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# Exact Devaney Chaos and Entropy

Dominik Kwietniak

*Institute of Mathematics, Jagiellonian University,*
*Reymonta 4, 30-059 Kraków, Poland*
E-mail: kwietnia@im.uj.edu.pl

and

Michał Misiurewicz*

*Department of Mathematical Sciences, IUPUI,*
*402 N. Blackford Street, Indianapolis, IN 46202-3216, USA*
E-mail: mmisiure@math.iupui.edu

We modify the definition of chaos in the sense of Devaney, by replacing the condition of topological transitivity by topological exactness. We study basic properties of exact Devaney chaos defined in such a way. We also investigate the infimum of topological entropies of exactly Devaney chaotic maps of a given space.

*Key Words:* Chaos, entropy, circle maps, hyperspaces

## 1. INTRODUCTION

In recent years many attempts have been made to formulate a mathematically rigorous definition of *chaos* in discrete dynamical systems. Each of the proposed definitions has its advantages and deficiencies depending on which concrete properties of the vague common notion of chaoticity are chosen as characteristics. As a result, there are many, sometimes even unrelated definitions of *chaotic systems*.

One of the most popular ones is *Devaney chaos* introduced in [10] (see Section 2). Another widely recognized indicator of chaotic behavior of the system is positivity of *topological entropy* (see, e.g., [12] or [19]).

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In general, these two notions are independent. The relations between them depend strongly on the phase space in the consideration.

It is known, e.g., that for the graph maps (including the circle and interval maps, see [14], [16] and [17]) positive topological entropy is equivalent to the existence of a subsystem chaotic in the sense of Devaney.

Therefore it is important to understand for which spaces chaos in the sense of Devaney forces the entropy to be large. It turns out (see [3]) that while this is true for interval or tree maps, it is not the case for many phase spaces.

The main question we investigate is whether strengthening the notion of Devaney chaos will yield different results. Let us specify that for us a *dynamical system* is a continuous map of a compact metric space into itself and that we are interested mainly in noninvertible systems. Thus, we strengthen the notion of Devaney chaos in a sense to the extreme, and introduce *exact Devaney chaos* by replacing in the definition topological transitivity by topological exactness (see Section 2). Then we investigate the quantity $I^{\mathrm{ED}}(X)$, which is the infimum of topological entropies of exactly Devaney chaotic maps $f\colon X \to X$. It is analogous to the quantity $I^{\mathrm{D}}(X)$, introduced in [3], where the infimum is taken over all Devaney chaotic maps.

We show that $I^{\mathrm{ED}}(X) = 0$ when $X$ is a circle or an $n$-dimensional torus (see Section 4). Moreover, knowing that $I^{\mathrm{ED}}(X) = 0$ for some space $X$ we are able to extend this result for the infinite collection of other spaces - *symmetric products* of the space $X$ (the Möbius band and the 3-dimensional sphere are among them). The key tool is topological theory of hyperspaces (see Section 5 and [13]). As a corollary we obtain an analogous result for the projective plane. Our constructions of exactly Devaney chaotic maps with small entropy are explicit. We also show that $I^{\mathrm{D}}(X) = 0$ if $X$ is the Klein bottle.

Those results provide affirmative partial answers to some questions from [3]. Namely, the authors asked there whether $I^{\mathrm{D}}(M) = 0$ for every compact manifold $M$ of dimension 2 or larger, and whether there are Devaney chaotic maps on $n$-dimensional sphere and other spaces with arbitrarily small positive topological entropies.

## 2. DEFINITIONS AND NOTATION

From now on, unless otherwise stated, $X$ stands for a compact metric space and all maps are assumed to be continuous . Let $f\colon X \to X$ be a map. We will write $f^{\times k}$ for the Cartesian product of $k$ copies of $f$, that is, $f^{\times k}\colon X^k \to X^k$ is given by $f^{\times k}(x_1, \ldots, x_k) = (f(x_1), \ldots, f(x_k))$.

Let us recall several notions that measure how a map is mixing the points of the space. A map $f$ is *transitive* if for any pair of nonempty open sets $U, V \subset X$ there exists $m > 0$ such that $f^m(U) \cap V \neq \emptyset$; it is *weakly mixing* if $f^{\times 2}$ is transitive; it is *mixing* if for any pair of nonempty open sets $U, V \in X$ there exists $m_0 > 0$ such that $f^m(U) \cap V \neq \emptyset$ for all $m \geq m_0$. Finally, $f$ is *locally eventually onto*, or shortly *leo*, if for every nonempty open set $U \subset X$ there exists a nonnegative integer $m$ such that $f^m(U) = X$. This property corresponds to exactness in ergodic theory, so it is also called *topological exactness*.

The following implications are elementary to check:

$$\text{leo} \implies \text{mixing} \implies \text{weak mixing} \implies \text{transitivity}.$$

The reverse implications do not hold in general.

Note that if $f$ is leo, mixing, weakly mixing, or has dense set of the periodic points, then the same property holds for any of its Cartesian powers $f^{\times k}$. However, this is not true for transitive maps. An irrational rotation on the circle $\mathbb{S}^1$ is an obvious counterexample.

According to a very general definition, $f$ has an *n-horseshoe* if there are pairwise disjoint compact sets $A_1, \dots, A_n$, $n \geq 2$ such that for any $1 \leq i \leq n$ we have $f(A_i) \supset \bigcup_{j=1}^{n} A_j$. If $f$ has an $n$-horseshoe then there exists an $f$-invariant compact set on which $f$ is semiconjugate to a full shift on $n$ symbols (see, e.g., [6]). If $X$ is a graph, then normally we assume that the sets $A_i$ are intervals (arcs), but it suffices to assume that they have pairwise disjoint interiors, i.e., they may have common endpoints.

According to the Devaney's definition of chaos [10], a map $f\colon X \to X$ is *chaotic* if it is transitive, periodic points are dense, and it is sensitive to initial conditions. However, if $X$ is infinite, then the third condition is redundant (see, e.g., [4]). Since we will restrict our attention to compact metric spaces without isolated points we will say that $f$ is *Devaney chaotic* if it is transitive with periodic points dense.

One can consider stronger notions of chaos by replacing transitivity by weak mixing, mixing or leo. In this paper we will go to the extreme and consider the strongest of those notions. We will say that $f$ is *exactly Devaney chaotic* if it is leo with periodic points dense. Note that every leo map is necessarily a noninvertible surjection and if a space with more than one point admits existence of a leo map then it has no isolated points. Hence, exact Devaney chaos implies Devaney chaos, but not conversely, as can be shown by simple examples of interval maps, or Devaney chaotic homeomorphisms, e.g. hyperbolic toral automorphisms (see [10]).

## 3. BASIC PROPERTIES OF EXACTLY DEVANEY CHAOTIC MAPS

First observe that the properties assumed in the definition of the exact Devaney chaos are not redundant. Clearly, density of periodic points does not imply leo (look for instance at the identity on any space consisting of more than one point). An example that leo does not imply density of periodic points is more complicated and it is due to Tomasz Downarowicz. Namely, let us take a countable product of exact Devaney chaotic maps $f_n$, such that $f_n^n$ does not have fixed points. It will be still leo, but will have no periodic points. As $f_n$ we can take for instance a circle map from Theorem 4 with rotation interval not containing any fraction with the denominator $n$.

We want to investigate the connections between exact Devaney chaos and topological entropy. It is known that a Devaney chaotic map can have zero topological entropy (see [11]), but for an exactly chaotic map it is impossible, since that every leo map has positive topological entropy.

Lemma 1. *Let $X$ be a compact metric space consisting of more than one point. Assume that $f\colon X \mapsto X$ is a leo map. Then $h(f) > 0$.*

*Proof.* We claim that there is $N > 0$ such that $f^N$ has a 2-horseshoe. Since $f$ is a leo map, for any disjoint closed sets $A_1$, $A_2$ with nonempty interiors there are positive integers $N_1$ and $N_2$ such that $f^{N_1}(A_1) = f^{N_2}(A_2) = X$. Now let $N = \max\{N_1, N_2\}$ and $A_1$, $A_2$ form a horseshoe for $f^N$ as claimed. Therefore there is an $f^N$-invariant compact set $S$ such that $f^N|_S$ is semiconjugate to a full shift on 2 symbols. Hence, $h(f) = (1/N)\cdot h(f^N) \geq (1/N)\log 2 > 0$. ∎

For a specific topological space $X$ not much is known about lower bounds for the entropy of exact Devaney chaotic maps, the only exception being a compact interval $I$. The following theorem is well known, see e.g. [20], Proposition 4.3.9 and Example 4.4.5.

Lemma 2. *Let $f\colon I \mapsto I$ be a mixing interval map. Then we have $h(f) > (1/2)\log 2$. Moreover, for every $\varepsilon > 0$ there exists an exactly Devaney chaotic interval map with topological entropy smaller than $(1/2)\log 2 + \varepsilon$. Therefore $I^{ED}(I) = (1/2)\log 2$.*

The next lemma follows immediately from the relevant definitions, so we leave its simple proof to the reader.

Lemma 3. *The Cartesian product of finitely many exactly Devaney chaotic maps is exactly Devaney chaotic, and a factor of an exactly Devaney chaotic map is exactly Devaney chaotic.*

## 4. ENTROPY OF EXACT CHAOTIC MAPS ON THE CIRCLE

Let us recall some facts from the theory of circle maps of degree 1 and their rotation theory. The reader can find this theory with all details, including proofs, for instance in [1].

Let $\mathbb{S}^1$ be the circle. We may assume that it is the unit circle in the complex plane and then we have the natural projection of the universal covering space to the circle, $p\colon \mathbb{R} \to \mathbb{S}^1$, given by $p(x) = e^{2\pi i x}$. Every continuous map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ has a *lifting* $F\colon \mathbb{R} \to \mathbb{R}$, which is a continuous map such that $p \circ F = f \circ p$. We will be interested in maps of the circle of *degree 1*. A simple characterization of such a map is that for its lifting $F$ we have $F(x+k) = F(x) + k$ for every integer $k$.

We will use for the circle maps the same terminology as for their liftings. For instance, we will speak of intervals (which are really arcs of the circle), monotonicity, local minima and maxima, derivatives, etc. A continuous map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ of degree 1 will be called *bimodal* if it has 2 local extrema. Then one of them has to be a local maximum (we will denote it $c$) and the other one a local minimum (we will denote it $d$). Such map will be called *piecewise expanding* if there is a constant $\alpha > 1$ such that for every interval $I$ containing no local extrema the length of $f(I)$ is larger than or equal to the length of $I$ multiplied by $\alpha$. Note that we really think about those intervals as living in the universal covering, because the length of $f(I)$ may be larger than 1 (the length of the whole circle). If, as it happens often, $f$ is smooth on $[c,d]$ and on $[d,c]$, then this condition is equivalent to $f' \le -\alpha$ on $[c,d]$ and $f' \ge \alpha$ on $[d,c]$.

For the lifting $F$ of $f$ we can define its *upper map* $F_u$ by $F_u(x) = \sup\{F(y)\colon y \le x\}$ and its *lower map* $F_l$ by $F_l(x) = \inf\{F(y)\colon y \ge x\}$. Those maps are liftings of continuous circle maps of degree 1, $f_u$ and $f_l$ respectively. We will call those maps the *upper map* and the *lower map* of $f$ respectively. Note that the maps $f_u$ and $f_l$ are nondecreasing. If $f$ is bimodal, then $f_u$ differs from $f$ only on one interval, whose left endpoint is $c$, and on which $f_u$ is constant. We will call this interval the *plateau* of $f_u$. Similarly, $f_l$ differs from $f$ only on its plateau, whose right endpoint is $d$.

The maps $f_u$ and $f_l$, as monotone circle maps of degree 1, have their *rotation numbers* $\rho_u$ and $\rho_l$ respectively. They are defined as

$$\rho_u = \lim_{n\to\infty} \frac{F_u^n(x) - x}{n}$$

for any $x \in \mathbb{R}$, and similarly for $\rho_l$. Note that if a monotone circle map of degree 1 has an irrational rotation number, it has no periodic points. Clearly, $\rho_l \le \rho_u$. The interval (perhaps degenerate) $[\rho_l, \rho_u]$ is the *rotation interval* of $f$.

THEOREM 4. *Assume that $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a bimodal piecewise expanding map and that the endpoints of its rotation interval are irrational. Then $f$ is exactly Devaney chaotic.*

*Proof.* We will use in the proof the notation ($\alpha, c, d$, etc.) introduced earlier.

Observe first that $\rho_l \neq \rho_u$. Indeed, since $f$ is piecewise monotone and piecewise expanding, then the topological entropy of $f$ is positive, so iterates of $f$ have horseshoes, so $f$ has periodic points (see, e.g., [15] or [1]). Therefore there are rational numbers in the rotation interval of $f$, so it cannot consist of one irrational number.

Let now $U$ be a nonempty open subset of $\mathbb{S}^1$. Suppose that $f^n(U) \neq \mathbb{S}^1$ for any $n$. The same holds for any subset of $U$, so we may assume that $U$ is an open interval. Look at the consecutive images of $U$ under the iterates of $f$. They are all intervals, and their lengths grow exponentially if none of them contains $c$ or $d$. Since their lengths are at most 1, there must be $n_1$ such that $f^{n_1}(U)$ contains $c$ or $d$ (we take the smallest $n_1$ with this property). We may assume that this is $c$; the proof for the other case is similar. If $f^{n_1}(U) = (a, b)$, we set $V = (a, c]$. All images of $V$ under the iterates of $f$ are intervals of length smaller than 1. By the same reason as above, there is $n_2 > 0$ such that $f^{n_2}(V)$ contains $c$ or $d$. Take the smallest $n_2$ with this property.

Since the rotation number of $f_u$ is irrational, the trajectories of $c$ for $f$ and for $f_u$ coincide (otherwise there would be $k > 0$ such that $f_u^k(c)$ belongs to the plateau of $f^u$, and therefore $f_u(c)$ would be periodic for $f_u$). Therefore, $f^i(c)$ does not belong to the plateau of $f_u$ for any $i$.

We claim that $f^{n_2}(V) = (f^{n_2}(a), f^{n_2}(c)]$. Otherwise, there is $n_3$ such that $0 < n_3 < n_2$ and $f^{n_3}(V) \subset [c, d]$. However, $[c, d]$ is contained in the plateau of $f_u$, so $f^{n_3}(c)$ belongs to this plateau, a contradiction. This proves our claim.

Therefore, if $f^{n_2}(V)$ contains $c$, then it contains the plateau of $f_u$, so in particular, it contains $d$. This proves that always $f^{n_2}(V)$ contains $d$. Thus, the images of $U$ under the iterates of $f$ contain $c$ and $d$, and those points have different rotation numbers. This proves that for a component $W$ of $p^{-1}(U)$, the lengths of $F^n(W)$ go to infinity as $n \to \infty$, a contradiction. Therefore $f$ is leo.

By [9], periodic points of $f$ are dense in $\mathbb{S}^1$. This completes the proof. ∎

THEOREM 5. *For every $\varepsilon > 0$ there exists an exactly Devaney chaotic circle map with topological entropy smaller than $\varepsilon$. Therefore $I^{ED}(\mathbb{S}^1) = 0$.*

*Proof.* Let us recall some results of [2] (see also [1]). Define the functions $R_{s,t}(x) = \sum a_{s,t}(n)x^{-n}$, where $s < t$ and $a_{s,t}(n)$ is the number of integers $k$ such that $s < k/n < t$. They are continuous and decreasing on $(1, \infty)$, with limits $\infty$ at 1 and 0 at $\infty$, and therefore there exists a unique root $\beta_{s,t}$ of the equation $R_{s,t}(x) = 1/2$. For each $s, t$ there exists a bimodal map $g_{s,t}$ with constant slope $\beta_{s,t}$ (and therefore piecewise expanding and with entropy $\log \beta_{s,t}$) and rotation interval $[s, t]$.

If $t < s+1$ then $a_{s,t}(n) \le n$. If additionally in $(s,t)$ there are no fractions $k/n$ with $n < N$, then for any $x > 1$ we have

$$R_{s,t}(x) \le \sum_{n=N}^{\infty} nx^{-n}. \tag{1}$$

Fix $x > 1$ such that $\log x < \varepsilon$. The series $\sum_{n=0}^{\infty} nx^{-n}$ is convergent, so the right-hand side of (1) goes to 0 as $N \to \infty$. In particular, if $N$ is large enough then $R_{s,t}(x) < 1/2$, so $\beta_{s,t} < x$. This proves that there exist irrational numbers $s < t$ and a bimodal piecewise expanding circle map $g_{s,t}$ with topological entropy smaller than $\varepsilon$ and rotation interval $[s,t]$. By Theorem 4, $g_{s,t}$ is exactly Devaney chaotic. ▌

Since any finite Cartesian product of exactly Devaney chaotic maps is exactly Devaney chaotic, we get the following corollary of Theorem 5 and Lemma 3.

Corollary 6. *Let $\mathbb{T}^k = (\mathbb{S}^1)^k$ be the k-dimensional torus. Then we have $I^{ED}(\mathbb{T}^k) = 0$ for any $k \ge 1$.*

Let us analyze to what degree our proof gives explicit examples. We understand the word "explicit" in the usual sense used in mathematics, and we do not want to wander into the realm of Constructive Real Analysis.

Clearly, given $x > 1$ with $\log x < \varepsilon$, we can explicitly find $N$ such that $\sum_{n=N}^{\infty} nx^{-n} < 1/2$. Then we can find explicitly irrational $s,t$ such that there are no fractions $k/n$ with $n < N$ in $[s,t]$ (provided we agree that we can find *any* irrational number explicitly). Then $\beta_{s,t}$ is the root of the equation $R_{s,t}(x) = 1/2$. This definition of $\beta_{s,t}$ looks already less explicit, but in fact it is not much worse than the definition of $\sqrt{2}$ as the positive root of the equation $x^2 - 2 = 0$. Given $\beta_{s,t}$ and $s$, we have an explicit formula for $g_{s,t}$ (which uses the sum of an infinite series; see [2] or [1]). Thus, we can conclude that getting a desired map in Theorem 5 is reasonably explicit.

## 5. MAPS INDUCED ON SYMMETRIC PRODUCTS

Let $(X,d)$ be a bounded metric space. A *hyperspace of* $X$ is a space whose points are (not necessarily all) subsets of $X$. For any nonempty subset $A$ of $X$ define a function $\operatorname{dist}(x,A) = \inf\{d(x,y) \colon y \in A\}$. The number $\operatorname{dist}(x,A)$ is called the *distance from* the point $x$ *to* the set $A$. For any $\varepsilon > 0$ we define an *$\varepsilon$-neighborhood of* $A$ as $N(A,\varepsilon) = \{x \in X \colon \operatorname{dist}(x,A) < \varepsilon\}$. Let $2^X$ be the hyperspace of all nonempty compact subsets of $X$ endowed with the Hausdorff metric, defined by

$$d_H(A,B) = \inf\{\varepsilon \ge 0 \colon A \subset N(B,\varepsilon) \text{ and } B \subset N(A,\varepsilon)\}.$$

The metric space $(2^X, d_H)$ is sometimes referred as "the space of fractals", i.e., it is the setting for iterated function systems (see [5]). It inherits many properties possessed by $X$ including compactness, completeness and connectedness. Let $X^{*k} \subset 2^X$ be the hyperspace of all nonempty subsets of $X$ having at most $k$ elements. The space $X^{*k}$ is closed in $2^X$ and is called *$k$-fold symmetric product of $X$*. These spaces were defined for the first time in 1931 by Borsuk and Ulam ([7]). There is an obvious identification of $X$ with $X^{*1}$, i.e, the set of all one-point subsets of $X$. Therefore we can write $X \subset X^{*k}$. A map $f\colon X \to X$ induces in a natural way a map $f^{*k}\colon X^{*k} \to X^{*k}$, by letting $f^{*k}(K) = f(K)$ for $K \in X^{*k}$. It is easy to check that $(f^{*k})^m(K) = f^m(K)$. Moreover $X$ is a closed, $f^{*k}$-invariant set for every $k$. For proofs and more details on hyperspaces see [13].

Although the $k$-fold symmetric product of $X$ and the Cartesian product of $k$ copies of $X$ are usually quite different spaces, topology given by Hausdorff metric on $X^{*k}$ and the quotient topology induced by the natural surjection

$$p\colon X^k \ni (x_1, \ldots, x_k) \mapsto \{x_1, \ldots, x_k\} \in X^{*k}$$

coincide. Much of the subsequent material in this section is based on the observation that $p$ semiconjugates $f^{\times k}$ with $f^{*k}$, i.e., $p \circ f^{\times k} = f^{*k} \circ p$. Summarizing the above considerations we can now formulate the following lemma.

LEMMA 7. *For every $k \geq 1$ the natural projection $p$ is a semiconjugacy from $f^{\times k}$ to $f^{*k}$.*

We can now state the main result of this section.

THEOREM 8. *If $I^{ED}(X) = 0$ then $I^{ED}(X^{*k}) = 0$ for every $k \geq 1$.*

*Proof.* Assume that $I^{\mathrm{ED}}(X) = 0$. By Lemma 3, $I^{\mathrm{ED}}(X^k) = 0$, so by Lemma 7 and again by Lemma 3, $I^{\mathrm{ED}}(X^{*k}) = 0$. ∎

By Theorems 8 and 5 we obtain the following corollary.

COROLLARY 9. *For every $k \geq 1$ we have $I^{ED}((\mathbb{S}^1)^{*k}) = 0$.*

## 6. APPLICATIONS

To decide if Theorem 8 allows us to say anything about $I^{\mathrm{ED}}(Y)$ for some interesting space $Y$ we have to know if there is $k > 1$ and a space $X$, for which we know $I^{\mathrm{ED}}(X)$ and such that $Y$ is homeomorphic to $X^{*k}$. Unfortunately, given topological space $X$ the characterizations of homeomorphic type of the spaces $X^{*k}$ are exceptional. We begin this section with a theorem that collects a number of known results about topological type of $X^{*k}$

for some non-trivial spaces. The proof of the following theorem can be found in [18]. Case (2) was proven first by R. Bott [8].

THEOREM 10.

*1.*$(\mathbb{S}^1)^{*2}$ *is homeomorphic to* $\mathbb{M}$*, where* $\mathbb{M}$ *denote the Möbius band. Moreover, points on the boundary of the band correspond to one-point subsets.*
*2.*$(\mathbb{S}^1)^{*3}$ *is homeomorphic to* $\mathbb{S}^3$*, where* $\mathbb{S}^3$ *denote the three dimensional sphere.*

Using Theorem 10 we can identify some spaces mentioned in Corollary 9 and we get immediately the following result.

THEOREM 11. *Let* $\mathbb{P}^2$ *be the projective plane. We have* $I^{ED}(\mathbb{P}^2) = I^{ED}(\mathbb{M}) = I^{ED}(\mathbb{S}^3) = 0$*.*

*Proof.* By Theorem 10 and Corollary 9 the last two equalities are true. To prove that $I^{\mathrm{ED}}(\mathbb{P}^2) = 0$, first observe that $\mathbb{P}^2$ is a quotient space of $\mathbb{M}$ obtained by collapsing the boundary circle of the band to a single point. Choose any $\varepsilon > 0$. By our previous considerations there is an exactly Devaney chaotic map $f\colon \mathbb{M} \mapsto \mathbb{M}$ with $h(f) < \varepsilon$ which leaves the boundary of $\mathbb{M}$ invariant. Therefore it induces a well defined map $\widehat{f}$ on the quotient space. Moreover canonical projection $\mathbb{M} \mapsto \mathbb{P}^2$ is a semiconjugacy from $f$ to $\widehat{f}$. Therefore $\widehat{f}$ is an exact Devaney chaotic map with $h(\widehat{f}) \leq h(f) < \varepsilon$. ▮

Let us also remark that the homeomorphisms which existence is stated in Theorem 10 can be constructed explicitly, i.e., it is possible to write analytic formulas for them.

Let us look now at the Klein bottle $\mathbb{K}$. Although we cannot prove an analogue of Theorem 5 for $\mathbb{K}$, we are able to show that $I^{\mathrm{D}}(\mathbb{K}) = 0$. To do this we need the following simple lemma.

LEMMA 12. *Let* $X$ *be a compact topological space and let* $X = X_1 \cup X_2$*, where* $X_1$ *and* $X_2$ *are closed sets with nonempty interiors in* $X$*. Assume that there is a homeomorphism* $\phi\colon X_1 \to X_2$ *such that* $\phi|_{X_1\cap X_2} = \mathrm{id}_{X_1\cap X_2}$*. If* $f\colon X_1 \to X_1$ *is a continuous bitransitive map (i.e., its second iterate* $f^2$ *is transitive) for which the set* $X_1 \cap X_2$ *is invariant then there exists continuous transitive, but not bitransitive map* $F\colon X \to X$*, such that*

$$h(F) = h(f).$$

*Moreover, periodic points of* $f$ *are dense in* $X_1$ *if and only if periodic points of* $F$ *are dense in* $X$*.*

*Proof.* Define a map $F\colon X \to X$ by

$$F(x) = \begin{cases} \phi(f(x)) & \text{if } x \in X_1, \\ f(\phi^{-1}(x)) & \text{otherwise.} \end{cases}$$

It is easy to see that $F$ is well defined and continuous.

We first observe that the sets $X_i$, $i = 1, 2$ are invariant for $F^2$ and (by assumption) have nonempty interiors in $X$. Hence $F^2$ cannot be transitive.

Note that $F^2|_{X_i}$ is conjugate to $f^2$ for $i = 1, 2$ via the identity and $\phi$ respectively.

For the proof of transitivity of $F$ let us take two nonempty open subsets $U$ and $V$ of $X$. There are two cases to consider:

*Case I.* There is $i \in \{1, 2\}$ such that $U, V \subset X_i$. Since $F^2|_{X_i}$ is conjugate to $f^2$ and the latter is transitive, there is $m > 0$ such that $F^{2m}(U) \cap V \neq \emptyset$, as required.

*Case II.* Suppose that Case I does not hold. We can assume (by replacing $U$ and $V$ by smaller sets if necessary) that $U \subset X_1$ and $V \subset X_2$. Then $V' = F^{-1}(V)$ is nonempty and open set contained in $X_1$ and we can proceed as in Case I. Hence, there is a point $x \in U$ and $m > 0$ such that $F^{2m}(x) \in V'$, so $F^{2m+1}(x) \in V$ and the proof of transitivity is complete.

If $x$ is a periodic point of $f$ with prime period $m$ then $x$ and $\phi(x)$ are periodic points of $F$ with prime periods at most $2m$. The equivalence of denseness of the sets of periodic points of $f$ and $F$ is now straightforward.

Since $X_1$, $X_2$ defined above are closed and invariant for $F^2$ and $X = X_1 \cup X_2$ then by [12], Proposition 8.2.9, we conclude that

$$h(F^2) = \max\{h(F^2|_{X_1}), h(F^2|_{X_2})\}.$$

Since conjugacy preserves topological entropy, we see that

$$h(F^2) = h(F^2|_{X_1}) = h(F^2|_{X_2}) = h(f^2).$$

It is now clear that $h(F) = h(f)$, which completes the proof. ∎

THEOREM 13. *We have $I^D(\mathbb{K}) = 0$.*

*Proof.* It is well known that the Klein bottle is homeomorphic to the quotient space obtained by gluing two copies of Möbius band $\mathbb{M}$ along their common boundary, i.e., $\mathbb{K} = \mathbb{M} \cup_{\mathrm{id}_{\partial\mathbb{M}}} \mathbb{M}$. By Theorem 11 we can find an exactly Devaney chaotic map $f\colon \mathbb{M} \to \mathbb{M}$ with arbitrary small positive topological entropy. Moreover, by the construction of this map and property of homeomorphism given by Theorem 10 (1) we may assume that the boundary of the band is an $f$-invariant subset. Clearly $f^2$ is also an exactly Devaney chaotic and therefore transitive map. Applying Lemma 12 we obtain a map $F\colon \mathbb{K} \to \mathbb{K}$ with arbitrary small positive topological entropy which is also Devaney chaotic. ∎

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