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Algebraic Aspects of Integrability for Polynomial Systems

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We present an introductory survey to the Darboux integrability theory of planar complex and real polynomial differential systems. Our presentation contains some improvements to the classical theory.

Key Words: Darboux integrability, exponential factors, invariant algebraic curves.

1. INTRODUCTION

The main part of these notes is devoted to explaining the fascinating connection between the local integrability of polynomial differential equations (a topological phenomena) and the existence of exact solutions for these equations (an algebraic one). However, having built up the appropriate methods, it seemed a good idea to digress into a few closely related areas.

There are now several expositions of the Darboux method of integration [9, 20, 1, 6], so our aim here is not at completeness, but at obtaining a general idea of the methods involved. Once the geometric ideas are grasped (and these are not difficult) the more technical theorems are mostly routine. Some of these have been left as exercises in the text.

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Unfortunately this is not a subject that lends itself nicely to hand calculations except in the simplest cases. However, we have tried to include a few of the simpler computations to give a flavour of the algebra involved. We have also mentioned a few of the more interesting research projects which suggested themselves as we went.

2. ALGEBRAIC CURVES AND DARBOUX INTEGRABILITY

Throughout these notes we will consider the differential equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}, \tag{1}$$

where P and Q are polynomials of degree at most m. Following Poincaré, we introduce a "time" parameter t, which enables us to express (1) as a first order autonomous system in the plane,

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y). \tag{2}$$

One of the most obvious questions to ask is whether the solutions to (1) or (2) are algebraic. By this we mean whether trajectories of (2) can be described implicitly by an algebraic formula, for example $F(x,y) = 0$, where F is a polynomial.

The answer is not easy. Jouanolou, for example, devotes a large section of his Lecture Notes [15], to showing that one particular system has no algebraic solutions. Even the much loved limit cycle in the van der Pol oscillator has only recently been shown to be non-algebraic [17].

The difficulty is that, although we suspect that the vast majority of systems have no algebraic solutions, whenever we find one we can integrate in closed form we find that it has algebraic solutions. The work of Prelle and Singer [18, 23] has shown that this is no coincidence, and that algebraic solutions are a necessary consequence of closed form solutions. A consequence of this is that the method of Darboux [12] which we shall explain below actually captures all of the closed form solutions of (2). We will return to this topic later on.

Suppose (2) has a trajectory (not a singular point) whose path is described by an algebraic curve. That is, it lies within the zero set of a polynomial, $F(x,y) = 0$. It is clear that the derivative of F with respect to time will not change along the curve $F = 0$. Since this derivative can be expressed as a polynomial in x and y which vanishes on $F = 0$, we are lead directly to the equation

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} P + \frac{\partial F}{\partial y} Q = FL, \tag{3}$$

where L must be a polynomial in x and y of degree $m-1$. We shall also write this as $DF = FL$, where

$$D \equiv P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}, \tag{4}$$

is the vector field associated to (2). Thus to each algebraic solution $F = 0$ of (2) we can associate its *cofactor*

$$L_F = \frac{DF}{F}.$$

Conversely, given a polynomial F which satisfies (3), it is easy to see that its zero set must be composed of trajectories of (2). To avoid any unnecessary circumlocution, we call a polynomial solution to (3) an *invariant algebraic curve* of (2).

The study of (3) and the possible degree of F is tackled in the papers of Campillo, Carnicer, Cerveau and Lins Neto [2, 3, 4]. Our interest here is rather in how the existence of algebraic solutions can be used to prove topological properties of the system.

Given several invariant algebraic curves, $C_i = 0$, with cofactors L_i, we find that

$$DB = B\left(\sum l_i L_i\right). \tag{5}$$

where

$$B = \prod C_i^{l_i}. \tag{6}$$

Of course, the above definition only makes sense in the case where $C_i > 0$. However we can revise equation (3) to give

$$\frac{d}{dt}|F| = |F|L, \tag{7}$$

and adapt equations (5) and (6) appropriately.

An alternative approach, which we prefer, is to work essentially over the complex numbers. In this case (6) will be a multivalued function. The advantage of this approach is that in many cases, there are solutions of (3) with complex coefficients even when the coefficients of (2) are real.

If the coefficients of P and Q in (2) are real valued, then we can show that $F = 0$ satisfies equation (3) with cofactor L if and only if $\bar{F} = 0$ satisfies (3) with cofactor $\bar{L}$. Here conjugation denotes conjugation of the coefficients of the polynomials only.

Thus when the coefficients of (3) are real we can combine complex invariant algebraic curves of (6) in complex conjugate pairs to obtain a real function

$$F^l \bar{F}^{\bar{l}} = \left[(\Re F)^2 + (\Im F)^2\right]^{\Re l} e^{-2\Im l \tan^{-1}(\Im F/\Re F)}. \tag{8}$$

When the complex curve $F = 0$ crosses the real plane, then $\Re F = \Im F = 0$ and the function can become multivalued. Some care therefore needs to be taken with using such functions.

For most of these notes we assume tacitly that the equations (3) are real, but that their invariant algebraic curves can be complex. However, when we form products of the form (6), we shall usually assume that the powers are chosen in complex conjugate pairs to obtain a real function generalising (8) above. The context will usually make this clear.

It is a simple matter to show that $FG = 0$ is an invariant algebraic curve if and only if both $F = 0$ and $G = 0$ are also.

We now come on to the main business of the section. Given a system (2), we are interested in the possibility of constructing two types of functions.

A *first integral* is a function $\phi(x, y)$ whose level curves $\phi = c$ describe the trajectories of the system; thus

$$D\phi = 0. \tag{9}$$

An (*reciprocal*) *integrating factor* is a function R, so that the vector field

$$\left(\frac{P}{R}\right)\frac{\partial}{\partial x} + \left(\frac{Q}{R}\right)\frac{\partial}{\partial y}, \tag{10}$$

is divergence free. That is

$$\frac{\partial}{\partial x}\left(\frac{P}{R}\right) + \frac{\partial}{\partial y}\left(\frac{Q}{R}\right) = 0, \tag{11}$$

Alternatively, we write this

$$DR = R\,(P_x + Q_y) = R\Delta, \tag{12}$$

where Δ is a convenient abbreviation for the divergence of (2). Interesting uses of the reciprocal integrating factor can be found in [14] and [5]. For convenience, we drop the term "reciprocal" and refer only to an integrating factor.

If we have a function R which satisfies (12) and which is well-defined at a point, then the system (10) is locally Hamiltonian, and we can find a local first integral ϕ so that

$$\frac{P}{R} = -\frac{\partial \phi}{\partial y}, \qquad \frac{Q}{R} = \frac{\partial \phi}{\partial x}. \tag{13}$$

Conversely, given such a ϕ, we can find an integrating factor R for which (12) holds.

It is interesting to relate this definition of integrating factor to the more familiar one used in solving the first order differential equation

$$\frac{dy}{dx} = p(x)y + q(x).$$

Transforming to the planar system

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = p(x)y + q(x),$$

we have the reciprocal integrating factor

$$e^{\int p(x)\,dx}.$$

The method of Darboux is as follows [12]. We suppose we have several invariant algebraic curves $C_i = 0$ with cofactors $L_i = L_{C_i}$, and we try to construct a function B of the form (6) which satisfies (9) or (12). From (5) and (12), we have

$$\begin{aligned} DB &= B\left(\sum l_i L_i\right), \\ DB - B\Delta &= B\left(\sum l_i L_i - \Delta\right). \end{aligned} \tag{14}$$

Thus the job of finding an explicit first integral or integrating factor of (2) is reduced to finding values of the l_i such that

$$\sum l_i L_i = 0, \quad \text{or} \quad \sum l_i L_i - \Delta = 0, \tag{15}$$

respectively. In the first case, we have a first integral of the system (possibly multivalued), and in the second case, we obtain a first integral after one quadrature. That is we have

$$\phi = \int \frac{P\,dy - Q\,dx}{B}. \tag{16}$$

Some interesting historical remarks on the method of Darboux can be found in [21]. Later on we shall discover that this method captures all solutions of (2) which can be found by quadratures.

The system (2) satisfies one of the conditions of (15) if and only if Δ and the L_i's are linearly dependent. Hence any system with more than $m(m+1)/2-1$ invariant algebraic curves must satisfy (15).

Given a family of functions of the form (6), it would be nice to extend our class of functions to include the limits of these functions. Suppose, for example, we have two invariant algebraic curves, $C=0$ and $C+\epsilon E=0$, where ϵ is a parameter in the family. We find that $L_{C+\epsilon E}=L_C+\epsilon L'+O(\epsilon^2)$, where L' is another polynomial of degree $m-1$. Now consider the expression

$$\left(\frac{C+\epsilon E}{C}\right)^{1/\epsilon},$$

with cofactor $L'+O(\epsilon)$. As ϵ tends to zero, the expression above tends to

$$F=e^{E/C},$$

and it is clear that this expression satifies

$$DF=L'F. \tag{17}$$

Thus, the function F satisfies the same equation as (3), with a cofactor of degree at most $m-1$.

DEFINITION 2.1. We call a (multi-valued) function of the form

$$e^{E/C}\prod C_i^{l_i} \tag{18}$$

with E, C and the C_i all polynomials, a *Darbouxian* function. A function of the form $F=e^{E/C}$ satisfying (17) with L a polynomial of degree at most $m-1$ will be called an *exponential factor*. We term L the *cofactor* of F as before.

An alternative term for exponential factor, used in [9], was “degenerate invariant algebraic curve”, to emphasise their origin from the coalescence of invariant algebraic curves. The name above seems preferable since $e^{E/C}$ is neither algebraic nor a curve!

Conversely, if $F=e^{E/C}$ is an exponential factor then C is an invariant algebraic curve, and E satisfies an equation

$$DE=EL_C+CL_F,$$

where L_C is the cofactor of C and L_F is the cofactor of F.

We shall show later on that, under some generic assumptions, all exponential factors arise as two invariant algebraic curves coalesce. There seems to be here a close analogy here with the theory of linear differential equations when two solutions $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ coalesce to give one solution $e^{\lambda x}$, and a degenerate one $xe^{\lambda x}$. In matrix notation we get

$$\frac{d}{dt}\begin{pmatrix} E \\ C \end{pmatrix} = \begin{pmatrix} L_C & L_F \\ 0 & L_C \end{pmatrix}\begin{pmatrix} E \\ C \end{pmatrix},$$

which is reminiscent of the degenerate block which arises in the Jordan normal form.

The theory in the previous section goes through essentially unchanged, with some of the algebraic curves replaced by exponential factors. So that the existence of more than $m(m+1)/2-1$ invariant algebraic curves or exponential factors implies the existence of a Darbouxian integrating factor, and more than $m(m+1)/2$ implies the existence of a Darbouxian first integral.

It seems that in all the applications to date, the Darbouxian functions are the natural completion of families of functions of the form (6). Is is possible to make this argument more formally or are there other families of functions which we need to take into account here?

We shall consider a simpler version of this problem later on. However we note here that generalising the matrix above gives no new results. That is, suppose we have a family of polynomials which satisfy

$$\begin{aligned} DC_0 &= C_0 L_0 \\ DC_1 &= C_1 L_0 + C_0 L_1 \\ DC_2 &= C_2 L_0 + C_1 L_1 + C_0 L_2 \\ \vdots & \qquad \vdots \end{aligned}$$

then we can consider the generating function

$$\psi = C_0 + zC_1 + z^2 C_2 + \cdots$$

which gives

$$D\,ln(\psi) = L_0 + zL_1 + zL_2 + \cdots .$$

Hence we can find exponential factors for each of the L_i. For example we have,

$$DF = FL_2, \qquad F = e^{(2C_2C_0 - C_1^2)/2C_0^2}.$$

Similar ideas were first considered in [13] which we have adapted here. We thank S. Walcher for pointing out this reference.

3. THE DARBOUX METHOD AND CENTRES

One of the main applications of the Darboux method is proving the existence of a centre. We pause here a little to explain this.

In the elementary theory of qualitative differential equations we identify three main types of behavior at a non-degenerate critical point: a node, a focus, or a saddle. All three are stable, in that a small perturbation will not change the stability of the critical point. Moreover, topologically, we can read off their behaviour from just their linear terms.

However, there is also the possibility that the critical point is a *fine focus* or a *centre*. That is, the divergence vanishes at that point. In this case the linear terms give a centre: a neighbourhood of the origin which consists of closed trajectories.

In this case, the non-linear terms must be examined in order to determine whether the point is stable or unstable. If it is neither and the system is analytic, then the critical point is also a centre for the nonlinear system. Without loss of generality, we can consider the critical point to be at the origin and to be in the form

$$\dot{x} = \lambda x - y + p(x,y), \qquad \dot{y} = x + \lambda y + q(x,y). \tag{19}$$

where p and q represent the nonlinear terms. The case of a fine focus or centre corresponds to $\lambda = 0$.

We can distinguish between a centre and a (fine) focus in a number of ways; we follow the most direct first. It can be shown that for any N there is a change of coordinates which brings the origin of (19) to the polar form

$$\dot{r} = c_3 r^3 + c_5 r^5 + \cdots + O(r^N),\ \dot{\theta} = 1 + d_2 r^2 + d_4 r^4 + \cdots + O(r^N), \tag{20}$$

If all the c_i are zero upto c_{2k+1}, then the system is said to have a *fine focus of order* k. The stability is given by the sign of c_{2k+1}.

It can also be shown that perturbations of the nonlinear terms of (19) can produce in this case at most k limit cycles bifurcating from the origin. Furthermore, if the class of systems (19) are sufficiently general, there are perturbations which produce this number of limit cycles in a multiple Hopf bifurcation. We call c_{2k+1} the k-th *Liapunov quantity*.

If all the c_i are zero, then it can be shown that there is an analytic change of coordinates which brings the system into the polar form

$$\dot{r} = 0, \qquad \dot{\theta} = 1 + d_2 r^2 + d_4 r^4 + \cdots .$$

The critical point is obviously a centre in this case.

Given a class of polynomial equations in the form (19), we are often interested in the sub-class with centres at the origin. The problem is that showing that we have a centre requires an infinite number of conditions. However, if we could find an analytic first integral in a neigbourhood of the critical point, then the critical point must be a centre. In fact, the linear terms of (19) implies that the first terms of such an integral are $a + b(x^2 + y^2)^s + \cdots$, and therefore trajectories close to the origin are closed.

Alternatively, from what was said in Section 2, we could find an integrating factor which is well-defined and non-zero in a neigbourhood of the critical point. In either case an obvious method for constructing such functions is the Darboux method. The surprising thing is that this method is so successful.

THEOREM 3.1. *All the non-degenerate centres of systems (19) with homogeneous quadratic or cubic p and q are integrable with Darbouxian first integrals. The same is true if p and q are of the form*

$$p = p_2 + xf, \qquad q = q_2 + yf,$$

where p_2, q_2 and f are all homogeneous quadratics.

For a proof of this theorem see [20, 19, 1].

The last system is the projective version of the quadratic systems and in fact was the system studied by Darboux.

As an example of this, consider the system

$$\begin{aligned} \dot{x} &= y + a_1x^2 + (a_2 + 2b_1)xy - a_1y^2 + x^2y, \\ \dot{y} &= -x + b_1x^2 + (b_2 - 2a_1)xy - b_1y^2 + xy^2, \end{aligned}$$

generically this has 4 invariant lines, whose cofactors are all of the form

$$\alpha x + \beta y + xy.$$

Hence we can conclude that the system has a centre at the origin.

Another method for distinguishing between a focus and a centre is to use a *Liapunov function.* This is a function $V = k + x^2 + y^2 + O((x^2 + y^2)^3/2)$ which satisfies

$$\frac{dV}{dt} = \eta_4(x^2 + y^2)^4 + \eta_6(x^2 + y^2)^6 + \ldots + O((x^2 + y^2)^{N/2}). \qquad (21)$$

Such a function can always be found in the neighbourhood of a fine focus. The origin is a centre if all the η_i vanish. If η_{2k+2} is the first non-zero term, then the origin is a fine focus of order k. Computationally, this method is easier to handle than the normal form (20). The coefficients η_{2i+2} are essentially positive multiples of the c_{2i+1} if we assume that the previous η_{2j+1}, $j < i$ vanish.

There is also a third method, closely related to the second. Here we seek a function which is almost an integrating factor. That is we look for a function $R = 1 + O(x, y)$ which satisfies

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{P}{R}\right) + \frac{\partial}{\partial y}\left(\frac{Q}{R}\right) &= \\ \zeta_2(x^2+y^2) \ + \ \zeta_4(x^2+y^2)^4 + \cdots &+ O((x^2+y^2)^{N/2-1}), \end{aligned} \tag{22}$$

Here the origin is a centre if all the ζ_i vanish, and a fine focus of order k if ζ_{2k} is the first non-zero term. The advantage here is that the degrees of the polynomials required here is less than (21). Again the coefficients ζ_{2i} are essentially positive multiples of the c_{2i+1} modulo the previous ζ_{2j}'s.

Now, suppose we are seeking conditions for a centre at the origin. Consider a Darbouxian function B which is composed of invariant algebraic curves which do not pass through the origin and exponential factors. Such a B is well-defined at the origin and

$$\begin{aligned} \frac{d}{dt}B &= B\left[\sum l_i L_i\right], \\ \frac{\partial}{\partial x}\left(\frac{P}{B}\right) + \frac{\partial}{\partial y}\left(\frac{Q}{B}\right) &= \frac{1}{B}\left[\sum l_i L_i - \Delta\right]. \end{aligned} \tag{23}$$

Since the invariant algebraic curves do not pass through the origin, then (3) implies that the cofactors L_i must vanish there. We also assume that the divergence is zero at the origin, or there would be no centre. If we have at least $m(m+1)/2-1-q$ invariant algebraic curves and exponential factors, $0 < q \leq (m-1)/2$, then we can choose the l_i non trivially so that the square brackets of one of the expressions above lies in the vector space generated by the polynomials $(x^2+y^2)^j$, $j = 1, \ldots q$. Comparing these expressions with (21) and (22), we obtain the following result.

THEOREM 3.2. *Suppose the origin is a fine focus, and that the first q Liapunov quantities at the origin vanish, $0 < q \leq (m-1)/2$. If there are at least $m(m+1)/2-1-q$ invariant algebraic curves or exponential factors not passing through the origin, then there is a local Darbouxian integrating factor. If there are at least $m(m+1)/2-q$, then there is a Darbouxian first integral. In either case the origin is a centre.*

The result was first noticed by Cozma and Şubă [11, 22] using different methods.

4. FURTHER RESULTS

We gather together some further results on the method of Darboux. The first is a very useful geometric consequence of a simple observation.

Assume $F = 0$ is an invariant algebraic curve or exponential factor. Since $dF/dt = 0$ at any critical point of (2) then, from (3), either F vanishes at the point or its cofactor does. Thus, given a collection of critical points $p_1, \ldots, p_q$, any invariant algebraic curve which does not pass through these points must have a cofactor which vanishes at these points. If the q points above are in general position with respect to polynomial functions of degree $m-1$ then the existence of more than $m(m+1)/2-q-1$ invariant algebraic curves or exponential factors implies a Darbouxian integrating factor; and more than $m(m+1)/2-q$ implies that there is a Darbouxian first integral.

For more details on this see [6].

What happens when we get more invariant algebraic curves than we need for a Darbouxian first integral? It turns out [15] that we get a rational first integral. To see this, suppose we have $m(m+1)/2+2$ invariant algebraic curves or exponential factors C_i, with cofactors L_i. It is clear that we get two linear dependencies, which after some linear algebra and relabeling, we can write as

$$L_1 + \sum_{i>2} l_i L_i = 0, \qquad L_2 + \sum_{i>2} m_i L_i = 0.$$

The functions

$$\phi_1 = \ln(C_1) + \sum_{i>2} l_i \ln(C_i), \qquad \phi_2 = \ln(C_2) + \sum_{i>2} m_i \ln(C_i),$$

are first integrals of the system, being logarithms of the associated Darbouxian first integrals. Each gives rise to an integrating factor, that is, the functions R_i so that

$$P = -\phi_{iy} R_i, \qquad Q = \phi_{ix} R_i,$$

where ϕ is the logarithm of the Darbouxian function. More detailed calculation shows that these integrating factors are rational and must be independent. Their quotient is a rational first integral of the system.

If R/S is the rational first integral as above, then all algebraic solutions of (2) must be a factor of one of the curves $R-cS=0$ with c some constant. Thus given a fixed system (2) the degrees of all the algebraic curves of the system is bounded, since either there is only finitely many of them, or there is a rational first integral and the degrees are bounded anyway.

Things are more complex, however, if we want a bound for the degrees of the algebraic curves dependent only on the degree of the system. In fact, there are simple examples to show that such a bound cannot exist. The best we can hope for is that the systems with high degree algebraic curves have rational first integrals for example. A similar conjecture would be:

Conjecture 4.1. There exists a bound $N(m)$ for which any system having an invariant algebraic curve has also an algebraic curve of degree $\leq N(m)$.

In particular, the set of coefficients of (2) with algebraic curves would lie in an algebraic subset. At the moment, the best we can say is that it lies in a union of a countable number of algebraic sets. The work of Lins Neto and others has shown that this set is not everywhere dense [4].

5. NON-EXISTENCE OF LIMIT CYCLES

We take a diversion from our main theme to examine a closely related problem. We rename a *Liapunov function* a function $\phi(x, y)$ for which

$$D\phi > 0$$

in the region of interest. Clearly the existence of such a function in a region precludes the existence of periodic solutions and, in particular, limit cycles.

In the same way, we define a *Dulac function* to be a function R such that

$$\frac{\partial}{\partial x}\left(\frac{P}{R}\right) + \frac{\partial}{\partial y}\left(\frac{Q}{R}\right) > 0.$$

By applying the divergence criterion, we can see that such a function also precludes the existence of periodic solutions or limit cycles in any simply connected region where R is well-defined and non-zero.

The analogy between these functions and the first integrals and integrating factors examined up to now is obvious. In particular, given a collection of invariant algebraic curves or exponential factors with cofactors L_i, if we can find constants l_i such that

$$\sum l_i L_i > 0, \quad \text{or} \quad \sum l_i L_i + \Delta > 0,$$

then there are no limit cycles in any simply connected region where the Darbouxian function is well-defined.

For example, the Lokta-Volterra equations are quadratic with two invariant lines. There is also a critical point which does not lie on either line around which any limit cycle must lie. Thus the cofactors of the two lines must vanish at this point. If the divergence also vanishes at the critical point, then we can find a linear dependency between the cofactors and the divergence which means that the system must be integrable and we have a family of closed orbits. If the divergence does not vanish then we can construct a Dulac function of Liapunov function as above. Thus there are no limit cycles in either case.

The example above shows in a simple way how detailed calculations can be reduced to simple geometric arguments instead. We could have replaced the lines above by invariant hyperbolas or parabolas with no increase in difficulty.

If one of the curves was an ellipse, however, we might have problems. First, the ellipse may be a limit cycle in its own right. Second, if the polynomial representing the ellipse appeared to a negative power in the Dulac function, then we can not apply Green's theorem since the region surrounding the ellipse is not simply connected. This can be overcome in certain cases by considering line integrals around the loop itself.

In order to get some deeper results, we need to allow some curves which are almost invariant. Rather than abstract things more, we give an example which is very representative of other results.

THEOREM 5.1. *Suppose a quadratic system has an invariant algebraic curve and a critical point not on this curve where the divergence vanishes, then the system has no limit cycles in any simply connected region of the complement of the curve.*

Proof. Any limit cycle in a quadratic system surrounds only one critical point which must be a focus. Suppose a limit cycle surrounds a critical point with non-zero divergence. Let $C = 0$ be the invariant algebraic curve with cofactor L. Thus

$$(PC^r)_x + (QC^r)_y = C^r(rL + \Delta).$$

Since both L and Δ must pass through the other critical point, and limit cycles of quadratic system must be convex, we can chose r so that $rL + \Delta$ does not pass through the limit cycle. Hence we have a Dulac function in this case, which contradicts our assumption of a limit cycle.

Suppose now that there is a limit cycle which surrounds the divergence free critical point. Since this point must be a focus we transform the system to the form

$$P = -y + ax^2 + bxy + cy^2, \qquad Q = x + dx^2 + exy + fy^2.$$

A further rotation allows us to set $c = 0$ without loss of generality. Consider the function $G = x - 1/b$ for $b \neq 0$ and $G = e^x$ if $b = 0$. We calculate

$$\frac{d}{dt}G = \begin{cases} byG + ax^2, & b \neq 0, \\ -yG + ax^2G, & b = 0. \end{cases}$$

Thus the line $G = 0$ is a transversal, and no limit cycle can cross it. Now, we calculate that

$$(PC^rG^s)_x + (QC^rG^s)_y = \begin{cases} C^rG^{s-1}(G[rL + sby + \Delta] + sax^2), & b \neq 0, \\ C^rG^{s-1}(G[rL - sy + \Delta] + sGax^2), & b = 0. \end{cases}$$

In either case we can find values of r and s to eliminate the term in square brackets. Once again we have a Dulac function. ▮

It seems that algebraic curve methods are the natural ones for proving non-existence of limit cycles. In Coppel's survey paper [10], for example, all the non-existence results are obtained this way except one which uses a Liénard system argument. We reprove this here using algebraic curves.

THEOREM 5.2. *A quadratic system with $rP + sQ = \Delta M$ for some polynomial M and real numbers r and s can have no limit cycles. In particular, a quadratic system with two critical points with zero divergence has no limit cycles.*

Proof. If Δ is a constant we have finished. If $\Delta = a(rx + sy + t)$ for some a and t, then $\Delta = 0$ would be an invariant line. All limit cycles would have to lie in one of the regions $\Delta > 0$ or $\Delta < 0$ which is not possible. Hence the linear terms of Δ must be linearly independent from $rx + sy$ for limit cycles to exist. We can therefore write $M = a\Delta + b(rx + sy) + c$ for some a, b and c.

If $b = 0$, then

$$\frac{d}{dt}(rx + sy) = a\Delta^2, \qquad (c = 0),$$

or

$$(e^{-(rx+sy)/c}P)_x + (e^{-(rx+sy)/c}Q)_y = -\frac{a}{c}\Delta^2 e^{-(rx+sy)/c}, \qquad (c \neq 0)$$

Hence there are no limit cycles. When $b \neq 0$ then

$$\frac{d}{dt}(rx + sy + \frac{c}{b}) = a\Delta^2 \quad on \quad rx + sy + \frac{c}{b} = 0,$$

so $rx + sy + c/b = 0$ is a transversal. Furthermore

$$(BP)_x + (BQ)_y = -\frac{a}{b(rx + sy + c/b)} B\Delta^2,$$

where

$$B = (rx + sy + \frac{c}{b})^{-1/b},$$

and so we have a Dulac function in this case too. ▮

Is it possible to extend these methods to prove the uniqueness of limit cycles if we allow the Dulac function to vanish at the critical point which the limit cycle surrounds?

It would be very nice if this was true as many of the uniqueness results for quadratic systems have a nice algebraic content. For example a quadratic system with an invariant line or an invariant parabola have at most one limit cycle. However, no such methods are known at the moment and other less direct methods need to be used.

We finish with a simple example of such a proof. We show that the van der Pol oscillator has a unique limit cycle for small values of the nonlinear terms.

THEOREM 5.3. *The system*

$$\dot{x} = y, \qquad \dot{y} = -x - \mu(1 - x^2)y$$

has at most one limit cycle for $|\mu| < \sqrt{3}$.

Proof. We first let $Y = y + \mu(x - x^3/3)$ to transform the system to the Liénard plane:

$$\dot{x} = Y - \mu(x - x^3/3), \qquad \dot{Y} = -x.$$

Now, taking $B = (x^2 - \mu x Y + Y^2)^{-1}$, we have

$$(B\dot{x})_x + (B\dot{y})_y = \mu B^2 x^2 (x^2/3 - 2\mu x Y/3 + Y^2).$$

For $|mu| < \sqrt{3}$, B is positive definite and we have a Dulac function defined in the whole of the plane except at the origin. Since each limit cycle must surround the origin, a simple application of Green's theorem shows that there can be at most one limit cycle. ∎

6. THE INVERSE PROBLEM

It is always helpful to look at a problem from many different angles. In this section, we take the alternative viewpoint of starting with a given set of algebraic curves and determining what form the systems which have such a set of invariant algebraic curves must look like. The details can be found in [7, 8], which are currently under revision. The latter also contains the details from the previous section.

It turns out that to do this we not only need to work over the complex numbers, but also need to give conditions about what the curve does "at infinity". That is, we need to consider our polynomials in the complex projecive plane $\mathbf{P}^2$. We therefore introduce a third coordinate z and say that two coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) are the same if they differ by a constant multiple. The points (x, y) in $\mathbf{C}^2$ correspond to the points $(x, y, 1)$ in $\mathbf{P}^2$.

To every curve $C(x, y) = 0$, we associate a homogenous polynomial

$$\tilde{C}(x, y, z) = z^{\deg C} C\left(\frac{x}{z}, \frac{y}{z}\right).$$

If $\tilde{C}(x,y,z)=0$ then $\tilde{C}(\lambda x,\lambda y,\lambda z)=0$ so the zero set of $\tilde{C}$ is well defined in $\mathbf{P}^2$.

An algebraic curve $C=0$ is said to be *non-singular* if there is no point on the curve $\tilde{C}$ where

$$\frac{\partial}{\partial x}\tilde{C}=\frac{\partial}{\partial y}\tilde{C}=\frac{\partial}{\partial z}\tilde{C}=0.$$

From the implicit function theorem, this means that at every point on the curve there is a local change of coordinates which would make the curve flat. For points in $\mathbf{C}^2$ the condition above reduces to

$$\frac{\partial}{\partial x}C=\frac{\partial}{\partial y}C=C=0,$$

by Euler's theorem.

THEOREM 6.1. *Suppose $C=0$ is a non-singular curve. Then any system having $C=0$ as an invariant algebraic curve is of the form*

$$\dot{x}=AC-DC_y, \qquad \dot{y}=BC+DC_x, \tag{24}$$

where $\deg(A)$, $\deg(B)\le m-\deg(C)+1$ *and* $\deg(D)\le m-\deg(C)+2$. *If the highest order terms of C are distinct (that is $C=0$ crosses the line at infinity transversally) then these bounds can be reduced by one.*

For example, a quadratic system with an invariant non-singular cubic curve must have $A=B=0$ in (24) provided the curve has distinct branches at infinity, and so the system must be Hamiltonian. We give more examples of the applications of this theorem in the next section.

We say that a collection of non-singular curves has *normal crossings* if all curves intersect transversally and no more than two meet at any point.

THEOREM 6.2. *Given a collection of non-singular curves $C_i=0$ with $N=\sum\deg(C_i)$ and with normal crossings, then any system which has these as invariant curves must be of the form*

$$\dot{x}=AK+\sum h_iK_iC_{iy}, \qquad \dot{y}=BK+\sum h_iK_iC_{ix}, \tag{25}$$

where we define

$$K=\prod C_i, \qquad K_j=\prod_{i\neq j}C_i.$$

Furthermore we can take $\deg(A)$, $\deg(B)\le m-N+1$ *and* $\deg(h_i)\le m-N+2$. *If all the curves cross the line at infinity in distinct places then we can reduce these bounds by one.*

The last remark is really just saying that the line at infinity has normal crossings with the other curves. We can allow the curves to be slightly more degenerate but then the theorem becomes correspondingly much weaker. See [8] or [25] for examples of this.

An interesting question here is whether we can generalise these results for simple singularities in an effectively computational way. Some preliminary results in this direction have been obtained by Walcher [24].

There are also versions of (25) for systems with exponential factors too. We give a simple example here as it illustrates that "generically" an exponential factor can always be considered as the result of two algebraic curves merging.

THEOREM 6.3. *Suppose a system has an exponential factor* $e^{E/C}$, *where* $E = 0$ *and* $C = 0$ *are non-singular and have normal crossings with themselves and the line at infinity. Then the system is of the form*

$$\begin{aligned} \dot{x} &= A_0C^2 - A_1CC_y - A_2(E_yC - EC_y), \\ \dot{y} &= A_3C^2 + A_1CC_x + A_2(E_xC - EC_x), \end{aligned} \tag{26}$$

where $\deg(A_i)$ *satisfy the expected bounds. In particular, the exponential can be seen to be the limit of the invariant curves* $C = 0$ *and* $C + \epsilon E$ *as* ϵ *tends to zero in the family of systems*

$$\begin{aligned} \dot{x} &= A_0\,C(C+\epsilon E) - (A_1 + \epsilon^{-1}A_2)C(C+\epsilon E)_y + \epsilon^{-1}A_2C_y(C+\epsilon E), \\ \dot{y} &= A_3\,C(C+\epsilon E) + (A_1 + \epsilon^{-1}A_2)C(C+\epsilon E)_x - \epsilon^{-1}A_2C_x(C+\epsilon E). \end{aligned}$$

with limit (26) as ϵ *tends to zero.*

Another result which is very useful for limiting the possibility of algebraic curves is the following. It follows directly from equating the highest order terms of (3).

THEOREM 6.4. *If the system (2) has an invariant algebraic curve* $C = 0$, *then the linear factors of the highest degree terms of* C *must divide the terms of degree* $m + 1$ *in* $xQ - yP$.

Are there useful generalisations of this theorem which relate the next highest order terms of P and Q with those of C? Is it possible to classify all algebraic curves in quadratic systems this way by taking enough of these higher order terms (a simple comparison of the number of unknowns would suggest it, but things are rarely that simple in this subject!) This result has been used in [16] to obtain a faster computational approach to the Prelle-Singer algorithm (see Section 8).

7. MORE NON-EXISTENCE RESULTS

We apply the work of the previous section to find conditions under which a system is Darboux integrable, or has a Dulac function. The first is immediate from the last section.

THEOREM 7.1. *Suppose a polynomial system of degree m with invariant algebraic curves $C_i = 0$ which are non-singular and have normal crossings then the system has a Darbouxian integrating factor of the form $\prod C_i$ when*
(i) $\sum \deg(C_i) = m + 2$, or
(ii) $\sum \deg(C_i) \geq m + 1$ and the C_i all have distinct factors in their highest order terms.
In both cases there is a Darbouxian first integral.

For example, a cubic system with two invariant lines cannot have an invariant ellipse without the system being integrable.

THEOREM 7.2. *A polynomial system of degree m with a non-singular algebraic curve C of degree $m + 1$ can have no limit cycles except for the curve itself. The same is true if the curve is of degree m and the highest order terms of C has distinct factors.*

Proof. From the previous section the system must be of the form

$$P = aC - (dx + ey + f)C_y, \qquad Q = bC + (dx + ey + f)C_x. \tag{27}$$

Note that the line $dx + ey + f = 0$ cannot be crossed by a limit cycle. This follows from the fact that the sign of C on the limit cycle cannot change and

$$\frac{d}{dt}(dx + ey + f) = C(da + eb) \quad on \quad dx + ey + f = 0.$$

We now find that $(dx + ey + f)C$ is a Dulac function with

$$\left(\frac{P}{(dx + ey + f)C}\right)_x + \left(\frac{Q}{(dx + ey + f)C}\right)_y = -\frac{ad + be}{(dx + ey + f)^2}.$$

Of course this only works if there are no components of $C = 0$ inside the limit cycle. If this is violated, we consider the curves $C = \pm\epsilon$ for ϵ small enough, and the line integral

$$\int_\gamma \frac{P}{(dx + ey + f)C}\, dy - \frac{Q}{(dx + ey + f)C}\, dx,$$

where γ is the components of $C = \pm\epsilon$ inside the limit cycle, with appropriate orientations. Clearly this integral is equivalent to

$$\int_\gamma \frac{a\,dy - b\,dx}{(dx + ey + f)},$$

which is well defined and must tend to zero as ϵ tends to zero. An application of Green's theorem completes the result. ∎

In fact, in the case where the curve is of degree $m+1$ the system is integrable, for the highest order terms in (27) imply that

$$aC^+ - (dx + ey)C^+_y = bC^+ + (dx + ey)C^+_x = 0,$$

where C^+ represents the highest order terms of C. This implies that $a = Ne$, $b = -Nd$ and $C^+ = r(dx + ey)^N$ where $N = \deg(C)$ and r is some constant. In turn this means that the line $dx + ey + f = 0$ is invariant. The Dulac function is in fact an integrating factor for the system.

Another generalisation which runs on similar grounds is

THEOREM 7.3. *Suppose a polynomial differential system of degree m has invariant algebraic curves $C_i = 0$ which are all non-singular with normal crossings and distinct factors in their highest order terms apart from one curve which is a parabola. If $\sum \deg(C_i) = m + 2$ then the system has no limit cycles.*

Thus a cubic system with two invariant lines and a parabola in general position can have no limit cycles. These theorems generalise many of the known non-existence theorems in quadratic systems.

Can these results be generalised to more singular systems? Unfortunately, it seems that relaxing these conditions just a little allows too many free parameters, and hence limit cycles. Maybe, however, there are results which can show that these limit cycles are unique.

8. ELEMENTARY AND LIOUVILLIAN FIRST INTEGRALS

We now examine the effectiveness of the Darboux method—what sort of integrals does it capture. The surprising result is that in some sense it captures every "closed form solution". However, we clearly need to make this idea precise before we can explain the known results.

The idea of calculating what sort of functions can arise as the result of evaluating an indefinite integral or solving a differential equation goes back to Liouville. The modern formulation of these ideas is usually done through differential algebra. The advantage over an analytic approach is first that the messy details of branch points etc, is hidden completely, and second that the way is open to apply these methods to symbolic computation.

We assume that the set of functions we are interested in form a field together with a number of *derivations*. We call such an object a *differential field*. The process of adding more functions to a given set of functions is described by a tower of such fields:

$$F_0 \subset F_1 \subset \cdots \subset F_n.$$

Of course, we must also specify how the derivations of F_0 are extended to derivations on each F_i.

The fields we are interested in arise by adding exponentials, logarithms or the solutions of algebraic equations based on the previous set of functions. That is we take

$$F_i = F_0(\theta_1, \ldots \theta_i),$$

where one of the following holds:

(i) $\delta\theta_i = \theta_i \delta g$, for some $g \in F_{i-1}$ and for each derivation δ.
(ii) $\delta\theta_i = g^{-1}\delta g$, for some $g \in F_{i-1}$ and for each derivation δ.
(iii) θ_i is algebraic over F_{i-1}. If we have such a tower of fields, F_n is called an *elementary* extension of F_0.

This is essentially what we mean by a function being expressible in closed form. We call the set of all elements of a differential field which are annihilated by all the derivations of the field the *field of constants*. We shall always assume that the field of constants is algebraically closed.

THEOREM 8.1. *(Liouville) If an element f in a differential field F is the derivative of an element g in an elementary extension field G with the same field of constants, then we must have*

$$f = h_0 + \sum c_i \ln(h_i),$$

where c_i are constants and all the h_i lie in F.

We say that our system (2) has an *elementary* first integral if there is an element u in an elementary extension field of the field of rational functions $\mathbf{C}(x, y)$ with the same field of constants such that $Du = 0$. The derivations on $\mathbf{C}(x, y)$ are of course d/dx and d/dy.

THEOREM 8.2. *(Prelle and Singer [18]) If the system (2) has an elementary first integral, then there is also an elementary first integral of the form*

$$v_0 + \sum c_i \ln(v_i),$$

where the c_i are constants and the v_i are algebraic functions over $\mathbf{C}(x, y)$.

It is known that we cannot strengthen this theorem to make all the v_i rational functions in x and y. By manipulating this formula and taking traces we obtain the following corollary.

COROLLARY 8.1. *In the situation above there is always an integrating factor of the form* $R^{1/N}$*, with* $R \in \mathbf{C}(x,y)$ *and* N *an integer.*

Thus the method of Darboux finds all elementary first integrals.

Another class of integrals we are interested in are the *Liouvillian* ones. Here we say that an extension F_n is a *Liouvillian* extension of F_0 if there is a tower of differential fields as above which satisfies conditions (i), (iii) or

$(ii)'$ $\delta_\alpha \theta_i = h_\alpha$ for some elements $h_\alpha \in F_{i-1}$ such that $\delta_\alpha h_\beta = \delta_\beta h_\alpha$.

This last condition, mimicks the introduction of line integrals into the class of functions. clearly (ii) is included in $(ii)'$.

This class of functions represents those functions which are obtainable "by quadratures". An element u of a Liouvillian extension field of $\mathbf{C}(x,y)$ with the same field of constants is said to be a *Liouvillian* first integral.

THEOREM 8.3. *(Singer [23]) If the system (2) has a Liouvillian first integral, then it has an integrating factor of the form*

$$e^{\int U dx + V dy}, \qquad U_y = V_x,$$

where U *and* V *are rational functions.*

It can be shown that this last expression can always be integrated to get a Darbouxian function. More specifically,

THEOREM 8.4. *Let* U*,* V *be two rational functions with*

$$U_y = V_x,$$

then

$$\int U\,dx + V\,dy = w_0 + \sum c_i \ln(w_i),$$

for some constants c_i *and* w_i *rational functions.*

Hence we have

COROLLARY 8.2. *If the system (2) has a Liouvillian first integral, then there is a Darbouxian integrating factor.*

Thus the method of Darboux finds all Liouvillian solutions. However, there is another surprising result.

THEOREM 8.5. *(Singer [23]) Suppose that a trajectory of (2) can be described by a function in a Liouvillian extension of* $\mathbf{C}(x,y)$. *Then either this function is a first integral of the system, or the function is a polynomial.*

Thus a system has a Darbouxian integrating factor, or the only trajectories that can be described by closed form solutions or quadratures are the polynomial ones.

What does the general first integral of a system with a Darbouxian integrating factor look like? Generically we can show that the first integral is also Darbouxian [7], but stranger things can happen. A reasonable conjecture which embodies all the cases which we know is that it is a sum of a Darbouxian function and terms of the form

$$\int^{R(x,y)} e^{s(u)} \prod k_i(u)^{l_i} \, du \ ,$$

where R, s and the k_i are rational functions. Several examples of these have been given by Żołądek.

Lets return to the problem we mentioned previously about the limits of Darbouxian functions. Suppose we have family of systems with Darbouxian first integrals where the C_i have fixed degrees, though their coefficients are parametrised by the family. Each member of the family must have an integrating factor of the form $\prod C_i/G$, where $G = 0$ is another invariant algebraic curve.

Suppose that this family is sufficiently well-behaved so that $G = 1$ and the C_i tend to definite polynomials at the limit points of the family. Now consider the system which is a limit of the family above. It is clear that this system will also have a polynomial integrating factor, the limit of the integrating factors in the family. However, as above, the system can be integrated using this factor to get a Darbouxian first integral. Thus in this case we get Darbouxian limits to Darbouxian functions.

9. THE CENTRE PROBLEM

We have seen how Darbouxian integrating factors account for all first integrals of polynomial systems which can be obtained by quadratures. We want to investigate to what extent they account for all first integrals. Since not even the Darbouxian first integrals are single-valued globally, it makes sense to ask this question locally. That is, we want to be able to classify all systems with a local first integral.

Standard theory tells us that there is always a local first integral about any point of (2) which is not a critical point, so we restrict our attention to critical points with local first integrals. In fact, we shall consider here only those critical points whose linear terms give centres.

As we saw before. A critical point like this is a centre if and only if has a local first integral. Thus there is an intimate connection between the local topological and local analytic properties of these critical points.

What mechanisms can be seen to imply the existence of a centre? We have already seen one—the existence of a Darbouxian first integral or integrating factor. Another one is symmetry. Let us consider this in a number of ways.

First, consider the system

$$\dot{x} = -y + p(x^2, y), \qquad \dot{y} = x + xq(x^2, y). \tag{28}$$

The critical point at the origin is clearly monodromic (locally the trajectories encircle the critical point). However a change of coordinates $(x, y, t) \mapsto (-x, y, -t)$ leaves the system invariant. Clearly the y-axis is a line of symmetry for the trajectories (ignoring time). Close to the origin the trajectories must therefore be closed.

A second view point is to note that the system (28) can be projected onto the system

$$\dot{u} = 2x(-y + p(u, y)), \qquad \dot{y} = x(1 + q(u, y)),$$

by the map $u = x^2$, and thence to the system

$$\dot{u} = -2y + 2p(u, y), \qquad \dot{y} = 1 + q(u, y), \tag{29}$$

by a non-linear time scaling. The trajectories of (29) close to the origin pass from the third to the second quadrant, moving anticlockwise around the origin. Working the map backwards, we see that the transformation "unfolds" these trajectories into closed loops surrounding the origin.

A third way to see the same result is to consider the first integral of (29) at the origin. This exists as the origin is no longer a critical point. Under the map above, this first integral is pulled back to a first integral of (28) and so the origin is a centre.

From Theorem 3.1 it is tempting to think that all centres that arise in polynomial systems are limits of those with Darbouxian first integrals. However, since any system can be transformed by one of these folding transformations to a system with a centre, that would imply that, generically, every polynomial system has a Darbouxian first integral.

Therefore, we need to consider those systems which arise by symmetries also. In the case of those systems considered under Theorem 3.1, the ones with symmetries are sufficiently simple that they can be integrated in closed form (and hence are Darboux integrabke.) For systems of higher degree, however, those centres which arise from symmetry will usually form a separate class of centres.

In fact, we generalise the situation above as follows. We say that a system (2) has a generalised symmetry if there exists a local algebraic transformation which is analytically equivalent to a transformation $(X, Y) \mapsto (-X, Y)$ which takes trajectories into trajectories. We call this map a *reversing transformation* for obvious reasons. In fact every centre has a generalised symmetry if we allow the transformations to be analytic rather than algebraic.

Conjecture 9.1. Every (non-degenerate) centre in a polynomial system comes from a Darbouxian integrating factor or a generalised symmetry.

It turns out that there is a large class of systems for which we can prove this conjecture to hold.

THEOREM 9.1. *If the system*

$$\dot{x} = P_3(x)y, \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2,$$

where the P_i are polynomials, has a centre at the origin then there is either a Darbouxian first integral or a generalised symmetry.

For Liénard systems, ($P_3 = 1$, $P_2 = 0$), all the centres arise from generalised symmetries. The $2:1$ map is rational in this case allthough the reversing transformation is not. However in the general case it seems that a rational $2:1$ map will not suffice, and we need to consider more general algebraic symmetries.

The proof of Theorem 9.1 borrows ideas from differential algebra and a clever transformation due to Cherkas which takes this system to a Liénard system with more general coefficients. The crux of the matter then lies in the analytic reversing transformation. If this is algebraic we get a generalised symmetry, if not then we have a Darbouxian integrating factor.

Are there other classes of system which can be analysed in this way? In particular what are the obstructions to transforming a general polynomial system to Liénard form? Cherkas has results which cover the case of a y^3 term in the $\dot{y}$ equation. The sticking point therefore is to be able to transform away the y^4 term. Maybe there is some differential invariant which does not allow this, like the genus which restricts the existence of birational maps from one curve to another.

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