given function can be obtained each one from another by a (right or left) composition with a linear function.*

Proof. Obviously, $r(t)$ from Corollary 11 defines rational common divisor under composition. If $\tilde{r}$ is another composition common divisor of P and Q, then $P = \tilde{P}(\tilde{r})$, $Q = \tilde{Q}(\tilde{r})$, so $\mathbb{C}(\tilde{r}(t)) \supseteq \mathbb{C}[P(t), Q(t)] = \mathbb{C}(r(t))$, hence $r(t) = R(\tilde{r}(t))$ for some rational function R. Therefore $r(t)$ is actually a CGCD.

If r and $\tilde{r}$ are two CGCD, then $r(t) = R(\tilde{r}(t))$, but $\tilde{r}(t) = \tilde{R}(r(t))$, so $R \circ \tilde{R} = id$, hence R is a linear function. ▮

The following facts are proved, for example, in [15]:

Lemma 19. *1) For a rational map $\gamma = (P, Q) : \mathbb{C} \to Y$ the number of preimages of almost each point is equal to* $\deg \gamma$.
2) $[\mathbb{C}(t) : \mathbb{C}(r(t))] = \deg r(t)$.

Corollary 20. *The degree of the map $\gamma = [P, Q]$ is equal to the degree of the rational CGCD of P and Q. If r is CGCD of P and Q, $P = \bar{P}(r)$, $Q = \bar{Q}(r)$, then $\bar{\gamma} : \mathbb{C} \to Y$ $z \mapsto (\bar{P}(z), \bar{Q}(z))$ is a minimal parameterization.*

The following two results show that allowing an analytic (and not a priori rational) composition does not add, in fact, anything new.

Lemma 21. $\deg \gamma > 1$ *if and only if there exists an analytic factorization $w : \mathbb{C} \to \tilde{X}$, where $\tilde{X}$ is a Riemann Surface, $\tilde{P}$ and $\tilde{Q}$ are analytic functions on $\tilde{X}$, such that $P(t) = \tilde{P}(w(t))$, $Q(t) = \tilde{Q}(w(t))$, and w is not 1-1.*

Proof. Let $\deg \gamma > 1$, then taking $w = r : \mathbb{C} \to \mathbb{C}$ we get the required analytic factorization. Vice versa, let there exist an analytic factorization $w : \mathbb{C} \to \tilde{X}$, $P(t) = \tilde{P}(w(t))$, $Q(t) = \tilde{Q}(w(t))$. Then $\mathbb{C}(P, Q) = \mathbb{C}(\tilde{P}(w), \tilde{Q}(w))$, which is a proper subset in $\mathbb{C}(t)$, because $w(t)$ glues some points in $\mathbb{C}$, but in $\mathbb{C}(t)$ there are functions which map these points into different points. Hence $[\mathbb{C}(t) : \mathbb{C}(P, Q)] > 1$. ▮

Corollary 22. *If there exists an analytic factorization of rational functions $P = \tilde{P}(w)$, $Q = \tilde{Q}(w)$ with w – not 1-1, then there exists a nontrivial CGCD of P and Q: $P = \bar{P}(r)$, $Q = \bar{Q}(r)$ for $\bar{r}$ of the degree greater than 1.*

3.2. Structure of composition in the case $X = \mathbb{C}$, P and Q – polynomials

We shall prove that the factorization in the form of *Polynomial Composition Condition* is essentially the only one natural factorization in the polynomial case, namely:

Theorem 23. *Assume there exists an analytic factorization $P = \tilde{P}(w)$, $Q = \tilde{Q}(w)$ with w – not 1-1. Then there exists a polynomial factorization*

$P(t) = \hat{P}(\hat{w}(t))$, $Q(t) = \hat{Q}(\hat{w}(t))$, with $\hat{P}$, $\hat{Q}$, $\hat{w}$ being polynomials, the degree of $\hat{w}$ is greater than 1 and $\deg(\hat{P}, \hat{Q}) = 1$.

The proof will follow from the lemma:

LEMMA 24. *If we have a factorization $P = \bar{P} \circ r$, $Q = \bar{Q} \circ r$ with $\bar{P}(t)$, $\bar{Q}(t)$, $r(t)$ - rational functions: $\mathbb{C} \to \mathbb{C}$, then there exists a linear rational function $\lambda : \mathbb{C} \to \mathbb{C}$, s.t. $\bar{P} \circ \lambda$, $\bar{Q} \circ \lambda$, $\lambda^{-1} \circ r$ are polynomials.*

Proof. This proof is contained essentially in [14]. Let $r(\infty) = a$. Let the degrees of the rational functions r, $\bar{P}$ be n, m respectively. The degree of the polynomial P will be mn. Then:
$r(\infty) = a$ with multiplicity not more than n, $\bar{P}(a) = \infty$ with multiplicity not more than m, but $\bar{P} \circ r(\infty) = \infty$ with multiplicity exactly mn, because $\bar{P} \circ r$ is a polynomial.

Hence we get that $r(\infty) = a$ with multiplicity n, $\bar{P}(a) = \infty$ with multiplicity m.

Now take $\lambda(z) = \frac{1}{z-a}$, i.e. $\lambda^{-1}(a) = \infty$. Then $\bar{P} \circ \lambda(\infty) = \infty$ with multiplicity n, $\lambda^{-1} \circ r(\infty) = \infty$ with multiplicity m, hence they are polynomials. Similarly $\bar{Q} \circ \lambda$ is a polynomial ▮

Proof. (theorem 23)
By corollary 22 there exists a factorization $P(t) = \bar{P}(r(t)$, $Q(t) = \bar{Q}(r(t)$, for some rational functions $\bar{P}(t)$, $\bar{Q}(t)$ and rational function $r(t)$ of degree greater than 1, such that $\deg(\bar{P}, \bar{Q}) = 1$. Then taking $\hat{P} = \bar{P} \circ \lambda$, $\hat{Q} = \bar{Q} \circ \lambda$, $\hat{w} = \lambda^{-1} \circ r$ we obtain the required polynomial factorization $P(t) = \hat{P}(\hat{w}(t))$, $Q(t) = \hat{Q}(\hat{w}(t))$ with $\hat{w}$ of degree greater than 1 and $\deg(\hat{P}, \hat{Q}) = 1$. ▮

The similar fact was proved by C. Christopher in [10], when he investigated polynomial case of Lienard system.

COROLLARY 25. *If* $\deg[P, Q] = s > 1$, *then the two polynomials P and Q have a nontrivial CGCD of the degree s in the algebra of polynomials under composition: $P(t) = \hat{P}(r(t))$, $Q(t) = \hat{Q}(r(t))$, with $\hat{P}$, $\hat{Q}$, r – algebraic polynomials and* $\deg r = s$. *The map $\gamma :\ z \mapsto (\hat{P}(z), \hat{Q}(z))$ defines a minimal polynomial parameterization of the algebraic curve $Y = \{(P(t), Q(t)), t \in \mathbb{C}\}$.*

3.3. Structure of composition in the case X – a neighborhood of the unit circle on $\mathbb{C}$, P and Q – Laurent polynomials

This case corresponds precisely to the classical Center problem for real polynomial vector fields on the real plane. In this case Laurent composition condition is not necessary for Abel equation to have a center (see subsection 2.2). Still an investigation of possible factorizations in the case of P, Q –

Laurent polynomials leads to some interesting new results. They will be used in the subsection 4.3

Definition 26. The composition representations $P = \bar{P}(\bar{r})$ and $P = \tilde{P}(\tilde{r})$ are *equivalent* if there exists a linear rational function λ, s.t. $\bar{P} = \tilde{P}(\lambda)$, $\bar{r} = \lambda^{-1}(\tilde{r})$.

Theorem 27. *Up to the equivalence relation of definition 26 there are only two types of composition representations of a Laurent polynomial P:*
(1) $P = \bar{P}(r)$, where $\bar{P}$ is a usual (algebraic) polynomial, and r is a Laurent polynomial.
(2) $P = \bar{P}(r)$, where $\bar{P}$ is a Laurent polynomial, and $r = z^k$ for some $k \in \mathbb{N}$, $k \geq 2$.
Any two composition representations of types (1) and (2) for $\deg r > 1$ *and* $k > 1$ *are not equivalent.*

Proof. P is a Laurent polynomial, hence ∞ has exactly two preimages – 0 and ∞. Assume we are given a composition in the class of rational functions: $P = \tilde{P}(\tilde{r})$. We shall show that by choosing a suitable linear rational function λ we obtain $P = (\tilde{P} \circ \lambda) \circ (\lambda^{-1} \circ \tilde{r})$, where $\bar{P} = \tilde{P} \circ \lambda$ and $r = \lambda^{-1} \circ \tilde{r}$ are of the required form (1) or (2).

$\tilde{P}$ may have either two or one preimage of ∞.

1) Assume first that ∞ has two preimages $a \neq b$ under the map $\tilde{P}$: $\tilde{P}(a) = \tilde{P}(b) = \infty$. Take a linear function $\lambda(z)$, s.t. $\lambda(0) = a$, $\lambda(\infty) = b$: $\lambda(z) = \dfrac{1}{z + \frac{1}{a-b}} + b$. Then $(\tilde{P} \circ \lambda)(0) = \infty$, $\quad (\tilde{P} \circ \lambda)(\infty) = \infty$, so $\bar{P} = \tilde{P}(\lambda)$ is a Laurent polynomial. Then necessary $(\lambda^{-1} \circ \tilde{r})(0) = 0$, $(\lambda^{-1} \circ \tilde{r})(\infty) = \infty$, so $r = \lambda^{-1}(\tilde{r})$ is an algebraic polynomial of the form z^k for some natural k.

2) If $\tilde{P}$ has only one preimage of ∞, then similarly to the lemma 19 there exists a linear rational function λ s.t. $\bar{P} = \tilde{P}(\lambda)$ is a polynomial. Then necessary $(\lambda^{-1} \circ \tilde{r})(0) = \infty$, $\quad (\lambda^{-1} \circ \tilde{r})(\infty) = \infty$, and there are no other points where $\lambda^{-1}(\tilde{r})$ takes values 0 and ∞, so $\lambda^{-1} \circ \tilde{r}$ is a Laurent polynomial.

Since under composition with a linear function the number of preimages of a given point can not change, (1) and (2) are not equivalent. ∎

Corollary 28. *If* $\deg(P, Q) > 1$*, then* $P = \bar{P}(r)$*,* $Q = \bar{Q}(r)$ *with either*
(1) Laurent polynomial composition*: $\bar{P}$, $\bar{Q}$ – algebraic polynomials, r – Laurent polynomial of degree greater than 1; or*
(2) $\bar{P}$, $\bar{Q}$ – Laurent polynomials, $r = z^k$ for $k \geq 2$.

4. MOMENTS OF P, Q ON S^1 AND CENTER CONDITIONS

4.1. Sufficient center condition for Abel equation with analytic p, q

We return to the case of Abel equation (A)

$$dy = y^2 dP + y^3 dQ \tag{A}$$

considered in a neighborhood of a unit circle $S^1 = \{|x| = 1\}$ in the complex plane $\mathbb{C}$, with P, Q – analytic functions in some neighborhood of S^1 (not necessary Laurent polynomials).

The following theorem is a summary of results due to J. Wermer ([17], [18]). The applicability of Wermer's results to Center problem was discovered by J.-P.Françoise ([11]):

Theorem (Wermer, 1958) *Let P, Q be a pair of functions on the unit circle $S^1 \subseteq \mathbb{C}$. Assume:*
(1) P and Q are analytic in an annulus containing S^1 and together separate points on S^1.
(2) $P' \neq 0$ on S^1.
(3) P takes only finitely many values more than once on S^1.
If $\int_{s^1} P^i Q^j dP = 0$ for all $i, j \geq 0$, then there exists a Riemann Surface X and a homeomorphism $\varphi : S^1 \to X$, such that $\varphi(S^1)$ is a simple closed curve on X bounding a compact region D, such that functions $\tilde{P}$, $\tilde{Q}$ defined on $\varphi(S^1)$ by $P = \tilde{P} \circ \varphi$, $Q = \tilde{Q} \circ \varphi$ can be extended inside D to be analytic there and continuous in $D \cup \varphi(S^1)$.

To use this theorem for our factorization, we need to replace "homeomorphism" by "analytic map of a certain neighborhood of S^1 into X".

Lemma 29. *Let $S^1 \subseteq \mathbb{C}$ be a unit circle, P and Q be analytic functions in a neighborhood U of S^1, such that $P' \neq 0$ on S^1. Let X be a Riemann Surface, $\tilde{P}$ and $\tilde{Q}$ – regular functions on X, and let $\varphi : S^1 \to X$ be a homeomorphism such that $P = \tilde{P} \circ \varphi$, $Q = \tilde{Q} \circ \varphi$ on S^1. Then φ can be extended as an analytic mapping of a certain neighborhood $V \subseteq U$ of S^1 into X, with the same property $P = \tilde{P} \circ \varphi$, $Q = \tilde{Q} \circ \varphi$ in V.*

Proof. $P'(s) \neq 0$, hence $\tilde{P}'(\varphi(s)) \neq 0$, so $\tilde{P}'(y) \neq 0$ in a neighborhood of $\varphi(s)$ on X. We define $\varphi(x)$ in this neighborhood as $y = \varphi(x) = \tilde{P}^{-1}(P(x))$. Locally φ exists and is well-defined. Since these local extensions agree on S^1, they in fact agree and define a required extension on a certain neighborhood of S^1. ∎

Corollary 30. *If in the Abel equation (A) on $\mathbb{C}$*

$$dy = y^2 dP + y^3 dQ \quad or \quad dy = y^2 dQ + y^3 dP$$

P and Q are functions, satisfying all the properties of Wermer's theorem and the domain D, provided by Wermer's theorem, is simply-connected, then the Abel equation (A) has a center.

Proof. If $P' \neq 0$ on S^1, we may apply lemma 29. Then the Abel equation (A) is induced by the analytic mapping φ from the Abel Equation on X

$$dy = y^2 d\tilde{P} + y^3 d\tilde{Q}. \qquad (\tilde{A})$$

Since $\tilde{P}$, $\tilde{Q}$ are analytic on a simply connected domain D bounded by $\varphi(S^1)$, the equation $(\tilde{A})$ has a center along $\varphi(S^1)$, and hence the equation (A) has a center along S^1. ▌

Notice, that this condition is a sufficient condition for center for Abel equation with arbitrary analytic coefficients. But it is symmetric with respect to P and Q, although some of the center conditions for Abel equation are known to be non-symmetric. Below we shall explain it for the case of P, Q – Laurent polynomials.

4.2. The degree of a rational mapping and an image of a circle on a rational curve

Consider two rational functions P, Q. The map $\gamma = [P, Q] : \mathbb{C} \to \mathbb{C}^2$ defines the rational curve $Y = \{(P(t), Q(t)) : t \in \mathbb{C}\}$. Image of a circle S^1 under the map γ is a closed curve on Y.

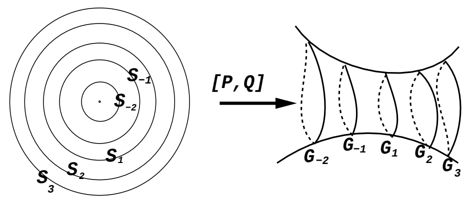
Theorem 31. *Let P, Q be rational functions without poles on S^1, s.t. at least one of them has a pole inside S^1 and at least one of them has a pole outside S^1 (for instance, Laurent polynomials). Let $\gamma(S^1)$ bound a compact domain in Y. Then* $\deg \gamma > 1$.

Proof. Assume that $\deg \gamma = 1$. Then consider a path χ in $\mathbb{C}$, joining two poles of P and Q inside and outside of S^1 (for simplicity 0 and ∞), and intersecting S^1 only once at a *regular* point $u \in \gamma$. We can assume also that χ does not contain preimages of double points in Y. So for any $x \in \chi$ there are no $y \neq x \in \mathbb{C}$ with $\gamma(x) = \gamma(y)$.

$\gamma(\chi(z))$ tends to ∞ as z tends to 0 and to ∞, so the image of $\chi(z)$ under the map $\gamma : \mathbb{C} \to Y$ can not stay inside a compact domain bounded by $\gamma(S^1)$. But it enters this domain, since u is a regular point of γ. Then it must intersect $\gamma(S^1)$ at another point $v \neq u$, and we get contradiction to the choice of the path γ. ▌

Example 1. $P(z) = z, \quad Q(z) = \dfrac{1}{z}.$

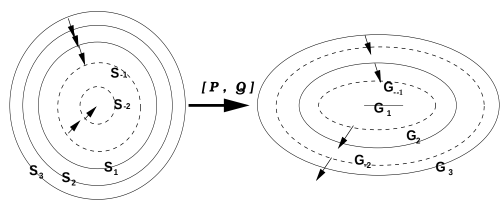
The rational curve Y is $\{xy = 1\} \subseteq \mathbb{C}^2$, and the curve $\gamma(S^1)$ does not bound a compact domain on it. The degree of the map $[P, Q]$ is one, and it is a general situation for a map of degree one: images of circles contracted to poles diverge on Y – see figure 1 with $S_k = \{|z| = k\}$, $S_{-k} = \{|z| = \frac{1}{k}\}$ ($k \in \mathbb{N}$), and G_k, G_{-k} – their images on Y under $\gamma = [P, Q]$.



Map of degree 1

Example 2. $P(z) = z + \dfrac{1}{z}, \quad Q(z) = z + \dfrac{1}{z}.$

The rational curve Y is $\{x = y\} \simeq \mathbb{C}$, the degree of the map γ is 2, and the curve $\gamma(S^1)$ bounds a compact domain on Y (in fact, $S_1 = \gamma(S^1) = [-1, 1]$). See figure 2 for illustration: images of the circles S_k cover Y twice, because they "have no space to diverge". Arrows indicate directions of "motion" of the curves S_k and G_k as k decreases from $+\infty$ to $-\infty$.



Map of degree 2

4.3. Center conditions for the case of P, Q – Laurent polynomials

Theorem 32. *If P and Q are Laurent polynomials, satisfying the condition*

$$\int_{|z|=1} P(z)^k Q(z)^n dP = 0$$

for all pairs (k,n) of nonnegative integers, then P and Q can be represented in the form of Laurent Polynomial composition, *and hence the equation (A) has a center.*

Proof. If both P and Q are algebraic polynomials in z, P and Q are represented as a composition with z, so we have a center.

Similarly, if both P and Q are algebraic polynomials in $\frac{1}{z}$, P and Q are represented as a composition with $\frac{1}{z}$, so we have a center. Otherwise P and Q have one pole inside (origin $z = 0$) and one pole outside (infinity) of S^1.

Obviously P, Q are analytic in a neighborhood of S^1 and take only finitely many values more than once on S^1.

Next, we always can assume that $P'(z) \neq 0$ for all $|z| = 1$. Indeed, (A) after the change of variables $z = \lambda u$ became

$$\frac{dy}{du} = d\hat{P}(u)y^2 + d\hat{Q}(u)y^3, \qquad (\hat{A})$$

where $\hat{P}(u) = \lambda P(\lambda u)$, $\hat{Q}(u) = \lambda Q(\lambda u)$, both (A) and $(\hat{A})$ have center simultaneously. But as P' has only finite number of zeroes on $\mathbb{C}$, by rescaling $z \mapsto \lambda z$, which does not change a center for (A), we can assure that there are no zeroes of $\hat{P}'$ on the circle S^1. For example, $P(z) = z + \frac{1}{z}$ has zeroes of P' on S^1, but $\hat{P}(z) = 4z - \frac{1}{z}$ has not. Under this change of variables the circle $|z| = 1$ goes to $|u| = \lambda$, but by Cauchy theorem for Laurent polynomials integrals along these circles coincide.

We believe that for the case of Laurent polynomials the Wermer's theorem remains valid without the assumption that P and Q together separate points on S^1. The proof together with a detailed investigation of Wermer's surface for the case of P, Q – Laurent polynomials will appear separately.

Hence by Wermer's theorem there exists a surface X and a homeomorphism $\varphi : S^1 \to X$, such that $\varphi(S^1)$ bounds a compact domain D on X and there exists functions $\tilde{P}$, $\tilde{Q}$ analytic inside D..

Remind that we have a map $\gamma = [P, Q] : \mathbb{C} \to Y = \{(P(t), Q(t)) : t \in \mathbb{C}\}$.

Lemma 33. *If the curve $\varphi(S^1)$ bounds a compact domain on X, then the curve $\gamma(S^1)$ bounds a compact domain on a rational curve Y.*

Proof. By lemma 29 $[P, Q] \circ \varphi$ is an analytic mapping, defined in a neighborhood of S^1, which coincides there with $\gamma = [P, Q]$. But hence $[\tilde{P}, \tilde{Q}] : X \to \mathbb{C}^2$ maps a neighborhood of $\varphi(S^1)$ into $Y \subseteq \mathbb{C}^2$. By analytic continuation, $[\tilde{P}, \tilde{Q}]$ maps X into Y. Hence $[\tilde{P}, \tilde{Q}]$ maps a compact domain D inside $\varphi(S^1)$ onto Y, and the image of a compact domain under continuous mapping is compact. Hence $\gamma(S^1)$ is contained in a compact $[\tilde{P}, \tilde{Q}](D)$. Now one can easily show that in fact $\gamma(S^1)$ bounds a compact domain in Y. ▌

Proof of theorem 32 (continue):
By theorem 31 $\deg[P, Q] > 1$, hence P and Q can be represented as a composition $P = \tilde{P}(w)$, $Q = \tilde{Q}(w)$. If $\tilde{P}$ and $\tilde{Q}$ are usual algebraic polynomials, we are done.

If not, then $P(z) = \tilde{P}(z^k)$, $Q(z) = \tilde{Q}(z^k)$ for Laurent polynomials $\tilde{P}$, $\tilde{Q}$. But then on the rational curve Y we get $[P, Q](S^1) = [\tilde{P}, \tilde{Q}](S^1)$, so $[\tilde{P}, \tilde{Q}](S^1)$ bounds a compact domain on $Y = \{(P(t), Q(t)) : t \in \mathbb{C}\} = \{(\tilde{P}(t), \tilde{Q}(t)) : t \in \mathbb{C}\}$. Therefore by theorem 31 $\deg[\tilde{P}, \tilde{Q}] > 1$, so they are represented as a composition.

If we again obtain their representation as a composition of Laurent polynomials with z^n, we repeat our considerations, and finally we are left with the composition $\tilde{P} = \tilde{\tilde{P}}(\tilde{w})$, $\tilde{Q} = \tilde{\tilde{Q}}(\tilde{w})$ with $\tilde{w}$ – Laurent polynomial, $\tilde{\tilde{P}}$ and $\tilde{\tilde{Q}}$ – algebraic polynomials. It gives us the composition $P = \tilde{\tilde{P}}(\tilde{w}(z^N))$, $Q = \tilde{\tilde{Q}}(\tilde{w}(z^N))$, and we are done. ▮

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