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Tracing Phase Portraits of Planar Polynomial Vector Fields with Detailed Analysis of the Singularities

Freddy Dumortier

Limburgs Universitair Centrum, Universitaire Campus, B-3590 Diepenbeek, Belgium.
E-mail: freddy.dumortier@luc.ac.be

and

Chris Herssens

Limburgs Universitair Centrum, Universitaire Campus, B-3590 Diepenbeek, Belgium.

The paper essentially aims at presenting a computer program drawing phase portraits of planar polynomial vector fields. It is essentially based on a detailed analysis of the singularities by means of desingularization using quasi-homogeneous blow-up and the Newton diagram, and good approximation of invariant manifolds. Phase portraits can be drawn locally near a singularity or globally on a Poincaré disc or a Poincaré-Lyapunov disc, compactifying the plane.

Key Words: ordinary differential equation, vector field, phase portrait, singularity, blow-up, Newton-diagram.

1. INTRODUCTION

Our aim is to study *ordinary differential equations* in two real variables

$$\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases}, \tag{1}$$

with P and Q both polynomial.

We will also call this a (polynomial) *vector field* on $\mathbf{R}^2$, emphasizing that the object under study can be defined in a coordinate-free way. Another way to express the vector field is by writing it as

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{2}$$

Both expressions (1) and (2) represent the vector field in the standard coordinates on $\mathbf{R}^2$, but during the analysis we will often use other coordinates, as well linear as non-linear ones, even not always globally defined. In fact our goal is surely not to look for an analytic expression of the global solution of (1). Not only would it be an impossible task for most equations but moreover even in the cases where a precise expression can be found it is not always clear what it really represents. Numerical analysis of (1) together with graphical representation, will be an essential ingredient in the analysis. We will however not limit our study to mere numerical integration. In fact in trying to do this one often encounters serious problems; calculations can take an enormous amount of time or even lead to erroneous results. Based however on a priori knowledge of some essential features of (1) these problems can often be avoided. Qualitative techniques are very appropriate to get such an overall understanding of the equation (1). A clear picture is achieved by drawing a phase portrait in which the relevant qualitative features are represented. Of course, for practical reasons, the representation may not be too far from reality and has to respect some numerical accuracy. These are, in a nutshell, the main ingredients in our approach. In section 5 we present a computer program based on them. The program is an extension of previous work due to J. C. Artés and J. Llibre. We have called it "Polynomial Planar Phase Portraits", which we abbreviate as **P4**.

We first start by studying the vector field near the singular points. Section 2 deals with the elementary singularities and section 3 with the non-elementary ones. In section 4 we introduce Poincaré and Poincaré-Lyapunov compactification in order to be able to study the vector fields near infinity. In section 5 we present the program P4, while in section 6 we treat some examples.

2. STUDY NEAR THE SINGULAR POINTS; THE ELEMENTARY CASE

Aiming at presenting some general methods to study singularities we suppose in this section that X is a C^∞ vector field defined on a neighbourhood of $0 \in \mathbf{R}^2$, with $X(0) = 0$. Let us first recall a number of general notions and results. If necessary we will indicate a precise reference, but often we will mention no reference at all if it is possible to find the information in a general reference work on dynamical systems like e.g. [15] or [20].

The study of a singularity starts by looking at the linear part $\mathrm{D}X(0) = A$. The linear part or *1-jet* represents a linear differential equation $\dot{x} = Ax$. It is called *hyperbolic* if all eigenvalues have a non zero real part.

The following theorem essentially says that all relevant information is contained in the eigenvalues of A if A is hyperbolic.

THEOREM 2.1 (Hartman-Grobman). *If X with $X(0) = 0$ is hyperbolic at 0 (which means that $\mathrm{D}X(0)$ is hyperbolic), then X is C^0-conjugate to its linear part. Moreover if two linear hyperbolic singularities have the same number of eigenvalues with negative real part, then they are C^0-conjugate.*

A *C^0-conjugacy* between two vector fields X and Y is a local homeomorphism $h : (V, 0) \to (V', 0)$ between two neighbourhoods V and V' of 0 with the property

$$h \circ X_t = Y_t \circ h,$$

where X_t and Y_t denote the respective flows of X and Y. In case the homeomorphism h does not conjugate the flows but only sends X-orbits to Y-orbits, in a sense preserving way, we speak about a *C^0-equivalence.*

In any case the singular point is isolated in a hyperbolic singularity. Three possibilities show up depending on the sign of the real parts α_1 and α_2 of the eigenvalues λ_1 and λ_2. If both α_1 and α_2 are negative (resp. positive) then all orbits have 0 as ω-limit (resp. α-limit). If $\alpha_1\alpha_2 < 0$, then we have a saddle.

In the saddle case there is a curve of points, whose orbit has 0 as ω-limit (resp. α-limit); it is called the *stable manifold W^s* of 0 (resp. *unstable manifold W^u* of 0).

Of course for an accurate numerical description of the singularity these manifolds W^s and W^u need to be positioned in a better way than by drawing merely the eigenspaces of the linear part $A = \mathrm{D}X(0)$.

The theoretical basis for such a positioning is provided by the following theorem.

THEOREM 2.2 (stable manifold theorem). *Let $(X, 0)$ be a singularity of a vector field on $\mathbf{R}^2$ of class C^r, respectively C^∞ or C^ω (i.e. analytic), with $r \geq 1$. Let $\mathrm{D}X(0)$ have eigenvalues $\lambda_1 < 0$ and $\lambda_2 \geq 0$. Let E^s be the eigenspace associated to λ_1. Then there exists a manifold W^s containing 0, invariant under the flow of X, of class C^r, respectively C^∞ or C^ω, with W^s tangent to E^s at 0 and $\mathrm{D}(X|_{W^s})(0)$ having λ_1 as eigenvalue.*

Applying this theorem to $-X$ it provides a similar result for the unstable manifold W^u. After applying a linear coordinate change, transforming the stable and unstable eigenspaces of $\mathrm{D}X(0)$ to respectively $\{y = 0\}$ and $\{x = 0\}$, we can express W^s and W^u as graphs of functions $y = f(x)$ and $x = g(y)$.

In working with polynomial vector fields we can in general not expect the functions f and g to be polynomial but they are at least analytic. Taylor approximations will be used to represent them in small neighbourhoods of 0. The precise way to do this will be presented in section 5. A finite Taylor approximation will depend on some finite jet of X at 0.

For the stable and the unstable hyperbolic points ($\alpha_1\alpha_2 > 0$) the only extra information we might need is whether orbits spiral around 0 (focus case) or whether orbits have a direction of approach (node case). This information is given by the eigenvalues λ_1 and λ_2.

The first case beyond hyperbolicity is given by $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Since one of the eigenvalues is non zero, we still speak about an *elementary singularity.* It is also called a *partially hyperbolic singularity* or *semi-hyperbolic singularity.* Because of the stable manifold theorem there can still be found a C^∞ (even analytic for analytic X) invariant manifold tangent to the eigenspace of λ_2; it is a "stable" one W^s in case $\lambda_2 < 0$ and an "unstable" one W^u in case $\lambda_2 > 0$. Moreover the eigenspace of λ_1 consists of zeroes for $\mathrm{D}X(0)$. We definitely need higher order jets to analyze the structure of the singularity. Following theorems provide the necessary information. In fact these theorems have interesting generalizations in $\mathbf{R}^n$, but we only state them in $\mathbf{R}^2$, referring to [15], [20] and also [12] for the n-dimensional version.

THEOREM 2.3 (center manifold theorem). *Let $(X,0)$ be a C^r-singularity of a vector field on $\mathbf{R}^2$, $r \in \mathbf{N} \setminus \{0\}$, with E^c the kernel of $A = \mathrm{D}X(0)$. Suppose* $\dim E^c = 1$. *Then there exists a 1-dimensional C^r manifold N^c containing 0, invariant under the flow of X with N^c tangent to E^c at 0 and $j_1(X|_{N^c})(0) = 0$.*

THEOREM 2.4 (reduction to the center manifold ([19],[16])). *Let X and N^c be as in the previous theorem, let λ denote the non zero eigenvalue of* $\mathrm{D}X(0)$. *Then the singularity $(X,0)$ is locally C^0-conjugate to the singularity at 0 of*

$$\begin{cases} \dot{y} &= \operatorname{sign}(\lambda) y \\ \dot{x} &= f(x) \end{cases},$$

where the second line expresses $X|_{N^c}$, with f of class C^r. Moreover, a local C^0-conjugacy (resp. C^0-equivalence) between two such expressions at the level of the center manifolds can be extended to a genuine C^0-conjugacy (resp. C^0-equivalence).

It will hence clearly suffice to study the behaviour on a *center manifold* in order to know the singularity completely.

The fact that we have stated the center manifold theorem for a finite class of differentiability is on purpose. Indeed in general the theorem is no longer true if we change C^r by C^∞ or C^ω.

In case some finite jet $j_n(X|_{N^c})(0)$ is non zero then the C^∞ version can be proven to be true (see [10]), although the C^ω-case however is still not true in general.

Starting with polynomial vector fields we can represent the center manifold by making a Taylor approximation. For a precise description we again refer to section 5. We will see that some problems can show up because of the disproportion between the center behaviour and the transverse hyperbolic one. In any case all necessary information is given by the non zero eigenvalue λ and its associated invariant (un)stable manifold on one hand, and the center manifold on the other hand.

For the latter we encounter two possibilities : either the center behaviour $X|_{N^c}$ has an isolated zero at 0 or not. In the first case one can prove that the center behaviour is given by

$$\dot{x} = x^m g(x),$$

for some $m \in \mathbf{N}_2$, with $\mathbf{N}_2 = \mathbf{N} \setminus \{0, 1\}$, and $g(0) \neq 0$. The topological structure of the singularity is then completely determined by $(m, \mathrm{sign}\lambda, \mathrm{sign}g(0$

In the second case it can be proved that the center manifold completely consists of singular points, meaning that for a vector field X described by

$$\begin{cases} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{cases},$$

the two polynomials P and Q have a common factor. We will show in section 5 how to deal with this case, by dividing out the common factor.

There remains however to study the non-elementary singularities, the ones for which $\mathrm{D}X(0)$ has both eigenvalues zero. In that case we use blow up.

3. BLOWING UP NON-ELEMENTARY SINGULARITIES

Before describing the effective algorithm that we use in the program P4, and which is based on the use of quasi-homogeneous blow up, let us first explain the basic ideas only introducing *homogeneous blow up*, which essentially means using polar coordinates. We will for a great part follow the introduction presented in [9].

Let $(X, 0)$ be a singularity of a C^∞ vector field on $\mathbf{R}^2$. Consider the map

$$\begin{array}{rccl} \phi : & S^1 \times \mathbf{R} & \to & \mathbf{R}^2 \\ & (\theta, r) & \mapsto & (r\cos\theta, r\sin\theta) . \end{array} \tag{3}$$

We can define a C^∞ vector field $\hat{X}$ on $S^1 \times \mathbf{R}$ such that $\phi_*(\hat{X}) = X$, in the sense that $\mathrm{D}\phi_v(\hat{X}(v)) = X(\phi(v))$. It is called the pull back of X by ϕ. It is nothing else but X written down in polar coordinates. If the k-jet $j_k(X)(0)$ is zero, then $j_k(\hat{X})(u) = 0$ for all $u \in S^1 \times \{0\}$.

In practice, however, we almost never use polar coordinates, but we use the so called *directional blow-up*

$$\text{in the } x\text{-direction: } (\bar{x}, \bar{y}) \mapsto (\bar{x}, \bar{y}\ \bar{x}), \text{ leading to } \hat{X}^x, \tag{4}$$

$$\text{in the } y\text{-direction: } (\bar{x}, \bar{y}) \mapsto (\bar{x}\ \bar{y}, \bar{y}), \text{ leading to } \hat{X}^y. \tag{5}$$

On $\{x \neq 0\}$, (4) up to an analytic coordinate change, is the same as *polar blow-up*, for $\theta \neq \pi/2, 3\pi/2$:

$$(\theta, r) \mapsto (r\cos\theta, \tan\theta) \mapsto (r\cos\theta, \tan\theta r\cos\theta) = (r\cos\theta, r\sin\theta).$$

In the case of (5), something analogous happens on $\{y \neq 0\}$. In case $j_k(X)(0) = 0$ and $j_{k+1}(X)(0) \neq 0$ we may gain information by considering $\bar{X}$ with

$$\bar{X} = \frac{1}{r^k}\hat{X}.$$

Then $\bar{X}$ also is a C^∞-vector field on $S^1 \times \mathbf{R}$. This division does not change the orbits of $\hat{X}$ nor their sense of direction, but only the parametrization by t.

For the related directional blow-up we use $(1/\bar{x}^k)\hat{X}^x$ in case (4) and $(1/\bar{y}^k)\hat{X}^y$ in case (5). On $\{x \neq 0\}$ (resp. $\{y \neq 0\}$) the vector fields $(1/r^k)\hat{X}$ and $(1/\bar{x}^k)\hat{X}^x$ (resp. $(1/\bar{y}^k)\hat{X}^y$) are the same up to analytic coordinate change and multiplication with a positive analytic function. Let us now treat two examples.

First we present an example where we use one blow-up to obtain quite easily the topological picture of the orbit structure of the singularity :

$$X = (x^2 - 2xy)\frac{\partial}{\partial x} + (y^2 - xy)\frac{\partial}{\partial y} + \mathrm{O}(\|(x, y)\|^3). \tag{6}$$

The formulas for (polar) blowing-up are

$$\bar{X} = \eta_1 \frac{\partial}{\partial \theta} + \eta_2 r \frac{\partial}{\partial r},$$

with

$$\begin{aligned}
\eta_1(\theta, r) &= \frac{1}{r^{k+2}} \left\langle X, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right\rangle (\phi(r,\theta)) \\
&= \frac{1}{r^{k+2}}(-r\sin\theta X_1(r\cos\theta, r\sin\theta) + r\cos\theta X_2(r\cos\theta, r\sin\theta)), \\
\eta_2(\theta, r) &= \frac{1}{r^{k+2}} \left\langle X, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right\rangle (\phi(r,\theta)) \\
&= \frac{1}{r^{k+2}}(r\cos\theta X_1(r\cos\theta, r\sin\theta) + r\sin\theta X_2(r\cos\theta, r\sin\theta)),
\end{aligned}$$

In our example $k = 1$ and the result is

$$\begin{aligned}
\bar{X}(\theta, r) &= (\cos\theta\sin\theta(3\sin\theta - 2\cos\theta) + \mathrm{O}(r))\frac{\partial}{\partial\theta} \\
&\quad + r(\cos^3\theta - 2\cos^2\theta\sin\theta - \cos\theta\sin^2\theta + \sin^3\theta + \mathrm{O}(r))\frac{\partial}{\partial r}.
\end{aligned}$$

Zeroes on $\{r = 0\}$ are located at

$$\theta = 0, \pi;\ \ \theta = \pi/2, 3\pi/2;\ \ \tan\theta = 2/3.$$

At these singularities, the radial eigenvalue is given by the coefficient of $r\partial/\partial r$ while the tangential eigenvalue can be found by differentiating the $\partial/\partial\theta$-component with respect to θ. One so finds Figure 1. All the singularities are hyperbolic. We say to have desingularized $(X, 0)$. The exact value of the eigenvalues at the different singularities only depends on the 2-jet of X. In [8] or [6] it can be seen how to prove that the singularity $(X, 0)$ is in fact C^0-conjugate to the singularity given by the 2-jet. The exact positioning of the invariant manifolds of the six hyperbolic singularities in the blow-up can be approximated by Taylor approximation using some finite jet. After blowing-down it leads to an accurate presentation of the six "separatrices" in the local phase portrait (see figure 2).

Secondly we present an example where blowing-up once is not sufficient to desingularize the singularity, but where we need to repeat the construction (successive blowing-up)

$$y\frac{\partial}{\partial x} + (x^2 + xy)\frac{\partial}{\partial y} + \mathrm{O}(\|(x,y)\|^3), \tag{7}$$

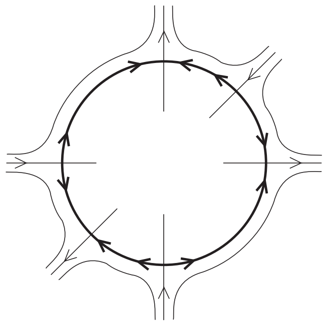

FIG. 1. Blow-up of example 1.

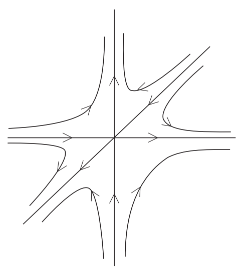

FIG. 2. Local phase portrait of example 1.

Blowing-up in the y-direction will give no singularities on $\{y = 0\}$; indeed the singularities (as well as their eigenvalues) only depend on the first non zero jet, hence on $y\partial/\partial x$. We perform a blow-up in the x-direction, but without using formulas like in the previous example. Writing

$$x = \bar{x}, \quad y = \bar{x}\bar{y},$$

or

$$\bar{x} = x, \quad \bar{y} = y/x,$$

we get

$$\begin{aligned}
\dot{\bar{x}} &= \dot{x} = y + \mathrm{O}(\|(x,y)\|^3) = \bar{y}\bar{x} + \mathrm{O}(|\bar{x}|^3), \\
\dot{\bar{y}} &= \frac{\dot{y}}{x} - y\frac{\dot{x}}{x^2} \\
&= (x+y) + \frac{1}{x}\mathrm{O}(\|(x,y)\|^3) - \frac{y^2}{x^2} - \frac{y}{x^2}\mathrm{O}(\|(x,y)\|^3)
\end{aligned}$$

$$= \bar{x} + \bar{y}\,\bar{x} - \bar{y}^2 + \mathrm{O}(|\bar{x}|^2).$$

The only singularity on $\bar{x} = 0$ occurs for $\bar{y} = 0$, where the 1-jet of the vector field $\bar{X}^x$ in this singularity is $\bar{x}\partial/\partial\bar{y}$.

As the singularity is neither hyperbolic, nor semi-hyperbolic (with a possible reduction to the center manifold) we are going to perform an extra blow-up in order to study it. Blowing-up in the $\bar{x}$-direction gives no singularities. Blowing-up in the $\bar{y}$-direction ($\bar{x} = \bar{\bar{y}}\,\bar{\bar{x}}, \bar{y} = \bar{\bar{y}}$) gives

$$\begin{aligned}
\dot{\bar{\bar{y}}} &= \dot{\bar{y}} = (\bar{x} + \bar{y}\,\bar{x} - \bar{y}^2 + \mathrm{O}(|\bar{x}|^2)) \\
&= \bar{\bar{x}}\,\bar{\bar{y}} - \bar{\bar{y}}^2 + \mathrm{O}(\|(\bar{\bar{x}},\bar{\bar{y}})\|^3), \\
\dot{\bar{\bar{x}}} &= \frac{\dot{\bar{x}}}{\bar{y}} - \bar{x}\frac{\dot{\bar{y}}}{\bar{y}^2} \\
&= \bar{x} + \frac{1}{\bar{y}}\mathrm{O}(|\bar{x}|^3) - \frac{\bar{x}}{\bar{y}^2}(\bar{x} + \bar{y}\bar{x} - \bar{y}^2 + \mathrm{O}(|\bar{x}|^2)) \\
&= \bar{\bar{y}}\bar{\bar{x}} - \bar{\bar{x}}^2 + \bar{\bar{y}}\bar{\bar{x}} + \mathrm{O}(\|(\bar{\bar{x}},\bar{\bar{y}})\|^2).
\end{aligned}$$

The 2-jet is now $(xy - y^2)\partial/\partial y + (2xy - x^2)\partial/\partial x$. As we have seen this singularity can be studied by blowing-up once. This succession of blowing-up is schematized in Figure 3 The reconstruction of the local phase portrait

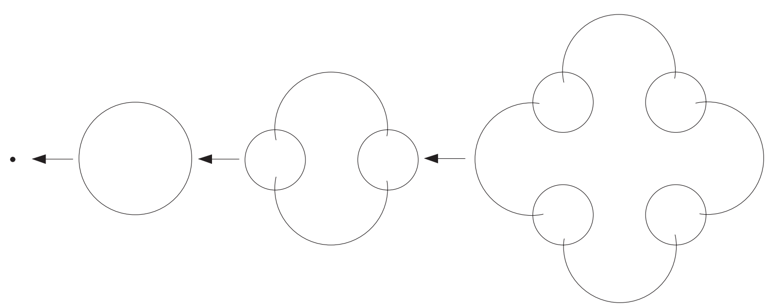

FIG. 3. Successive blowing-up.

is represented in figure 4. As a result we also obtain that the singularities are topologically determined by the 2-jet. A precise drawing of the two separatrices of the *cusp* can be obtained by using Taylor approximations of the invariant manifolds in the desingularization followed by a blowing-down, like shown in Figure 5. The procedure of successive blowing-up can

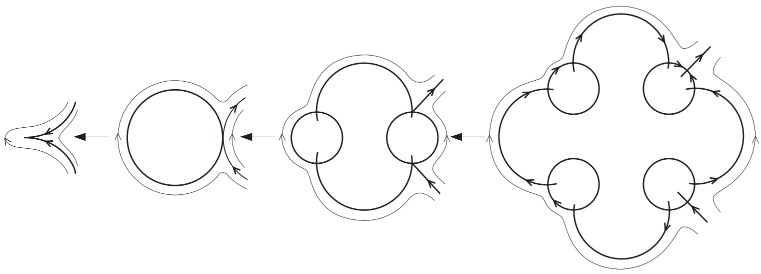

FIG. 4. Blowing-up example 2.

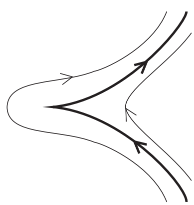

FIG. 5. Local phase portrait of example 2.

be formalised as follows, providing an overall geometric view. Instead of using ϕ and dividing by some power of r, we use the map

$$\tilde{\phi} : \{z \in \mathbf{R}^2 | \, \|z\| > \frac{1}{2}\} \subset \mathbf{R}^2 \to \mathbf{R}^2, z \mapsto z - \frac{z}{\|z\|},$$

and divide by the same power of $(\|z\| - 1)$.

The vector fields we so obtain are analytically equivalent, but the second is now defined on an open domain in $\mathbf{R}^2$ and therefore it becomes easier to visualize how we can blow up again in some point $z_0 \in \{z \in \mathbf{R}^2 | \, \|z\| = 1\}$: we just use the mapping $T_{z_0} \circ \phi$ where T_{z_0} denotes the translation $z \mapsto z + z_0$.

As we again end up on an open domain of $\mathbf{R}^2$ we can repeat the construction if necessary. For simplicity in notation we denote the first blow-up by ϕ_1, the second by ϕ_2 and so on.

After a sequence of n-times blowing-up we find some C^∞-vector field $\bar{X}^n$ defined on a domain $U_n \subset \mathbf{R}^2$. $\bar{X}^n$ is even analytic if we start with an analytic X. We write $\Gamma_n = (\phi_1 \circ \ldots \circ \phi_n)^{-1}(0) \subset U_n$. Only one of the connected components of $\mathbf{R}^2 \backslash \Gamma_n$, call it A_n, has a non-compact closure.

Furthermore $\partial A_n \subset \Gamma_n$ and ∂A_n, which is homeomorphic to S^1, consists of a finite number of analytic regular closed arcs meeting transversally. The mapping $(\phi_1 \circ \ldots \circ \phi_n)|_{A_n}$ is an analytic diffeomorphism sending A_n onto $\mathbf{R}^2 \backslash \{0\}$. There exists a strictly positive function F_n on A_n such that $\hat{X}^n = F_n \cdot \bar{X}^n$ and $\hat{X}^n|_{A_n}$ is analytically diffeomorphic to $X|_{\mathbf{R}^2 \backslash \{0\}}$ by means of the diffeomorphism $(\phi_1 \circ \ldots \circ \phi_n)|_{A_n}$. The function F_n extends in a C^ω way to ∂A_n where in general it is 0.

To control whether the succession of blowing-up finally leads to a tractable result we use the notion of *Lojasiewicz-inequality*. We say that a vector field X on $\mathbf{R}^2$ satisfies a Łojasiewicz-inequality at 0 if there is a $k \in \mathbf{N}^*$, with $\mathbf{N}^\star = \mathbf{N} \setminus \{0\}$, and a $c > 0$ such that $\|X(x)\| \geq c\|x\|^k$ on some neighbourhood of 0.

For analytic vector fields at isolated singularities, a Łojasiewicz-inequality always holds. In [7] it has been proven that if X satisfies a Łojasiewicz-inequality, there exists a finite sequence of blowing-up $\phi_1 \circ \ldots \phi_n$ leading to a vector field $\bar{X}^n$ defined in the neighbourhood of ∂A_n such that the singularities of $\bar{X}^n$ on ∂A_n are elementary.

These elementary singularities can be as follows:

(i) Isolated singularities p which are hyperbolic or semi-hyperbolic with the property that $j_\infty(\bar{X}^n|_{N^c})(p) \neq 0$ if N^c is a center manifold for $\bar{X}^n$ in p, or;

(ii) Regular analytic closed curves (or possibly the whole ∂A_n in case $n = 1$) along which $\bar{X}^n$ is normally hyperbolic.

The position and the properties of the singularities mentioned above only depend on a finite jet of X. Unless the singularity is a focus or a center it is always possible to find a finite number of C^∞-lines (stable, unstable or center manifolds, sometimes one has to choose an ordinary trajectory as boundary of two elliptic sectors), each cutting ∂A_n in one point, and dividing small neighbourhoods of ∂A_n into a finite number of zones which, after blowing-down, provide a decomposition of small neighbourhoods of the singularity into hyperbolic (or saddle) sectors, elliptic sectors and parabolic sectors of attracting (or stable) or repelling (or unstable) type (see [7, 8]) In figure 6 we represent the typical (topological) picture of such sectors, not representing fully attracting or repelling singularities.

The invariant C^∞-lines in the boundary of these sectors blow down to so called *characteristic orbits* (or *characteristic lines*), i.e. orbits (or an orbit together with the singularity) tending to the singularity with a well defined slope, the time tending to $+\infty$ or to $-\infty$. Not all these invariant curves are relevant, but only those which separate different topological

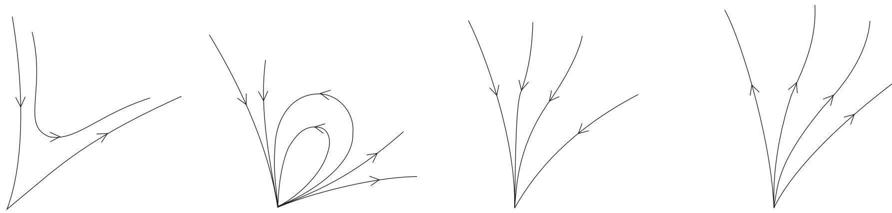

FIG. 6. Sectors near a singular point.

behaviour. There is e.g. no need to draw a separation between two adjacent parabolic sectors, or between an elliptic sector and an adjacent parabolic one. It suffices to draw the boundary curves of the hyperbolic sectors and to draw some characteristic lines between two adjacent elliptic sectors. The remaining characteristic lines are often called *separatrices*; the ones bordering a hyperbolic sector are of *finite type* in the sense that they possess a C^∞ parametrization $\gamma : [0,\varepsilon] \mapsto \mathbf{R}^2$ with $j_r\gamma(0) \neq 0$ for some $r \in \mathbf{N}$. They can also be seen as graphs of a C^∞ function in the variable $x^{1/n}$ for some $n \in \mathbf{N}_1$ in suitable C^∞ coordinates (x, y) (see [10]). The separatrices between two elliptic sectors do not need to have this property (see [10]).

Although the method of successively using homogeneous blow up is sufficient to study isolated singularities of an analytic vector field, it reveals to be much more efficient to include *quasi-homogeneous blow up*. In fact the algorithm that we have implemented relies on the systematic approach presented in [17], and which is based on the use of quasi-homogeneous blow up (see also [5] and [4]). Let us first present the technique before describing the algorithm.

Let $(X, 0)$ be a singularity of a C^∞ vector field on $\mathbf{R}^2$. Consider the map

$$\begin{array}{rcl} \phi : \ S^1 \times \mathbf{R} & \to & \mathbf{R}^2 \\ (\theta, r) & \mapsto & (r^\alpha \cos\theta, r^\beta \sin\theta) \ , \end{array} \tag{8}$$

for some well chosen $(\alpha, \beta) \in \mathbf{N}^* \times \mathbf{N}^*$. Exactly like in the " homogeneous case ", where $(\alpha, \beta) = (1, 1)$, we can define a C^∞ vector field $\hat{X}$ on $S^1 \times \mathbf{R}$ with $\phi_*(\hat{X}) = X$. We will divide it by r^k, for some k, in order to get a C^∞ vector field $\bar{X} = \frac{1}{r^k}\hat{X}$, which is as non-degenerate as possible along the invariant circle $S^1 \times \{0\}$.

In practice one again uses directional blow-ups:

$$
\begin{array}{llll}
\text{positive } x\text{-direction:} & (\bar{x},\bar{y}) \mapsto (\bar{x}^{\alpha},\bar{x}^{\beta}\bar{y}), & \text{leading to } \hat{X}^{x}_{+}, \\
\text{negative } x\text{-direction:} & (\bar{x},\bar{y}) \mapsto (-\bar{x}^{\alpha},\bar{x}^{\beta}\bar{y}), & \text{leading to } \hat{X}^{x}_{-}, \\
\text{positive } y\text{-direction:} & (\bar{x},\bar{y}) \mapsto (\bar{x}\bar{y}^{\alpha},\bar{y}^{\beta}), & \text{leading to } \hat{X}^{y}_{+}, \\
\text{negative } y\text{-direction:} & (\bar{x},\bar{y}) \mapsto (\bar{x}\bar{y}^{\alpha},-\bar{y}^{\beta}), & \text{leading to } \hat{X}^{y}_{-}.
\end{array}
$$

In case α is odd (resp. β is odd), the information found in the positive x-direction (resp. y-direction) also covers the one in the negative x-direction (resp. y-direction).

To show on an example that this technique can be quite efficient, we again study the cusp-singularity

$$y\frac{\partial}{\partial x} + (x^2 + xy)\frac{\partial}{\partial y} + \mathrm{O}(\|(x,y)\|^3),$$

this time using a quasi-homogeneous blowing up with $(\alpha,\beta) = (2,3)$.

In the x-direction we consider the transformation $(x,y) = (\bar{x}^2,\bar{x}^3\bar{y})$. In this case we have $\dot{x} = 2\bar{x}\dot{\bar{x}} \Rightarrow \dot{\bar{x}} = \frac{\bar{x}^2\bar{y}}{2} + \mathrm{O}(\bar{x}^3)$ and $\dot{y} = 3\bar{x}^2\bar{y}\dot{\bar{x}} + \bar{x}^3\dot{\bar{y}} \Rightarrow \dot{\bar{y}} = (1 - \frac{3}{2}\bar{y}^2)\bar{x} + \mathrm{O}(\bar{x}^2)$. We divide by $\bar{x}$ and find

$$\begin{cases} \dot{\bar{x}} = \frac{\bar{x}\bar{y}}{2} + \mathrm{O}(\bar{x}^2) \\ \dot{\bar{y}} = 1 - \frac{3}{2}\bar{y}^2 + \mathrm{O}(\bar{x}) \end{cases}$$

We find two hyperbolic singularities of saddle type, situated at the points $(\bar{x},\bar{y}) = (0,\pm\sqrt{2/3})$.

Similar calculations in the negative $\bar{x}$-direction, as well as in the positive $\bar{y}$-direction show that no other singularities show up.

As such blowing-up once suffices to desingularize the singularity leading to the picture in figure 7.

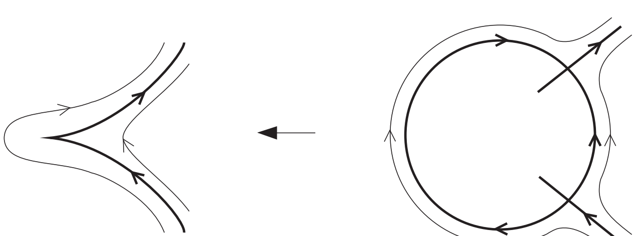

FIG. 7. Quasi-homogeneous blow up of the cusp singularity.

Again an accurate positioning of the invariant separatrices can be obtained by Taylor approximation of the stable and unstable manifolds.

A question one might ask is how to find effectively the coefficient (α, β) to use in a quasi-homogeneous blow up. This can be obtained by using the so called Newton diagram. It is also essential in the formulation of an effective desingularization algorithm based on the use of successive quasi-homogeneous blowing up. Let us first define the *Newton diagram.*

Let $X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ be a polynomial vector field with an isolated singularity at the origin.

Let $P(x,y) = \sum_{i+j\geq 1} a_{ij}x^iy^j$ and $Q(x,y) = \sum_{i+j\geq 1} b_{ij}x^iy^j$. The *support* of X is defined to be

$$S = \{(i-1,j)|a_{ij} \neq 0\} \cup \{(i,j-1)|b_{ij} \neq 0\} \subset \mathbf{R}^2, \tag{9}$$

and the *Newton polyhedron* of X is the convex hull Γ of the set

$$P = \bigcup_{(r,s)\in S} \{(r,s) + \mathbf{R}^2_+\}. \tag{10}$$

The *Newton diagram* of X is the union γ of the compact faces γ_k of the Newton polyhedron Γ, which we enumerate from the left to the right. If there exists a face γ_k which lies completely on the half-plane $\{r \leq 0\}$, then we start the enumeration with $k=0$, otherwise we start with $k=1$. Since the origin is an isolated singularity we have that at least one of the points $(-1,s)$ or $(0,s)$ is an element of S for some s, and also at least one of the points $(r,0)$ or $(r,-1)$ is an element of S for some r. Hence there always exists a face γ_1 in the Newton diagram.

Suppose that γ_1 has equation $\alpha r + \beta s = d$, with $\gcd(\alpha,\beta) = 1$. As a first step in the desingularization process we use a quasi-homogeneous blow up of degree (α,β). Denote $X = \sum_{j\geq d} X_j$, with $X_j = P_j(x,y)\frac{\partial}{\partial x} + Q_j(x,y)\frac{\partial}{\partial y}$ the *quasi-homogeneous component* of type (α,β) and (quasi-homogeneous) degree j, i.e. $P_j(r^\alpha x, r^\beta y) = r^{j+\alpha}P_j(x,y)$ and $Q_j(r^\alpha x, r^\beta y) = r^{j+\beta}Q_j(x,y)$. We will divide by r^d. In practice we first blow up the vector field in the positive x-direction, yielding, after multiplying the result with $\alpha\bar{x}^{-d}$:

$$\bar{X}^x_+ : \begin{cases} \dot{\bar{x}} = \sum_{\delta\geq d} \bar{x}^{\delta+1-d}P_\delta(1,\bar{y}) \\ \dot{\bar{y}} = \sum_{\delta\geq d} \bar{x}^{\delta-d}(\alpha Q_\delta(1,\bar{y}) - \beta\bar{y}P_\delta(1,\bar{y})) \end{cases} \tag{11}$$

We determine the singularities on the line $\{\bar{x} = 0\}$.

1) If $\alpha Q_d(1,\bar{y}) - \beta\bar{y}P_d(1,\bar{y}) \not\equiv 0$, the points $(0,\bar{y}_0)$ satisfying the equation $\alpha Q_d(1,\bar{y}) - \beta\bar{y}P_d(1,\bar{y}) = 0$ are isolated singularities of $\bar{X}$ on the line $\{\bar{x} = 0\}$, at which

$$\mathrm{D}(\bar{X}^x_+)_{(0,\bar{y}_0)} = \begin{pmatrix} P_d(1,\bar{y}_0) & 0 \\ \star & \alpha\frac{\partial Q_d}{\partial\bar{y}}(1,\bar{y}_0) - \beta(P_d(1,\bar{y}_0) + \bar{y}_0\frac{\partial P_d}{\partial\bar{y}}(1,\bar{y}_0)) \end{pmatrix},$$

providing immediately the eigenvalues on the diagonal. In case the singularity is hyperbolic, we are done. In case the singularity is semi-hyperbolic, we have to determine the behaviour on the center manifold. In case the singularity is non-elementary, we introduce $\tilde{y} = \bar{y} - \bar{y}_0$, and blow up this vector field again in the positive $\bar{x}$-direction as well as in the positive and negative $\tilde{y}$-direction with a certain degree (α', β'), which we determine from the Newton diagram associated to the vector field.

2) If $\alpha Q_d(1,\bar{y}) - \beta\bar{y}P_d(1,\bar{y}) \equiv 0$, we have a line of singularities. Since

$$\mathrm{D}(\bar{X}^x_+)_{(0,\bar{y}_0)} = \begin{pmatrix} P_d(1,\bar{y}_0) & 0 \\ \star & 0 \end{pmatrix},$$

all the singularities are semi-hyperbolic, except those singularities $(0,\bar{y}_0)$ for which $P_d(1,\bar{y}_0) = 0$. The latter will require further blow up.

Next we blow up the vector field in the negative x-direction and study this vector field in the same way as in the previous case.

Finally we have to blow up the vector field in the positive and the negative y-direction, and determine whether or not $(0,0)$ is a singular point, since the others have been studied in the previous charts.

It is easy to see that $(0,0)$ is a singularity iff γ_1 lies completely in the half-plane $\{r \geq 0\}$. If this is the case then $(0,0)$ is elementary. Indeed, blowing up the vector field in the positive y-direction yields, after multiplying the result with $\beta\bar{y}^{-d}$:

$$\bar{X}^y_+ : \begin{cases} \dot{\bar{x}} = \sum_{\delta\geq d} \bar{y}^{\delta-d}(\beta P_\delta(\bar{x},1) - \alpha\bar{x}Q_\delta(\bar{x},1)) \\ \dot{\bar{y}} = \sum_{\delta\geq d} \bar{y}^{\delta+1-d}Q_\delta(\bar{x},1) \end{cases} . \tag{12}$$

Hence $(0,0)$ is a singular point if $P_d(0,1) = 0$, i.e. if $P_d(x,y) = xF(x,y)$, implying that γ_1 lies completely in the half-plane $\{r \geq 0\}$. Suppose now that $(0,0)$ is a singular point of $\bar{X}^y_+$, then we have

$$\mathrm{D}(\bar{X}^y_+)_{(0,0)} = \begin{pmatrix} \beta\frac{\partial P_d}{\partial \bar{x}}(0,1) - \alpha Q_d(0,1) & \star \\ 0 & Q_d(0,1) \end{pmatrix} .$$

Let $(0,s)$ be the intersection of the line γ_1 and the line $r = 0$, then $P_d(x,y) = axy^s + G(x,y)$ and $Q_d(x,y) = by^{s+1} + H(x,y)$, with $a^2 + b^2 \neq 0$, $\deg_x G(x,y) \geq 2$ and $\deg_x H(x,y) \geq 1$. Hence $\beta\frac{\partial P_d}{\partial \bar{x}}(0,1) - \alpha Q_d(0,1) = a\beta - b\alpha$. So, if $a\beta - b\alpha \neq 0$ then $(0,0)$ is non-elementary. if $a\beta - b\alpha = 0$, then $Q_d(0,1) = b \neq 0$, and $(0,0)$ is elementary too.

In [17] it has been proven that the algorithm, as presented here, leads to a desingularization. It is also more efficient than the usual one.

In the program P4 we will not only perform a detailed study near the singular points in $\mathbf{R}^2$, but also near singular points *at infinity* . Let us now describe how polynomial vector fields on $\mathbf{R}^2$ can be extended to *infinity.*

4. POINCARÉ AND POINCARÉ-LYAPUNOV COMPACTIFICATION

If we study a vector field, we also have to determine what happens near infinity. In case of polynomial vector fields this can be done in two ways, namely we can extend the vector field on the Poincaré disc or on a Poincaré-Lyapunov disc. In both cases one compactifies $\mathbf{R}^2$ by adding a circle, and one extends the polynomial vector field to an analytic one on the disc. Let us first describe how to extend to a Poincaré disc. Essentially near infinity one uses

$$(x, y) = (\cos\theta/s, \sin\theta/s),$$

and one multiplies the resulting vector field by s^{d-1}, where d is the degree of the vector field. There is however a more geometric way to describe the Poincaré disc, as e.g. presented in [1] and [18]. As it is this construction that we implement in our program, let us describe it in full detail.

Let X be a polynomial vector field of degree d on the plane. We consider the unit sphere $S^2 = \{(y_1, y_2, y_3) \in \mathbf{R}^3 | y_1^2 + y_2^2 + y_3^2 = 1\}$ and denote by $T_{(y_1,y_2,y_3)}S^2$ the tangent space to S^2 at the point (y_1, y_2, y_3). Consider the two central projections $p^+ : T_{(0,0,1)}S^2 \to S^2_+$ and $p^- : T_{(0,0,1)}S^2 \to S^2_-$, where $S^2_+ = \{(y_1, y_2, y_3) \in S^2 | y_3 > 0\}$ and $S^2_- = \{(y_1, y_2, y_3) \in S^3 | y_3 < 0\}$. These maps define two copies of X, $(p^+)_*X$ on the northern hemisphere and $(p^-)_*X$ on the southern hemisphere. Let $f : S^2 \to \mathbf{R}$ be defined by $f(y_1, y_2, y_3) = y_3^{d-1}$, then the vector fields $f \cdot (p^+)_*X$ and $f \cdot (p^-)_*X$ can be extended to an analytic vector field $p(X)$ on S^2. The vector field $p(X)$ is often called the *Poincaré compactification of* X. It is defined on S^2, but is equivariant under the point-reflection $(y_1, y_2, y_3) \mapsto (-y_1, -y_2, -y_3)$. For the flow of $p(X)$, the equator $S^1 = \{(y_1, y_2, y_3) | y_3 = 0\}$ is invariant and the equator corresponds to the circle at infinity of $\mathbf{R}^2$. The projection of the closure of S^2_+ on the plane $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the *Poincaré disc.*

To make calculations concerning $p(X)$ we consider the following six local charts $U_i = \{(y_1, y_2, y_3) \in S^2 | y_i > 0\}$ and $V_i = \{(y_1, y_2, y_3) \in S^2 | y_i < 0\}$ where $i = 1, 2, 3$ and the diffeomorphisms $F_i : U_i \to \mathbf{R}^2$ and $G_i : V_i \to \mathbf{R}^2$, with $F_i(y_1, y_2, y_3) = G_i(y_1, y_2, y_3) = (y_j y_i^{-1}, y_k y_i^{-1})$ for $j < k$ and $j, k \neq i$. It is easy to see that these maps are the inverse of the central projections from the planes tangent to S^2 at the points $(1,0,0)$, $(-1,0,0)$, $(0,1,0)$, $(0,-1,0)$, $(0,0,1)$, $(0,0,-1)$ respectively. We denote by $z=(z_1,z_2)$ the value of $F_i(y_1,y_2,y_2)$ or $G_i(y_1,y_2,y_3)$ for any $i=1,2,3$. Let $X=P(x,y)\frac{\partial}{\partial x}+Q(x,y)\frac{\partial}{\partial y}$, then some easy computations give for $p(X)$ the following expressions on the local charts:

On the U_1 chart we have

$$\left\{\begin{array}{l} \dot{z_1}=z_2^d g(z)(-z_1P(\frac{1}{z_2},\frac{z_1}{z_2})+Q(\frac{1}{z_2},\frac{z_1}{z_2})) \\ \dot{z_2}=-z_2^{d+1}g(z)P(\frac{1}{z_2},\frac{z_1}{z_2}) \end{array}\right. , \tag{13}$$

with $g(z)=(1+z_1^2+z_2^2)^{(1-d)/2}$.

On the U_2 chart we have

$$\left\{\begin{array}{l} \dot{z_1}=z_2^d g(z)(P(\frac{z_1}{z_2},\frac{1}{z_2})-z_1Q(\frac{z_1}{z_2},\frac{1}{z_2})) \\ \dot{z_2}=-z_2^{d+1}g(z)Q(\frac{z_1}{z_2},\frac{1}{z_2}) \end{array}\right. , \tag{14}$$

and on the U_3 chart we have

$$\left\{\begin{array}{l} \dot{z_1}=g(z)P(z_1,z_2) \\ \dot{z_2}=g(z)Q(z_1,z_2) \end{array}\right. . \tag{15}$$

The expression for the vectorfield $p(X)$ on V_i is equal to the one on U_i multiplied by $(-1)^{d-1}$. Since the factor $g(z)$ is strictly positive, we can omit this factor by rescaling the vector field $p(X)$. So, in each chart we get a polynomial vector field.

A singular point of X is called *infinite* (resp. *finite*) if it is a singular point of $p(X)$ in S^1 (resp. $S^2\setminus S^1$). It is easy to see that the infinite singular points of X are the points $(z_1,0)$ satisfying

$$Q_d(1,z_1)-z_1P_d(1,z_1)=0 \text{ if } (z_1,0)\in U_1,$$

$$P_d(z_1,1)-z_1Q_d(z_1,1)=0 \text{ if } (z_1,0)\in U_2,$$

where P_d and Q_d are the homogeneous part of degree d of P and Q.

Sometimes, it is better to work with a *Poincaré-Lyapunov compactification*, i.e. we use a quasi-homogeneous compactification at infinity essentially given near infinity by

$$\left\{\begin{array}{l} x=\cos\theta/s^\alpha \\ y=\sin\theta/s^\beta \end{array}\right. , \tag{16}$$

for some well chosen powers $(\alpha, \beta) \in \mathbf{N}^* \times \mathbf{N}^*$. Again the precise calculations are not really worked out with the expression (16). Sometimes one prefers not to use the usual functions $(\cos\theta, \sin\theta)$ but to work with the periodic functions $\mathrm{Cs}\theta$ and $\mathrm{Sn}\theta$, solution of the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{Cs}\theta = -\mathrm{Sn}^{2\alpha-1}\theta \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{Sn}\theta = \mathrm{Cs}^{2\beta-1}\theta \\ \mathrm{Cs}0 = 1,\ \mathrm{Sn}0 = 0 \end{cases}, \tag{17}$$

and satisfying the relation $\beta\mathrm{Sn}^{2\alpha}\theta + \alpha\mathrm{Cs}^{2\beta}\theta = \alpha$. Using such a transformation for well chosen α and β, make it possible in many cases that instead of getting a non-elementary singular point at infinity (in a Poincaré compactification) one finds only elementary singular points. For the calculations it is again better to work in different charts and this will be done in section 5.

5. THE PROGRAM P4

P4 is a tool which can be used in the study of a polynomial planar differential equation. Depending on the user's choice it draws the phase portraits on either the Poincaré disc, or on a Poincaré-Lyapunov disc, or near a singular point. P4 is partly written in C and partly written in REDUCE [11]. It is possible to work in numerical mode or in mixed mode, i.e. if possible, the calculations are done in algebraic mode. We shall now describe the structure and possibilities of P4.

First it checks whether or not the vector field has a continuous set of finite singular points, that is, if whether or not the two polynomial components of the vector field have a common factor. If they have a common factor, we divide the vector field by this common factor and study the new vector field. Sometimes the used computer algebra package (i.e. Reduce) cannot find this common factor. In such cases also P4 works incorrectly. If the user knows the common factor (e.g. by means of another computer algebra package such as Maple, Mathematica, Axiom, ...), he can avoid this problem by giving this factor, together with the reduced vector field (i.e. the vector field after division by the common factor), to P4.

So, in what follows let $X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ with $\gcd(P,Q) = 1$. Now we will determine the finite isolated singular points. This can be done in algebraic or numeric mode. In both cases P4 will ask REDUCE to solve the problem. If the degree of the vector field is high, determining these singularities can take a lot of time, in such cases it is better to work numerically.

For each singular point (x_0, y_0), P4 determines the local phase portrait in the following way. First it computes the jacobian matrix at each singular point, i.e.

$$\mathrm{D}X_{(x_0,y_0)} = \begin{pmatrix} \frac{\partial P}{\partial x}(x_0, y_0) & \frac{\partial P}{\partial y}(x_0, y_0) \\ \frac{\partial Q}{\partial x}(x_0, y_0) & \frac{\partial Q}{\partial y}(x_0, y_0) \end{pmatrix},$$

and evaluates its eigenvalues λ_1 and λ_2. We have to distinguish different cases, depending on whether both eigenvalues are real, both eigenvalues are purely imaginary or both eigenvalues are complex.

1) λ_1 and λ_2 are real. If λ_1 and λ_2 have the same sign then (x_0, y_0) is a stable (unstable) node and we are done. If they have different sign, then (x_0, y_0) is a saddle, and we compute a Taylor approximation of order n of the stable and unstable manifold as follows.

Consider the transformations

$$\begin{cases} \bar{x} = x - x_0 \\ \bar{y} = y - y_0 \end{cases},$$

and

$$\begin{cases} \bar{x} = w_{11}u + w_{21}v \\ \bar{y} = w_{12}u + w_{22}v \end{cases},$$

with (w_{11}, w_{12}) (resp. (w_{21}, w_{22})) an eigenvector associated to the eigenvalue λ_1 (resp. λ_2).

Using these transformations yields the vector field

$$\begin{cases} \dot{u} = \lambda_1 u + p(u, v) \\ \dot{v} = \lambda_2 v + q(u, v) \end{cases}, \tag{18}$$

with $\deg(p) \geq 2$ and $\deg(q) \geq 2$. Writing the invariant manifold as a graph $(u, f(u))$ and using the invariance of the flow, we have that

$$f(u) = \sum_{i=2}^{n} a_i u^i + \mathrm{o}(u^n), \tag{19}$$

with

$$a_i = \frac{b_i}{(i\lambda_1 - \lambda_2)}, \quad i = 2, \ldots, n,$$

where b_i is the coefficient of u^i in the expression $q(u, f(u)) - f'(u)p(u, f(u))$. The manifold $(v, g(v))$ is computed in the same way.

If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ then the singularity (x_0, y_0) is semi-hyperbolic. In this case there is a center manifold which is tangent to the line $v_2(x - x_0) - v_1(y - y_0) = 0$, with (v_1, v_2) an eigenvector associated to the zero eigenvalue. To compute the center manifold, we simplify the vector field in the same way as in the saddle case. Hence the new vector field satisfies

$$\left\{ \begin{array}{l} \dot{u} = p(u, v) \\ \dot{v} = \lambda_2 v + q(u, v) \end{array} \right. , \tag{20}$$

with $\deg(p) \geq 2$ and $\deg(q) \geq 2$. Writing the center manifold as a graph $(u, f(u))$, and using the invariance of the flow, we have

$$f(u) = \sum_{i=2}^{n} a_i u^i + \mathrm{o}(u^n),$$

with a_i the coefficient of u^i in the expression $-[q(u, f(u)) - f'(u)p(u, f(u))]/\lambda_2$. This results in the behaviour

$$\dot{u} = c_m u^m + \mathrm{o}(u^m).$$

Using this information we find that the origin is

(i) a stable node if $c_m < 0, m$ odd and $\lambda_2 < 0$,
(ii) an unstable node if $c_m > 0, m$ odd and $\lambda_2 > 0$.
(iii) a saddle-node if m even,
(iv) a saddle if $c_m > 0, m$ odd and $\lambda_2 < 0$ or $c_m < 0, m$ odd and $\lambda_2 > 0$.

If the singularity is a saddle-node or a saddle then we also compute a Taylor approximation for the unstable or stable manifold.

In case the two eigenvalues are zero, the point (x_0, y_0) is non-elementary. To study the vector field near the singularity, we desingularize the singularity by means of quasi-homogeneous blow up.

The desingularization algorithm consists in constructing a list S of elementary singularities, together with the invariant manifolds, on the blow-up locus which we order counter-clockwise. Each element of S is of the form

$$[[T_1, \ldots, T_m], x, y, Y, \mathit{sep}, \mathit{type}],$$

where (x, y) is an elementary singularity on the blow-up locus, Y is the blow-up vector field. The variable m is the number of blow-up levels we needed and $T_1, \ldots, T_m$ are the transformations, i.e. T_i is of the form

$(x, y) \mapsto (c_1 x^{d_1} y^{d_2} + x_{i-1}, c_2 x^{d_3} y^{d_4} + y_{i-1})$, with (x_{i-1}, y_{i-1}) the non-elementary singularity at blow-up level $i-1$. The variable *sep* is the Taylor approximation of the invariant manifold and *type* is the type of singularity we have (see figure 8).

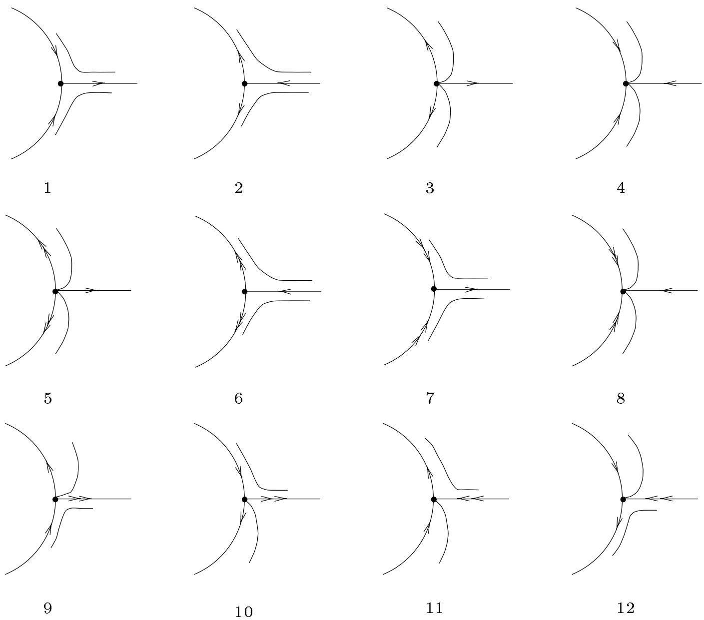

FIG. 8. Different types of singularities on the blow-up locus.

In the following construction we will use "Gosub" followed by a Roman number, meaning that one first has to elaborate the procedure indicated by the Roman number, before continuing the next line. The construction of the set S is as follows.

I. Input: vector field X with a non-elementary singularity (x_0, y_0).

• If $(x_0, y_0) \neq (0, 0)$ then consider the transformation $\bar{x} = x - x_0, \bar{y} = y - y_0$.

• Determine the Newton diagram and $\gamma_1 : \alpha r + \beta s = d$, with $\gcd(\alpha, \beta) = 1$.

• Let $N_p = 0, l = 1$ and $T_1 : (x, y) \mapsto (x^\alpha + x_0, x^\beta y + y_0)$.

• Blow up in the positive x-direction. This gives us a vector field Y.

• Gosub II.

• Let $N_n = 0, l = 1$ and $T_1 : (x, y) \mapsto (-x^\alpha + x_0, x^\beta y + y_0)$.

- Blow up in the negative x-direction. This gives us a vector field Y.
- Gosub III.
- If γ_1 lies completely in the half-plane $\{r \geq 0\}$ then

  – Let $T : (x,y) \mapsto (xy^\alpha + x_0, y^\beta + y_0)$.

  – Blow up in the positive y-direction. This gives us a vector field Y with $(0,0)$ an elementary singularity. In the same way as in II we construct a list $V = [[T], 0, 0, Y, \mathit{sep}, \mathit{type}]$.

  – Let $T : (x,y) \mapsto (xy^\alpha + x_0, -y^\beta + y_0)$.

  – Blow up in the negative y-direction. This gives us a vector field Y with $(0,0)$ an elementary singularity. In the same way as in II we construct a list $W = [[T], 0, 0, Y, \mathit{sep}, \mathit{type}]$.

  – $S = [W, L_1^p, \ldots, L_{N_p}^p, V, L_1^n, \ldots, L_{N_n}^n]$.

else $S = [L_1^p, \ldots, L_{N_p}^p, L_1^n, \ldots, L_{N_n}^n]$.
- Print out all the separatrices and the type of sectors as follows

  – For $i = 2$ to length(S) do

    * If $S[i-1][6] \in \{1,7,10\}$ and $S[i][6] \in \{2,6,12\}$ then we have a hyperbolic sector.
    * If $S[i-1][6] \in \{2,6,11\}$ and $S[i][6] \in \{1,7,9\}$ then we have a hyperbolic sector.
    * If $S[i-1][6] \in \{3,5,9\}$ and $S[i][6] \in \{4,8,11\}$ then we have a elliptic sector.
    * If $S[i-1][6] \in \{4,8,12\}$ and $S[i][6] \in \{3,5,10\}$ then we have a elliptic sector.
    * If $S[i-1][6] \in \{2,6,11\}$ and $S[i][6] \in \{4,8,11\}$ then we have an attracting sector.
    * If $S[i-1][6] \in \{4,8,12\}$ and $S[i][6] \in \{2,6,12\}$ then we have an attracting sector.
    * If $S[i-1][6] \in \{1,7,10\}$ and $S[i][6] \in \{3,5,10\}$ then we have a repelling sector.
    * If $S[i-1][6] \in \{3,5,9\}$ and $S[i][6] \in \{1,7,9\}$ then we have a repelling sector.

  – Determine the type of sector between the last element of S and the first one.

- End.

II. Input: vector field Y, the blow-up level l and the list $[T_1, \ldots, T_l]$.

(1) If $x = 0$ is not a line of singularities then determine the singularities of Y on the line $x = 0$.

- Sort the singularities such that $[y_1, \ldots, y_n]$ are in increasing order.
- For $i = 1$ to n do

– Let λ_1 and λ_2 be the eigenvalues of $\mathrm{D}Y(0, y_i)$.

– Translate the point $(0, y_i)$ to the origin. This gives us the vector field $\bar{Y}$.

– If $\lambda_1 = \lambda_2 = 0$ then we need to blow up $\bar{Y}$ at the origin. Gosub IV.

else

∗ If $\lambda_1 > 0$ and $\lambda_2 < 0$ then *sep* is the Taylor approximation of the unstable manifold and *type*=1.

∗ If $\lambda_1 < 0$ and $\lambda_2 > 0$ then *sep* is the Taylor approximation of the stable manifold and *type*=2.

∗ If $\lambda_1 = 0$ then *sep* is the Taylor approximation of the center manifold. Depending on the behaviour on the center manifold we have *type*=5 or 6 (resp. 7 or 8) if $\lambda_2 > 0$ (resp. $\lambda_2 < 0$).

∗ If $\lambda_2 = 0$ then *sep* is the Taylor approximation of the unstable (resp. stable) manifold and *type*=1,3,9 or 10 (resp. 2,4,11 or 12) if $\lambda_1 > 0$ (resp. $\lambda_1 < 0$).

∗ If $\lambda_1 > 0$ and $\lambda_2 > 0$ then *type*=3. If $\lambda_1 \neq \lambda_2$ then *sep* is a Taylor approximation of a orbit which is tangent with the line $y = vx$, with v a eigenvector associated to the eigenvalue λ_1. If $\lambda_1 = \lambda_2$ then *sep* is the line $y = 0$.

∗ If $\lambda_1 < 0$ and $\lambda_2 < 0$ then *type*=4. If $\lambda_1 \neq \lambda_2$ then *sep* is a Taylor approximation of a orbit which is tangent with the line $y = vx$, with v a eigenvector associated to the eigenvalue λ_1. If $\lambda_1 = \lambda_2$ then *sep* is the line $y = 0$.

∗ $N_p = N_p + 1, L^p_{N_p} = [[T_1, \ldots, T_l], 0, y_i, \bar{Y}, sep, type]$.

- Return.

(2) If $x = 0$ is a line of singularities then determine all the non-elementary singularities on the line $x = 0$.

- Sort the singularities such that $[y_1, \ldots, y_n]$ are in increasing order.
- For $i = 1$ to n do

– Translate the point $(0, y_i)$ to the origin. This gives us the vector field $\bar{Y}$.

– Determine the Newton diagram of $\bar{Y}$ and $\gamma_1 : \alpha r + \beta s = d$.

– Let $T_{l+1} : (x, y) \mapsto (x^\alpha, x^\beta y + y_i)$.

– Blow up in the positive x-direction.

– Gosub II with $l \to l + 1$.

- Return.

III. Same as II, but we sort the singularities in decreasing order. Change the variables N_p and $L^p_{N_P}$ with N_n and $L^n_{N_n}$, and II and IV with III and V.

IV. Input vector field $\bar{Y}$, the point $(0, y_i)$ and $[T_1, \ldots, T_l]$.

- Determine the Newton Diagram and $\gamma_1 : \alpha r + \beta s = d$.
- Blow up in the positive y-direction. This gives us the vector field Y^p.
- Determine the behaviour of Y^p near the origin.
- If the behaviour near the origin is like in figure 9(a) then

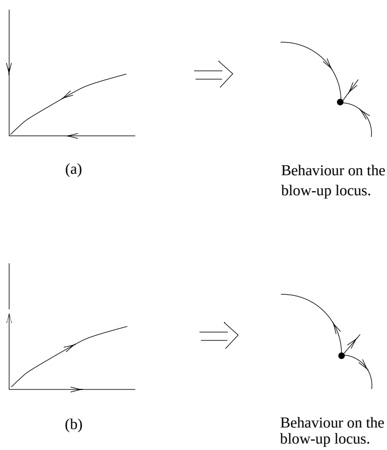

FIG. 9. Second blow-up in the y-direction.

  - *type*=4 and *sep* is the line $y = x$.
  - $N_p = N_p + 1$.
  - $L^p_{N_p} = [[T_1, \ldots, T_l, (x, y) \mapsto (xy^\alpha, y^\beta + y_i)], 0, 0, Y^p, sep, type]$.
- If the behaviour near the origin is like in figure 9(b) then
  - *type*=3 and *sep* is the line $y = x$.
  - $N_p = N_p + 1$.
  - $L^p_{N_p} = [[T_1, \ldots, T_l, (x, y) \mapsto (xy^\alpha, y^\beta + y_i)], 0, 0, Y^p, sep, type]$.
- Let $T_{l+1} : (x, y) \mapsto (x^\alpha, x^\beta y + y_i)$.
- Blow up in the positive x-direction. This gives us a vector field Y.

- Gosub II with $l \to l+1$.
- Blow up in the negative y-direction. This gives us the vector field Y^n.
- Determine the behaviour of Y^n near the origin.
- If the behaviour near the origin is like in figure 9(a) then
  - *type*=4 and *sep* is the line $y = x$.
  - $N_p = N_p + 1$.
  - $L^p_{N_p} = [[T_1, \ldots, T_l, (x, y) \mapsto (xy^\alpha, -y^\beta + y_i)], 0, 0, Y^n, sep, type]$.
- If the behaviour near the origin is like in figure 9(b) then
  - *type*=3 and *sep* is the line $y = x$.
  - $N_p = N_p + 1$.
  - $L^p_{N_p} = [[T_1, \ldots, T_l, (x, y) \mapsto (xy^\alpha, -y^\beta + y_i)], 0, 0, Y^n, sep, type]$.
- Return.

V. Same as IV, but first we blow up in the negative y-direction and than in the positive y-direction. Change the variables N_p and $L^p_{N_p}$ with N_n and $L^n_{N_n}$ and II with III.

2) If the eigenvalues are purely imaginary, then the point (x_0, y_0) is a weak focus. To determine its type, we compute the Lyapunov constants using the technique developed by Gasull and Torregrosa [21]. In case of a quadratic vector field or a linear plus homogeneous cubic vector field, P4 is able to determine whether or not the point is a center, an unstable or a stable weak focus of a certain order. In all other cases P4 evaluates by default the first four Lyapunov constants. If they are all zero we have an undetermined weak focus, in the other case we have a stable or an unstable weak focus. The algorithm is written in C and hence the computations are done numerically. So, the Lyapunov constants are calculated up to a certain precision. By default we say that a Lyapunov constant V is zero if $|V| < 10^{-8}$.

3) In case the eigenvalues are complex but not purely imaginary, the point (x_0, y_0) is a strong stable (resp. unstable) focus if $\mathrm{Tr}(\mathrm{D}X_{(x_0,y_0)}) < 0$ (resp. > 0).

Now we determine the singularities at infinity. By default we study the vector field on the Poincaré disc. First we transform the vector field using the transformation

$$\left\{ \begin{array}{l} x = \frac{1}{z_2} \\ y = \frac{z_1}{z_2} \end{array} \right. .$$

This yields the vector field (after multiplying the result with z_2^{d-1})

$$\begin{cases} \dot{z_1} = z_2^d(-z_1 P(\frac{1}{z_2}, \frac{z_1}{z_2}) + Q(\frac{1}{z_2}, \frac{z_1}{z_2})) \\ \dot{z_2} = -z_2^{d+1} P(\frac{1}{z_2}, \frac{z_1}{z_2}) \end{cases},$$

with d the degree of the vector field. Suppose that $Q_d(1, z_1) - z_1 P_d(1, z_1) \not\equiv 0$. The points $(z_1, 0)$ which sastisfy $Q_d(1, z_1) - z_1 P_d(1, z_1) = 0$ are infinite singular points of X. These points are studied in the same way as the finite ones. In case that $Q_d(1, z_1) - z_1 P_d(1, z_1) \equiv 0$, the line at infinity is a line of singularities. To study the behaviour near infinity we divide the vector field by z_2, and study this vector field near the line $\{z_2 = 0\}$.

Secondly we transform the vector field using the transformation

$$\begin{cases} x = \frac{z_1}{z_2} \\ y = \frac{1}{z_2} \end{cases}.$$

This yields the vector field (after multiplying the result with z_2^{d-1})

$$\begin{cases} \dot{z_1} = z_2^d(P(\frac{z_1}{z_2}, \frac{1}{z_2}) - z_1 Q(\frac{z_1}{z_2}, \frac{1}{z_2})) \\ \dot{z_2} = -z_2^{d+1} Q(\frac{z_1}{z_2}, \frac{1}{z_2}) \end{cases}.$$

We only have to determine whether or not the point $(0, 0)$ is a singular point, since the others have been studied in the first chart.

If there is a singularity at infinity which is non-elementary, it is sometimes better to study the vector field on a Poincaré-Lyapunov disc of some degree (α, β), i.e. we use a transformation of the form

$$\begin{cases} x = \frac{\cos\theta}{r^\alpha} \\ y = \frac{\sin\theta}{r^\beta} \end{cases},$$

for the study near infinity, which yields the vector field (after multiplying the result with r^c)

$$\begin{cases} \dot{r} = -r^{c+1} \sum_{\delta \le c} r^{-\delta}(\cos\theta P_\delta(\cos\theta, \sin\theta) + \sin\theta Q_\delta(\cos\theta, \sin\theta)) \\ \dot{\theta} = r^c \sum_{\delta \le c} r^{-\delta}(-\beta \sin\theta P_\delta(\cos\theta, \sin\theta) + \alpha\cos\theta Q_\delta(\cos\theta, \sin\theta)) \end{cases} \tag{21}$$

with $P_\delta(x, y)\frac{\partial}{\partial x} + Q_\delta(x, y)\frac{\partial}{\partial y}$ the quasi-homogeneous component of type (α, β) and quasi-homogeneous degree δ; c is chosen to be the maximal δ.

With an appropriate choice of (α, β) we will often only encounter elementary singularities at infinity. To simplify the calculations we prefer to work with charts.

First we transform the vector field using the transformation

$$\begin{cases} x = \frac{1}{z_2^\alpha} \\ y = \frac{z_1}{z_2^\beta} \end{cases} .$$

This yields the vector field (after multiplying the result with αz_2^c)

$$\begin{cases} \dot{z_1} = z_2^c \sum_{\delta \leq c} z_2^{-\delta}(\alpha Q_\delta(1, z_1) - \beta z_1 P_\delta(1, z_1)) \\ \dot{z_2} = -z_2^{c+1} \sum_{\delta \leq c} z_2^{-\delta} P_\delta(1, z_1) \end{cases} . \tag{22}$$

If $\alpha Q_c(1, z_1) - \beta z_1 P_c(1, z_1) \not\equiv 0$, then the points $(z_1, 0)$ which satisfy $\alpha Q_c(1, z_1) - \beta z_1 P_c(1, z_1) = 0$ are infinite singular points of X. These points are studied in the same way as the finite ones. In cases that $\alpha Q_c(1, z_1) - \beta z_1 P_c(1, z_1) \equiv 0$, the line at infinity is a line of singularites. To study the behaviour near infinity we divide the vector field by z_2 and study this vector field near the line $\{z_2 = 0\}$.

Next we transform the vector field using the transformation

$$\begin{cases} x = \frac{-1}{z_2^\alpha} \\ y = \frac{z_1}{z_2^\beta} \end{cases} .$$

This yields the vector field (after multiplying the result with αz_2^c)

$$\begin{cases} \dot{z_1} = z_2^c \sum_{\delta \leq c} z_2^{-\delta}(\alpha Q_\delta(-1, z_1) + \beta z_1 P_\delta(-1, z_1)) \\ \dot{z_2} = z_2^{c+1} \sum_{\delta \leq c} z_2^{-\delta} P_\delta(-1, z_1) \end{cases} . \tag{23}$$

This vector field can be studied in the same way as the previous one.

Finally we consider the two transformations

$$\begin{cases} x = \frac{z_1}{z_2^\alpha} \\ y = \frac{1}{z_2^\beta} \end{cases} ,$$

and

$$\begin{cases} x = \frac{z_1}{z_2^\alpha} \\ y = \frac{-1}{z_2^\beta} \end{cases} .$$

For these two vector fields we only have to determine whether or not the point $(0, 0)$ is a singular point, since the others have been studied in the first two charts.

At this stage we are ready to draw a large part of the phase portrait of the vector field. First we draw the invariant separatrices in the following way. In case the singularity is a saddle or a saddle-node, we use the Taylor approximation of the invariant manifold until it meets the boundary of a circle of radius ε, for a certain choice of $\varepsilon \geq 0$. From this point on we integrate the separatrices with the multi-step Runge-Kutta method of orders 7 and 8. To prevent numeric overflow in the Taylor approximation, we normalize the vector fields (18) and (20) before we compute the Taylor approximation as follows. Let a be the largest coefficient in absolute value of the vector field. We rescale the time such that this coefficient becomes equal to $1000 \cdot \mathrm{sign}(a)$. At the beginning of the numerical integration of the separatrices we have an error which comes from the Taylor approximation. By default we take $\varepsilon = 0.01$ and as order of approximation $n = 6$. So we have an error of order 10^{-14}. To make sure that this error is not too large, we do a test to decide whether or not the Taylor approximation "fits" the real invariant manifold. Let $f(t)$ be the Taylor approximation of the invariant manifold, which is tangent to the line $v = 0$. Suppose that $t_1^2 + f(t_1)^2 = \varepsilon^2$ and consider the points $(ih, f(ih))$, $i = 1, \ldots, 100$, with $h = t_1/100$. Consider the angles $\alpha_i = \arctan(f'(ih))$ and $\beta_i = \arctan\left(\frac{\dot{v}(ih, f(ih))}{\dot{u}(ih, f(ih))}\right)$, $i = 1, \ldots, 100$. If $|\alpha_i - \beta_i| < 10^{-8}, \forall i = 1, \ldots, 100$, we accept the Taylor approximation, otherwise we compute the Taylor approximation one order higher and do the test again. By default we take as maximum order $n = 20$. In this case the error is of order 10^{-42}. This test works very well for the stable and unstable manifolds, but for the center manifolds it sometimes fails, especially if the non zero eigenvalue is large in absolute value.

If the singularity is non-elementary, we split the point into several singularities which are elementary. For each of these points we draw the invariant manifold (which correspond to a separatrix of the non-elementary singularity) as follows. First we use the Taylor approximation in the blow up chart which corresponds to the elemantary singularity, up to distance ε from the singularity. Then we extend the separatrix in this chart by numeric integration, up to distance 1 from the singularity. Next we extend by numeric integration in the real plane. The number of steps has to be decided in an interactive way by the user.

To prevent numerical overflow when integrating the vector field, we do not always integrate the vector field in the real plane and project it on the Poincaré sphere, but we use different charts which cover the Poincaré sphere as follows. Let (X, Y, Z) be a point on the Poincaré sphere with $Z > 0$, and let (θ, φ) be the sphere coordinates of the point, i.e. $X = \cos\theta \sin\varphi$, $Y = \sin\theta \sin\varphi$, and $Z = \cos\varphi$.

If $0 \leq \varphi \leq \frac{\pi}{4}$ we transform the point to the real plane, i.e. we consider the point $(\frac{X}{Z}, \frac{Y}{Z})$ and integrate the original vector field. If $\varphi > \frac{\pi}{4}$ then we consider the following four cases.

(i) If $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, we consider the point $(z_1, z_2) = (\frac{Y}{X}, \frac{Z}{X})$ and integrate the vector field

$$\begin{cases} \dot{z_1} = z_2^d(-z_1 P(\frac{1}{z_2}, \frac{z_1}{z_2}) + Q(\frac{1}{z_2}, \frac{z_1}{z_2})) \\ \dot{z_2} = -z_2^{d+1} P(\frac{1}{z_2}, \frac{z_1}{z_2}) \end{cases}, \tag{24}$$

(ii) If $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, we consider the point $(z_1, z_2) = (\frac{X}{Y}, \frac{Z}{Y})$ and integrate the vector field

$$\begin{cases} \dot{z_1} = z_2^d(P(\frac{z_1}{z_2}, \frac{1}{z_2}) - z_1 Q(\frac{z_1}{z_2}, \frac{1}{z_2})) \\ \dot{z_2} = -z_2^{d+1} Q(\frac{z_1}{z_2}, \frac{1}{z_2}) \end{cases}, \tag{25}$$

(iii) If $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$, we consider the point $(z_1, z_2) = (\frac{Y}{X}, \frac{Z}{X})$ and integrate the vector field

$$\begin{cases} \dot{z_1} = (-1)^{d-1} z_2^d(-z_1 P(\frac{1}{z_2}, \frac{z_1}{z_2}) + Q(\frac{1}{z_2}, \frac{z_1}{z_2})) \\ \dot{z_2} = (-1)^d z_2^{d+1} P(\frac{1}{z_2}, \frac{z_1}{z_2}) \end{cases}, \tag{26}$$

(iv) If $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$, we consider the point $(z_1, z_2) = (\frac{X}{Y}, \frac{Z}{Y})$ and integrate with the vector field

$$\begin{cases} \dot{z_1} = (-1)^{d-1} z_2^d(P(\frac{z_1}{z_2}, \frac{1}{z_2}) - z_1 Q(\frac{z_1}{z_2}, \frac{1}{z_2})) \\ \dot{z_2} = (-1)^d z_2^{d+1} Q(\frac{z_1}{z_2}, \frac{1}{z_2}) \end{cases}. \tag{27}$$

The pattern of singularities, as well finite as infinite ones, together with their separatrices will already give a very good idea of the global phase portrait (see [13, 14]). We of course do not see the exact number and location of the closed orbits, but we have confined the regions in which limit cycles or annuli of closed orbits can occur. If one has the impression that closed orbits and especially limit cycles will occur, one can ask P4 to find these limit cycles as follows. First one has to select two points x and y. The two points should be close to the region where one expects to find a limit cycle, and the line L joining both points should cut the expected limit cycle. P4 tries to determine the limit cycle as follows. First it divides the line in segments $[p_i, p_{i+1}]$ of length h and starts integrating from the one end of the line L to the other. Every orbit close to the limit cycle is supposed to cut the line L again. ¿From this we detect the existence of the limit cycle when we find a change in the Poincaré Return Map. P4 detects such change as follows. Suppose that we start integrating from a point p_i on L, and that the orbit cuts the line L again in a point q_i with $p_i < q_i$. P4 takes now the point p_j nearest to q_i with $p_j > q_i$ and starts integrating in the same direction. If this orbit cuts L in a point q_j with $q_j < p_j$ then there is a limit cycle between the points q_i and q_j. By default we take $h = 10^{-4}$. Of course in this way we only can say that in a region of length 10^{-4} there exists at least one limit cycle. Sometimes it is possible that P4 finds non-existent limit cycles. The reason is that in these cases the Poincaré Return Map is very close to the identity.

In case we study the vector field on a Poincaré-Lyapunov disc of degree (α, β), P4 draws the orbits of the vector field as follows (see figure 10).

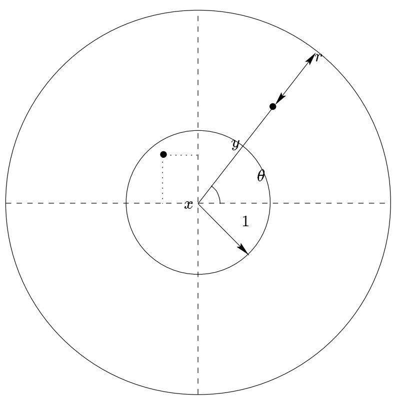

FIG. 10. Representation of the Poincaré-Lyapunov disc of degree (α, β).

Let $(x, y) \in \mathbf{R}^2$. If $x^2 + y^2 \leq 1$ then (x, y) will be plot in the interior of the unit circle around the origin, by integrating the original vector field (of course making the detailed analysis of the finite singularities as presented in the case of Poincaré compactification). If $x^2 + y^2 > 1$, P4 makes a transformation of the form $x = \cos\theta / r^\alpha$ and $y = \sin\theta / r^\beta$ in order to plot in the annulus limited by the finite circle of radius 1 and the infinity one, integrating the vector field (21), to extend the information near the singularities. Unfortunately orbits crossing the circle of radius 1 give the impression to have a non-continuous derivative. This is due to the fact that we are using two different transformations which do not match in a differentiable way on the unit circle.

6. TREATMENT OF EXAMPLES

Let us consider the vector field

$$X : \begin{cases} \dot{x} &= y \\ \dot{y} &= -x - y(1+2x) \end{cases} .$$

First we study this vector field on the Poincaré disc. Using the tool P4 we find that the origin is a singularity of the vector field which is a strong stable focus. For the study at infinity, we first consider the vector field in the U_1 chart. This vector field has equation

$$X_1 : \begin{cases} \dot{z_1} &= -2z_1 - z_2 - z_1 z_2 - z_2^2 z_2 \\ \dot{z_2} &= -z_1 z_2^2 \end{cases} .$$

We have that the origin is a singularity which is a semi-hyperbolic saddle. Now we consider the vector field in the U_2 chart which has equation

$$X_2 : \begin{cases} \dot{z_1} &= z_2 + z_1 z_2 + 2z_1^2 + z_1^2 z_2 \\ \dot{z_2} &= 2z_1 z_2 + z_2^2 + z_1 z_2^2 \end{cases} .$$

We have that the origin is a non-elementary singularity. To know the behaviour near the origin we perform a desingularization at the origin, using a quasi-homogeneuos blowing up of degree $(1,2)$. This procedure is describe in figure 11

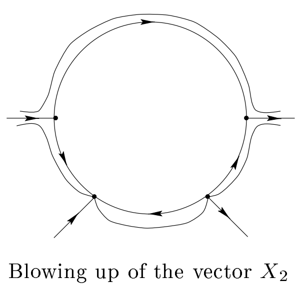

FIG. 11. Blowing up of the vector X_2 near the origin.

Using this information, we can now draw the phase portrait on the Poincaré disc (see figure 12)

Since there is a non-elementary singular point at infinity, we can study the vector field on a Poincaré-Lyapunov disc. Using a Poincaré-Lyapunov disc of degree $(1,2)$ we get four singularities at infinity, two nodes and two semi-hyperbolic saddles. Using this information, we can now draw the phase portrait on the Poincaré-Lyapunov disc of degree $(1,2)$ (see figure 13).

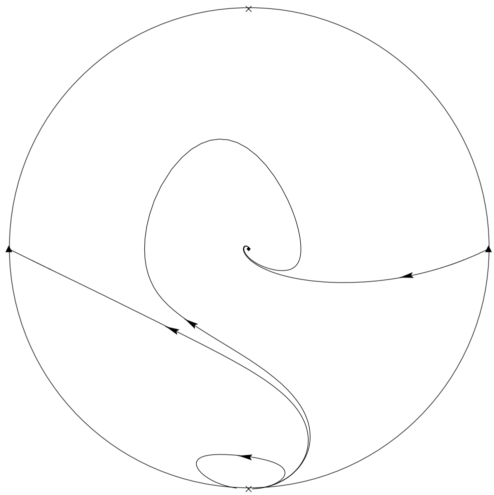

FIG. 12. Phase portrait of the vector field X on the Poincaré disc.

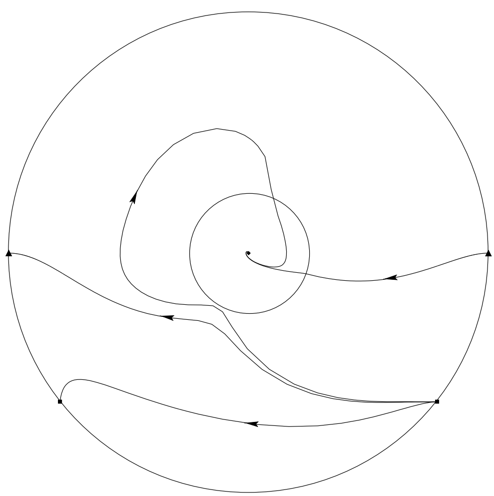

FIG. 13. Phase portrait of the vector field X on the Poincaré-Lyapunov disc of degree $(1, 2)$.

Consider the vector field

$$Y : \left\{ \begin{array}{rcl} \dot{x} & = & y - x^2 \\ \dot{y} & = & -\frac{1}{20} - x - x^2 \end{array} \right. .$$

We study this vector field on the Poincaré-Lyapunov disc of degree $(1,2)$. Using $P4$ we find that $((5+\sqrt{30})/10, (11+2\sqrt{(30)})/20)$ is a saddle point and $((5-\sqrt{30})/10, (11-2\sqrt{(30)})/20)$ is a strong unstable focus

At infinity we find four singular points, two points in the U_1 chart, namely $(0,0)$ which is an repelling node and $(1,0)$ which is a saddle-node and two in the U_2 chart, namely $(0,0)$ which is an attracting node and $(1,0)$ which is a saddle-node

Using this information, we can now draw the phase portrait on the Poincaré-Lyapunov disc of degree $(1,2)$ (see figure 14). We also find that the vector field has one limit cycle (see figure 15). ΩΩ Ω

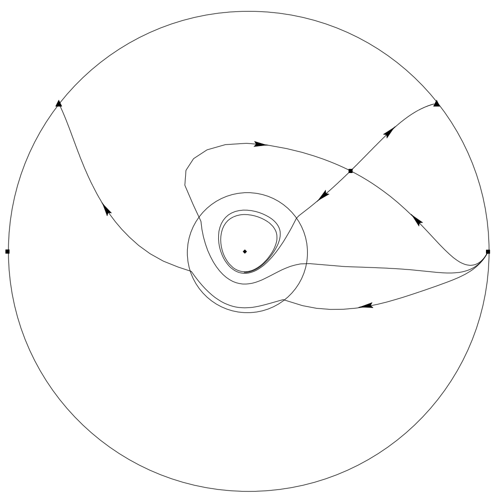

FIG. 14. Phase portrait of the vector field Y on the Poincaré-Lyapunov disc of degree $(1,2)$.

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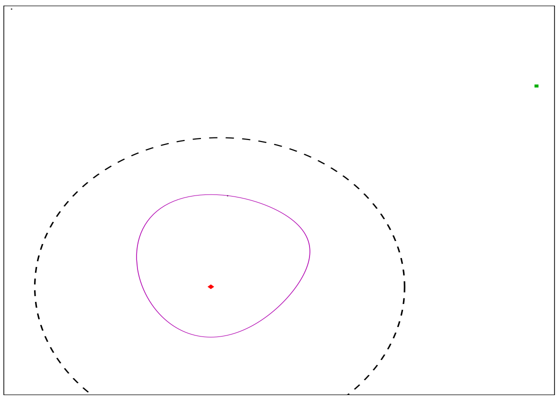

FIG. 15. Limit cycle of the vector field Y.

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