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Qualitative Classification of Singular Points

Qibao Jiang

Department of Mathematics and Mechanics, Southeast University, Nanjing 210018, China.

and

Jaume Llibre

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra 08193, Barcelona, Spain
E-mail: jllibre@mat.uab.es

We study in this paper the qualitative classification of isolated singular points of analytic differential equations in the plane. Two singular points are said to be qualitatively equivalent if they are topologically equivalent and furthermore, two orbits start or end in the same direction at one singular point if and only if the equivalent two orbits start or end in the same direction at the other singular point. The degree of the leading terms in the Taylor expansion of a differential equation at a singular point will be called the degree of this singular point. The qualitative equivalence divides the set of singular points of degree m into equivalence classes. The main problems studied here are the characterization of all qualitative equivalence classes and then, to determine to which class a singular point will belong to. We remark that up to now these problems have been solved only in the case $m = 1$ before.

The difficulty for this classification problem is that the number of blowing-ups necessary for the analysis of a singular point is unbounded (although it is finite) when this singular point varies in the set of singular points of degree m. To overcome this difficulty, we associate an oriented tree to the blowing-up process of any singular point such that each vertex represents some singular point. Then we prove that: (i) the above unboundedness comes exactly from the arbitrary length of an equidegree path; and (ii) the local phase portraits of the starting and the ending singular points in such a path are closely related by a simple rule which depends on the parity of the length of the path, but not on the length itself. Thus we obtain a successful method for this classification which can be applied in principle to the general m-degree case.

As application of our method, we get the precise list of qualitative equivalence classes of singular points of degree 2 (Theorem D). Further, we prove (Theorem F) that there are finitely many qualitative equivalence classes in the set of singular points of degree m, and there are a finite set of quantities, which are computed by a bounded number of operations, such that they determine to which class a given singular point belongs.

As a by-product we obtain the topological classification of singular points of degree 2. A topological equivalence class is determined by the number of elliptic, hyperbolic and parabolic sectors (denoted by e, h and p respectively) of the local phase portraits and their arrangement. The precise conditions for the tripe (e, h, p) have been given by Sagalovich. But this result is not sufficient for the topological classification problem. In fact, according to Sagalovich's theorem, there are 15 possible topological equivalence classes in the set of all singular points of degree 2; our result (Corollary E) shows that there are exactly 14 classes.

Key Words: Singular points, blowing-up, local phase portrait

1. INTRODUCTION

1.1. The problem of classifying singular points

Let (P, Q) be a C^1 mapping from $\mathbb{R}^2$ into itself. Instead of giving explicit solutions of the differential system

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \qquad (x, y) \in \mathbb{R}^2, \tag{1}$$

the qualitative theory of ordinary differential equations in the plane tries to provide a qualitative description of the behavior of each orbit (i.e., a planar curve represented by a solution of system (1)). More exactly, if $(x(t), y(t))$ is an orbit of (1) with maximal interval of definition (α, ω), one of the main objectives is to describe its behavior when $t \to \alpha$ and $t \to \omega$; i.e., the α- and ω-limit sets of this orbit. To this end, it suffices (see, for instance, [13] or [11]): (i) to describe the local phase portraits of singular points; (ii) to determine the number and the location of limit cycles; (iii) to determine the α- and ω-limit sets of all separatrices of the differential system. This paper deals with the first one. First we recall some concepts and basic facts of the qualitative theory, for some details see, for example, [2, 18].

By definition a point $p = (x_0, y_0)$ is called a *singular point* of system (1) if $P(x_0, y_0) = Q(x_0, y_0) = 0$. Its *local phase portrait* is a picture which describes the configuration of the orbits of system (1) in some neighborhood of this point. As one can see from the pictures in Figure 1, the local phase portrait of a singular point consists of this singular point and several orbits around it. As usual the study of the local phase portraits of singular points can be treated as a classification problem; i.e., we must classify some specific set of singular points according to some equivalence relation which will be defined below.

Two singular points p_1 and p_2 (possibly of different differential systems) are called C^r*-equivalent* if and only if p_i has a neighborhood U_i for $i = 1, 2$, and there is a homeomorphism $\Psi : U_1 \to U_2$ mapping orbits to orbits, where Ψ, Ψ^{-1} are C^r, and $r \in \{0, 1, 2, \cdots, \infty, \omega\}$ (C^ω means real analytic). The C^0-equivalence is also called the *topological equivalence*. We say p_1 and p_2 are *qualitatively equivalent* if (i) they are topologically equivalent through a homeomorphism Ψ; and (ii) two orbits are tangent to the same straight line at p_1 if and only if the corresponding two orbits through Ψ are tangent to the same straight line at p_2. If two singular points are equivalent in some of the sense defined above, we also say that their local phase portraits are equivalent in the same sense.

Only the qualitative equivalence and the topological equivalence will be considered in the rest of this paper. Evidently the qualitative equivalence is strictly finer than the topological equivalence. On the other hand, as it is shown in Lemma 25, it is strictly coarser than the C^1-equivalence.

The qualitative equivalence tells us not only when the local phase portraits of two singular points are topologically the same, but the information on the directions along which orbits enter the singular point. This kind of information is important in the qualitative analysis of system (1). As an example, one can find in [10] applications of our results to polynomial foliations in the plane.

Now we specify the set of singular points which we will study. The singular point $p = (x_0, y_0)$ of system (1) is called *analytic and isolated* if there is a neighborhood of (x_0, y_0) in which P and Q are analytic and system (1) has no other singular points. This kind of singular points have some good properties as we will see later. *In what follows a singular point means an analytic and isolated singular point except in the Desingularization Theorem.*

To study an arbitrary singular point of system (1), we move it to the origin. Then we consider the Taylor expansion of P and Q. Thus we obtain a system of the form

$$\begin{aligned} \dot{x} &= P_m(x,y) + P_{m+1}(x,y) + P_{m+2}(x,y) + \cdots, \\ \dot{y} &= Q_m(x,y) + Q_{m+1}(x,y) + Q_{m+2}(x,y) + \cdots, \end{aligned} \tag{2}$$

where $P_m^2(x,y) + Q_m^2(x,y) \not\equiv 0$, $P_i(x,y)$ and $Q_i(x,y)$ are homogeneous polynomials of degree i. The integer m is called the *degree* of the singular point $O = (0,0)$. This concept provides a natural partition to the set of all singular points. We consider the set of all singular points of a fixed degree m. The topological equivalence or qualitative equivalence divides this set into equivalence classes. The *classification problem* for degree m is just a solution to the following two questions:

(**A**) *How to characterize each equivalence class?*

(**B**) *Given a singular point of degree m, how to determine its equivalence class?*

To answer (A) we simply assign a local phase portrait to each equivalence class. Thus, to obtain all equivalence classes, it suffices to give the precise list of all topologically or qualitatively different local phase portraits. In particular, we have two classes which exist if and only if m is odd (see Lemma 26 for a proof): the class of all centers and the class of all foci. Their union is just the set of all singular points without any characteristic orbit. We recall that an orbit is a *characteristic orbit* of a singular point if it is tangent to some straight line at this point (for more details see [3, 8]). The problem of distinguishing between the center class and the focus class is not easy. Since here we are not interested in this problem, as usual we will not consider these two classes separately in the rest of this paper, and take their union as a single class in the above classification problem. This class is called the *center-focus class*.

By the Desingularization Theorem (§1.4), we know that the local phase portrait of any given singular point can be determined by doing finitely many blowing-ups (recall that we do not distinguish between centers and foci). So question (B) can always be solved by using the blowing-up method. From this point of view, no further consideration is needed for (B). But, in fact, if we look into the case of degree 1, we find that a solution to the following question is useful:

(**B′**) *How to find a finite set of quantities which will effectively solve (B) and can be easily computed?*

In the 1-degree case, there are 9 quantities which satisfy the requirement of (B′) (see Theorem A below). The computation of these quantities is usually easier than doing repeated blowing-ups.

1.2. Background

Until now $m = 1$ is the only case in which the classification problem stated above has been completely solved. This result will be summarized in Theorem A.

Assume that O is a singular point of the following real analytic system

$$\begin{aligned} \dot{x} &= ax + by + P_2(x,y) + P_3(x,y) + \cdots , \\ \dot{y} &= cx + dy + Q_2(x,y) + Q_3(x,y) + \cdots , \end{aligned} \tag{3}$$

where a, b, c and d are real numbers such that $a^2+b^2+c^2+d^2 \neq 0$, $P_i(x,y)$ and $Q_i(x,y)$ are homogeneous polynomials of degree i. For convenience, we introduce two analytic functions $F(x,y)$ and $G(x,y)$ defined by

$$F(x,y) = P_2(x,y) + P_3(x,y) + \cdots , \quad G(x,y) = Q_2(x,y) + Q_3(x,y) + \cdots .$$

Let λ_1 and λ_2 be the two roots of the equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

If exactly one of the roots λ_1 and λ_2 is non-zero, then it is easy to see that with an affine change of coordinates and a rescaling of the time variable (if necessary), we can write the linear part of system (3) in the form $a = b = c = 0$ and $d = 1$. Similarly we can take $a = c = d = 0$, $b = 1$ in system (3) if $\lambda_1 = \lambda_2 = 0$. In the case $\lambda_1 = \lambda_2 \neq 0$, an affine change of coordinates will reduce the linear part of system (3) into the Jordan canonical form, i.e., $c = 0, a = d = \lambda_1 = \lambda_2, b = \gamma$, where $\gamma = 0$ or 1.

In the case that $a = b = c = 0$ and $d = 1$, we can find the solution $y = f_1(x)$ of the equation $y + G(x, y) = 0$ by the Implicit Function Theorem. Let

$$F(x, f_1(x)) = a_m x^m + \cdots, \quad a_m \neq 0.$$

If $a = c = d = 0$ and $b = 1$, we have an analytic function $y = f_2(x)$ such that $f_2(x) + F(x, f_2(x)) \equiv 0$. Suppose that

$$G(x, f_2(x)) = \bar{a} x^\alpha + \cdots, \ \bar{a} \neq 0; \quad \Phi(x) = \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right)(x, f_2(x)) = \bar{b} x^\beta + \cdots.$$

Theorem A states that the above quantities $\lambda_1, \lambda_2, \gamma, m, a_m, \bar{a}, \alpha, \bar{b}, \beta$ can successfully determine the qualitative class of the local phase portrait of system (3) at the singular point $x = y = 0$.

Theorem A. *The following statements provide a complete description of the qualitative equivalence class of the local phase portrait of system (3) at the origin.*

(i) If $\lambda_1 \cdot \lambda_2 \neq 0$, the local phase portrait is determined by λ_1, λ_2 and γ. More precisely, the local phase portrait is qualitatively equivalent to

(a) the saddle of Figure 1(1) if $\lambda_1 \lambda_2 < 0$,

(b) the node of Figure 1(2) if $\lambda_1 \lambda_2 > 0$ and $\lambda_1 \neq \lambda_2$,

(c) the starlike node of Figure 1(3) if $\lambda_1 = \lambda_2 \neq 0$, $\gamma = 0$,

(d) the node of Figure 1(4) if $\lambda_1 = \lambda_2 \neq 0$, $\gamma \neq 0$,

(e) either the center (Figure 1(5)) or the focus (Figure 1(6)) if the imaginary parts of λ_1 and λ_2 are non-zero.

(ii) If $a = b = c = 0$ and $d = 1$, then the local phase portrait is determined by m and a_m. More concretely, the local phase portrait is qualitatively equivalent to

(a) the node of Figure 1(2) if m is odd and $a_m > 0$,

(b) the saddle of Figure 1(1) if m is odd and $a_m < 0$,

(c) the saddle-node of Figure 1(7) if m is even.

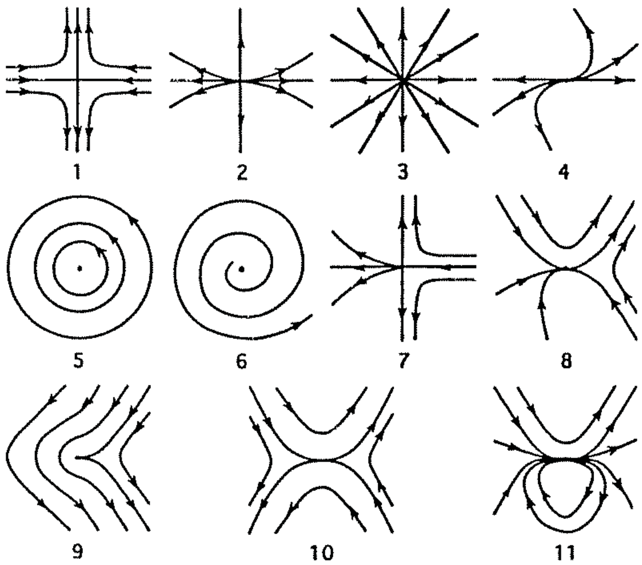

FIG. 1. The qualitatively different local phase portraits of singular points of degree 1.

(iii) If $a = c = d = 0$ *and* $b = 1$*, then the local phase portrait is determined by* $\bar{a}, \alpha, \bar{b}, \beta$*. More precisely, the local phase portrait is qualitatively equivalent to*

(a) the saddle-node of Figure 1(8) if α *is even and* $\alpha > 2\beta + 1$*,*

(b) the cusp of Figure 1(9) if α *is even and either* $\alpha < 2\beta + 1$ *or* $\Phi(x) \equiv 0$*,*

(c) the saddle of Figure 1(10) if α *is odd and* $\bar{a} > 0$*,*

(d) the node of Figure 1(4) if α *is odd,* β *is even and* $\bar{a} < 0$*, and further, either* $\alpha > 2\beta + 1$*, or* $\alpha = 2\beta + 1$ *and* $\bar{b}^2 + 4\bar{a}(\beta + 1) \geq 0$*,*

(e) Figure 1(11) if both α *and* β *are odd and* $\bar{a} < 0$*, and, moreover, either* $\alpha > 2\beta + 1$*, or* $\alpha = 2\beta + 1$ *and* $\bar{b}^2 + 4\bar{a}(\beta + 1) \geq 0$*,*

(e) the center (Figure 1(5)) or the focus (Figure 1(6)) if α *is odd and* $\bar{a} < 0$*, and either* $\alpha = 2\beta + 1$ *and* $\bar{b}^2 + 4\bar{a}(\beta + 1) < 0$*, or* $\alpha < 2\beta + 1$*, or* $\Phi(x) \equiv 0$*.*

Theorem A is due to several authors. Part (i) was known to Poincaré; part (ii) was shown by Bendixson [5]; and part (iii) was proved by Andreev [1]. For a proof of Theorem A see, for instance, [1, 2, 18].

When the degree $m > 1$ there are no works devoted to the qualitative classification problem, but some results are available for the topological classification. We account for this below.

It is well-known that the local phase portrait of a singular point that is different from a center or a focus, can be decomposed into a finite union of *elliptic, hyperbolic* and *parabolic sectors* (for precise definition and a proof see, for example, [2]). And the topological equivalence classes are characterized by the number of elliptic, hyperbolic and parabolic sectors (denoted by e, h, p respectively) and the arrangement of these sectors. Thus in order to give the precise list of topological equivalence classes of degree m, it suffices to determine which triple of non-negative integers can be taken as the triple (e, h, p) and which arrangement can be realized for each of these triples. Although there are no results available for the second part of this question, the first part has been completely solved.

Theorem B. *Assume that (e, h, p) is a triple of non-negative integers such that $e + h + p > 0$. Then, there is a singular point of degree m such that its local phase portrait has e elliptic sectors, h hyperbolic sectors and p parabolic sectors if and only if*

(i) $e + h \equiv 0 \pmod 2$;
(ii) $e \le 2m - 1$ *and* $e + h \le 2m + 2$;
(iii) if $e \ne 0$, *then* $e + h \le 2m$;
(iv) if $h = 2m + 2$, *then* $p = 0$*;*
(v) if $h = 2m$, *then* $p \le 1$;
(vi) if $e = 1$ *and* $h = 2m - 1$, *then p is even;*
(vii) $e + \operatorname{sgn}(e \cdot h) \le p \le e + h + 1 - \operatorname{sgn}(e + h)$.

The "only if" part of this theorem is due to several authors. Condition (i) was obtained by Bendixson [5]; (ii) and (iii) were proved by Berlinskii [6, 7]; (iv), (v) and (vi) are Sagalovich's results [14, 15]; whereas (vii) is obvious and can be seen from the definition of the number p. The "if" part was proved in [15].

We remark that Theorem B does not give any information about centers or foci, i.e., the case $e + h + p = 0$. But we have the well-known fact that the center-focus class is non-empty if and only if m is odd (see Lemma 26 for a proof).

By using Theorem B, we can easily get a finite list of pictures for any given m such that the local phase portrait of a singular point of degree m is topologically equivalent to one of them. To do this, it suffices to draw all

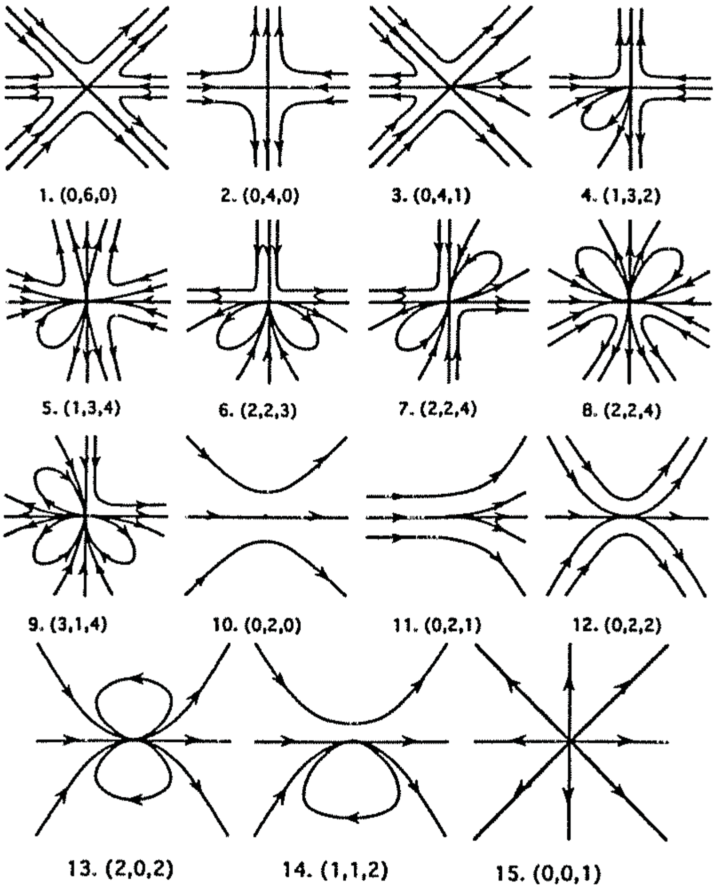

FIG. 2. The topologically different local phase portraits of singular points of degree 2.

possible topologically different local phase portraits for any triple (e, h, p) satisfying Theorem B. Notice that this list may contain some pictures which cannot be realized. As an example, we consider the case $m = 2$.

Corollary C. *The local phase portrait of a singular point of degree* 2 *is topologically equivalent to one of the pictures in Figure 2.*

In Figure 2, all pictures except B(7) and B(8) bijectively correspond to the triples (e, h, p). Whereas B(7) and B(8) correspond to the same triple

$(e, h, p) = (2, 2, 4)$. As we have pointed out, in order to solve the topological classification problem, we should determine which arrangements can be realized for any given triple (e, h, p). In our particular case, we know from Theorem B that at most one picture in Figure 2 cannot be realized, and that if one picture of Figure 2 cannot be realized, it must be B(7) or B(8). If we check the proof of Theorem B in [15], then it is easy to find that B(8) can be realized. So in order to complete the topological classification of the singular points of degree 2, it is necessary to answer the following question: Can Figure 2(7) be realised? As far as we know, there were no results which can answer this question.

1.3. Our Results

Since the qualitative classification is finer than the topological classification, we will only study the qualitative classification problem in the following. As it is explained in §1.1, this problem is divided into two parts, i.e., the questions (A) and (B′). The answer to the first part is the following theorem.

Theorem D. *The set of all singular points of degree* 2 *has 65 qualitative equivalence classes given in Figure 3.*

As an easy corollary we get the topological classification of singular points of degree 2.

Corollary E. *The set of singular points of degree* 2 *has* 14 *topological equivalence classes given in Figure 2 except Figure 2(7).*

To prove Corollary E it is sufficient to check the pictures in Figure 3. On the other hand, it is interesting to note that Corollary E strictly shows that Theorem B is not sufficient for solving the topological classification problem even in the 2-degree case.

In order to prove Theorem D we develop a method in §3 which can be applied to obtain the qualitative classification of singular points of any given degree m. Here only the case $m = 2$ is treated, because there are too many computations involved when m increases.

The following theorem answers the second question of the qualitative classification problem (question (B′) of §1.1). As a motivation we consider the 1-degree case (Theorem A). There the question is solved by giving 9 quantities $(\lambda_1, \lambda_2, \gamma, m, a_m, \bar{a}, \alpha, \bar{b}, \beta)$, which are obtained by applying to system (3) a bounded number of operations of the following types: (A) linear change of variables; (B_1) solving a polynomial equation of degree 2; (C) finding the leading term of the function $G(x, f(x))$ in which $y = f(x)$ is the solution of the equation $y + F(x, y) = 0$, where F and G are analytic functions without linear part; (D) computing $\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}$ for analytic functions $F(x, y)$ and $G(x, y)$. On the other hand, these 9 quantities are also the

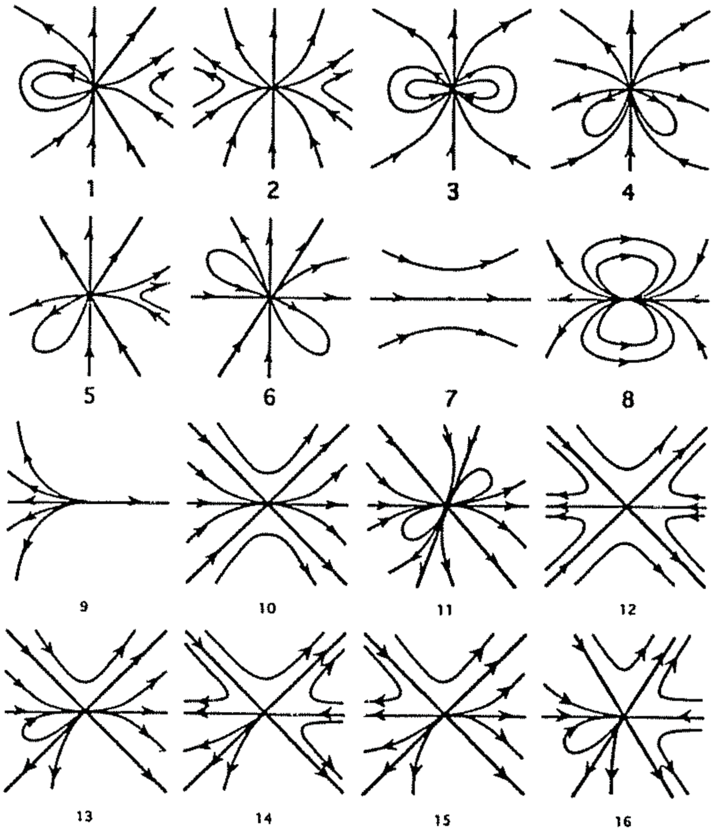

FIG. 3. The qualitative classification of singular points of degree 2. Each picture here represents a qualitative equivalence class such that the local phase portraits of singular points in this class are qualitatively equivalent to this picture.

results of a series of additions and multiplications of real numbers. For any given singular point of degree 1, the number of such elementary operations is finite since any analytic singular point is finitely determined [8], i.e., it is determined by a finite jet of the vector field at the singular point. However this number is unbounded.

The situation for the m-degree case is similar. But more types of operations are needed: (B_m) solving a polynomial equation of degree not higher than $m+1$; (E) moving a singular point to the origin; (F) the blowing-up transformation (see §2); (G) a special kind of change of variables which is defined below.

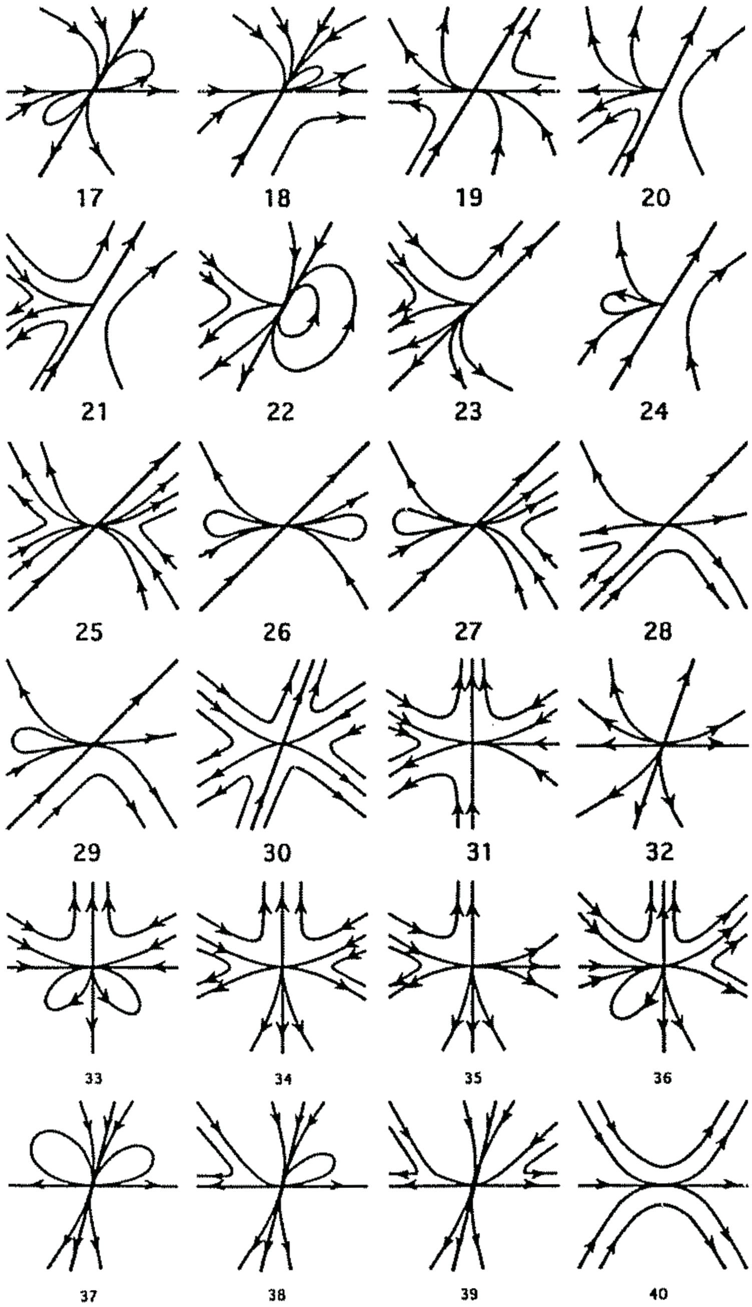

FIG. 4. Continuation of Figure 3.

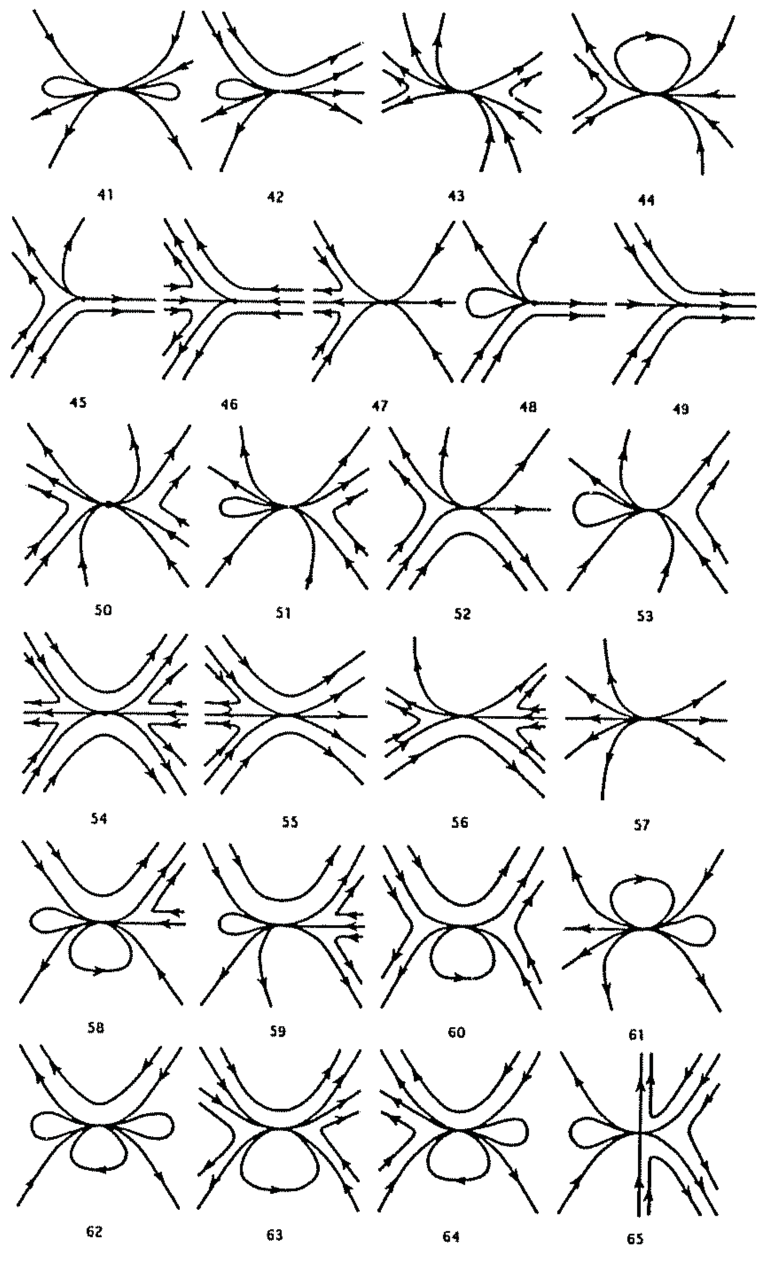

FIG. 5. Continuation of Figure 3.

Definition 1. *Assume that in system (2), $P_m(x,y) = x[ay^{m-1} + \cdots]$, $Q_m(0,y) = by^m, a^2 + b^2 \neq 0$. Let*

$$\begin{aligned} P(x,y) &= P_m(x,y) + P_{m+1}(x,y) + \cdots, \\ Q(x,y) &= Q_m(x,y) + Q_{m+1}(x,y) + \cdots; \\ P^j(x;\delta_0,\delta_1,\cdots,\delta_{j-1}) &= P(x, \delta_0 x + \delta_1 x^2 + \cdots + \delta_{j-1}x^j), \\ E^j(x;\delta_0,\delta_1,\cdots,\delta_{j-1};\delta) &= Q(x, \delta_0 x + \delta_1 x^2 + \cdots + \delta_{j-1}x^j + \delta x^{j+1}) - \\ & \quad [\delta_0 + 2\delta_1 x + \cdots + j\delta_{j-1}x^{j-1} + j\delta x^j] \cdot \\ & \quad P(x, \delta_0 x + \delta_1 x^2 + \cdots + \delta_{j-1}x^j + \delta x^{j+1}), \end{aligned}$$

where j is a positive integer; $\delta_0, \delta_1, \cdots, \delta_{j-1}, \delta$ are real parameters. Then the operation of type (G) is applied to system (2) if and only if there exists a real number d_0 such that $x^{1-2m} \cdot P^2(x; d_0, \delta)$ and $x^{-2m}E^1(x; d_0; \delta)$ are analytic at $x = 0$ for any δ. If this condition is satisfied, then the operation of type (G) is an operation with the following two steps: (i) we inductively find the longest sequence $d_0, d_1, \cdots, d_{r-1}$; and (ii) we apply the change of variables $(x, y, t) \to (x, u, \tau)$ defined by

$$y = x^r u + d_0 x + d_1 x^2 + \cdots + d_{r-1}x^r, \qquad d\tau = x^{r(m-1)}dt$$

to system (2), where t is the time variable of system (2), and τ is the new time variable. The induction is described as follows. Assume that we have found $d_0, d_1, \cdots, d_{j-1}$, then d_j is a real number such that

$$x^{(j+2)(1-m)-1}P^{j+2}(x; d_0, d_1, \cdots, d_j, \delta), \quad x^{-m(j+2)}E^{j+1}(x; d_0, d_1, \cdots, d_j; \delta)$$

are both analytic at $x = 0$ for all δ. If such a number d_j does not exist, then $r = j$.

Lemma 13 will explain how the operation of type (G) appears. Now we can state the following result.

Theorem F. *For any positive integer m the set of singular points of degree m is divided into finitely many qualitative equivalence classes. Moreover, there is a finite set of quantities, which can be computed by applying to system (2) a bounded number of operations of types (A),(B_m),(C),(D),(E),(F) and (G), such that they determine the class to which a given singular point belongs.*

No explicit list of the quantities in Theorem F will be given in this paper in order that this paper is not too long. We remark that there are no essential difficulties for doing this. The method developed in §3 is sufficient.

As in the 1-degree case there are finitely many additions and multiplications among real numbers involved in the computation of those quantities in Theorem F for a fixed singular point of degree m, since any analytic singular point is determined by a finite jet of the vector field at this singular point. Also the number of such elementary operations is unbounded.

1.4. Various Remarks

We first summarize briefly the contents of the following sections.

In §2, we review the concept "blowing-up". This classical construction is one of the most powerful tools in the local study of singular points since according to the following Desingularization Theorem, any fixed isolated analytic singular point can be successfully analyzed if we ignore the problem of distinguishing between centers and foci.

Desingularization Theorem. *Assume that $x = y = 0$ is a singular point of the system*

$$\dot{x} = X_1(x, y), \qquad \dot{y} = X_2(x, y).$$

Then in the following three cases, after applying finitely many blowing-ups to the above system, we get only singular points with at least one non-zero eigenvalue. The cases are

(i) (X_1, X_2) *is analytic at* $x = y = 0,\ (x, y) \in \mathbb{R}^2$;

(ii) (X_1, X_2) *is* $C^\infty,\ (x, y) \in \mathbb{R}^2$, *and satisfies a Łojasiewicz inequality, i.e., there exist* $k, c, \delta > 0$ *such that* $X_1^2 + X_2^2 \geq c(x^2 + y^2)^k$ *for* $x^2 + y^2 < \delta$;

(iii) (X_1, X_2) *is a formal power series in* x *and* y *with coefficients in some field, e.g., the field* $\mathbb{C}$.

The case (i) was due to Bendixson [5]; (ii) was shown by Dumortier [8]; (iii) was proved by Seidenberg [17].

Due to its importance, the blowing-up concept has several treatments in the literature (see, for instance, [4, 8, 12]), but there are some differences among them. In fact, sometimes the same thing receives several different names. In order to fix terminology and make the material of the following sections readable, it seems necessary to treat it here since it is also the technical basis of our method.

As we note in §3 the number of blowing-ups necessary for the analysis of singular points of fixed degree m is unbounded. To overcome this unboundedness, we investigate carefully the blowing-up process of a singular point in §3. Our analysis shows that the above unboundedness comes from the arbitrary length of an equidegree path (Lemma 10), whereas this length is not important since there is a simple relation between the local phase portrait of the beginning singular point and the local phase portrait of the ending singular point. Thus we get the right method for solving the qualitative classification problem. In the end of §3, there is also a proof of Theorem F.

In the rest of the sections we prove Theorem D which gives the qualitative classification of the 2-degree singular points. In §4 we divide our analysis into seven cases according to the blowing-up process. The following sections (§§5–9) are devoted to the case-by-case analysis.

At the end of the paper there is an appendix (§10) which contains the proof of three facts used in the main body of this paper.

Finally there are some remarks about the notations. We denote by P_i and Q_i (possibly with some superscript) homogeneous polynomials of degree i in many places. In different places the same symbol may denote things without any relation. This remark is also applied to lower-case letters (possibly with some subscript or superscript) which denote some real constants or parameters.

We always use $\dot{\zeta}$ to denote the derivation of ζ with respect to a time variable in a differential equation, where ζ stands for a lower-case letter. But when we apply a change of variables to a system of differential equations, we often want to change the time variable. In this case, the time variables need to be mentioned explicitly. In order to simplify the notions in this case, we make the convention that if we do not say any other thing explicitly, we always use $d\tau = *dt$ to denote the change of the time variables, where dt is the differential of the original time variable t, whereas $d\tau$ denote the differential of the new time variable τ, and $*$ is some expression.

Usually we use "$+\cdots$" to denote all the higher order terms in a formula.

2. THE BLOWING UP OF A SINGULAR POINT

The concept blowing-up of a singular point is classical. Several treatments (with some differences) of this concept are available in the literature (e.g. [8, 12]). We review it here since it is the basis of our method developed in §3.

2.1. Blowing up a disc at its center

Blowing up a manifold at a point is a well-known geometric construction (see, for example, [9]). Shortly speaking, blowing up an n-dimensional manifold at a point p is just replacing p by the $(n-1)$-dimensional projective space. For our later use, we look carefully at the following special case, that is, blowing up an open disc at the center. Let

$$U = \{(x,y) \in \mathbb{R}^2 |\ x^2 + y^2 < r_0^2\}, \qquad \tilde{U} = \{(x,y) \in \mathbb{R}^2 |\ 0 < x^2 + y^2 < r_0^2\},$$

with $r_0 > 0$. Then the blowing-up of U at $O = (0,0)$ is just the pair (U_O, π), where U_O is an analytic manifold, $\pi : U_O \longrightarrow U$ is an analytic map. Both of them are defined as follows.

Let

$$U_1 = \{(x,y) \in U|\ x \neq 0\} \cup \{[x,y] \in \mathbb{PR}^1|\ x \neq 0\},$$
$$U_2 = \{(x,y) \in U|\ y \neq 0\} \cup \{[x,y] \in \mathbb{PR}^1|\ y \neq 0\},$$

where $[x,y]$ denotes the homogeneous coordinate in the projective line $\mathbb{PR}^1$. The maps $\pi_i : U_i \longrightarrow \mathbb{R}^2,\ i = 1,2$, are defined as

$$\pi_1|_{\tilde{U}\cap U_1} : (x,y) \longmapsto (x,u), u = \tfrac{y}{x}; \qquad \pi_1|_{\mathbb{PR}^1\cap U_1} : [1,u] \longmapsto (0,u);$$
$$\pi_2|_{\tilde{U}\cap U_2} : (x,y) \longmapsto (v,y), v = \tfrac{x}{y}; \qquad \pi_2|_{\mathbb{PR}^1\cap U_2} : [v,1] \longmapsto (v,0);$$

These two pairs $(U_i, \pi_i),\ i = 1,2$, generate an analytic manifold structure on the set $\tilde{U} \cup \mathbb{PR}^1$. This manifold is just U_O. The map $\pi : U_O \longrightarrow U$ is defined as

$$\pi|_{\tilde{U}} = \mathrm{id}, \qquad \pi(\mathbb{PR}^1) = \{(0,0)\}.$$

Via π, $\tilde{U}$ will be taken as an open subset of U_O.

The manifold U_O can be thought of in a more intuitive way. Consider the annulus $T = \{(x,y) \in \mathbb{R}^2|\ 1 \le x^2+y^2 < (1+r_0)^2\}$. We identify antipodal points on the unit circle $\{(x,y) \in \mathbb{R}^2|\ x^2+y^2 = 1\}$ of T, and denote the corresponding quotient space by $\tilde{T}$. There is a bijection $\tilde{\phi} : \tilde{T} \longrightarrow U_O$ which is defined by the map $\phi : T \longrightarrow U_O$:

$$\phi(x,y) = \begin{cases} ((x^2+y^2-1)x, (x^2+y^2-1)y) & \text{if } x^2+y^2 > 1; \\ \text{[x,y]} & \text{if } x^2+y^2 = 1. \end{cases}$$

Evidently ϕ and $\tilde{\phi}$ are continuous maps. Thus we have a unique analytic structure on $\tilde{T}$ such that $\tilde{\phi}$ is an analytic diffeomorphism. That is, we can say U_O is just $\tilde{T}$. With this identification, we investigate π_i more carefully.

The y-axis cuts T into two parts: $\{(x,y) \in T|\ x > 0\}$, $\{(x,y) \in T|\ x < 0\}$. We call them as the *right part* and the *left part* respectively. Similarly, we use the x-axis to cut T into the *upper part* and the *lower part.* The composition $\pi_1 \circ \phi$ maps the right (left) part of T onto the right (respectively, left) part of $\pi_1(U_1)$, i.e., the set $\pi_1(U_1) \cap \{(x,u) \in \mathbb{R}^2|\ x \ge 0\}$ (respectively, $\pi_1(U_1) \cap \{(x,u) \in \mathbb{R}^2|\ x \le 0\}$); the points of T on a line of slope u_0 through O are mapped to the points of $\pi_1(U_1)$ on the line $u = u_0$. Similarly, The map $\pi_2 \circ \phi$ maps the upper (lower) part of T onto the upper (respectively, lower) part of $\pi_2(U_2)$, i.e., the set $\pi_2(U_2) \cap \{(v,y) \in \mathbb{R}^2|\ y \ge 0$ (respectively, $y \le 0)\}$; the points of T on a line of slope $\frac{1}{v_0}$ through O are mapped to the points of $\pi_2(U_2)$ on the line $v = v_0$. Notice that if restricted to one of these four parts, $\pi_1 \circ \phi$ or $\pi_2 \circ \phi$ is a homeomorphism. We also identify the open annulus $\{(x,y)|\ 1 < x^2+y^2 < (1+r_0)^2\}$ with $\tilde{U}$ which has been looked as an common open subset of U and U_O. We remark that the above correspondence and identification will be important in the following discussion. Special attention shall be paid to the behavior of the map $\pi_1 \circ \phi$ on the left part of T and the map $\pi_2 \circ \phi$ on the lower part of T.

2.2. The foliation induced by a blowing-up

Now assume that the analytic system (2) is defined on U with O being its unique singular point in U (and we will keep this assumption till the end of this section). Since $\tilde{U}$ is the common open subset of U and U_O, the system (4) has defined a flow on an open subset of U_O (here we mean by the word "flow" just a foliation (with singularity) by oriented curves, but we can make it a flow in the usual sense by rescaling the time variable). The problem is how to extend it to the whole manifold U_O. Certainly, we can define a vector field on U_O by taking its value on $\mathbb{PR}^1$ to be zero. But for our purpose, this way is not useful since all points on $\mathbb{PR}^1$ would be singular points. In the following, we will show that there is a natural foliation (in the usual sense, but with singularity) on U_O, which is generated by the two vector fields on the charts $(U_i, \pi_i),\ i = 1, 2$, and coincides with system (2) if we restrict it to $\tilde{U}$. To do this, we distinguish two cases: the *dicritical case* ($D(x, y) = xQ_m(x, y) - yP_m(x, y) \equiv 0$) and the *non-dicritical case* ($D(x, y) \not\equiv 0$).

In the dicritical case, we apply the following two changes of variables

$$y = xu,\ d\tau = x^m dt; \qquad x = vy,\ d\tau = y^m dt$$

to system (2). We obtain the following two systems respectively:

$$\begin{aligned} \dot{x} &= P_m(1, u) + xP_{m+1}(1, u) + x^2 P_{m+2}(1, u) + \cdots , \\ \dot{u} &= Q_{m+1}(1, u) - uP_{m+1}(1, u) + x[Q_{m+2}(1, u) - uP_{m+2}(1, u)] + \cdots ; \end{aligned} \tag{4}$$

and

$$\begin{aligned} \dot{y} &= Q_m(v, 1) + yQ_{m+1}(v, 1) + y^2 Q_{m+2}(v, 1) + \cdots , \\ \dot{v} &= P_{m+1}(v, 1) - vQ_{m+1}(v, 1) + y[P_{m+2}(v, 1) - vQ_{m+2}(v, 1)] + \cdots . \end{aligned} \tag{5}$$

These two systems induce two foliations on U_1 and U_2 respectively. If we restrict them to $\tilde{U}$, then they coincide with the one induced by system (2). Thus we get a foliation globally defined on U_O.

In the non-dicritical case, we apply the following two changes of variables

$$y = xu,\ d\tau = x^{m-1} dt; \qquad x = vy,\ d\tau = y^{m-1} dt$$

to system (2). The resulting systems are

$$\begin{aligned} \dot{x} &= xP_m(1, u) + x^2 P_{m+1}(1, u) + \cdots , \\ \dot{u} &= D(1, u) + x[Q_{m+1}(1, u) - uP_{m+1}(1, u)] + \cdots ; \end{aligned} \tag{6}$$

and

$$\begin{aligned} \dot{y} &= yQ_m(v,1) + y^2 Q_{m+1}(v,1) + \cdots, \\ \dot{v} &= -D(v,1) + y[P_{m+1}(v,1) - vQ_{m+1}(v,1)] + \cdots. \end{aligned} \tag{7}$$

By the same reason as in the dicritical case, these two systems determine a global foliation on U_O.

2.3. The flow induced by a blowing-up

Generally speaking, the foliation on U_O which we have constructed above need not be a flow. But there is a flow naturally defined on T. We show this below.

To start, we consider the dicritical case. Firstly, since $\pi_1 \circ \phi$ is a homeomorphism on the right part and the left part of T, so we get two flows on these two parts of T from system (4) through the map $\pi_1 \circ \phi$. Similarly we can obtain two flows on the upper part and the lower part of T from system (5) through the map $\pi_2 \circ \phi$. Our next step is a modification of the orientations of these flows. This is done according to the time rescaling $d\tau = x^m dt$ or $d\tau = y^m dt$ which we have made when system (2) is changed into system (4) or (5). For the flows on the right and upper part of T, the orientations keeps unchanged. For the flows on the left and lower part of T, the orientations depend on the parity of m : keeping unchanged if m is even; being reversed if m is odd. Now these four flows will produce a global flow on T because, if we restrict them to the open annulus $\{(x,y) |\, 1 < x^2 + y^2 < (1+r_0)^2\}$, each of them coincides with the flow given by (2) (recall that this open annulus is identified with $\tilde{U}$).

In the non-dicritical case, the flow on T can be constructed similarly. Firstly, in each of the four parts of T there is a flow on it which is determined by systems (6) and (7). Secondly, we modify the orientation for each of these four flows. The rule is similar: for the flows on the right and the upper parts, the orientation keeps unchanged; for the flows on the left and the lower parts, the orientation is reversed if $m-1$ is odd, and keeps unchanged if $m-1$ is even. Finally, by the same reason as in the dicritical case, we get a global flow on T from these four flows.

Now we can determine when the foliation on U_O $(= \tilde{T})$ is a flow, that is, finding conditions under which we still get a flow from the flow on T after identifying the antipodal points on the unit circle of T. To do this, it suffices to check the orbits on the unit circle (which is invariant) in the non-dicritical case and the orbits passing through the unit circle in the dicritical case. We can see easily that if m is odd in the non-dicritical case and m is even in the dicritical case, we get a flow on $\tilde{T}$ from the flow on T; if "even" is interchanged with "odd", we get only a foliation on $\tilde{T}$ which is not a flow.

2.4. Definitions

The construction from the flow on U to the foliation on U_O described in §2.2 is called a *blowing-up*. The foliation on U_O is called the blowing-up of the flow on U defined by system (2) at O, or simply the *blowing-up of the singular point O*. But when we blow up a singular point in practice, we just write down the systems (4) and (5) in the dicritical case, or (6) and (7) in the non-dicritical case, since these systems determine the foliation on U_O. The corresponding changes of variables are called as the *blowing-up transformations*. But for simplicity, we shall call a blowing-up transformation as a blowing-up. (Here there is an abuse of terminology: a blowing-up may stand for a particular kind of changes of variables; it may also denote the construction in §2.2. The authors apologize for this and hope that no confusion will appear.) The phase portrait of the flow on T is called the *phase portrait of O on the unit circle*, from which we get the local phase portrait of system (2) in a neighborhood of O just by contracting the unit circle to O. The polynomial $D(x, y)$ will be called the *characteristic polynomial* of the singular point O of system (2). A *characteristic direction* of the singular point O of system (2) is just a ray starting at O and satisfying $D(x, y) = 0$.

The blowing-up defined here is called *σ-process* in Russian literature (e.g. [3, 4]), and *directional blowing-up* by Dumortier [8].

2.5. Singular points after a blowing-up

Notice that if $D(x, y) \equiv 0$, there is a homogeneous polynomial $f(x, y)$ of degree $m - 1$ such that

$$P_m(x, y) = xf(x, y), \qquad Q_m(x, y) = yf(x, y). \tag{8}$$

Let S be the set of singular points of the foliation on U_O obtained by applying a blowing-up to system (2). Recall that O is the unique singular point of system (2) in U. We have the following lemma.

LEMMA 1. *The following statements hold.*

(i)Assume $D(x, y) \not\equiv 0$. Then $S = \{[x, y] \in \mathbb{PR}^1 |\ D(x, y) = 0\}$. Furthermore for any point $p_0 = [x_0, y_0] \in S$, the degree of p_0 is not larger than the multiplicity of the linear factor $y_0x - x_0y$ in the factorization of $D(x, y)$.

(ii)Assume $D(x, y) \equiv 0$, and the polynomial $f(x, y)$ is defined by (8). Then

$$S = \{[x, y] \in \mathbb{PR}^1 |\ f(x, y) = 0 \text{ and } xQ_{m+1}(x, y) - yP_{m+1}(x, y) = 0\}.$$

Moreover the degree of the point $p_0 = [x_0, y_0] \in S$ is not larger than the multiplicity of $y_0x - x_0y$ in the greatest common factor of $f(x, y)$ and $xQ_{m+1}(x, y) - yP_{m+1}(x, y)$.

Proof. It is clear that $S \subset \mathbb{PR}^1$ because we have assumed that O is the unique singular point of system (2) in U. Therefore the points in S correspond to the singular points of system (4) or (6) on the u-axis, and to the singular points of system (5) or (7) on the v-axis. Thus the statements of Lemma 1 are obtained by directly checking these four systems. ∎

Since O is the unique singular point of system (2) in U, all the singular points of the flow on T are on the unit circle. We know that each singular point $[x_0, y_0]$ of the foliation on U_O corresponds to a pair of antipodal singular points (i.e., (x_0, y_0) and $(-x_0, -y_0)$ with $x_0^2 + y_0^2 = 1$) of the flow on T. Furthermore, if $x_0 > 0$ and $x_0^2 + y_0^2 = 1$, then the local phase portrait of (x_0, y_0) is the same as the right part of the local phase portrait of the singular point $x = u - \frac{y_0}{x_0} = 0$ divided by the u-axis (more exactly, these two phase portraits are analytically equivalent via the map $\pi_1 \circ \phi$); and the local phase portrait of $(-x_0, -y_0)$ is obtained by the map $\pi_1 \circ \phi$ from the left part of the local phase portrait of the singular point $x = u - \frac{y_0}{x_0} = 0$ and a suitable choice of orientation of orbits (i.e., reversing when $m - 1$ is odd in the non-dicritical case and m is odd in the dicritical case; keeping unchanged in all other cases). In the case $y_0 > 0$ and $x_0^2 + y_0^2 = 1$, the local phase portraits of (x_0, y_0) and $(-x_0, -y_0)$ are obtained similarly from the local phase portrait of the singular point $y = v - \frac{x_0}{y_0} = 0$ via $\pi_2 \circ \phi$.

The local phase portrait of a singular point of the flow on T is different from the usual one. It is just one half of the local phase portrait of a usual singular point. In the following sense, we say a singular point of the flow on T is a *saddle*, or a *node*, or a *saddle-node* if its local phase portrait is composed by two hyperbolic sectors, or one parabolic sector, or one hyperbolic sector and one parabolic sector respectively.

2.6. The blowing-up technique

Since we are interested only in the local phase portrait at the singular point $x = y = 0$ of system (2), the positive number r_0 can be chosen so small that the foliation on U_O and the flow on T are all determined by the local phase portrait of its singular points. In the above description, we have shown that if we know the local phase portraits of the singular points of system (4) or (6) on u-axis and those of the singular points of system (5) or (7) on v-axis, then the phase portrait of the flow on T can also be obtained according to the rule described above. By contracting the unit circle to the point $x = y = 0$, we obtain the local phase portrait of system (2) at $x = y = 0$. Hence instead of studying the singular point O, we study the singular points of system (4) or (6) on u-axis and the singular points of system (5) or (7) on v-axis, or equivalently, we study the singular points obtained by applying a blowing-up to O. Generally speaking, the later ones are simpler than the original singular point. Thus it is reasonable to use repeated blowing-ups for the qualitative analysis of a singular point. Actually this is a successful way. We explain this in the rest of this section.

We first divide the set of all singular points (recall that they are assumed to be analytic and isolated) into two classes according to certain conditions which will be specified below. One of these two classes contains all *simple* singular points to which we shall not apply blowing-ups. Singular points in the other class will be the *complex* ones, and blowing-ups will be needed in order to analyze them.

Now consider the singular point O of system (2). After applying a blowing-up to it, we get a foliation on U_O. The manifold U_O is more complex than U. But the foliation is simpler in the sense that the singular points on U_O will be simpler than the original one. Assume that these singular points are $p_1^1, p_1^2, \cdots, p_1^{s_1}$. Among these singular points, some are simple singular points, the others are complex. Assume that $p_1^j,\ 1 < j \leq t_1$, are all the complex ones. Now we apply a blowing-up to each singular point $p_1^j,\ 1 < j \leq t_1$. After this, the manifold U_O is changed to be a more complicated manifold (it is obtained from U_O by replacing t_1 distinct points with t_1 projective lines). The original foliation is changed into a foliation on this manifold. The singular points $p_1^j,\ 1 < j \leq t_1$, do not exist. They are changed into a new set of singular points, say, $p_2^1, p_2^2, \cdots, p_2^{s_2}$. Among them, some are simple, and the others, say, $p_2^1, p_2^2, \cdots, p_2^{t_2}$, are complex. We apply a blowing-up to each of these t_2 singular points $p_2^1, p_2^2, \cdots, p_2^{s_2}$. Repeating this process, the manifold is changed to be more and more complicated; the foliation on it will be simpler and simpler. At last, when the foliation contains only simple singular points, this process is stopped. We call the whole process described above as the *blowing-up process* associated to the singular point O.

In order that the blowing-up process is useful, we must make a proper choice of the class of simple singular points such that they are really simple and, furthermore, the blowing-up process is finite, i.e. containing only finitely many blowing-ups. This is provided by the Desingularization Theorem, which says the blowing-up process associated to any given singular point is finite if we choose the class of simple singular points as singular points with at least one non-zero eigenvalue. By Theorem A, we know any singular point with at least one non-zero eigenvalue can be successfully analyzed. As a corollary, any given singular point can be analyzed by a finite blowing-up process (if we do not distinguish between a center and a focus).

In our following discussion the class of simple singular points is taken as the set of all singular points of degree one plus those singular points from which we will get no singular points when we apply a single blowing-up to them. Once again the blowing-up process for a given singular point is finite. The simple singular points in a blowing-up process are called the *terminals* of the blowing-up process since the blowing-up process will stop at them.

3. THE BLOWING-UP PROCESS OF A SINGULAR POINT

We have seen in the last section that it suffices to apply repeated blowing-ups in order to get the local phase portrait of a given isolated analytic singular point. But if one wants to classify topologically or qualitatively all singular points of degree m, it is not sufficient to apply repeated blowing-ups to the general system (2) of singular points of degree m since the number of blowing-ups necessary for the analysis of an m-degree singular point is unbounded (although finite). This can be seen in the following example.

EXAMPLE 2. *The system* $\dot{x} = y^m,\ \dot{y} = x^{km+k-1}$ *needs* k *blowing-ups for any* $k > 1,\ m > 1$. *These blowing-ups are* $y = xu_0,\ d\tau_0 = x^{m-1}dt$ *and* $u_i = xu_{i-1},\ d\tau_i = x^m d\tau_{i-1}$ *for* $1 \leq i \leq k-1$, *where* $\tau_j,\ 0 \leq j \leq k-1$, *are time variables, and* t *is the time variable of the original system.*

In spite of this example the blowing-up method is still the successful way to obtain the qualitative classification of all m-degree singular points. But we must go further and study the blowing-up process associated to each singular point, as we will do in this section.

We first show that the blowing-up process of a given singular point corresponds naturally to an oriented tree as was observed in [16]. This tree grows at the given singular point (the *initial vertex*) where our blowing-up process starts. We often call it the *starting singular point.* All other *vertices* of this tree are just all the singular points obtained during the blowing-up process. The *edges* of this oriented tree are some arrows. Assume that A and B are two vertices. Then there is an arrow from A to B if and only if B is one of the singular points obtained by applying just one blowing-up to A. In this case, B is called a *successor* of A; whereas A is called the *predecessor* of B. As in the last section, we take the set of terminals as the set of all singular points of degree one plus those singular points from which we will get no singular points when we apply blowing-ups to them. In this way, we have defined an oriented tree, which, in the following, will be called the *blowing-up tree* of our starting singular point. We know that this tree is finite by the Desingularization Theorem in §1.4.

Since our tree is oriented, there is a natural partial ordering on the set of vertices. The terminologies "predecessor" and "successor" just come from this ordering. We are also interested in some *oriented subgraphs* of this tree, which are formed by a subset of vertices and a subset of edges such that (i) this subgraph is connected; (ii) any two vertices joined by an edge from the edge set of this subgraph must belong to the set of vertices of this subgraph. A subgraph is called a *path* if the set of vertices of this subgraph is a totally-ordered subset. A path is called *complete* if it is maximal, i.e., it cannot be extended. The *length* of a path is just the number of vertices in this path. An oriented subgraph is called an *oriented subtree* if any nonterminal vertex in this subgraph is followed by the same set of successors as in the original tree. Notice that the terminals of an oriented subgraph need not be the terminals of the original blowing-up tree. Since all our trees, subtrees, subgraphs are oriented, sometimes we will omit the word "oriented" for simplicity.

In the following we shall show some properties of a blowing-up tree.

LEMMA 3. *The following statements hold.*

(i)If $B^1, B^2, \cdots, B^s$ *are successors of a vertex* A*, then* $\sum_{i=1}^{s} \deg(B^i) \le \deg(A)+1$*. Here* $\deg(C)$ *denotes the degree of the singular point* C*.*

(ii)For any path $B_1 \longrightarrow B_2 \longrightarrow B_3$*, if* $\deg(B_2) < \deg(B_3)$*, then* $\deg(B_1) > \deg(B_2)$*.*

(iii)If the starting singular point is of degree m*, then all the singular points in its blowing-up tree are of degrees smaller than* $m+2$*.*

Proof. (i) This follows immediately from Lemma 1.

(ii) Without loss of generality, we assume that B_1 is the singular point $x = y = 0$ of system (2). If the statement (ii) is not true, then $\deg(B_2) \ge \deg(B_1) = m$. Hence, by Lemma 1 there is a linear factor which is of degree at least m in the factorization of $D(x, y) = xQ_m(x, y) - yP_m(x, y)$, i.e., we have the following factorization

$$D(x, y) = xQ_m(x, y) - yP_m(x, y) = (c_1y + d_1x)^m(c_2y + d_2x),$$

where $(c_1^2 + d_1^2) \cdot (c_2^2 + d_2^2) \ne 0$. Again without loss of generality, we can assume $c_1 = 1, d_1 = 0$. Now apply the change of variables $y = xu,\ d\tau = x^{m-1}dt$ to system (2). Then B_2 is just the singular point $x = u = 0$ of the following system

$$\begin{aligned} \dot{x} &= xP_m(1, u) + x^2P_{m+1}(1, u) + \cdots , \\ \dot{u} &= u^m(c_2u + d_2) + x[Q_{m+1}(1, u) - uP_{m+1}(1, u)] + \cdots . \end{aligned} \tag{9}$$

If $\deg(B_2) = m + 1$, then $d_2 = 0, c_2 \ne 0, P_m(1, u) = -c_2u^m$, and the characteristic polynomial of B_2 is $x[2c_2u^{m+1} + \cdots]$, where "$\cdots$" denotes the terms containing the factor x. So, by Lemma 1, we cannot get a singular point whose degree is larger than $m+1 = \deg(B_2)$ when we apply a blowing-up to B_2. This contradicts the assumption of (ii).

If $\deg(B_2) = m$ and $\deg(B_3) \geq m+1$, then the characteristic polynomial of B_2 must be not identically zero and of degree $m+1$. In fact, it is clear from system (9) that x is its factor; since $\deg(B_3) \geq m+1$, this polynomial can be written as $c_3 x^{m+1}$ according to Lemma 1. But when we apply the change of variables $x = uv,\ d\tau = u^{m-1}dt$ to system (9), we get only one singular point whose degree is at most 2 because the term $u^m(c_2u + d_2)$ in system (9) will be changed into a non-zero 2-degree polynomial in the new system. Since $\deg(B_2) < \deg(B_3)$ by assumption, we get $\deg(B_2) < 2$. But B_2 is not a terminal. Thus we get a contradiction, and the statement (ii) is proved.

(iii) Let A_1 be the starting singular point. Suppose $A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_k$ is an arbitrary path. By assumption, $\deg(A_1) = m$. To prove (iii), it is sufficient to show that $\deg(A_i) \leq m+1$ for any $i \leq k$. This is proved in the following by induction on the degree m of the starting singular point and the length k of the path. Notice that if $k = 2$, this statement follows easily from (i). Thus it suffices to prove the case $m = m_0$ and $k = k_0 > 2$ under the hypothesis that this statement is true if $m < m_0$, or $m = m_0$ but $k < k_0$.

From (i), we know $\deg(A_2) \leq m_0 + 1$. If $\deg(A_2) \leq m_0$, then take A_2 as the starting singular point, our assertion follows from the inductive hypothesis. Hence we need only to consider the case $\deg(A_2) = m_0+1$. By (ii), we know $\deg(A_3) \leq \deg(A_2) = m_0+1$. Repeating the above argument, it suffices to consider the case $\deg(A_3) = m_0+1$. Repeating this reasoning, we know our assertion is true since k_0 is a fixed integer. ∎

Now we consider a fixed complete path $A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_k$. By definition, A_1 is the starting singular point, A_k is a terminal. In this path, a subpath

$$A_i \longrightarrow A_{i+1} \longrightarrow \cdots \longrightarrow A_{i+r} \tag{10}$$

such that

$$\deg(A_i) = \deg(A_{i+1}) = \cdots = \deg(A_{i+r}) \tag{11}$$

is important as we will show in the rest of this section.

Lemma 4. *Consider the path (10) which satisfies (11). Assume A_{i-1} is the predecessor of A_i, and $\deg(A_i) > \deg(A_{i-1})$. Then there exist r saddles $B_j, 1 \leq j \leq r$, such that*

$$A_{i-1} \longrightarrow A_i \overset{\nearrow B_1}{\longrightarrow} A_{i+1} \overset{\nearrow B_2}{\longrightarrow} \cdots \longrightarrow A_{i+r-1} \overset{\nearrow B_r}{\longrightarrow} A_{i+r}$$

is a subtree. Moreover, the local phase portrait of A_{i-1} is determined by the local phase portrait of A_{i+r} and the parity of r.

Proof. Assume that A_{i-1} is the singular point $x = y = 0$ of system (2). From the hypothesis and Lemma 3, we have $\deg(A_i) = m + 1 = \deg(A_{i-1}) + 1$. By Lemma 1, it follows that

$$D(x, y) = xQ_m(x, y) - yP_m(x, y) = c_0(cy + dx)^{m+1}$$

for some real constants c_0, c and d such that $c_0 \neq 0, c^2 + d^2 \neq 0$. Without loss of generality, we can assume $d = 0, c_0 \cdot c^{m+1} = -1$, that is, $D(x, y) = -y^{m+1}$. This implies $P_m(0, y) = y^m$. Now we apply the following change of variables $y = xu_1,\ d\tau_1 = x^{m-1}dt$ to system (2). Then we get

$$\begin{aligned} \dot{x} &= xP_m^{(1)}(x, u_1) + xP_{m+1}^{(1)}(x, u_1) + xP_{m+2}^{(1)}(x, u_1) + \cdots , \\ \dot{u}_1 &= -u_1^{m+1} + xQ_m^{(1)}(x, u_1) + xQ_{m+1}^{(1)}(x, u_1) + xQ_{m+2}^{(1)}(x, u_1) + \cdots , \end{aligned} \tag{12}$$

where $P_l^{(1)}(x, u_1)$ and $Q_l^{(1)}(x, u_1)$ are homogeneous polynomials of degree l, and $P_m^{(1)}(0, u_1) = u_1^m$. The singular point $x = u_1 = 0$ is A_i, which is the unique successor of A_{i-1}. Since A_{i+1} is a singular point of degree $m + 1$, the characteristic polynomial of A_i must be

$$-u_1xP_m^{(1)}(x, u_1) + x[-u_1^{m+1} + xQ_m^{(1)}(x, u_1)] = -2x(u_1 - c_1x)^{m+1}$$

according to Lemma 1, where c_1 is some real constant. Hence A_i has two successors which are obtained by applying the following two changes of variables

$$u_1 = xu_2 + c_1x,\ d\tau_2 = x^m d\tau_1; \qquad x = vu_1,\ d\tau_2' = u_1^m d\tau_1$$

to system (12). The resulting systems are

$$\begin{aligned} \dot{x} &= xP_m^{(2)}(x, u_2) + xP_{m+1}^{(2)}(x, u_2) + xP_{m+2}^{(2)}(x, u_2) + \cdots , \\ \dot{u}_2 &= -2u_2^{m+1} + xQ_m^{(2)}(x, u_2) + xQ_{m+1}^{(2)}(x, u_2) + xQ_{m+2}^{(2)}(x, u_2) + \cdots , \end{aligned}$$

and

$$\dot{u}_1 = -u_1 + \cdots , \qquad \dot{v} = 2v + \cdots ,$$

where, as before, $P_l^{(2)}(x, u_2)$ and $Q_l^{(2)}(x, u_2)$ are homogeneous polynomial of degree l. From these two systems, we know that A_{i+1} is the singular point $x = u_2 = 0$, the singular point $v = u_1 = 0$ (denoted by B_1) is a saddle. Repeating this process, we will find by induction on j that, for any $j \leq r,\ A_{i+j}$ is the singular point $x = u_{j+1} = 0$ of the system

$$\begin{aligned} \dot{x} &= xP_m^{(j+1)}(x, u_{j+1}) + xP_{m+1}^{(j+1)}(x, u_{j+1}) + xP_{m+2}^{(j+1)}(x, u_{j+1}) + \cdots , \\ \dot{u}_{j+1} &= -(j + 1)u_{j+1}^{m+1} + xQ_m^{(j+1)}(x, u_{j+1}) + xQ_{m+1}^{(j+1)}(x, u_{j+1}) + \cdots ; \end{aligned}$$

and B_j is the singular point $v = u_j = 0$ of the system

$$\dot{u}_j = -ju_j + \cdots, \qquad \dot{v} = (j+1)v + \cdots.$$

Moreover the singular points A_{i+j} and B_j are obtained from A_{i+j-1} by applying the following two changes of variables

$$u_j = xu_{j+1} + c_j x,\ d\tau_{j+1} = x^m d\tau_j; \qquad x = vu_j,\ d\tau'_{j+1} = u_j^m d\tau_j$$

to the system for A_{i+j-1} respectively. Here τ_{j+1} and τ'_{j+1} are the time variables of the systems for A_{i+j} and B_j, c_j is some real constant such that the characteristic polynomial of A_{i+j-1} is

$$-u_j \cdot xP_m^{(j)}(x, u_j) + x[-ju_j^{m+1} + xQ_m^{(j)}(x, u_j)] = -(j+1)x(u_j - c_j x)^{m+1}.$$

This factorization follows from Lemma 1 and the fact $\deg(A_{i+j}) = m+1$, as we have seen in the case $j = 1$.

Obviously B_j is a saddle. Since $\deg(A_{i+j-1}) + 1 = \deg(A_{i+j}) + \deg(B_j)$, A_{i+j} and B_j are the only two successors of A_{i+j-1} by Lemma 3(i). Hence, by definition, the subgraph in Lemma 4 is a subtree.

To prove the second part, we first see how to get the local portrait of A_{i+r-1} from that of A_{i+r}. Let us draw the local phase portrait of A_{i+r-1} on the unit circle (see §2.4). We know that there are four singular points on the unit circle; that is, the top and the bottom (i.e., $(x, u_r) = (0, 1)$ and $(x, u_r) = (0, -1)$ respectively), plus $(x, u_r) = \frac{1}{\sqrt{1+c_r^2}}(1, c_r)$ and $(x, u_r) = \frac{-1}{\sqrt{1+c_r^2}}(1, c_r)$. The first two correspond to the saddle B_r. The last two correspond to A_{i+r}. According to the rule of §2.3, the local phase portrait of the top is the same as the upper part of the local phase portrait of B_r (i.e., the part in the region $u_{r+1} \geq 0$). The local phase portrait of the point $(\frac{1}{\sqrt{1+c_r^2}}, \frac{c_r}{\sqrt{1+c_r^2}})$ is just the right part of the local phase portrait of A_{i+r} (i.e., the part in the region $x \geq 0$). The local phase portraits at the bottom and at $(-\frac{1}{\sqrt{1+c_r^2}}, -\frac{c_r}{\sqrt{1+c_r^2}})$ come from the lower part of B_r and the left part of A_{i+r} respectively. The rule is described in §2.3. Now we contract the unit circle to the point $x = u_r = 0$. Since the top and the bottom are two saddles, we find that the right part of A_{i+r-1} and the right part of A_{i+r} are the same; if we reflect the left part of the local phase portrait of A_{i+r} across the x-axis, then the resulting picture is just the left part of the local phase portrait of A_{i+r-1} if m is even; if, furthermore, we reverse the orientation of orbits, then the resulting picture is just the left part of A_{i+r-1} in the case that m is odd. Now our conclusion is easily obtained by repeating this process. More explicitly, we have the following rule:

(i) The local phase portrait of A_{i-1} on the unit circle has only two singular points, one is on the right semi-circle, the other is on the left semi-circle. The right one has the same phase portrait as the right part of A_{i+r}.

(ii) If $(r+1)m-1$ is even, then the local phase portrait of the singular point on the left semi-circle is obtained from the left part of the local phase portrait of A_{i+r} by $r+1$ reflections across the x-axis.

(iii) If $(r+1)m-1$ is odd, then the local phase portrait of the singular point on the left semi-circle is obtained from the left part of the local phase portrait of A_{i+r} by $r+1$ reflections across the x-axis and a reversing of the orientation of orbits.

Obviously this rule implies the second part of Lemma 4. The proof is finished. ∎

Lemma 5. *In the path (10) we assume that A_i is the singular point $x=y=0$ of system (2) with $P_m(x,y)=x[ay^{m-1}+\cdots]$, $Q_m(0,y)=by^m$, $a^2+b^2\neq 0$, and $\deg(A_j)=m$ for $i\le j\le i+r$, where "$\cdots$" denotes the terms containing the factor x. Then there exist $B_1,B_2,\cdots,B_r$ such that $\deg(B_j)=1$ for all $1\le j\le r$, and among them, there is at most one anti-saddle, the others are saddles. Furthermore the following subgraph*

$$A_i \overset{\nearrow B_1}{\longrightarrow} A_{i+1} \overset{\nearrow B_2}{\longrightarrow} \cdots \longrightarrow A_{i+r-1} \overset{\nearrow B_r}{\longrightarrow} A_{i+r}$$

is a subtree. Moreover, the local phase portrait of A_i is determined by the local phase portrait of A_{i+r}, the parity of r and the values of a and b.

Remark 6. If $a=1, b=-1$, Lemma 5 is just the case considered in Lemma 4.

Proof. The proof is similar to the one of Lemma 4. Firstly we can inductively find all the systems for A_{i+j} and B_j. For any $0<j\le r$, A_{i+j} is just the singular point $x=u_j=0$ of the system

$$\begin{aligned}\dot{x} &= x\bar{P}^{(j)}_{m-1}(x,u_j)+x\bar{P}^{(j)}_{m}(x,u_j)+x\bar{P}^{(j)}_{m+1}(x,u_j)+\cdots,\\ \dot{u}_j &= (b-ja)u_j^m+x\bar{Q}^{(j)}_{m-1}(x,u_j)+x\bar{Q}^{(j)}_{m}(x,u_j)+x\bar{Q}^{(j)}_{m+1}(x,u_j)+\cdots,\end{aligned} \tag{13}$$

where $\bar{P}^{(j)}_l$ and $\bar{Q}^{(j)}_l$ are homogeneous polynomials of degree l, $\bar{P}^{(j)}_{m-1}(0,u_j)=au_j^{m-1}$; and B_j is the singular point $v=u_{j-1}=0$ of the system

$$\dot{u}_{j-1}=[b-(j-1)a]u_{j-1}+\cdots,\qquad \dot{v}=(ja-b)v+\cdots, \tag{14}$$

where we identify $u_0=y$. We obtain A_{i+j} and B_j by applying the following two changes of variables

$$u_{j-1}=xu_j+d_{j-1}x,\ d\tau_j=x^{m-1}d\tau_{j-1};\qquad x=vu_{j-1},\ d\tau'_j=u^{m-1}_{j-1}d\tau_{j-1}$$

to the system for A_{i+j-1} respectively, where τ_j is the time variable of system (13), τ_j' is the time variable of system (14), and d_{j-1} is a real constant such that the characteristic polynomial of A_{i+j-1} is

$$\begin{aligned}&-u_{j-1}\cdot x\bar{P}_{m-1}^{(j-1)}(x,u_{j-1})+x[(b-(j-1)a)u_{j-1}^m+x\bar{Q}_{m-1}^{(j-1)}(x,u_{j-1})]\\&=x(b-ja)(u_{j-1}-d_{j-1}x)^m\end{aligned}$$

when $j>1$; if $j=1$, then d_0 is the real constant such that

$$xQ_m(x,y)-yP_m(x,y)=bx(y-d_0x)^m=bx(u_0-d_0x)^m.$$

The claims in the last paragraph are proved by induction. Since all A_{i+j} have the same form, it suffices to look at A_i and see what will happen after applying a blowing-up to it.

Since $\deg(A_{i+1})=m$, by Lemma 1, the characteristic polynomial

$$D(x,y)=xQ_m(x,y)-yP_m(x,y)=x[(b-a)y^m+\cdots]$$

has a factor $(\alpha x+\beta y)^m$ with $\alpha^2+\beta^2\neq 0$ if $r\geq 1$ in the path (10). There are two possibilities: (i) $b-a\neq 0$ and $D(x,y)=(b-a)x(y-d_0x)^m$; (ii) $b-a=0$, and consequently $D(x,y)=x^m(\delta x-\gamma y)$. In the first case, we get A_{i+1} and B_1 by the above-mentioned two changes of variables. In the second case, $b=a\neq 0$, and one can show by easy calculation that all the successors of A_i are singular points of degree 1. This is a contradiction. Therefore in the case $j=1$, the singular points A_{i+j} and B_j are described by systems (13) and (14) respectively. Consequently this is true for general j by induction on j.

As proved in the last paragraph for the case $j=1$, we have $b-ja\neq 0$ for $j=1,2,\ldots,r$. Now the following conclusion for B_j follows directly from system (14):

(i) If $b=0$, then $a\neq 0$, $B_2,\cdots,B_r$ are all saddles, and B_1 is a singular point of degree 1. It can be a saddle, a node, or a saddle-node.

(ii) If $b>0$, then B_j is a node if $a\in(\frac{b}{j},\frac{b}{j-1})$ for $j>1$, and $a\in(b,+\infty)$ for $j=1$. Otherwise B_j is a saddle. So if $b>0$, then B_j is the unique node if and only if $a>\frac{b}{r}$ and $a\neq\frac{b}{l}$ for $l=1,\cdots,r$. If $a<\frac{b}{r}$, then all B_j are saddles.

(iii) If $b<0$, then B_j is a node if $a\in(\frac{b}{j-1},\frac{b}{j})$ for $j>1$, and $a\in(-\infty,b)$ for $j=1$. Otherwise B_j is a saddle. So if $b<0$, then B_j is the unique node if and only if $a<\frac{b}{r}$ and $a\neq\frac{b}{l}$ for $l=1,\cdots,r$. If $a>\frac{b}{r}$, then all B_j are saddles.

From these facts, we know that the subgraph in Lemma 5 is a subtree. And the first part is proved.

Using the proof of Lemma 4, we know that the second part of Lemma 5 is true in the cases that $b = 0$, or that all B_j are saddles. So we need only to consider the case that there is a unique node B_j. We know that the others are saddles.

Assume that B_{j_0} is a node. The local phase portrait of A_{i+j_0} can be obtained as follows. The right part (in the region $x \geq 0$) is the same as the right part of the local phase portrait of A_{i+r} (in the region $x \geq 0$); the left part is obtained from the same part of the local phase portrait of A_{i+r} by making $r - j_0$ changes of the following type: reflection across the x-axis if $m - 1$ is even; the above reflection and reversing the orientation of orbits if $m - 1$ is odd. Now we consider the local phase portrait of A_{i+j_0-1} on the unit circle. The top and the bottom are two nodes. On the right semi-circle, there is a unique singular point whose phase portrait is the same as the right part of A_{i+j_0}. On the left semi-circle, there is still a unique singular point whose phase portrait is obtained from the left part of the phase portrait of A_{i+j_0} by the above change (i.e. the reflection across the x-axis and the modification of the orientation of orbits). Now we make a modification of this local phase portrait like the following. The singular points on the left and right semi-circles keep unchanged, but the top and the bottom are changed to be two saddles, and on each side of them we add a node. The orientation of orbits near these saddles and nodes is chosen so that we still get a flow. Note that after and before this modification, the local phase portraits which we get by contracting the unit circle to a point are topologically the same. So in the following, we use this modified phase portrait on the unit circle to obtain the local phase portraits of $A_{i+j_0-2}, \cdots, A_{i+1}$ and A_i. According to the rule described in §2, we know that the local phase portrait of A_{i+j_0-2} on the unit circle is obtained from that of A_{i+j_0-1} by a certain change. Since B_{j_0-1} is a saddle, the saddles and nodes will keep unchanged, the other two singular points are changed in the same way as in the case that all B_j are saddles. Repeating this process, we can obtain the local phase portrait of A_i directly from that of A_{i+r}. The rule is described as follows:

(i) If $b = 0$, the local phase portrait of A_i on the unit circle has four singular points. The top and the bottom correspond to B_1 which can be a saddle, a node or a saddle-node. On the right semi-circle, there is a unique singular point whose phase portrait is the same as the right part of A_{i+r}. On the left semi-circle, the local phase portrait of the unique singular point is obtained from the left part of A_{i+r} by r reflections across x-axis if $r(m - 1)$ is even, and by the above reflections and reversing orientation of orbits if $r(m - 1)$ is odd.

(ii) If all B_j are saddles, then the local phase portrait of A_i on the unit circle is the same as the case that $b = 0$ and B_1 is saddle.

(iii) If $b \neq 0$, and a unique B_{j_0} is a node, then the local phase portrait of A_i comes from the following flow on T. On the unit circle of T there are 8 singular points. The top and the bottom are two saddles. On each side of them, there is a node. The remaining two singular points which are on the right and left semi-circles are obtained from A_{i+r} in the same way as in (i). If $j_0 = 1$, we accumulate the saddle at the top and the two nodes around it into one singular point located at the top, and do the same at the bottom. Then we get a node at the top as well as at the bottom. The phase portrait of the resulting flow is just the local phase portrait of A_i on the unit circle. If $j_0 > 1$, we accumulate all singular points on the right (or left) semi-circle at the point $(\frac{1}{\sqrt{1+d_1^2}}, \frac{d_1}{\sqrt{1+d_1^2}})$ (respectively, $(-\frac{1}{\sqrt{1+d_1^2}}, -\frac{d_1}{\sqrt{1+d_1^2}})$). Then the phase portrait of the resulting flow is just the local phase portrait of A_i on the unit circle.

Certainly this rule implies the second part of Lemma 5. The proof is finished. ∎

Definition 7. A path $A_i \longrightarrow A_{i+1} \longrightarrow \cdots \longrightarrow A_{i+r}$ is called an *equidegree path* if $\deg(A_i) = \deg(A_{i+1}) = \cdots = \deg(A_{i+r})$. It is called *maximal* if the degrees of all the predecessors of A_i and all the successors of A_{i+r} are different from $\deg(A_{i+r})$.

Assume that $A_i \longrightarrow A_{i+1} \longrightarrow \cdots \longrightarrow A_{i+r}$ is a maximal equidegree path such that $\deg(A_i) = m$ and A_i is described by system (2) with

$$P_m(x, y) = a_0 y^m + a_1 x y^{m-1} + \cdots + a_m x^m,$$
$$Q_m(x, y) = b_0 y^m + b_1 x y^{m-1} + \cdots + b_m x^m.$$

We suppose $r \geq 1$ in the following. Since $\deg(A_{i+1}) = \deg(A_i)$, so, by Lemma 1, the characteristic polynomial $D(x, y) = xQ_m(x, y) - yP_m(x, y)$ has the following factorization

$$D(x, y) = (c_1 x + d_1 y)^m (\alpha x + \beta y),$$

where $(c_1^2 + d_1^2) \cdot (\alpha^2 + \beta^2) \neq 0$. In the following we discuss when this occurs. We have the following 3 cases:

(i) $a_0 = 0,\ a_1^2 + b_0^2 \neq 0$. This is the case studied in Lemma 5.

(ii) $a_0 = a_1 = b_0 = 0$. Then $d_1 = 0$. Without loss of generality, we can take $c_1 = 1$. Applying the change of variables $x = yv, d\tau = y^{m-1} dt$ to system (2), we have

$$\begin{aligned} \dot{y} &= yQ_m(v, 1) + y^2 Q_{m+1}(v, 1) + \cdots , \\ \dot{v} &= -D(v, 1) + y[P_{m+1}(v, 1) - vQ_{m+1}(v, 1)] + \cdots . \end{aligned} \tag{15}$$

Since $\deg(A_{i+1}) = m,\ A_{i+1}$ must be the singular point $y = v = 0$. In order that this singular point is of degree m, we must have

$$b_0 = b_1 = \cdots = b_{m-2} = 0, \quad a_0 = a_1 = \cdots = a_{m-1} = 0.$$

Hence

$$D(v,1) = b_m v^{m+1} + (b_{m-1} - a_m)v^m, \quad Q_m(v,1) = b_m v^m + b_{m-1}v^{m-1}.$$

Since $D(x,y) \not\equiv 0$, there are 2 cases which allow that $\deg(A_{i+1}) = m$: (1) $b_{m-1}^2 + a_m^2 \neq 0$; (2) $a_m = b_{m-1} = 0$ but $b_m \neq 0$. In the first case, the path $A_{i+1} \longrightarrow \cdots \longrightarrow A_{i+r}$ is the one considered in Lemma 5. In the second case, it follows from system (15) that y^2 is a factor of the characteristic polynomial of the singular point $y = v = 0$. So, by Lemma 1, if one of the successors of the singular point $y = v = 0$ is of degree larger than $m - 1$, this polynomial must have the following factorization

$$y^m(\delta v + \gamma y).$$

To see what happens to this factorization, we apply the change of variables: $y = vy_1, d\tau = v^{m-1}dt$ to system (15). The system we obtain has the following form

$$\begin{aligned} \dot{v} &= vy_1\bar{P}_{m-1}(1,y_1) + v^2[-b_m + \cdots], \\ \dot{y}_1 &= y_1^m(\delta + \gamma y_1) + vy_1[2b_m + \cdots], \end{aligned} \tag{16}$$

where $y\bar{P}_{m-1}(v,y)$ is the m-degree part of $\dot{v}$ in system (15). By Lemma 1, we know $v = y_1 = 0$ in system (16) is the singular point corresponding to the factor y^m of the characteristic polynomial of $y = v = 0$. But, since $b_m \neq 0$, this singular point is of degree at most 2. Consequently either all the successors of A_{i+1} are of degree smaller than m, or $m = 2$ ($m = 1$ is impossible since 1-degree singular points are the terminals of blowing-up trees). In the first case, the length of our path is 2 (i.e., there are only 2 vertices in this path). In the second case we consider the characteristic polynomial of $v = y_1 = 0$:

$$v[2b_m vy_1 + \delta y_1^2] - y_1[-b_m v^2 + \bar{P}_{m-1}(1,0)vy_1] = 3b_m v^2 y_1 + [\delta - \bar{P}_{m-1}(1,0)]vy_1^2.$$

Since $b_m \neq 0$, it must be the product of 3 different linear factors if $\delta - \bar{P}_{m-1}(1,0) \neq 0$. In this case, the length of our path is 3 since we obtain only 3 different singular points of degree 1 after applying a blowing-up to A_{i+2} (i.e., the singular point $v = y_1 = 0$ of system (16)). If $\delta - \bar{P}_{m-1}(1,0) = 0$, then the singular point A_{i+2}, which is the point $v = y_1 = 0$ of system (16), is just the point A_i considered in Lemma 5. One can see this by taking (y_1, v) in system (16) as (x, y) in Lemma 5.

(iii) $a_0 \neq 0$. In this case, $d_1 \cdot \beta \neq 0$ since $D(0,1) = a_0 \neq 0$. Thus A_{i+1} is just the singular point $x = u = 0$ of system (6). We can easily check that the path $A_{i+1} \longrightarrow \cdots \longrightarrow A_{i+r}$ is the one considered in the above two cases. Thus we have proved:

LEMMA 8. *For any equidegree path $A_i \longrightarrow A_{i+1} \longrightarrow \cdots \longrightarrow A_{i+r}$, there exists an integer j_0 with $0 \leq j_0 \leq 3$ such that the path $A_{i+j_0} \longrightarrow \cdots \longrightarrow A_{i+r}$ satisfies the assumptions of the path considered in Lemma 5.*

Remark 9. Lemma 5 says that the length is not important for the paths studied there. Now Lemma 8 generalizes this fact to arbitrary equidegree paths.

We note that the above results provide a successful method for the qualitative classification of singular points of any given degree. We argue this below.

First of all we note that the unboundedness of the number of blowing-ups necessary for the analysis of a singular point of a fixed degree comes just from the arbitrary length of equidegree paths. To see this, we need the following definition: two blowing-up trees are said to be *the same* if there are a bijection between their vertex sets and a bijection between their edge sets such that (a) any edge connecting two vertices corresponds to the edge connecting the corresponding pair of vertices; (b) the corresponding two vertices have the same property, i.e., the same degrees, and the same local phase portraits if their degrees are equal to 1. Now the above statement is exactly shown in the following lemma.

LEMMA 10. *There is a function $\Lambda : \mathbb{N} \longrightarrow \mathbb{N}$, $\mathbb{N}$ is the set of natural numbers, such that there are at most $\Lambda(m)$ different trees satisfying*

(i)each represents the blowing-up process of a singular point of degree m;
(ii)the lengths of the maximal equidegree paths are not larger than 4.

Proof. We use induction on m. That is, we assume that we have proved that the number $\Lambda(k)$ exists for $k = 2, 3, \cdots, m-1$.

Assume that A_1 is our starting singular point which is the singular point $x = y = 0$ of system (2). In the following we divide our discussion into 3 cases according to the successors of A_1.

In the first case all the successors of A_1 are of degree smaller than m. By the inductive hypothesis, it is clear that there are finitely many different trees satisfying the conditions of this lemma in this case.

In the second case there is one successor of A_1 which has degree m. By Lemma 3(i), we know that A_1 has at most 2 successors. The other, if exists, must be of degree 1. Let A_2 denote the one of degree m. By Lemma 1, we know that A_1 must be non-dicritical, i.e., its characteristic polynomial $D(x, y) \not\equiv 0$. Therefore A_2 is one of the singular points of system (6) on the u-axis (here, without loss of generality, we assume $D(0, y) \neq 0$). Without loss of generality, we assume that A_2 is just $x = u = 0$. Now it is easy to find that A_2 can be taken as the singular point A_i studied in the case (i) or (ii) of the proof of Lemma 8. In the case (i), we get the following subtree

$$A_2 \overset{\nearrow B_2}{\longrightarrow} \cdots \longrightarrow A_{k_1-1} \overset{\nearrow B_{k_1-1}}{\longrightarrow} A_{k_1}$$

such that B_j is of degree 1 for any $2 \leq j \leq k_1 - 1$, $\deg(A_2) = \cdots = \deg(A_{k_1-1}) = \deg(A_{k_1}) = m$, but all the successors of A_{k_1} are of degree smaller than m (see the proof of Lemma 5). We know $k_1 \leq 4$ from the condition (ii) of this lemma. So, for trees in this lemma, the part before A_{k_1} has finitely many possibilities. According to the inductive hypothesis, the part after A_{k_1} has also finitely possibilities. Therefore in this case there are finitely many different trees satisfying the conditions of our lemma. In the case (ii), we know from the arguments of the proof of Lemma 8, that this statement is still true.

In the third case A_1 has only one successor A_2 which is of degree $m + 1$. We know from system (6) that either all the successors of A_2 are of degree smaller than $m + 1$, or they are a saddle and a singular point of degree $m + 1$, which is denoted by A_3. The same situation happens to A_3. So we get generally a subtree like the following

$$A_1 \longrightarrow A_2 \overset{\nearrow B_2}{\longrightarrow} A_3 \overset{\nearrow B_3}{\longrightarrow} \cdots \longrightarrow A_{k_2-1} \overset{\nearrow B_{k_2-1}}{\longrightarrow} A_{k_2},$$

where $k_2 \leq 4$, $\deg(A_2) = \cdots = \deg(A_{k_2-1}) = \deg(A_{k_2}) = m + 1$ and $B_2, \cdots, B_{k_2-1}$ are all saddles. Furthermore we require that all the successors of A_{k_2} are of degree smaller than $m + 1$. Two cases may happen. The first case is that all the successors of A_{k_2} are of degree smaller than m. Again by the inductive hypothesis, we know in this case that there are finitely many trees satisfying the conditions of this lemma. The second case is that there is one successor of degree m which is denoted by A_{k_2+1}. There are at most two other successors which must be of degree 1 (see the system for the singular point A_r in Lemma 4). The singular point A_{k_2+1} can be taken as the singular point A_i studied in the case (i) of (ii) of the proof of Lemma 8. As in the last paragraph we know by induction that there are still finitely many such trees. The proof is finished. ▌

Remark 11. Lemma 10 is still true if the number 4 in the assumption (ii) is replaced by any other fixed positive integer.

In the following, any path satisfying Lemma 5 will be assumed to be maximal in the sense that any extension will not be a path studied in Lemma 5. The following corollary is a consequence of Lemma 10.

COROLLARY 12. *The following statements hold.*

(i)The number of essential terminals in a blowing-up tree of a singular point of degree m is bounded by an integer which depends only on m. Here a terminal is called essential if it is not a saddle which appears in a path studied in Lemma 5.

(ii)The number of maximal paths in a blowing-up tree of a singular point of degree m is bounded by an integer depending only on m.

(iii)If we ignore the length of the paths studied in Lemma 5, then there are finitely many blowing-up trees of singular points of degree m.

(iv)There are finitely many qualitative equivalence classes in the set of all singular points of degree m.

Proof. The statements (i), (ii) and (iii) are immediate consequences of Lemma 10. To prove (iv), we note that the local phase portrait of the starting point of a blowing-up tree is determined by local phase portraits of essential terminals and the parities of the lengths of the paths studied in Lemma 5. They all have bounded number of possibilities. Therefore the local phase portrait of the starting singular point has also finitely many possibilities. That is, there are finitely many qualitatively different local phase portraits for the set of singular points of degree m. ∎

The rest of this section is the proof of Theorem F, and is also a description of our method for the qualitative classification problem. As a preparation, we have the following lemma which can be directly verified.

LEMMA 13. *Suppose that $A_i \longrightarrow A_{i+1} \longrightarrow \cdots \longrightarrow A_{i+r}$ is an equidegree path such that A_i satisfies the conditions in Lemma 5, and the degrees of the successors of A_{i+r} are smaller than $\deg(A_{i+r})$. Then we apply the operation of type (G) to the singular point A_1 if and only if $r \geq 1$; and if $r \geq 1$, the system for A_{i+r} is obtained from the system for A_i by applying the operation of type (G) in Definition 1. Furthermore, the constants $d_0, d_1, \cdots, d_{r-1}$ which appear in the operation of type (G) are just those given in the proof of Lemma 5.*

In fact, if we have found the sequence $d_0, d_1, \ldots, d_{j-1}$, then the existence of the number d_j is equivalent to that $r \geq j+1$. The condition for d_j in Definition 1 is equivalent to the fact that $\deg(A_{i+j+1}) = m$ where A_{i+j+1} is the singular point $x = u = 0$ of the system obtained by applying the change of variables $y = ux^{j+1} + d_0x + d_1x^2 + \cdots + d_jx^{j+1}, d\tau = x^{(j+1)(1-m)}dt$ to system (2).

Proof. *(Theorem F)* Consider system (2) which is the general system for singular points of degree m. Our aim is to determine the local phase portrait of the singular point $x = y = 0$ of system (2), which is denoted by A_1. Let us proceed the first step, i.e. applying a single blowing-up or one operation of type G to system (2).

If there is an equidegree path $A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_r$ studied in Lemma 5 with $r \geq 1$, then we apply the operation of type (G) to A_1. We know that the local phase portrait of A_1 is determined by the local phase portrait of A_r and the quantities r, a, b (see the rules described in the proof of Lemma 5).

In all other cases, we apply a blowing-up to A_1. Then the number of successors, their ordering in the unit circle of the annulus T, the degree of each one and the local phase portraits of those terminals all have finitely many possibilities. Consequently there are only finitely many possibilities happening to the successors. In order to distinguish these possibilities, a finite set of quantities are employed. These quantities can be computed from system (2) by finitely many operations of type (A)–(F). And the local phase portrait of A_1 is determined by this set of quantities and the local phase portraits of the non-terminal successors of A_1. But we have finitely many cases in all to be considered separately.

Therefore after we finished the first step, we have finitely many cases which are divided by a finite set of quantities. In some cases, there will be no non-terminals and no further analysis is needed. In each of all other cases, there are some non-terminals, and we should proceed further analysis. Let us consider the collection of these cases and proceed the second step, i.e., for each case in this collection, applying the first step to each of those non-terminals.

Similarly, after we finished the second step, we still get finitely many cases which are distinguished by a finite set of quantities. In each case, the local phase portrait of A_1 is still determined by the local phase portraits of a finite set of non-terminals and a finite set of quantities. Moreover all these quantities can be computed by applying a finite set of operations of types (A)–(G) to system (2).

As we continue this process, by Corollary 12, we will eventually arrive at the situation in which there are no non-terminals in each case after finitely many steps. Our analysis is stopped here. At this endpoint, we get finitely many cases which are distinguished by a finite set of quantities. In each case, the local phase portrait of A_1 is determined by a finite set of quantities. All these quantities can be computed by applying a finite number of operations of types (A)–(G) to system (2). Since system (2) is the general system for singular points of degree m, the above finiteness of the amount of operations implies the boundedness in Theorem F. We have finished the proof of Theorem F. ∎

4. THE BLOWING-UP PROCESS OF SINGULAR POINTS OF DEGREE 2

From this section onwards, we proceed the analysis of the singular points of degree 2, which will provide a proof of Theorem D. By definition, a singular point of degree 2 can be taken as the singular point $O = (0, 0)$ of the system

$$\begin{aligned} \dot{x} &= P_2(x, y) + P_3(x, y) + P_4(x, y) + \cdots , \\ \dot{y} &= Q_2(x, y) + Q_3(x, y) + Q_4(x, y) + \cdots , \end{aligned} \tag{17}$$

where $P_2^2(x, y) + Q_2^2(x, y) \not\equiv 0$, P_i and Q_i are homogeneous polynomials of degree i. Of course, O is assumed to be an isolated singular point.

Two possibilities should be distinguished when we apply a blowing-up to system (17), namely,

$$xQ_2(x, y) - yP_2(x, y) \equiv 0 \quad \text{or} \quad xQ_2(x, y) - yP_2(x, y) \not\equiv 0.$$

In the second case, we can assume (after an affine change of the coordinate system) that $\Delta(u) = Q_2(1, u) - uP_2(1, u)$ is a polynomial of degree 3. According to the zeros of $\Delta(u)$, we have the following cases: (i) $\Delta(u) = 0$ has no multiple roots; (ii) $\Delta(u) = 0$ has a double root; (iii) $\Delta(u) = 0$ has a triple root.

Evidently the dicritical case (i.e., $xQ_2(x, y) - yP_2(x, y) \equiv 0$) and the case (i) are very simple: their associated blowing-up process contains at most one blowing-up.

Let us consider the case (iii). Evidently O has only one successor, say A_1, which corresponds to the unique triple root of $\Delta(u) = 0$. By Lemma 1, $\deg(A_1) \le 3$. If $\deg(A_1) = 3$, we apply a blowing-up to A_1, then either all successors of A_1 are of degree less than 3, or they are one saddle B_1 and a singular point of degree 3, say, A_2. If we apply a blowing-up to A_2, we still have the above possibilities. As we continue this argument, we eventually arrive at the following subtree

$$O \longrightarrow A_1 \overset{\nearrow^{B_1}}{\longrightarrow} \cdots \longrightarrow A_{l-1} \overset{\nearrow^{B_{l-1}}}{\longrightarrow} A_l,$$

where $\deg(A_1) = \cdots = \deg(A_{l-1}) = 3$, $B_1, \dots, B_{l-1}$ are all saddles, and the integer l is uniquely determined by the requirement that either (a) A_l is a singular point of degree less than 3 which corresponds to a triple zero of the characteristic polynomial of A_{l-1}; or (b) $\deg(A_l) = 3$, and the characteristic polynomial of A_l has no triple zero (so, by Lemma 1, each successor of A_l is of degree less than 3). The existence of l is an immediate consequence of the Desingularization Theorem in §1.4. Notice that $l \ge 1$.

More explicitly, let us apply Lemma 4 to this case. We first assume $l > 1$. Then we know that A_{l-1} can be taken as the singular point $x = y = 0$ of the system

$$\begin{aligned} \dot{x} &= x\bar{P}_2(x,y) + x\bar{P}_3(x,y) + x\bar{P}_4(x,y) + \cdots, \\ \dot{y} &= \bar{Q}_3(x,y) + \bar{Q}_4(x,y) + \bar{Q}_5(x,y) + \cdots, \end{aligned} \tag{18}$$

where $\bar{Q}_3(0,1) = -(l-1)\bar{P}_2(0,1) = -a(l-1)$, $a = P_2(0,1) \neq 0$. The characteristic polynomial of A_{l-1} is just

$$x\bar{Q}_3(x,y) - y \cdot x\bar{P}_2(x,y) = x[\bar{Q}_3(x,y) - y\bar{P}_2(x,y)],$$

where the factor x corresponds to the singular point B_{l-1}, the part $[\dots]$ must be the third power of some linear factor, which corresponds to the singular point A_l. Without loss of generality, we assume that this linear factor is y (i.e., $x\bar{Q}_3(x,y) - y \cdot x\bar{P}_2(x,y) = -laxy^3$). Now apply the blowing-up $y = xu$, $d\tau = x^2dt$ to system (18), and we obtain

$$\begin{aligned} \dot{x} &= x\bar{P}_2(1,u) + x^2\bar{P}_3(1,u) + x^3\bar{P}_4(1,u) + \cdots, \\ \dot{u} &= -lau^3 + x[\bar{Q}_4(1,u) - u\bar{P}_3(1,u)] + x^2[\bar{Q}_5(1,u) - u\bar{P}_4(1,u)] + \cdots. \end{aligned} \tag{19}$$

The singular point $x = u = 0$ is just A_l. By the requirement for l, either $\deg(A_l) \leq 2$ or $\deg(A_l) = 3$. In the second case, we rewrite (19) as

$$\begin{aligned} \dot{x} &= x\tilde{P}_2(x,u) + x\tilde{P}_3(x,u) + x\tilde{P}_4(x,u) + \cdots, \\ \dot{u} &= \tilde{Q}_3(x,u) + \tilde{Q}_4(x,u) + \tilde{Q}_5(x,u)) + \cdots. \end{aligned} \tag{20}$$

Let $\bar{\Delta}(u) = \tilde{Q}_3(1,u) - u\tilde{P}_2(1,u) = -(l+1)au^3 + \cdots$, where "$\cdots$" is some 2-degree polynomial in u. Then by the requirement on l, we know that $\bar{\Delta}(u) = 0$ either has only simple roots or has a unique double root.

In the case $l = 1$, we apply a blowing-up to system (17). Then we find that the resulting system is still described by system (19) if we set in (19) $\bar{P}_i = P_i, \bar{Q}_{i+1} = Q_i$ for any $i \geq 2$ (here we keep the requirement that the singular point $x = u = 0$ of system (19) is just $A_l = A_1$). Thus the discussion on A_l in the last paragraph is still valid in the case $l = 1$.

To sum up, we have the following classification. The notations here are the same as those defined above, and will be used in the following sections (§§5–9).

PROPOSITION 14. *The singular point O of system (17) must satisfy one of the following conditions:*

(i)$xQ_2(x,y) - yP_2(x,y) \equiv 0$;
(ii)$\Delta(u) = 0$ *has no multiple roots;*
(iii)$\Delta(u) = 0$ *has a double root;*

(iv)$\Delta(u) = 0$ *has a triple root,* $\deg(A_l) = 1$;

(v)$\Delta(u) = 0$ *has a triple root,* $\deg(A_l) = 2$;

(vi)$\Delta(u) = 0$ *has a triple root,* $\deg(A_l) = 3$; $\bar{\Delta}(u) = 0$ *has only simple roots;*

(vii)$\Delta(u) = 0$ *has a triple root,* $\deg(A_l) = 3$; $\bar{\Delta}(u) = 0$ *has a double root.*

Based on this classification, our analysis of the singular point O of system (17) in the following sections is divided into 7 cases, among which the cases (i),(ii),(iii) are studied in §5,§6,§7 respectively; the cases (vi) and (vii) are studied in §8, whereas §9 devotes to the cases (iv) and (v). Since the constant a in systems (18) and (19) is different from zero, we can assume, without loss of generality, that $a = 1$ in the following sections.

5. THE CASE $XQ_2(X,Y) - YP_2(X,Y) \equiv 0$

In this case after a linear coordinate change and a rescaling of the time variable if necessary, we can assume

$$P_2(x,y) = xy, \qquad Q_2(x,y) = y^2.$$

Applying the blowing-up $y = xu, d\tau = x^2 dt$ to system (17), we get

$$\begin{aligned} \dot{x} &= u + xP_3(1,u) + x^2P_4(1,u) + \cdots , \\ \dot{u} &= Q_3(1,u) - uP_3(1,u) + x[Q_4(1,u) - uP_4(1,u)] + \cdots . \end{aligned} \tag{21}$$

For convenience we rewrite system (21) as

$$\begin{aligned} \dot{x} &= a_1x + u + a_3x^2 + a_2xu + \cdots , \\ \dot{u} &= b_0 + b_2x + b_1u + \cdots . \end{aligned} \tag{22}$$

The local phase portrait of system (17) at O is determined by the local phase portrait of system (22) on the u-axis, where there is only one singular point $x = u = 0$ if $b_0 = 0$, and no singular point if $b_0 \neq 0$. The details of our analysis are given below according to the values of the parameters a_i and b_i.

In system (22), if $b_0 \neq 0$, then the local phase portrait of system (17) at $x = y = 0$ is given in Figure 3(1). If $b_0 = 0$ and $x = u = 0$ is a saddle, then it is Figure 3(2). If $x = u = 0$ is a node, then we get Figure 3(4). If $x = u = 0$ is a center or a focus, then we have Figure 3(3). If $x = u = 0$ is a saddle-node, we obtain Figure 3(5). If $x = u = 0$ is a singular point with one hyperbolic sector and one elliptic sector, then the local phase portrait of system (17) at O is Figure 3(6). If $x = u = 0$ is a union of two hyperbolic sectors, we still get Figure 3(1). From Theorem A we know that we have considered all possibilities. The following two examples realize these 6 pictures.

EXAMPLE 15. *If we apply the change of variables* $y = xu,\ d\tau = x^2 dt$ *to the system*

$$\dot{x} = xy + a_1x^3, \qquad \dot{y} = y^2 + b_0x^3 + (a_1 + b_1)x^2y + b_2x^4 + b_3x^3y,$$

we obtain

$$\dot{x} = a_1x + u, \qquad \dot{u} = b_0 + b_2x + b_1u + b_3xu.$$

If $b_0 = 1$ *and* $a_1 = b_1 = b_2 = b_3 = 0$*, then the local phase portrait of* $x = y = 0$ *is Figure 3(1). Similarly, we obtain Figure 3(2) if* $b_0 = a_1 = b_1 = b_3 = 0$ *and* $b_2 = 1$*; Figure 3(3) if* $b_1 = a_1 = b_0 = b_3 = 0$ *and* $b_2 = -1$*; Figure 3(4) if* $b_0 = b_2 = b_3 = 0$ *and* $a_1 = b_1 = 1$*; and Figure 3(5) if* $b_0 = b_1 = b_2 = 0$, $a_1 = b_3 = -1$.

EXAMPLE 16. *The local phase portait of the system* $\dot{x} = xy,\ \dot{y} = y^2 + x^3y - x^8$ *at* $x = y = 0$ *is Figure 3(6). In fact, if we apply the change of variables* $y = xu,\ d\tau = x^2dt$ *to this system, we obtain* $\dot{x} = u,\ \dot{u} = xu - x^5$. *According to Theorem A, the local phase portrait of the singular point* $x = u = 0$ *is the union of one hyperbolic sector and one elliptic sector. Now the assertion follows from the above analysis.*

6. THE CASE $\Delta(U) = 0$ HAS NO MULTIPLE ROOTS

Making a linear change of variables if necessary, we can assume

$$P_2(1, u) = u^2 + a_1u + a_2$$

in system (17). So, when applying the blowing-up $y = ux,\ d\tau = xdt$ to it, we obtain

$$\begin{aligned}
\dot{x} &= x(u^2 + a_1u + a_2) + x^2P_3(1, u) + x^3P_4(1, u) + \cdots, \\
\dot{u} &= \Delta(u) + x[Q_3(1, u) - uP_3(1, u)] + x^2[Q_4(1, u) - uP_4(1, u)] + \cdots,
\end{aligned} \tag{23}$$

where $\Delta(u) = Q_2(1, u) - uP_2(1, u) = -u^3 + \cdots$ with "$\cdots$" being some 2-degree polynomial in u. By the theory of blowing-ups (§2), the local phase portrait of system (17) at $x = y = 0$ is obtained from the phase portrait of system (23) on the u-axis. But, instead of studying system (23), we prefer to discuss the following more general system which will be used in §8:

$$\begin{aligned} \dot{x} &= x(u^2 + a_1 u + a_2) + x^2 P_3(1,u) + x^3 P_4(1,u) + \cdots, \\ \dot{u} &= B(u) + x[Q_3(1,u) - uP_3(1,u)] + x^2[Q_4(1,u) - uP_4(1,u)] \cdots, \end{aligned} \tag{24}$$

where $P_i(x, y)$ and $Q_i(x, y)$ are homogeneous polynomials of degree i, the function $B(u) = -(l+1)u^3 + \cdots$ is a polynomial of degree 3, and the "$\cdots$" here is some 2-degree polynomial in u. We are interested in two cases for system (24). The first is that $l = 0$, $B(u) = \Delta(u)$, P_i and Q_i are those in system (23). The second is that $l > 0$, $B(u) = \bar{\Delta}(u)$, P_i and Q_i are correspondingly the polynomials $\tilde{P}_{i-1}$ and $\tilde{Q}_i$ in system (20). We study in §6 and §7 the phase portrait of system (24) on the u-axis. Then the results in the first case are used in the same sections to obtain the local phase portrait of system (17) at $x = y = 0$ in the cases (ii) and (iii) of Proposition 14. The results in the second case are used in §8 to analyze the cases (vi) and (vii) in Proposition 14 by using the method developed in §3.

In the rest of this section we will study the phase portrait of system (24) near the u-axis in the case that $B(u) = 0$ has no multiple roots. We divide our discussion into two cases, see §6.1 and §6.2. We first give all pictures for the phase portrait of system (24) near the u-axis. And then we give the corresponding pictures for the local phase portrait of system (17) at $x = y = 0$ in the case $l = 0$ by using the theory of blowing-ups (§2). For the case $l > 0$, see §8.2.

6.1. $B(u) = 0$ has only one real root

In this situation we can write $B(u)$ as

$$B(u) = -(l+1)(u - b_1)[(u - b_2)^2 + b_3^2], \quad b_3 \neq 0.$$

Here b_1, b_2, b_3 and a_1, a_2 in system (24) are all real constants, which have nothing to do with those of the previous section. We know that on the u-axis, system (24) has a unique singular point $x = u - b_1 = 0$, which has at least one non-zero eigenvalue. Thus, by Theorem A, its local phase portrait is either a saddle, or a node, or a saddle-node as is shown in Figure 6. These four pictures are realized in Example 17.

EXAMPLE 17. *The phase portrait of the system*

$$\dot{x} = x(u^2 + a_1 u + a_2), \qquad \dot{u} = -(l+1)u(u^2 + 1) + a_3 x^3$$

on the u-axis is given in Figure 6(a) if we take $a_2 = 1, a_1 = a_3 = 0$. We obtain Figure 6(b) if $a_2 = -1, a_1 = a_3 = 0$; we have Figure 6(c) if $a_2 = 0, a_1 = a_3 = 1$. Finally, to obtain Figure 6.d), we take $a_2 = 0, a_1 = 1, a_3 = -1$.

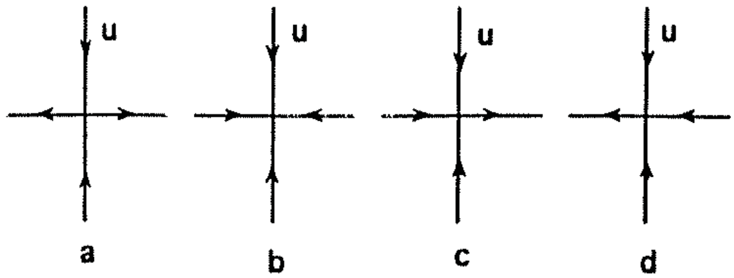

FIG. 6. The phase portrait of system (24) on the u-axis in the case that $B(u) = 0$ has only one real root.

In the case $l = 0$, by using the theory of blowing-ups (§2), we obtain 3 pictures (Figures 3(7), 3(8), 3(9)) for the local phase portrait of system (17) at $x = y = 0$ from Figure 6. More accurately, from Figure 6(a) we obtain Figure 3(7); from Figure 6(b) we get Figure 3(8); from Figures 6(c) and 6(d), we get Figure 3(9). These 3 pictures can be realized since the system in Example 17 is obtained by applying the blowing-up $y = ux$, $d\tau = xdt$ to the system

$$\dot{x} = y^2 + a_1xy + a_2x^2, \qquad \dot{y} = a_1y^2 + (a_2 - 1)xy + a_3x^5$$

if $l = 0$. The values for the parameters a_1, a_2, a_3 are: $a_2 = 1, a_1 = a_3 = 0$ for Figure (7); $a_2 = -1, a_1 = a_3 = 0$ for Figure (8); $a_2 = 0, a_1 = a_3 = 1$ for Figure (9).

6.2. $\boldsymbol{B(u)} = 0$ has 3 different real roots

In this case we can assume that

$$B(u) = -(l+1)(u - b_{01})(u - b_{02})(u - b_{03}), \quad b_{01} < b_{02} < b_{03}.$$

Then $c_i = P_2(1, b_{0i})$ is the eigenvalue in the x-direction for the singular point $(0, b_{0i})$. So the phase portrait of system (24) on the u-axis is determined by the signs of c_i. The following pictures are drawn according to the location of the two zeros of the polynomial $P_2(1, u)$ on the u-axis. If no c_i is zero for $i = 1, 2, 3$, then both eigenvalues at $(0, b_{0i})$ are different from zero. Consequently there are 7 possible pictures for the local phase portrait of system (24) on the u-axis shown in Figure 7. They are realized in Example 18.

Example 18. *In the system*

$$\dot{x} = x(u - a)(u - b), \qquad \dot{u} = -(l+1)u(u - 1)(u + 1),$$

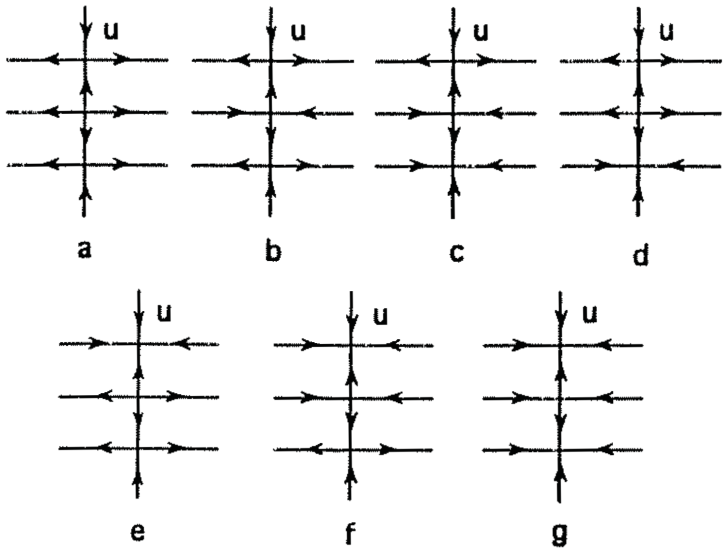

FIG. 7. The phase portrait of system (24) on the u-axis where there are 3 hyperbolic singular points.

let a and b belong to some of the intervals $(\infty, 1), (1, 0), (0, -1), (-1, -\infty)$. *All the combinations will give the 7 pictures of Figure 7, for example, if* $a = b = 2$, *we have Figure 7(a); if* $a = \frac{1}{2}, b = -\frac{1}{2}$, *we have Figure 7(b); if* $a = \frac{1}{2}, b = -2$, *we get Figure 7(c); if* $a = -\frac{1}{2}, b = -2$, *we obtain Figure 7(d); if* $a = 2, b = \frac{1}{2}$, *we get Figure 7(e); if* $a = 2, b = -\frac{1}{2}$, *we obtain Figure 7(f); and if* $a = 2, b = -2$, *we have Figure 7(g).*

Now let $l = 0$. We get 3 pictures (Figures 3(10), 3(11), 3(12)) for the local phase portrait of system (17) at $x = y = 0$ from Figure 7. Figure 3(10) corresponds to Figures 7(a),(c),(f); Figure 3(11) corresponds to Figures 7(d),(e),(g); and Figure 3(12) corresponds to Figure 7(b). The following system realize these 3 pictures

$$\dot{x} = (y - ax)(y - bx), \qquad \dot{y} = -(a + b)y^2 + (1 + ab)xy,$$

since the system in Example 18 is obtained by applying the blowing-up $y = ux,\ d\tau = xdt$ to it if $l = 0$.

Now assume that one c_i is zero. The only interesting case is when the corresponding singular point $(0, b_{0i})$ is a saddle-node of system (24). By the symmetry with respect to the u-axis, we can fix the direction of the orbits

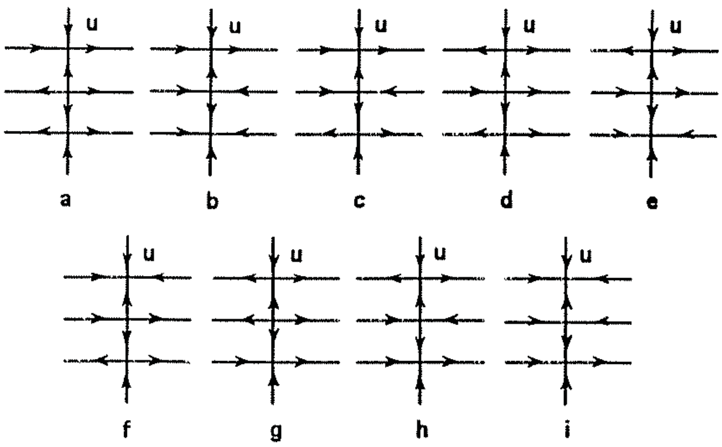

FIG. 8. The phase portrait of system (24) on the u-axis where there are two hyperbolic singular points and one saddle-node.

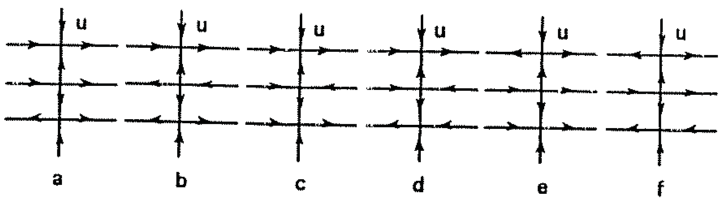

FIG. 9. The phase portrait of system (24) on the u-axis where there are one hyperbolic singular points and two saddle-node.

entering at this saddle-node along the characteristic direction which is not the u-axis. As in Figure 7, we can get all possible pictures according to the location of the two zeros of $P_2(1,u)$. This is Figure 8, which is realized by Example 19 below. We keep in mind that Figure 8 contains all possibilities modulo the symmetry with respect to the u-axis. Similarly Figures 9, 12, 17 below contain also all possibilities modulo this symmetry.

EXAMPLE 19. *In the system*

$$\dot{x} = x(u-a)(u-b) + x^2, \qquad \dot{u} = -(l+1)u(u-1)(u+1),$$

if we choose $a = 1$*, and* b *belongs to one of the intervals* $(0,\infty), (-1,0)$*,* $(-\infty,-1)$*, then we get Figures 8(a),(c),(b) respectively. Similarly if* $a = 0$*,*

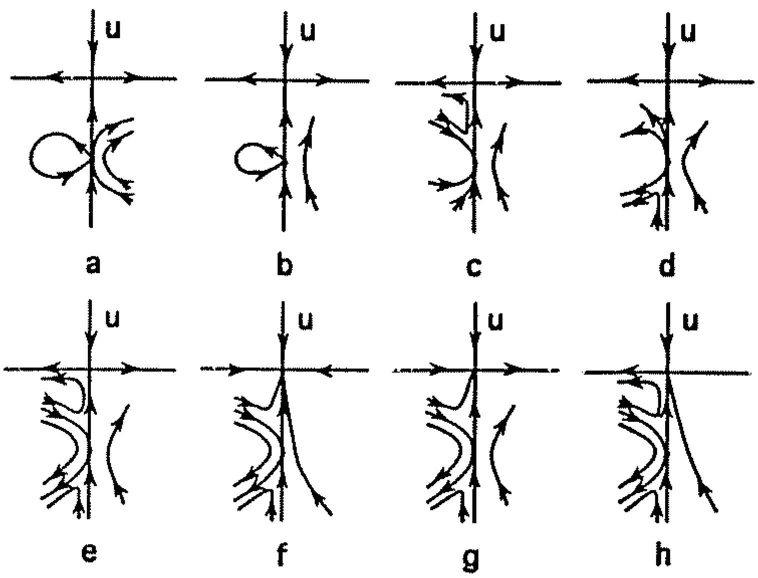

FIG. 10. The phase portrait of system (25) on the u-axis in the case that $a_2 = 0$ and $Q_3(1, 0) \neq 0$.

and b belongs to one of the intervals $(-1, 1), (1, \infty), (-\infty, -1)$*, we will get Figures 8(d),(f),(e) respectively; if* $a = -1$*, and b belongs to one of the intervals* $(-\infty, 0), (0, 1), (1, \infty)$*, then we obtain Figures 8(g),(h),(i) correspondingly.*

In the case $l = 0$, there are 2 pictures (Figures 3(13), 3(14)) for the local phase portrait of system (17) at $x = y = 0$. From Figures 8(a), (b), (e), (f), (g), (i), we obtain Figure 3(13); the other 3 pictures correspond to Figure (14). By the same reason, Figures 3(13) and 3(14) can be realized by the following system

$$\dot{x} = (y - ax)(y - bx) + x^3, \qquad \dot{y} = -(a + b)y^2 + (ab + 1)xy + x^2y.$$

For example, we can choose $a = 1, b = \frac{1}{2}$ for Figure 3(13), $a = 1, b = -\frac{1}{2}$ for Figure 3(14).

Now assume two c_i are zero. As in the above cases, Figure 9 shows all the possibilities for the phase portrait of system (24) near the u-axis. They can be realized by the following system

$$\dot{x} = x(u - a)(u - b) + (u - c)x^2, \qquad \dot{u} = -(l + 1)u(u - 1)(u + 1).$$

The values of the parameters a, b and c can be taken as: $a = 1, b = 0, c = -1$; $a = 1, b = 0, c = \frac{1}{2}$; $a = 1, b = -1, c = -2$; $a = 1, b = -1, c = 0$; $a = 0, b = -1, c = -2$; $a = 0, b = -1, c = -\frac{1}{2}$ for Figures 9(a),(b),(c),(d),(e),(f) respectively.

In the case $l = 0$, we get Figure 3(15) from Figures 9(a),(c),(e); and Figure 3(16) from Figures 9(b),(d),(f). By the same reason as in the above cases, these two pictures are realized by the system

$$\dot{x} = (y-ax)(y-bx)+(y-cx)x^2, \quad \dot{y} = -(a+b)y^2+(ab+1)xy+(y-cx)xy,$$

from which we obtain Figure 3(15) if we take $a = 1, b = 0, c = -1$; and Figure 3(16) if $a = 1, b = 0, c = \frac{1}{2}$.

We remark that at most two c_i can be zero because P_2 is a polynomial of degree 2. Thus we have considered all possibilities.

7. THE CASE $\Delta(U) = 0$ HAS A DOUBLE ROOT

In this case, we can assume

$$\Delta(u) = -u^2(u-1).$$

To obtain all possible pictures for the local phase portrait of system (17) at $x = y = 0$, we need to analyze the system (23) on the u-axis. As we explained at the beginning of §6, we study the more general system (24), which can be written as

$$\begin{aligned} \dot{x} &= x(u^2 + a_1 u + a_2) + x^2 P_3(1,u) + \cdots , \\ \dot{u} &= -(l+1)u^3 + u^2 + x[Q_3(1,u) - uP_3(1,u)] + \cdots , \end{aligned} \tag{25}$$

after a rescaling of the variables x, u and t, since $B(u) = 0$ has a double root. We divide the analysis of system (25) into several cases.

7.1. The case $a_2 \neq 0$

In this case there are two singular points on the u-axis, namely, $(0,0)$ and $(0, \frac{1}{l+1})$. Each of them has at least one non-zero eigenvalue. Thus Figure 10 contains all the possible pictures for the phase portrait of system (25) on the u-axis. They can be realized by the system

$$\dot{x} = x(u-a)(u-b) + \alpha x^2, \qquad \dot{u} = u^2 - (l+1)u^3.$$

The values of the parameters α, a and b can be taken as: $a = b = 2, \alpha = 0$ for Figure 10(a); $a = 2, b = \frac{1}{l+2}, \alpha = 0$ for Figure 10(b); $a = b = \frac{1}{l+1}, \alpha = 1$ for Figure 10(c); $a = b = \frac{1}{l+1}, \alpha = -1$ for Figure 10(d); $a = \frac{1}{l+2}, b =$

$-1, \alpha = 0$ for Figure 10(e); $a = 2, b = -1, \alpha = 0$ for Figure 10(f); $a = \frac{1}{l+1}, b = -1, \alpha = -1$ for Figure 10(g); $a = \frac{1}{l+1}, b = -1, \alpha = 1$ for Figure 10(h).

In the case $l = 0$ we obtain 3 pictures (Figures 3(17), 3(18), 3(19)) for the phase portrait of system (17) at $x = y = 0$. We get Figure 3(17) from Figures 10(b),(f). Figure 3(18) is obtained from Figures 10(c),(d),(g),(h). Whereas Figure 3(19) corresponds to Figures 10(a),(e). These 3 pictures can also be realized since the above system is obtained by applying the blowing-up $y = ux$, $d\tau = xdt$ to the system

$$\dot{x} = (y - ax)(y - bx) + \alpha x^3, \quad \dot{y} = (1 - a - b)y^2 + abxy + \alpha x^2 y$$

if $l = 0$. The values of the parameters a, b and α can be taken as: $a = 2, b = \frac{1}{l+2}, \alpha = 0$ for Figure 3(17); $a = b = \frac{1}{l+1}, \alpha = 1$ for Figure 3(18); $a = b = 2, \alpha = 0$ for Figure 3(19).

7.2. The case $a_2 = 0$, $Q_3(1, 0) \neq 0$

In this case we can always take $Q_3(1, 0) > 0$ in system (25) by using the rescaling: $(x, u, t) \to (-x, u, t)$. There are two singular points on the u-axis for system (25). The singular point $x = u - \frac{1}{l+1} = 0$ has at least one non-zero eigenvalue. So it is either a saddle, or a node, or a saddle-node. The singular point $x = u = 0$ is just the one studied in §10.2 because, if the variables x and u in system (25) are changed to y and x respectively, we then get system (50) in §10.2. So Figure 32 has shown all possible pictures for the local phase portrait of system (25) at $x = u = 0$. Furthermore, we must have in system (25) that $a_1 \geq 0$ if $x = u = 0$ is not a saddle. Correspondingly, $x = u - \frac{1}{l+1} = 0$ must be a saddle. So Figure 11 contains all possible pictures for the phase portrait of system (25) on the u-axis.

We use the following system to realize the pictures in Figure 11:

$$\dot{x} = x(u^2 + a_1 u) + \alpha x^2, \qquad \dot{u} = u^2 - (l + 1)u^3 + x.$$

To choose the values of the parameters α and a_1 for each picture in Figure 11, we can use the analysis in §10.2. For example, we can choose $a_1 = 3, \alpha = 0$ for Figure 11(a); $a_1 = 1, \alpha = 0$ for Figure 11(b); $a_1 = 0, \alpha = 2$ for Figure 11(c); $a_1 = \alpha = 0$ for Figure 11(d); $\alpha = 0, a_1 = -\frac{1}{l+2}$ for Figure 11(e); $\alpha = 0, a_1 = -2$ for Figure 11(f); $\alpha = 0, a_1 = -\frac{1}{l+1}$ for Figure 11(g); and $\alpha = -2, a_1 = -\frac{1}{l+1}$ for Figure 11(h).

In the case $l = 0$ we get 6 pictures (Figures 3(20),(21),(22), (23), (24),(27)) for the local phase portrait of system (17) at $x = y = 0$, among which Figure 3(27) comes from Figure 11(a); Figure 3(24) corresponds to Figure 11(b); Figure 3(20) comes from Figures 11(c) and (d); Figure 3(21) comes from Figure 11(e); Figure 3(22) corresponds to Figure 11(f); Figure 3(23)

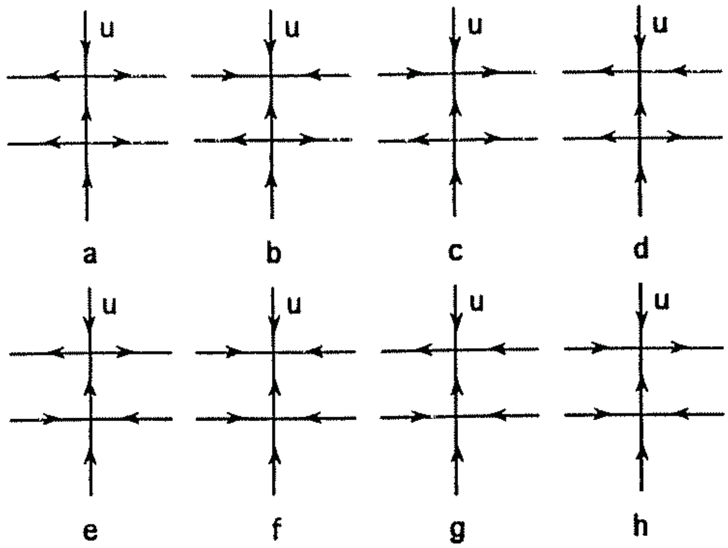

FIG. 11. Figure The phase portrait of system (25) on the u-axis in the case $a_2 \neq 0$.

corresponds to Figures 11(g) and (h). Since the above system is obtained in the case $l = 0$ by applying the blowing-up $y = ux$, $d\tau = xdt$ to the system

$$\dot{x} = y^2 + a_1 xy + \alpha x^3, \qquad \dot{y} = (1 + a_1)y^2 + \alpha x^2 y + x^3,$$

these 6 pictures can be realized by this system. The corresponding values for the parameters can be chosen as those for the realization of Figure 11 according to the above correspondence between these 6 pictures and those in Figure 11.

7.3. The case $\boldsymbol{a}_2 = \boldsymbol{Q}_3(1, 0) = 0$

Consider the following system

$$\begin{aligned} \dot{x} &= a_1 xu + a_3 x^2 + xP_2^*(x, u) + xP_3^*(x, u) + \cdots, \\ \dot{u} &= u^2 + b_1 xu + b_2 x^2 + Q_3^*(x, u) + Q_4^*(x, u) + \cdots, \end{aligned} \tag{26}$$

where P_i^* and Q_i^* are homogeneous polynomials of degree i, $P_2^*(0, u) = u^2$, $Q_3^*(0, u) = -(l + 1)u^3$; $P_i^*(0, u) = Q_{i+1}^*(0, u) = 0$ when $i > 2$. Notice that system (26) is a rewritten form of system (25). We will use it to study the singular point $x = u = 0$ of system (25). The other singular point of

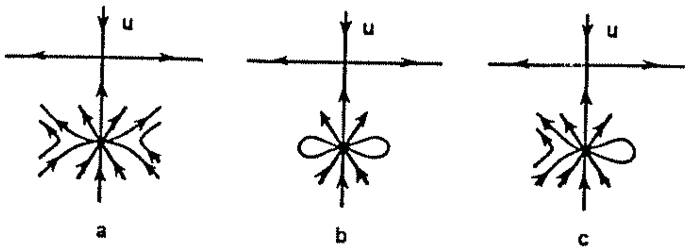

FIG. 12. The phase portrait of system (25) on the u-axis in the case studied in §7.3.1.

system (25) on u-axis is $x = u - \frac{1}{l+1} = 0$. Its local phase portrait has 4 possibilities: it is a saddle if $(l+1)a_1+1 > 0$; it is a node if $(l+1)a_1+1 < 0$; if $(l+1)a_1+1 = 0$, it is a saddle-node, but this time there are two possibilities which are distinguished by the orientation of the characteristic orbits that are not on the u-axis.

Let

$$D^*(x,u) = x[(1-a_1)u^2 + (b_1 - a_3)xu + b_2x^2].$$

The singular point $x = u = 0$ will be studied in the rest of this section according to the values of the parameters a_i and b_i.

7.3.1 The case $1 - a_1 = b_1 - a_3 = b_2 = 0$

In this case $D^*(x,u) \equiv 0$. The singular point $x = u - \frac{1}{l+1} = 0$ is a saddle. According to §5, there are 3 topologically different pictures for the local phase portrait of system (26) at $x = u = 0$. So there are 3 pictures for the phase portrait of system (25) on the u-axis, which are shown in Figure 12, and realized by Example 20.

EXAMPLE 20. *Consider the system*

$$\dot{x} = ux + a_1x^3 + xu^2, \qquad \dot{u} = u^2 - (l+1)u^3 + (a_1+b_1)x^2u + b_3x^3u + b_2x^4.$$

Applying the blowing-up $u = xu_1$, $d\tau = x^2dt$ *to it, we obtain*

$$\dot{x} = u_1 + a_1x + xu_1^2, \qquad \dot{u}_1 = -(l+2)u_1^3 + b_1u_1 + b_2x + b_3xu_1.$$

The point $x = u_1 = 0$ *is a saddle if we choose* $b_2 = b_3 = 0, a_1 = 1, b_1 = -1$*; a focus if* $a_1 = b_1 = 1, b_2 = -1, b_3 = 0$*; a saddle-node if* $a_1 = b_3 = -1, b_1 = b_2 = 0$*. The corresponding pictures for the phase portrait of system (25) on the u-axis are Figures 12(a),(b) and (c) respectively.*

In the case $l = 0$ we get 3 pictures (Figures 3(25),(26),(27)) for the phase portrait of system (17) at $x = y = 0$. From Figure 12(a) we get Figure

3(25); Figure 3(26) comes from Figure 12(b), and Figure 12(c) corresponds to Figure 3(27). These 3 pictures can be realized by the system

$$\dot{x} = y^2 + xy + a_1x^4, \qquad \dot{y} = 2y^2 + (2a_1 + b_1)x^3y + b_2x^6 + b_3x^4y,$$

since the system in Example 20 is obtained by applying the blowing-up $y = xu,\ d\tau = xdt$ to it when $l = 0$. And the values of the parameters can be chosen as those for the corresponding picture in Figure 12.

7.3.2 The case $a_1 = 1,\ D^*(x, u) \not\equiv 0$

Apply the following two changes of variables

$$x = us,\ d\tau = udt; \qquad u = xv,\ d\tau = xdt,$$

to system (26), we obtain the following two systems

$$\dot{u} = (1 + \cdots)u, \qquad \dot{s} = (a_3 - b_1)s^2 - b_2s^3 + u[\cdots], \tag{27}$$

and

$$\dot{x} = (v + a_3)x + x^2[\cdots], \qquad \dot{v} = (b_1 - a_3)v + b_2 + x[\cdots], \tag{28}$$

where $[\cdots]$ is some analytic function. Now we use these two systems to obtain all possible pictures for the phase portrait of system (25) at $x = u = 0$ modulo the symmetry with respect to the u-axis. We note that we can assume $a_3 - b_1 \geq 0$. To see this, we know that the above symmetry is generated by the change $(x, u, t) \to (-x, u, t)$ in system (25). This is equivalent to the transformation $(u, s, \tau) \to (u, -s, \tau)$ in system (27) and $(x, v, \tau) \to (-x, -v, -\tau)$ in system (28). After these changes $a_3 - b_1$ is changed to $b_1 - a_3$. This implies that it is sufficient to consider the case $a_3 - b_1 \geq 0$.

Now we can show that the 6 pictures of Figure 13 are all the possibilities for the phase portrait of system (25) at $x = u = 0$. First we assume $a_3 - b_1 = 0$. Then system (28) has no singular points on the v-axis. Moreover $u = s = 0$ is a saddle if $b_2 > 0$; a node if $b_2 < 0$. Correspondingly the phase portrait of system (25) at $x = u = 0$ is Figure 13(a) or (b). Now we assume $a_3 - b_1 > 0$. Then $u = s = 0$ is a saddle-node. System (28) has a unique singular point $x = v - \frac{b_2}{a_3 - b_1} = 0$ on the v-axis. It is a saddle if $b_2 + a_3^2 - a_3b_1 > 0$; a node if $b_2 + a_3^2 - a_3b_1 < 0$; a saddle-node, or a saddle, or a node if $b_2 + a_3^2 - a_3b_1 = 0$. The first two cases correspond to Figures 13(c) and (d). In the third case, only the saddle-node is interesting since we have considered the saddle and the node. This case has two possibilities which are Figures 13(e) and (f).

Since the singular point $x = u - \frac{1}{l+1} = 0$ is always a saddle in this case, the corresponding pictures for the phase portrait of system (25) on the u-axis are those in Figure 14.

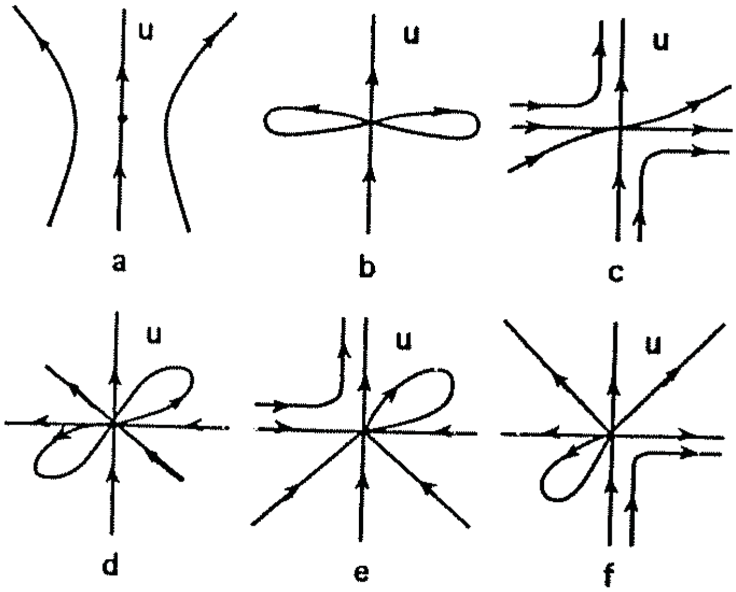

FIG. 13. The local phase portrait of system (25) at $x = u = 0$ in the case studied in §7.3.2.

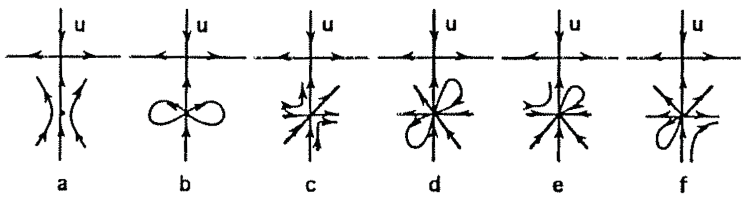

FIG. 14. The phase portrait of system (25) on the u-axis in the case studied in §7.3.2.

Example 21. *Consider the following system*

$$\dot{x} = xu + a_3x^2 + a_4x^3 + xu^2, \qquad \dot{u} = u^2 + b_1xu - (l+1)u^3 + a_4x^2u + b_2x^2.$$

If we apply a blowing-up to it, then the system corresponding to (28) is the following

$$\dot{x} = x(v + a_3) + a_4x^2 + x^2v^2, \qquad \dot{v} = (b_1 - a_3)v + b_2 - (l+2)xv^3.$$

From this system, we can easily find the values of the parameters a_i and b_i to realize all the pictures in Figure 14. Thus we can choose $b_1 = a_3 = a_4 = 0$ and $b_2 = 1$ for Figure 14(a); $b_1 = a_3 = a_4 = 0, b_2 = -1$ for Figure 14(b); $b_1 = a_4 = 0, a_3 = b_2 = 1$ for Figure 14(c); $b_1 = a_4 = 0, a_3 = 1, b_2 = -2$ for Figure 15(d); $b_1 = 0, a_3 = 1, b_2 = -1, a_4 = -l - 4$ for Figure 14(e); and $b_1 = a_4 = 0, a_3 = 1, b_2 = -1$ for Figure 14(f).

In the case $l = 0$ we get four pictures for the phase portrait of system (17) at $x = y = 0$. That is, we obtain Figure 3(7) from Figure 14(a); Figure 3(26) from Figures 14(b) and (d); Figure 3(28) from Figure 14(c); Figure 3(29) from Figures 14(e) and (f). To realize these pictures, we can use the system

$$\dot{x} = xy + y^2 + a_3x^3 + a_4x^4, \qquad \dot{y} = 2y^2 + (a_3 + b_1)x^2y + 2a_4x^3y + b_2x^4,$$

since when applying the blowing-up $y = xu,\ d\tau = xdt$ to it, we obtain the system of Example 21 if $l = 0$. The values of the parameters can be chosen those for the corresponding pictures in Figure 14.

7.3.3 The case $1 - a_1 \neq 0,\ D^*(1, u)$ has no real roots

Apply the following changes of variables

$$u = xv,\ d\tau = xdt; \qquad x = us,\ d\tau = udt,$$

to system (26), we obtain two systems

$$\dot{x} = x(a_1v + a_3) + x^2P_2^*(1, v) + \cdots, \qquad \dot{v} = D^*(1, v) + x[\cdots] \tag{29}$$

and

$$\dot{u} = u + \cdots, \qquad \dot{s} = (a_1 - 1)s + \cdots, \tag{30}$$

where $[\cdots]$ denotes some analytic function. In the present case $D^*(1, v)$ has no real zeros, so there are no singular points on the v-axis. When $a_1 > 1$, the point $u = s = 0$ is a node and, correspondingly, the point $x = u - \frac{1}{l+1} = 0$ in system (25) is a saddle. If $a_1 < 1$, then $u = s = 0$ is a saddle. Correspondingly, the point $x = u - \frac{1}{l+1} = 0$ can be a node,

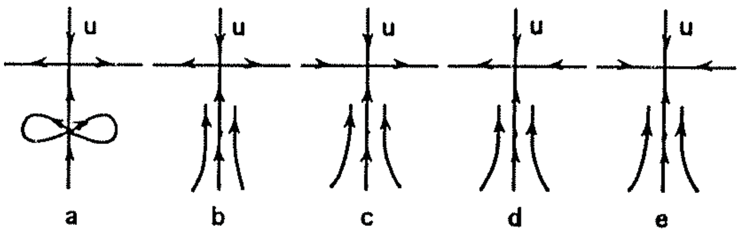

FIG. 15. The phase portrait of system (25) on the u-axis in the case studied in §7.3.3.

a saddle or a saddle-node. So Figure 15 contains all possible pictures for the phase portrait of system (25) on the u-axis, which are realized by the following example.

Example 22. *For the system*

$$\dot{x} = a_1xu + a_3x^2 + xu^2, \qquad \dot{u} = u^2 + a_3xu + (1-a_1)x^2 - (l+1)u^3,$$

we have $D^*(1,u) = (1-a_1)(u^2+1)$. *We can choose* $a_1 = 2$ *and* $a_3 = 0$ *for Figure 15(a);* $a_1 = a_3 = 0$ *for Figure 15(b);* $a_1 = -\frac{1}{l+1}, a_3 = 1$ *for Figure 15(c);* $a_1 = -\frac{1}{l+1}, a_3 = -1$ *for Figure 15(d); and* $a_1 = -2, a_3 = 0$ *for Figure 15(e).*

In the case $l = 0$ there are no new pictures for the phase portrait of system (17) at $x = y = 0$. We get Figure 3(26) from Figure 15(a); Figure 3(7) from Figure 15(b); Figure 3(9) from Figures 15(c) and (d); and Figure 3(8) from Figure 15(e).

7.3.4 The case that $1 - a_1 \neq 0$ and $D^*(1,u) = 0$ has two different roots

Let $v_2^* > v_1^*$ be the two roots of the equation $D^*(1,v) = 0$. Then we have

$$D^*(1,v) = (1-a_1)(v-v_1^*)(v-v_2^*).$$

Now we can completely analyze this case by using the systems (29) and (30). The discussion is divided into 4 cases.

The case $a_1 > 1$. In this case the phase portrait of system (29) on the v-axis has 7 possibilities which are shown in Figure 16. Correspondingly Figure 17 contains all the possibilities for the phase portrait of system (25) on the u-axis which are realized by the following example.

Example 23. *Applying the blowing-up* $u = xv$, $d\tau = xdt$ *to the system*

$$\dot{x} = a_1xu + a_3x^2 + xu^2 + a_4x^3, \qquad \dot{u} = u^2 - (l+1)u^3 + (a_3+a_1-1)xu + a_4x^2u,$$

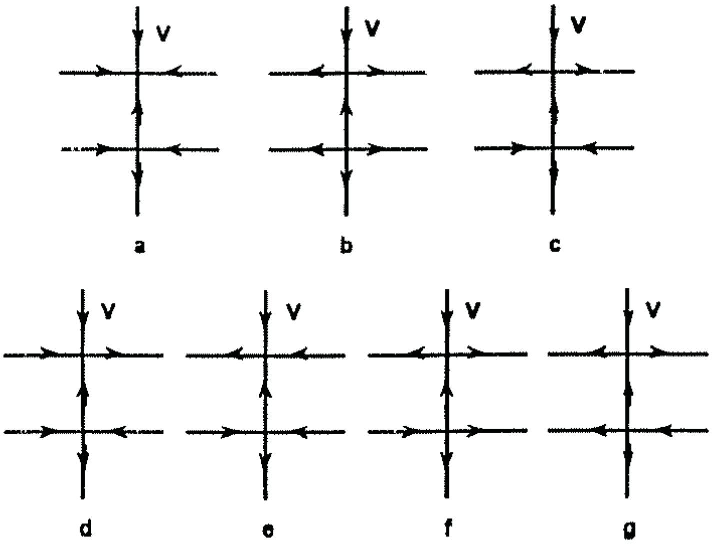

FIG. 16. The phase portrait of system (29) on the v-axis in the case studied in §7.3.4.

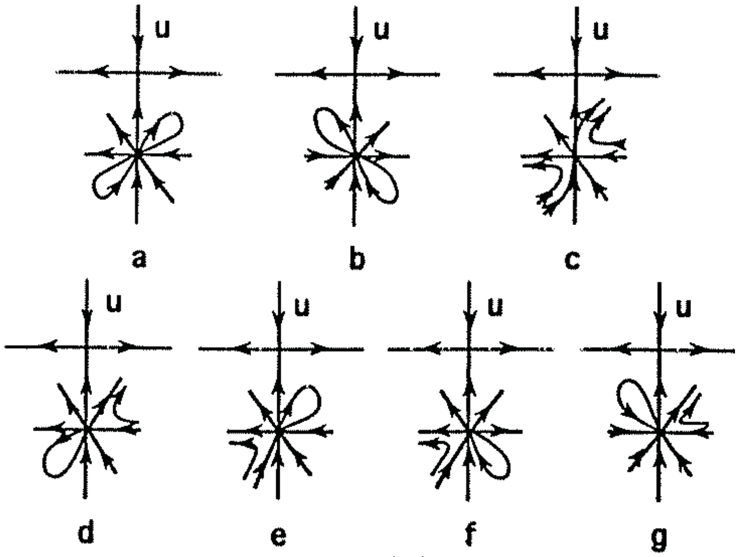

FIG. 17. The phase portrait of system (25) on the u-axis in the case studied in §7.3.4 with $a_1 > 0$.

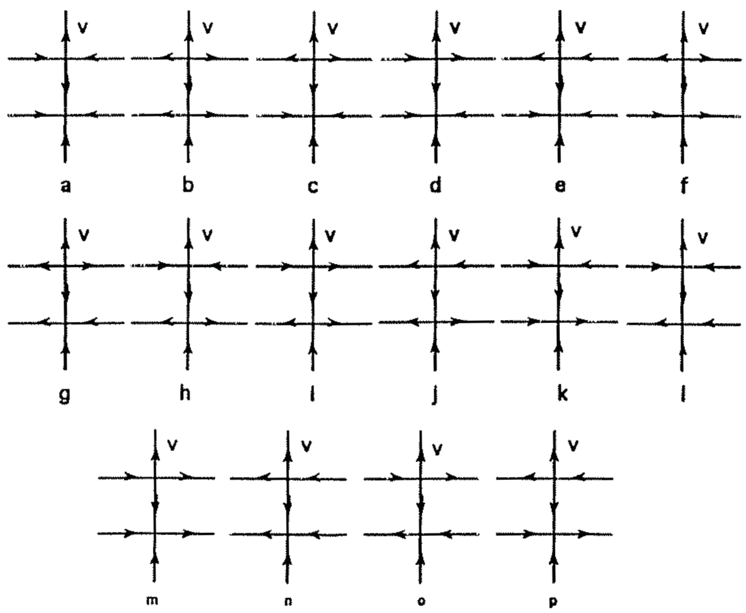

FIG. 18. The phase portrait of system (29) on the v-axis in the case studied in §7.3.4 with $-\frac{1}{l+1} < a_1 < 1$.

we get

$$\dot{x} = x(a_1 v + a_3) + x^2(v^2 + a_4), \qquad \dot{v} = (1 - a_1)v(v - 1) - (l + 2)xv^3.$$

Let $a_1 = 2$*. Then if* $a_3 = -4$*, we obtain Figure 17(a); if* $a_3 = 1$*, we have Figure 17(b); if* $a_3 = -1$*, we get Figure 17(c); if* $a_3 = -2, a_4 = 2l + 4$*, we have Figure 17(d); if* $a_3 = -2, a_4 = 2l$*, we obtain Figure 17(e); if* $a_3 = 0, a_4 = 1$*, we get Figure 17(f); and if* $a_3 = 0, a_4 = -1$*, we have Figure 17(g).*

In the case $l = 0$ we get Figures 3(25),(26) and (27) from Figure 17 for the phase portrait of system (17) at $x = y = 0$. More precisely, we get Figure 3(26) from Figures 17(a) and (b); we get Figure 3(25) from Figure 17(c); and we get Figure 3(27) from Figures 17(d),(e),(f) and (g).

The case $-\frac{1}{l+1} < a_1 < 1$**.** In this case $u = s = 0$ in system (30) is a saddle, and $x = u - \frac{1}{l+1} = 0$ in system (25) is also a saddle. The phase portrait of system (29) on the v-axis has 16 possibilities which are shown in Figure 18 among which (a)–(g) appear in the case $a_1 \geq 0$; whereas (a),(b),(h)–(l) appear in the case $a_1 < 0$; (m)–(p) only appear in the case $a_1 = a_3 = 0$.

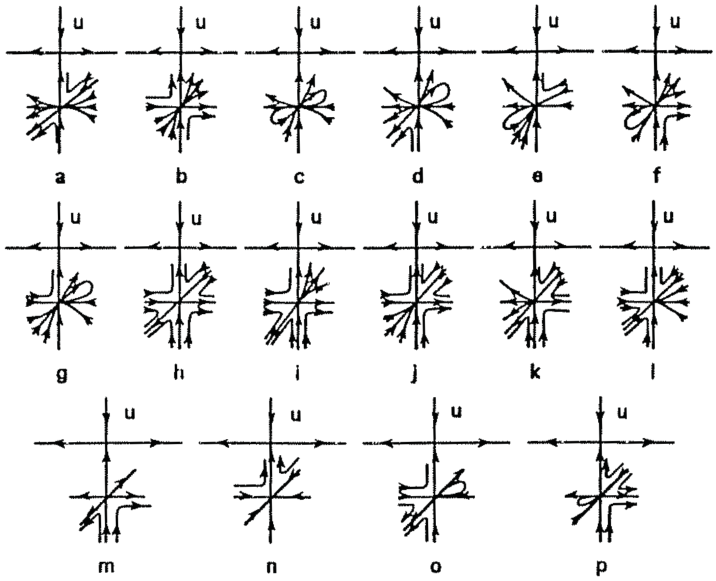

FIG. 19. The phase portrait of system (25) on the u-axis in the case studied in §7.3.4 with $-\frac{1}{l+1} < a_1 < 1$.

Correspondingly we have 16 pictures for the phase portrait of system (25) on the u-axis which are shown in Figure 19.

All the pictures in Figure 19 are realized by Example 23. We can choose $a_1 = 0, a_3 = -1, a_4 = 0$ for Figure 19(a); $a_1 = 0, a_3 = 1, a_4 = 0$ for Figure 19(b); $a_1 = \frac{1}{2}, a_3 = -\frac{1}{4}, a_4 = 0$ for Figure 19(c), $a_1 = \frac{1}{2}, a_3 = -\frac{1}{2}, a_4 = 0$ for Figure 19(d); $a_1 = \frac{1}{2}, a_3 = -\frac{1}{2}, a_4 = -l-4$ for Figure 19(e); $a_1 = \frac{1}{2}, a_3 = 0, a_4 = 1$ for Figure 19(f); $a_1 = \frac{1}{2}, a_3 = 0, a_4 = -1$ for Figure 19(g); $a_1 = -\frac{1}{l+2}, a_3 = \frac{1}{2l+4}, a_4 = 0$ for Figure 19(h); $a_1 = -\frac{1}{l+2}, a_3 = \frac{1}{l+2}, a_4 = 0$ for Figure 19(i); $a_1 = -\frac{1}{l+2}, a_3 = \frac{1}{l+2}, a_4 = -\frac{2}{l+3}$ for Figure 19(j); $a_1 = -\frac{1}{l+2}, a_3 = 0, a_4 = 1$ for Figure 19(k); $a_1 = -\frac{1}{l+2}, a_3 = 0, a_4 = -1$ for Figure 19(l); $a_1 = a_3 = 0, a_4 = 1$ for Figure 19(m); $a_1 = a_3 = 0, a_4 = -2$ for Figure 19(n); $a_1 = a_3 = 0, a_4 = -\frac{1}{2}$ for Figure 19(o). Figure 19(p) is realized by the system

$$\dot{x} = xu^2 + a_4x^3 + a_5x^2u, \qquad \dot{u} = u^2 - (l+1)u^3 - xu + a_4x^2u + a_5xu^2,$$

if we choose $a_4 = 1$ and $a_5 = -3$.

In the case $l = 0$ there are 7 pictures for the phase portrait of system (17) at $x = y = 0$ corresponding to Figure 19, among which we have

Figure 3(28) from Figures 19(a) and (b); Figure 3(26) comes from Figure 19(c); Figure 3(29) corresponds to Figures 19(d),(e),(f),(g); we get Figure 3(30) from Figure 19(h), and Figure 3(31) from Figures 19(i)–(l); we obtain Figure 3(19) from Figures 19(m) and (n), and Figure 3(65) from Figures 19(o) and (p). There are 3 new among these 7 pictures, and all of them can be realized by the system

$$\begin{aligned}\dot{x} &= a_1xy + y^2 + a_3x^3 + a_4x^4 + a_5x^2y,\\ \dot{y} &= (1+a_1)y^2 + (2a_3 + a_1 - 1)x^2y + 2a_4x^3y + 2a_5xy^2,\end{aligned}$$

Since the systems which realize Figure 19 are obtained by applying the blowing-up $y = xu$, $d\tau = xdt$ to it if $l = 0$. So we have Figure 3(30) if $a_1 = -\frac{1}{2}, a_3 = \frac{1}{4}, a_4 = a_5 = 0$; Figure 3(31) if $a_1 = -\frac{1}{2}, a_3 = \frac{1}{2}, a_4 = a_5 = 0$; and Figure 3(65) if $a_1 = a_3 = 0, a_4 = 1, a_5 = -3$.

The case $a_1 = -\frac{1}{l+1}$**.** In this case the point $x = u - \frac{1}{l+1} = 0$ of system (25) can be a saddle, a node or a saddle-node; and the pictures for the phase portrait of system (29) on the v-axis are just Figures 18(a),(b),(h)–(l). Thus the phase portrait of system (25) on the u-axis has been shown in Figure 19 if the point $x = u - \frac{1}{l+1} = 0$ is a saddle. If this point is a node, then the corresponding pictures will appear in Figure 21 below. In the case that $x = u - \frac{1}{l+1} = 0$ is a saddle-node, modulo the symmetry with respect to the u-axis, Figure 20 contains all the possibilities for the phase portrait of system (25) on the u-axis. They are realized by the next example.

Example 24. *Applying the change of variables* $u = xv$, $d\tau = xdt$ *to the system*

$$\begin{aligned}\dot{x} &= -\frac{1}{l+1}xu + a_3x^2 + xu^2 + a_4x^3 + a_5x^2u,\\ \dot{u} &= u^2 - (l+1)u^3 + (a_3 - \frac{l+2}{l+1})xu + a_4x^2u + a_5xu^2,\end{aligned}$$

we obtain

$$\dot{x} = -\frac{1}{l+1}xv + a_3x + x^2[v^2 + a_4 + a_5v], \qquad \dot{v} = \frac{l+2}{l+1}v(v-1) - (l+2)xv^3.$$

To realize the pictures of Figure 20, the values of the parameters should be chosen such that the point $x = u - \frac{1}{l+1} = 0$ *is a saddle-node, and the phase portrait of the last system on the* v*-axis is the corresponding picture in Figure 18. Thus we can choose* $a_3 = -1, a_4 = 0, a_5 = l+3$ *for Figure 20(a);* $a_3 = \frac{2}{l+1}, a_4 = a_5 = 0$ *for Figure 20(b);* $a_3 = \frac{1}{2l+2}, a_4 = 0, a_5 = 1$ *for Figure 20(c);* $a_3 = \frac{1}{l+1}, a_4 = 0, a_5 = 1$ *for Figure 20(d);* $a_3 = \frac{1}{l+1}, a_4 = -2, a_5 = 1$ *for Figure 20(e);* $a_3 = 0, a_5 = 2, a_4 = 1$ *for Figure 20(f); and* $a_3 = a_5 = 0, a_4 = -1$ *for Figure 20(g).*

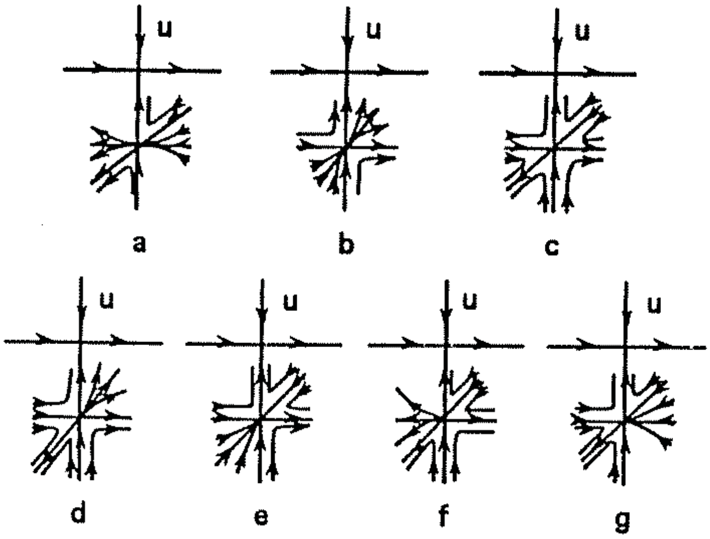

FIG. 20. The phase portrait of system (25) on the u-axis in the case studied in §7.3.4 with $a_1 = -\frac{1}{l+1}$.

In the case $l = 0$ there are 5 pictures for the phase portrait of system (17) at $x = y = 0$. We obtain Figures 3(32), 3(32) and 3(34) from Figures 20(a),(b) and (c) respectively; we get Figure 3(35) from Figures 20(d) and (e), and Figure 3(36) from Figures 20(f) and (g). These 5 pictures can all be realized by the system

$$\dot{x} = -\frac{1}{l+1}xy + y^2 + a_3x^3 + a_4x^4 + a_5x^2y, \;\; \dot{y} = (2a_3-2)x^2y + 2a_4x^3y + 2a_5xy^2,$$

because we can obtain the system of Example 14 by applying the blowing-up $y = xu, \; d\tau = xdt$ to it. The values of the parameters can be chosen as those for the corresponding pictures in Figure 20.

The case $a_1 < -\frac{1}{l+1}$**.** In this case the point $x = u - \frac{1}{l+1} = 0$ of system (25) is a node. The pictures for the phase portrait of system (29) on the v-axis are Figures 18(a),(b),(h)–(l). Correspondingly we obtain 7 pictures for the phase portrait of system (25) on the u-axis, which are shown in Figure 21 and realized by Example 23. Let $a_1 = -2$. Then the values of the other parameters in Example 23 can be chosen as $a_3 = -1, a_4 = 0$ for Figure 21(a); $a_3 = 4, a_4 = 0$ for Figure 21(b); $a_3 = 1, a_4 = 0$ for Figure 21(c); $a_3 = 2, a_4 = \frac{2l+2}{3}$ for Figure 21(d); $a_3 = 2, a_4 = 0$ for Figure 21(e); $a_3 = 0, a_4 = 1$ for Figure 21(f); and $a_3 = 0, a_4 = -1$ for Figure 21(g).

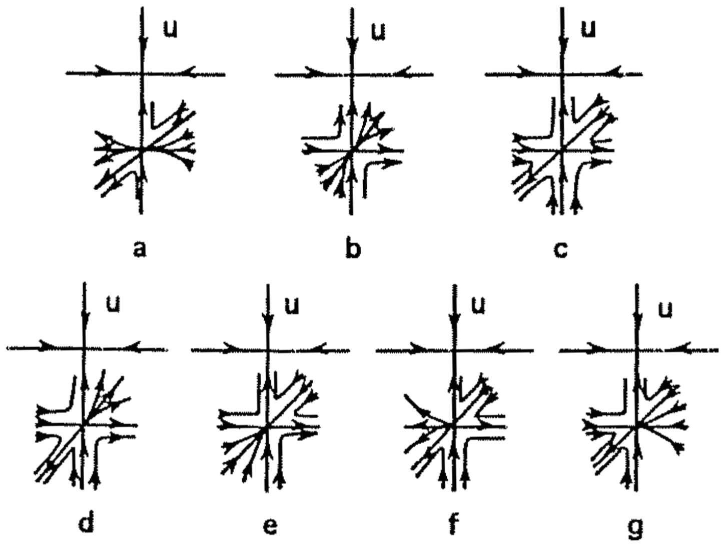

FIG. 21. The phase portrait of system (25) on the u-axis in the case studied in §7.3.4 with $a_1 < -\frac{1}{l+1}$.

In the case $l = 0$ there are 3 pictures for the phase portrait of system (17) at $x = y = 0$ corresponding to Figure 21. We get Figure 3(37) from Figures 21(a) and (b); Figure 3(39) from Figure 21(c); Figure 3(38) from Figures 21(d)–(g). By a reason similar to the above cases, we know that all of them can be realized.

7.4. $1 - a_1 \neq 0$ and $D^*(1, u)$ has a double root

This case is similar to the one that we have seen in §4. We have in general the following blowing-up tree

$$A_1^* \overset{\nearrow B_1^*}{\longrightarrow} \cdots \longrightarrow A_{k-1}^* \overset{\nearrow B_{k-1}^*}{\longrightarrow} A_k^*,$$

where A_1^* is the singular point $x = u = 0$ of system (26), $\deg(A_1^*) = \cdots = \deg(A_{k-1}^*) = 2$, and $B_1^*, \cdots, B_{k-1}^*$ are singular points of degree 1. This tree is uniquely determined by the requirement on the singular point A_k^*. Roughly speaking, A_k^* is a singular point that can be identified with the singular point $x = u = 0$ of system (25) analyzed in §§7.1–7.3.4. To make this requirement precise, we apply Lemma 5 to the present case. As in the proof of Lemma 5, we can inductively find the systems governing the singular points in this tree. Let $u = u_1$. We get the system for A_1^* by replacing u with u_1. For $2 \le i \le k-1$, we can inductively show that A_i^* is the singular point $x = u_i = 0$ of the system

$$\dot{x} = a_1 x u_i + x^2[\cdots], \qquad \dot{u}_i = [1-(i-1)a_1]u_i^2 + x[\cdots],$$

where $[\cdots]$ denotes some analytic function at $x = u_i = 0$. For any $1 \le i \le k-2$, B_i^* is the singular point $v = u_i = 0$ of the system

$$\dot{u}_i = [1-(i-1)a_1]u_i + \cdots, \qquad \dot{v} = (ia_1 - 1)v + \cdots. \tag{31}$$

The points A_{i+1}^* and B_i^* are obtained by applying the following changes of variables to A_i^*

$$u_i = (u_{i+1} + d_i)x,\ d\tau = xdt; \qquad x = vu_i,\ d\tau = u_i dt,$$

where d_i is a real constant such that for any $1 \le i \le k-2$, the characteristic polynomial of the singular point A_i^* can be written as $(1-ia_1)x(u_i - d_i x)^2$ with $1 - ia_1 \ne 0$. Up to now all the statements follow directly from Lemma 5. Now we begin to formulate our requirement. We first need that the characteristic polynomial of A_{k-1}^* can be written as $[1-(k-1)a_1]x(u_{k-1} - d_{k-1}x)^2$ such that $1-(k-1)a_1 \ne 0$. Thus if we apply the above two changes of variables (with $i = k-1$) to A_{k-1}^*, then B_{k-1}^* is still described by system (31) with $i = k-1$, and A_k^* is the singular point $x = u_k = 0$ of the system

$$\dot{x} = a_1 x u_k + \bar{a}_2 x + x^2 P(x, u_k), \qquad \dot{u}_k = [1-(k-1)a_1]u_k^2 + xQ(x, u_k). \tag{32}$$

The difference between (31) and (32) is that $\bar{a}_2$ and $Q(0,0)$ need not be zero. If $\bar{a}_2 = Q(0,0) = 0$, then $\deg(A_k^*) = 2$. In this case, we require that the degrees of the successors are smaller than 2.

Now we begin the investigation of the local phase portrait of A_1^*, which is the unique 2-degree singular point of system (25) on the u-axis. For this system, the u-axis is an invariant line on which the regular orbits can be easily determined from system (26). But for convenience we prefer to consider the following local phase portrait instead of studying A_1^* directly: the orbits on the u-axis are the same as those of A_1^*; the left and right parts are topologically (i.e., need not be qualitatively) the same as the corresponding parts of the local phase portrait of A_1^*. It is easy to check that both local phase portraits will produce the same local phase portrait of system (17) at $x = y = 0$ in the case $l = 0$ (the same is true for the case $l > 0$ in system (25) which will be used in §8). Due to this fact we identify these two local phase portraits below. And both of them are called as the local phase portrait of A_1^*.

Now we apply Lemma 5 to the present case. Then we can easily conclude that the local phase portrait of A_1^* can be obtained from that of A_k^*, and the rule is just the following (see the proof of Lemma 5):

(i) Since the present case is just the case $m = 2$ and $b = 1$ in Lemma 5, we conclude that some B_i^* is a node if and only if a_1 belongs to one of the following intervals: $(1, +\infty), (\frac{1}{2}, 1), (\frac{1}{3}, \frac{1}{2}), \ldots, (\frac{1}{k-1}, \frac{1}{k-2})$. We know also from Lemma 5 that if one B_i^* is an anti-saddle, then it must be a node and all other B_j^*, $1 \le j \le k-1$, $j \ne i$, are saddles.

(ii) From system (32), we know that the local phase portrait of A_k^* is represented by the pictures obtained in §7.3.1 and §7.3.2 if and only if $a_1 = \frac{1}{k}$ and $\bar{a}_2 = Q(0,0) = 0$ in system (32).

(iii) If all the B_i^*, $1 \le i \le k-1$, are saddles, the rule to get the local phase portrait of A_1^* from that of A_k^* is: (1) the invariant u-axis cuts the local phase portrait of A_1^* into two parts, i.e., the right part and the left part; (2) the right part is the same as that of A_k^*; (3) the left part has 2 possibilities, namely, it is the same as the left part of A_k^* if k is odd; if k is even, it is obtained by a reflection across the x-axis of the left part of A_k^* and then reversing the orientation of the orbits.

(iv) If one B_i^* is a node, we draw the local phase portrait of A_1^* on the unit circle (see §2.4). The top and the bottom are two nodes, the stability is determined by the two regular orbits of system (26) on the u-axis. The other 2 singular points on the unit circle are the intersection points of the x-axis with this circle. The local phase portrait of the point on the right semi-circle is the same as the right part of A_k^*. The left one is either the same as the left part of A_k^* (in the case that k is odd), or obtained by a reflection of the left part of A_k^* across the x-axis and then reversing the directions of the orbits (in the case that k is even).

By our assumption that $1-(k-1)a_1 \ne 0$, we apply the time rescaling $d\tau = [1-(k-1)a_1]dt$ to system (32). Then we obtain

$$\begin{aligned} \dot{x} &= \frac{a_1}{1-(k-1)a_1} x u_k + \frac{\bar{a}_2}{1-(k-1)a_1} x + \frac{1}{1-(k-1)a_1} x^2 P(x, u_k), \\ \dot{u}_k &= u_k^2 + \frac{1}{1-(k-1)a_1} x Q(x, u_k). \end{aligned}$$

This system has the same form as system (25) at $(0,0)$. Thus we conclude that the local phase portrait of A_k^* is qualitatively equivalent to the local phase portrait of system (25) at $x = u = 0$ discussed in §§7.1–7.3.4. By the above rule, it follows that the phase portrait of system (25) on the u-axis in the present case can be obtained from those pictures in the previous parts of this section (§§7.1–7.3.4) with a slight change. Therefore all we need to do is to check these pictures, then to list by using the above rule all pictures for the phase portrait of system (25) on the u-axis, and finally to realize them by examples. To do this, we first assume that A_k^* is a singular point of degree 2 (i.e., $Q(0,0) = \bar{a}_2 = 0$ in system (32)). We should check the pictures in §§7.3.1–7.3.4.

If $a_1 = \frac{1}{k}$, then all B_i^* are saddles, and A_k^* is the same as the singular point $x = u = 0$ of system (25) analyzed in §7.3.1 and §7.3.2. The point $x = u - \frac{1}{l+1} = 0$ is also a saddle. If k is odd, then all pictures in Figures 12 and 14 keep unchanged. If k is even, Figure 14(c) is changed to Figure 7.1(a); modulo the symmetry with respect to the u-axis, Figures 14(e) and (f) are interchanged; all other pictures in Figures 12 and 14 keep unchanged.

Corresponding to the situation considered in §7.3.3, the local phase portrait of A_k^* is topologically either a union of 2 elliptic sectors or a union of two hyperbolic sectors. The former appears exactly when $a_1 \in (\frac{1}{k}, \frac{1}{k-1})$ (see system (32)). In this case all B_i^* and the point $x = u - \frac{1}{l+1} = 0$ are saddles. So the phase portrait of system (25) on the u-axis is given by Figure 30(a). When $a_1 > \frac{1}{k-1}$, then some B_i^* is a node, but $x = u - \frac{1}{l+1} = 0$ is a saddle, so we still get Figure 15(a). If $a_1 < \frac{1}{k}$, then the phase portraits of A_1^* and A_k^* are the same since all B_i^* are saddles. The point $x = u - \frac{1}{l+1} = 0$ can be a saddle, a node or a saddle-node. Respectively we get Figures 15(b)–(e) as the phase portrait of system (25) on the u-axis.

Now we consider the situation corresponding to §7.3.4. If $a_1 \in (\frac{1}{k}, \frac{1}{k-1})$, then the pictures for the local phase portrait of A_k^* are just the same as the pictures for the singular point $x = u = 0$ in Figure 17; if $a_1 \in (\frac{1}{k}, -\frac{1}{l+1})$, $a_1 = -\frac{1}{l+1}$, or $a_1 < -\frac{1}{l+1}$, then the phase portrait of A_k^* is represented by the pictures of Figures 19, 20 or 21 respectively (at $x = u = 0$). In these cases, all B_i^* are saddles, $x = u - \frac{1}{l+1} = 0$ in system (25) is the same as in the corresponding figures. So, to obtain the phase portrait of system (25) on the u-axis in each case, it suffices to check the corresponding changes by applying the point (iii) of our rule to the above pictures. More precisely, in Figure 17, all pictures keep unchanged. In Figure 19, if k is even, (j) and (k) are interchanged; modulo the symmetry with respect to the u-axis, (d) and (g) will be changed to (e) and (f); (a) and (b) are interchanged with (n) and (m) respectively; all other pictures keep unchanged. In Figure 20, if k is even, (a) and (b) are changed to Figure 7.1(h) and (c); (e) and (f) are interchanged; all other pictures keep unchanged. In Figure 21, if k is even, (a) and (b) are changed to Figures 7.1 (f) and (b); (e) and (f) are interchanged; all other pictures keep unchanged. In the case that $a_1 > \frac{1}{k-1}$, we take u as u_k in the changes of variables in §7.3.3, and apply these changes of variables to system (32). Then we get two systems corresponding to systems (29) and (30). The phase portrait of the first system on the v-axis has the same possibilities as in Figure 16; corresponding to system (30), we get a saddle. However there is a unique node B_i^*. Thus the pictures we

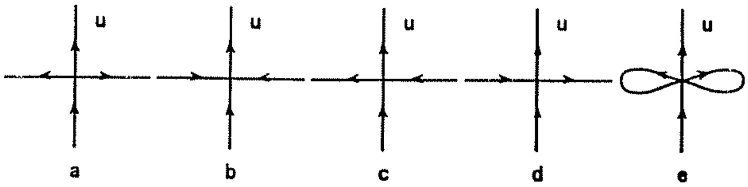

FIG. 22. The local phase portrait of A_1^* when $\bar{a}_2 \neq 0$ in system (32).

get for the phase portrait of system (25) on the u-axis in this case are the same as those in Figure 17.

Now we assume A_k^* is of degree 1.

If $\bar{a}_2 \neq 0$ in system (32), the local phase portrait of A_1^* has 5 possibilities which are shown in Figure 22. Among these 5 pictures, we get Figures 22(a)–(d) if all B_i^* are saddles (i.e., $a_1 < \frac{1}{k-1}$); we always obtain Figures 22(e) if one B_i^* is a node (namely, $a_1 > \frac{1}{k-1}$). In the former case the point $x = u - \frac{1}{l+1}$ can be a saddle, a node or a saddle-node; in the later case, this point must be a saddle. So, corresponding to Figure 22(a), the pictures for the phase portrait of system (25) on the u-axis are Figures 7.1(a)–(d); corresponding to Figure 22(b), the pictures are Figures 7.1(e)–(h); corresponding to Figure 22(c), the pictures are Figures 19(a), 20(a) and 21(a); corresponding to Figure 22(d), the pictures are Figures 19(b), 20(b) and 21(b); corresponding to Figure 22(e), it is Figure 19(c).

Now we assume that $\bar{a}_2 = 0$ but $Q(0,0) \neq 0$. If we compare systems (32) and (50), and take x and u_k in system (32) as y and x in system (50) respectively, we find that (32) is equivalent to (50) with the condition $a_2 = b_3 = 0$. We can assume that $Q(0,0) > 0$ in system (32). By the remark on system (50), we can assume $1 - (k-1)a_1 > 0$. So all B_i^* are saddles. According to the analysis in §10.2, we know that the phase portrait of A_k^* has 5 possibilities which are shown in Figure 32. But Figures 32(a),(b),(d) and (e) apppear only when $a_1 \geq 0$. So in this situation the singular point $x = u - \frac{1}{l+1} = 0$ in system (25) is a saddle. The phase portrait of A_1^* is one of the 5 pictures in Figure 32. Actually, if the phase portrait of A_k^* is Figures 32(a),(b),(c),(d) and (e), then the phase portrait of A_1^* is Figures 32(a),(b),(c),(e) and (d) respectively if k is even; if k is odd, these two phase portraits are the same. So in the case that $\bar{a}_2 = 0$ and $Q(0,0) \neq 0$, we still have Figure 11 as the pictures for the phase portrait of system (25) on the u-axis.

To sum up, we have no new pictures for the phase portrait of system (25) on the u-axis in this subsection in the sense that any picture obtained in the present case corresponds to a picture obtained in the previous part such that the local phase portraits of the point $x = u - \frac{1}{l+1}$ are the same; whereas the local phase portraits of the point $x = u = 0$ are topologically equivalent. As a corollary, it follows that no new pictures for the phase portrait of the point $x = y = 0$ of system (17) will be produced in this case.

8. THE CASE $\Delta(U) = 0$ HAS A TRIPLE ROOT AND $\deg(A_L) = 3$

In this case we have the following tree (see §4)

$$O(0,0) \longrightarrow A_1 \overset{\nearrow B_1}{\longrightarrow} \cdots \longrightarrow A_{l-1} \overset{\nearrow B_{l-1}}{\longrightarrow} A_l,$$

where $\deg(A_1) = \cdots = \deg(A_l) = 3$, $B_1, B_2, \cdots, B_{l-1}$ are all saddles, and A_l is described by system (20). Recall that in system (20), $\tilde{Q}_3(0,y) = -ly^3$, $\tilde{P}_2(0,y) = y^2$, $\bar{\Delta}(u) = \tilde{Q}_3(1,u) - u\tilde{P}_2(1,u)$ is a polynomial of degree 3. Now we apply the following two changes of variables

$$y = xu,\ d\tau = x^2dt; \qquad x = vy,\ d\tau = y^2dt,$$

to system (20), we obtain

$$\begin{aligned} \dot{x} &= x\tilde{P}_2(1,u) + x^2\tilde{P}_3(1,u) + \cdots, \\ \dot{u} &= \bar{\Delta}(u) + x(\tilde{Q}_4(1,u) - u\tilde{P}_3(1,u)) + \cdots, \end{aligned} \tag{33}$$

and

$$\dot{y} = -ly + \cdots, \qquad \dot{v} = (l+1)v + \cdots.$$

By our requirement on A_l, $\bar{\Delta}(u) = 0$ has either a simple real root and 2 complex roots, or 3 different real roots, or a double real root and a simple real root. In the first 2 cases, system (33) is in fact system (24) with $l > 0$. In the third case, it is just system (25) with $l > 0$ (see §8.1 below). So the pictures for the phase portrait of system (33) on the u-axis have been shown in §6 and §7 in all these cases. All we need to do is to get the pictures for the local phase portrait of system (17) at $x = y = 0$ from the pictures in §6 and §7. We do this below.

8.1. The case $\bar{\Delta}(u) = 0$ has a double root

The unique double root can be assumed to be $u = 0$. So $\bar{\Delta}(u) = u^2(-(l+1)u + c)$ for some real constant $c \neq 0$. Without loss of generality, we can take $c = 1$ since c is changed to be 1 after applying the change of variables: $(x, u, t) \mapsto (cx, cu, c^{-2}t)$ to system (33) (notice that such a kind of rescaling is obtained by a rescaling of (x, y, t) in system (17) of the form

$(x, y, t) \mapsto (d_1 x, d_2 y, d_3 t)$, where d_1, d_2 and d_3 are non-zero real constants). Now (33) is system (25) with $l > 0$, which has been studied in §7.

Applying Lemma 4 to the present case, we can easily conclude that the local phase portrait of system (17) at $x = y = 0$ can be obtained from that of system (33) on the u-axis, and the rule is the following:

(i) The phase portrait of system (33) on the u-axis contains two singular points: $x = u = 0$ and $x = u - \frac{1}{l+1} = 0$. The singular point $x = u - \frac{1}{l+1} = 0$ moves along the u-axis in such a way that these 2 singular points meet at $x = u = 0$, and collapse into one singular point which is denoted by A_{l+1}. The invariant u-axis cuts the phase portrait into 2 parts, namely, the left part and the right part.

(ii) The local phase portrait at $x = y = 0$ (of system (17)) on the unit circle (see §2.4) has a pair of singular points which correspond to the unique triple root of $\Delta(u) = 0$.

(iii) The phase portrait at the singular point on the right semi-circle is the same as the right part of A_{l+1}.

(iv) If l is odd, the phase portrait at the singular point on the left semi-circle is the left part of the local phase portrait of A_{l+1} but with an inverse orientation. If l is even, it is obtained from the left part of the local phase portrait of A_{l+1} by reversing the orientation and a reflection across the x-axis.

To prove this rule, we first note that the local phase portrait of A_{l+1} is closely related to the phase portrait of A_l. This relation is described by the theory of blowing-ups in §2. Then this rule follows easily from the rule described in the proof of Lemma 4.

Now all we need to do is to draw new pictures from the phase portrait of system (33) on the u-axis according to the above rule, and then to find examples to realize them. To this end, it is sufficient to consider the figures in §7, since we have listed all possible pictures modulo the symmetry with respect to the u-axis, but this symmetry is precisely the change of variables $(x, u, t) \to (-x, u, t)$ in system (33), which is obtained by a rescaling of the variables x, y and t in system (17). For the realization problem, we note that system (33) is obtained from system (17) by applying a change of variables of the following form

$$y = ux^{l+1} + d_1 x + d_2 x^2 + \cdots + d_{l+1} x^{l+1}, \ d\tau = x^{2l+1} dt.$$

Conversely, this change of variables can also be used to obtain system (17) from system (33). For example, we can take $d_1 = d_2 = \cdots = d_{l+1} = 0$. We see from the above change of variables that

$$\frac{dy}{dt} = x^{l+1}\frac{du}{dt} + x^l \cdot (l+1) u \frac{dx}{dt} = x^{3l+2}\frac{du}{d\tau} + (l+1) u x^{3l+1} \frac{dx}{d\tau}, \quad \frac{dx}{dt} = x^{2l+1}\frac{dx}{d\tau}.$$

The functions $\frac{dx}{d\tau}$ and $\frac{du}{d\tau}$ are given in system (33). Thus system (17) is obtained. Since we have realized all pictures in §7, there are only trivial calculations left for the present realization problem. We omit the details, just point out all the pictures can be realized. This remark is still true for the realization problem in §8.2, and we will not touch it there.

Now we start to check the pictures in §7. First we assume that l is even. By the above rule, we obtain 22 new pictures (Figures 3(40)–3(61)). The details are given in the rest of this paragraph: we get Figures 3(40), (41), (42), (42), (43), (8), (44), (44) from Figures 10(a)–(h) respectively; from the pictures in Figure 11, we get Figures 3(51), (48), (45), (49), (46), (44), (45), (47); we obtain Figures 3(50), (41), (51) from Figure 12; corresponding to the pictures in Figure 14, we obtain Figures 3(7), (41), (52), (41), (53), (42); from Figure 15, we obtain Figures 3(41), (7), (9), (9), (8); from Figure 17, we get Figures 3(41), (41), (50), (51), (51), (51), (51); from Figure 19, we have Figures 3(52), (52), (41), (42), (53), (42), (53), (54), (55), (56), (55), (56), (40), (43), (59), (59); from Figure 20, we obtain Figures 3(58), (57), (56), (52), (47), (59), (60); and from Figure 21, we get Figures 3(61), (61), (43), (53), (44), (53), (44).

Now assume that l is odd. We get three new pictures (Figures 3(62), (63), (64)). The details are the following. From Figure 10, we get Figures 3(40), (62), (42), (42), (60), (8), (44), (44); from Figure 11, we get Figures 3(64), (48), (45), (49), (46), (44), (45), (47); from Figure 12, we have Figures 3(63), (62), (64); from Figure 14, we get Figures 3(7), (62), (52), (62), (58), (42); from Figure 15, we obtain Figures 3(62), (7), (9), (9), (8); from Figure 17, we have Figures 3(62), (62), (63), (64) ,(64), (64), (64); from Figure 19, we get Figures 3(52), (52), (62), (42), (58), (42), (58), (54), (55), (56), (55), (56), (40), (60), (59), (59); from Figure 20, we obtain Figures 3(53), (57), (56), (52), (47), (59), (43); from Figure 21, we have Figures 3(61), (61), (60), (58), (44), (58), (44).

8.2. The case $\bar{\Delta}(\boldsymbol{u}) = 0$ has only simple roots

In this case we have a similar rule to obtain the phase portrait of system (17) at $x = y = 0$ from the phase portrait of system (33) on the u-axis. The only change is the point (i). Here all singular points on the u-axis accumulate at one point, and form a new singular point A_{l+1}. By using this rule and the results in §6, we can get all pictures for the phase portrait of system (17) at $x = y = 0$. The details are given below.

If l is even, we get Figures 3(7), (8), (9), (9) from Figure 6; Figures 3(40), (54), (43), (41), (41), (43), (8) from Figure 7; Figures 3(42), (44), (56), (55), (53), (53), (42), (56), (44) from Figure 8; and Figures 3(52), (59), (47), (60), (52), (59) from Figure 9.

Assume l is odd. We obtain Figures 3(7), (8), (9), (9) from Figure 6; Figures 3(40), (54), (60), (62), (62), (60), (8) from Figure 7; Figures 3(42), (44), (56), (55), (58), (58), (42), (56), (44) from Figure 8; Figures 3(52), (59), (47), (43), (52), (59) from Figure 9.

To sum up, no new pictures for the local phase portrait of system (17) at $x = y = 0$ appear in this subsection.

9. THE CASE $\Delta(U)$ HAS A TRIPLE ROOT AND $\deg(A_L) \leq 2$

Let us go back to §4. We know that A_{l-1} is described by system (18), and A_l is just the singular point $x = u = 0$ of system (19). Since $\deg(A_l) \leq 2$, system (19) can be rewritten as

$$\begin{aligned} \dot{x} &= x[u^2 + \tilde{a}_1 u + \tilde{a}_2] + x^2[\tilde{a}_3 + \cdots], \\ \dot{u} &= -lu^3 + x[\tilde{b}_0 + \tilde{b}_1 u + \tilde{b}_2 x + \cdots]. \end{aligned} \tag{34}$$

As in §8 we can obtain the local phase portrait of system (17) at $x = y = 0$ directly from the local phase portrait of A_l. To do this, we draw the phase portrait of system (17) at $x = y = 0$ on the unit circle. The rule is the following, which follows immediately from Lemma 4 and its proof.

(i) The phase portrait of system (17) at $x = y = 0$ on the unit circle has a pair of singular points which correspond to the unique triple root of $\Delta(u) = 0$ (see Lemma 1).

(ii) The local phase portrait at the singular point on the right semi-circle is the same as the right part of A_l.

(iii) If l is even, the local phase portrait at the singular point on the left semi-circle is the left part of the local phase portrait of A_l but with inverse orientation. If l is odd, it is obtained from the left part of the local phase portrait of A_l by a reflection across the x-axis and then reversing the orientation of orbits.

In the following we will analyze the local phase portrait of system (34) at $x = u = 0$, and then find the corresponding pictures for the local phase portrait of system (17) at $x = y = 0$ by using the above rule, and finally find examples to realize those new local phase portraits.

9.1. The case $\deg(A_l) = 1$

By assumption, $\tilde{a}_2^2 + \tilde{b}_0^2 \neq 0$. If $\tilde{a}_2 \neq 0$, then $x = u = 0$ is either a saddle or a node (Figure 6(a) or (b) respectively). Correspondingly, the local phase portrait of system (17) at $x = y = 0$ is Figure 3(7) or 3(8). Now we assume $\tilde{a}_2 = 0$. Then $\tilde{b}_0 \neq 0$. With the help of the transformation $(x, u, t) \to (-x, u, t)$, we can even assume $\tilde{b}_0 > 0$ in system (34). Now it is easy to check that $x = u = 0$ is a saddle-node if $\tilde{a}_1 \neq 0$; and a saddle if $\tilde{a}_1 = 0$. The corresponding pictures for the local phase portrait of system (17) at $x = y = 0$ are Figures 3(9) and 3(7).

9.2. The case $\deg(\boldsymbol{A}_l) = 2$

In this case $\tilde{a}_2 = \tilde{b}_0 = 0$. We rewrite system (34) as

$$\begin{aligned} \dot{x} &= \tilde{a}_1 xu + \tilde{a}_3 x^2 + xP_2'(x,u) + xP_3'(x,u) + \cdots, \\ \dot{u} &= \tilde{b}_1 xu + \tilde{b}_2 x^2 + Q_3'(x,u) + Q_4'(x,u) + \cdots. \end{aligned} \tag{35}$$

Where, as before, $P_i'(x,u)$ and $Q_i'(x,u)$ are homogeneous of degree i, moreover, $P_2'(0,u) = u^2$, $Q_3'(0,u) = -lu^3$, and $P_i'(0,u) = Q_{i+1}'(0,u) = 0$ for any $i > 2$. Let

$$\Delta'(x,u) = x[\tilde{b}_1 xu + \tilde{b}_2 x^2] - u[\tilde{a}_1 xu + \tilde{a}_3 x^2] = x[\tilde{b}_2 x^2 + (\tilde{b}_1 - \tilde{a}_3)xu - \tilde{a}_1 u^2].$$

Applying the following changes of variables

$$x = us,\ d\tau = udt; \qquad u = xv,\ d\tau = xdt,$$

to system (35), we obtain

$$\begin{aligned} \dot{u} &= \tilde{b}_1 us + \tilde{b}_2 us^2 + u^2(-l + \cdots), \\ \dot{s} &= \tilde{a}_1 s + (\tilde{a}_3 - \tilde{b}_1)s^2 - \tilde{b}_2 s^3 + us[(l+1) + \cdots] \end{aligned} \tag{36}$$

and

$$\dot{x} = x(\tilde{a}_1 v + \tilde{a}_3) + x^2(\cdots), \qquad \dot{v} = -\tilde{a}_1 v^2 + (\tilde{b}_1 - \tilde{a}_3)v + \tilde{b}_2 + x(\cdots), \tag{37}$$

where $(\cdots)$ denotes some analytic function. The local phase portrait of system (35) at $x = u = 0$ is obtained by studying systems (36) and (37). We divide our discussion into 4 cases according to the form of Δ'.

9.2.1 The case $\tilde{a}_1 \neq 0$

Without loss of generality we can assume $\tilde{a}_1 > 0$ (see the remark on the constant c in the first paragraph of §8.1). From system (36), we know that $u = s = 0$ is a saddle-node. The phase portrait of system (37) on the v-axis is discussed according to the roots of $\Delta'(1,v) = -\tilde{a}_1 v^2 + (\tilde{b}_1 - \tilde{a}_3)v + \tilde{b}_2 = 0$.

If $\Delta'(1,v) = 0$ has no real roots, then $x = u = 0$ of system (35) is a node. Correspondingly the phase portrait of system (17) at $x = y = 0$ is Figure 3(8).

If $\Delta'(1,v) = 0$ has two different real roots, the possibilities for the phase portrait of system (37) on the v-axis are shown in Figure 23. Correspondingly, we have the pictures in Figure 24 as the local phase portrait of system (35) at $x = u = 0$.

By using the above rule, we can obtain the corresponding pictures for the local phase portrait of system (17) at $x = y = 0$ from Figure 24. If l is

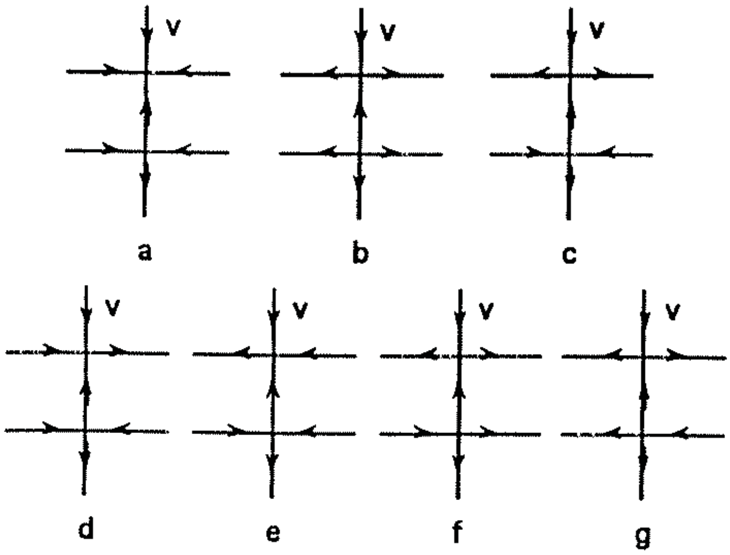

FIG. 23. The phase portrait of system (37) on the v-axis in §9.2.1.

even, we get Figures 3(61), (61), (60), (58), (44), (58), (44) respectively. If l is odd, we obtain Figures 3(61), (61), (43), (53), (44), (53), (44).

If $\Delta'(1,v)=0$ has a double root which is not equal to $-\frac{\tilde{a}_3}{\tilde{a}_1}$, then system (37) has a unique singular point on the v-axis, where the eigenvalue of the linear part of system (37) in the x-direction is non-zero. So the local phase portrait of system (35) at $x=u=0$ is topologically either Figure 24(a) or 24(b). Consequently the local phase portrait of system (17) at $x=y=0$ is Figure 3(61).

Now we assume that the double root of $\Delta'(1,v)=0$ is equal to $-\frac{\tilde{a}_3}{\tilde{a}_1}$. Moving this double root to $v=0$, then we can assume $\tilde{b}_1=\tilde{b}_2=\tilde{a}_3=0$. If the degree of the singular point $x=v=0$ is 1, it is easy to check that it is a saddle. Correspondingly, the local phase portrait of system (17) at $x=y=0$ is Figure 3(44). If the singular point $x=v=0$ of system (37) is of degree 2, then it can be studied in a way similar to §7.3.5. Denote this singular point by A'_{l+1}. We have in general the following subtree

$$A'_{l+1} \overset{\nearrow B_{l+1}}{\longrightarrow} A'_{l+2} \overset{\nearrow B_{l+2}}{\longrightarrow} \cdots \longrightarrow A'_{l+l_0} \overset{\nearrow B_{l+l_0}}{\longrightarrow} A'_{l+l_0+1}.$$

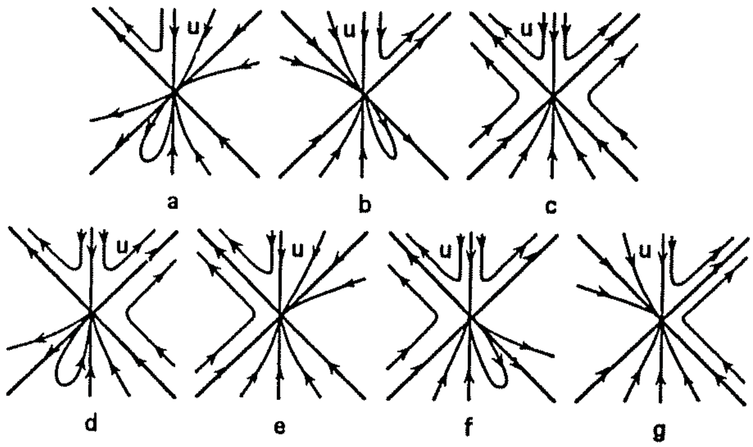

FIG. 24. The phase portrait of system (35) at $x = u = 0$ in §9.2.1.

Here $\deg(A'_{l+i}) = 2$ and B_{l+i} is a saddle for $1 \leq i \leq l_0$. The singular point A'_{l+l_0+1} can be taken as the singular point $x = y = 0$ of the system

$$\begin{aligned} \dot{x} &= (\tilde{a}_1 y + \hat{a}_2)x + x^2(\hat{a}_3 + \cdots), \\ \dot{y} &= -(l_0+1)\tilde{a}_1 y^2 + x(\hat{b}_1 + \hat{b}_2 x + \hat{b}_3 y + \cdots). \end{aligned} \tag{38}$$

If $\hat{a}_2^2 + \hat{b}_1^2 \neq 0$, then A'_{l+l_0+1} is a singular point of degree 1. If $\hat{a}_2 = \hat{b}_1 = 0$, then A'_{l+l_0+1} is a singular point of degree 2. In the second case we apply the following changes of variables

$$y = xu, \ d\tau = xdt; \qquad x = yv, \ d\tau = ydt,$$

to system (38), we have

$$\begin{aligned} \dot{x} &= (\tilde{a}_1 u + \hat{a}_3)x + x^2(\cdots), \\ \dot{u} &= -(l_0+2)\tilde{a}_1 u^2 + (\hat{b}_3 - \hat{a}_3)u + \hat{b}_2 + x[\cdots], \end{aligned} \tag{39}$$

and

$$\dot{y} = -(l_0+1)\tilde{a}_1 y + \ldots, \qquad \dot{v} = (l_0+2)\tilde{a}_1 v + \cdots . \tag{40}$$

The requirement for the above subtree is that $-(l_0+2)\tilde{a}_1 u^2 + (\hat{b}_3 - \hat{a}_3)u + \hat{b}_2 = 0$ has two different (real or complex) roots. Using an argument similar to that in §4, we can easily show that the above cases cover all the possibilities for the singular point A'_{l+1}.

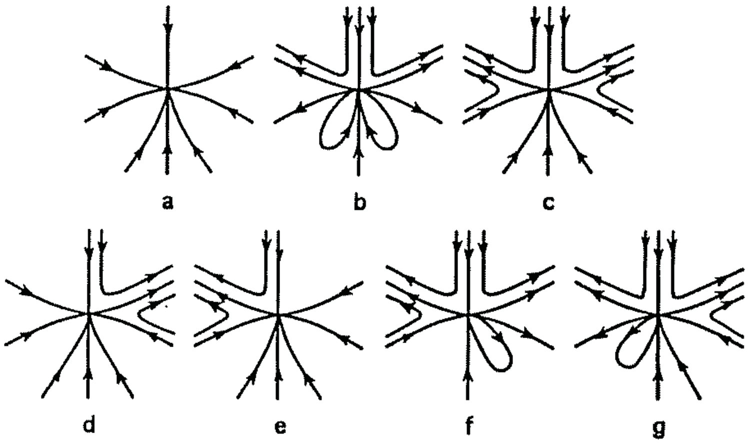

FIG. 25. The phase portrait of A_l in §9.2.1.

The local phase portrait of A'_{l+l_0+1} can always be determined by systems (38), (39) and (40). To obtain the local phase portraits of A_l and the point $x = y = 0$ of system (17), we use Lemma 5 as in some previous cases. We draw the local phase portrait of A_l on the unit circle. Then the rule is as follows.

(i) The top is a saddle, the bottom is a node. The orientation of orbits at them is determined by system (36).

(ii) The other two singular points are the intersection points of the circle with the x-axis. Their local phase portraits are obtained from the right and the left parts of the local phase portrait of A'_{l+l_0+1} which are obtained by cutting this phase portrait along the invariant y-axis.

(iii) The local phase portrait of the singular point on the right semi-circle is just the right part of the the local phase portrait of A'_{l+l_0+1}.

(iv) If l_0 is odd, then the one on the left semi-circle is the same as the left part of the local phase portrait of A'_{l+l_0+1}. If l_0 is even, it is obtained from the left part of the local phase portrait at A'_{l+l_0+1} by a reflection across the x-axis and then reversing the orientation of orbits.

In the case that $\hat{a}_2 = 0$ but $\hat{b}_1 \neq 0$, the singular point $x = y = 0$ of system (38) is a saddle. So A'_{l+1} is also a saddle, and Figure 3(44) is the local phase portrait of system (17) at $x = y = 0$. In the case that $\hat{a}_2 = \hat{b}_1 = 0$ and $-(l_0 + 2)\tilde{a}_1 u^2 + (\hat{b}_3 - \hat{a}_3)u + \hat{b}_2 = 0$ has no real roots, there are no singular points on the u-axis in system (39). So A_l is a node. Correspondingly the local phase portrait of system (17) at $x = y = 0$ is Figure 3(8). The case $\hat{a}_2 \neq 0$ is equivalent to (i.e., it will produce the same

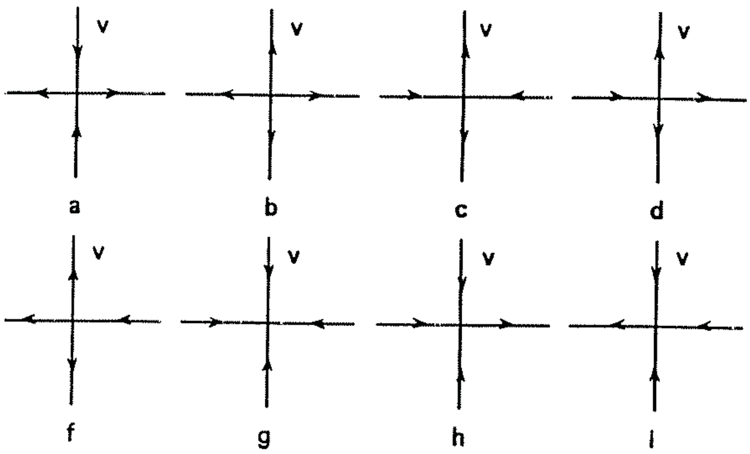

FIG. 26. The local phase portrait of system (37) at $x = v + \frac{\tilde{b}_2}{\tilde{b}_1 - \tilde{a}_3} = 0$ in §9.2.2.

pictures for the local phase portrait of A_l as) the case in which $\hat{a}_2 = \hat{b}_1 = 0$ and $-(l_0+2)\tilde{a}_1 u^2 + (\hat{b}_3 - \hat{a}_3)u + \hat{b}_2 = 0$ has 2 different real roots, and the eigenvalues in the characteristic direction different from the u-axis at the 2 singular points of system (39) on the u-axis have the same sign. Therefore in the following it suffices to consider the case that $\hat{a}_2 = \hat{b}_1 = 0$ and $-(l_0+2)\tilde{a}_1 u^2 + (\hat{b}_3 - \hat{a}_3)u + \hat{b}_2 = 0$ has 2 different real roots.

The phase portrait of system (39) on the u-axis has the same possibilities as the phase portrait of system (37) on the v-axis, which have been shown in Figure 23. If l_0 is odd, then the corresponding pictures for the local phase portrait of A_l are shown in Figure 24. If l_0 is even, the corresponding pictures are those of Figure 25.

Corresponding to Figure 25, we obtain Figures 3(8), (62), (60), (44), (44), (58), (58) for the local phase portrait of system (17) at $x = y = 0$ if l is even; if l is odd, then the corresponding pictures are Figures 3(8), (41), (43), (44), (44), (53), (53).

To sum up, we have shown that if $\tilde{a}_1 \neq 0$, there are no new pictures for the local phase portrait of system (17) at $x = y = 0$.

9.2.2 The case that $\tilde{a}_1 = 0$ and $\tilde{b}_1 \neq \tilde{a}_3$

In this case and the following two subsections (§§9.2.2–9.2.4), we can assume $\tilde{a}_3 \geq 0$ in system (35) with the help of the transformation $(x, u, t) \to (-x, u, t)$. The singular point $x = v - \frac{\tilde{b}_2}{\tilde{b}_1 - \tilde{a}_3} = 0$ in system (37) is of degree 1. Its phase portrait has 8 possibilities as shown in Figure 26. The condition on the parameters $\tilde{b}_1$ and $\tilde{a}_3$ for each picture of Figure 26 is as follows: $\tilde{b}_1 < \tilde{a}_3$ and $\tilde{a}_3 \geq 0$ for Figure 26(a); $\tilde{b}_1 > \tilde{a}_3 \geq 0$ for Figure 26(b); $\tilde{b}_1 > \tilde{a}_3 = 0$ for Figures 26(c), (d) and (e); $\tilde{b}_1 < \tilde{a}_3 = 0$ for Figures 26(f), (g) and (h).

To study the singular point $u = s = 0$, we apply the following two changes of variables

$$u = su_1, \ d\tau = sdt, \qquad s = us_1, \ d\tau = udt,$$

to system (36), we have

$$\begin{aligned} \dot{s} &= (\tilde{a}_3 - \tilde{b}_1)s + (l+1)su_1 + s^2(-\tilde{b}_2 + \cdots), \\ \dot{u}_1 &= (2\tilde{b}_1 - \tilde{a}_3)u_1 - (2l+1)u_1^2 + u_1 s(2\tilde{b}_2 + \cdots), \end{aligned} \tag{41}$$

and

$$\dot{u} = \tilde{b}_1 us_1 - lu + u^2[\cdots], \qquad \dot{s}_1 = (2l+1)s_1 + (\tilde{a}_3 - 2\tilde{b}_1)s_1^2 + u[\cdots], \tag{42}$$

where $[\cdots]$ is some analytic function. The singular point $u = s_1 = 0$ is a saddle. The phase portrait of system (41) on the u_1-axis is determined by the values of the parameters. It is easy to know that there are two singular points on the u_1-axis, namely, $u_1 = s = 0$ and $u_1 - \frac{2\tilde{b}_1 - \tilde{a}_3}{2l+1} = s = 0$. At $u_1 = s = 0$, the two eigenvalues are $\tilde{a}_3 - \tilde{b}_1$ and $2\tilde{b}_1 - \tilde{a}_3$. At $u_1 - \frac{2\tilde{b}_1 - \tilde{a}_3}{2l+1} = s = 0$, the eigenvalue in the direction of the u_1-axis is $\tilde{a}_3 - 2\tilde{b}_1$, the other eigenvalue is $\frac{l\tilde{a}_3 + \tilde{b}_1}{2l+1}$. So the pictures of Figure 27 cover all the possibilities for the local phase portrait of system (36) at $u = s = 0$. The corresponding condition on the parameters is listed below: $\tilde{a}_3 > 2\tilde{b}_1$ and $\tilde{b}_1 + l\tilde{a}_3 \geq 0$ for Figure 27(a); $\tilde{a}_3 > 2\tilde{b}_1$ and $\tilde{b}_1 + l\tilde{a}_3 \leq 0$ for Figure 27(b); $\tilde{a}_3 > 2\tilde{b}_1$ and $\tilde{b}_1 + l\tilde{a}_3 = 0$ for Figure 27(c) and (d); $\tilde{a}_3 < \tilde{b}_1 < 2\tilde{b}_1$ for Figure 27(e); $\tilde{b}_1 < \tilde{a}_3 < 2\tilde{b}_1$ for Figure 27(f); $\tilde{a}_3 = 2\tilde{b}_1$ for Figure 27(g). In Figures 27(g) and (a)–(d), we have $\tilde{a}_3 > \tilde{b}_1$ because $\tilde{a}_3 \geq 2\tilde{b}_1$ and $\tilde{a}_3 \geq 0$.

Corresponding to the pictures in Figure 27, we obtain 12 pictures (Figure 28) for the local phase portrait of system (35) at $x = u = 0$. Among these pictures, if $\tilde{a}_3 > 2\tilde{b}_1$ and $\tilde{b}_1 + l\tilde{a}_3 > 0$, we get Figure 28(a); if $\tilde{a}_3 > 0 > \tilde{b}_1$ and $\tilde{b}_1 + l\tilde{a}_3 < 0$, then we get Figure 28(b); if $\tilde{a}_3 = 0 > \tilde{b}_1$, we have Figures 28(b),(c),(d),(e); if $\tilde{a}_3 = -\frac{\tilde{b}_1}{l} > 0$, then we have Figures 28(f),(g),(a),(b); if $\tilde{b}_1 > \tilde{a}_3 = 0$, we have Figures 28(i),(j),(k),(l); if $0 < \tilde{a}_3 < \tilde{b}_1$, we get Figure 28(j); if $0 < \tilde{b}_1 < \tilde{a}_3 < 2\tilde{b}_1$, we have Figure 28(h); and if $\tilde{a}_3 = 2\tilde{b}_1 > 0$, we get Figure 28(a). The above analysis shows all the possibilities since $\tilde{a}_3 \geq 0$.

Corresponding to Figure 28, we still have no new pictures for the phase portrait of system (17) at $x = y = 0$. Actually, if l is even, we get Figures

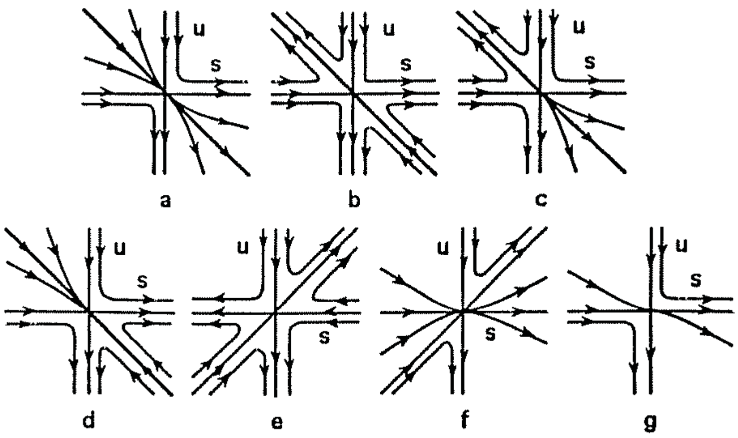

FIG. 27. The local phase portrait of system (36) at $u = s = 0$ in §9.2.2.

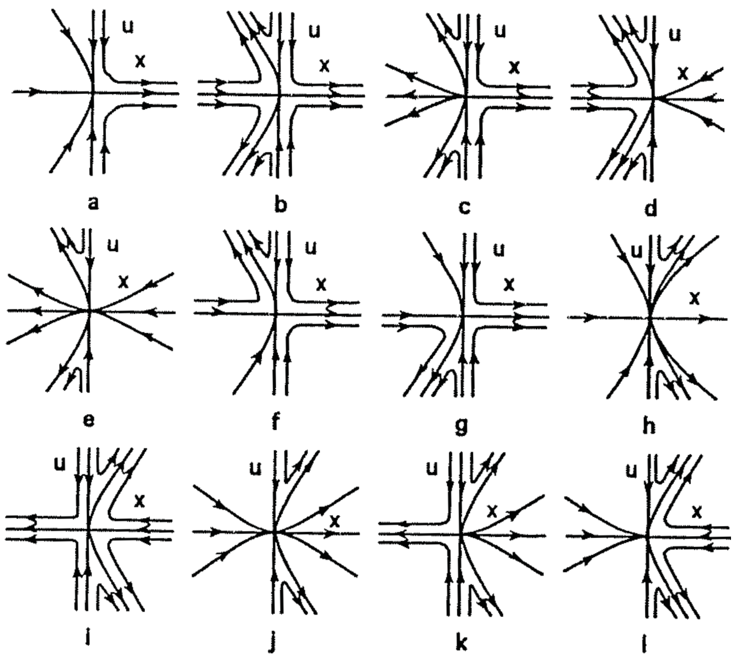

FIG. 28. The local phase portrait of system (35) at $x = u = 0$ in §9.2.2.

3(9), (46), (49), (47), (57), (45), (45), (57), (46), (57), (49), (47). If l is odd, we get the same.

9.2.3 The case $\tilde{a}_1 = \tilde{b}_1 - \tilde{a}_3 = 0,\ \tilde{b}_2 \neq 0$

In this case there are no singular points on the v-axis in system (37). So, in order to analyze the singular point $x = u = 0$ of system (35), it is sufficient to analyze $u = s = 0$ in system (36). We first assume $2\tilde{b}_1 - \tilde{a}_3 \neq 0$ (this is equivalent to $\tilde{a}_3 = \tilde{b}_1 \neq 0$). Since $\tilde{a}_3 \geq 0$ (see §9.2.2), it is easy to see from systems (41) and (42) that the local phase portrait of system (36) at $u = s = 0$ is Figure 29(a) or (b) according as $\tilde{b}_2 < 0$ or $\tilde{b}_2 > 0$. If $\tilde{a}_3 = \tilde{b}_1 = 0$, we apply the following changes of variables

$$u_1 = su_2,\ d\tau = sdt; \qquad s = u_1 s_2,\ d\tau = u_1 dt,$$

to system (41) and obtain

$$\dot{s} = s[(l+1)u_2 - \tilde{b}_2] + s^2[\cdots], \qquad \dot{u}_2 = 3\tilde{b}_2 u_2 - (3l+2)u_2^2 + su_2[\cdots],$$

and

$$\dot{u}_1 = u_1[2\tilde{b}_2 s_2 - (2l+1)] + u_1^2[\cdots], \qquad \dot{s}_2 = (3l+2)s_2 - 3\tilde{b}_2 s_2^2 + u_1[\cdots],$$

where $[\cdots]$ denotes some analytic function. From these two system, we get Figures 29(c) (if $\tilde{b}_1 > 0$) and 29(d) (if $\tilde{b}_2 < 0$) as the local phase portrait of system (41) at $u_1 = s = 0$. Correspondingly, we get Figures 29(e) and (f) for the local phase portrait of system (35) at $x = u = 0$. By using our rule, we get Figure 3(57) for the phase portrait of system (17) at $x = y = 0$ from Figures 29(a) and (b). From Figures 29(e) and (f), we get Figure 3(7) for the local phase portrait of system (17) at $x = y = 0$.

9.2.4 The case $\tilde{a}_1 = \tilde{b}_1 - \tilde{a}_3 = \tilde{b}_2 = 0$

We must have $\tilde{a}_3 = \tilde{b}_1 \neq 0$. As before, we can assume $\tilde{a}_3 > 0$. Applying the transformation $x = us,\ d\tau = u^2 dt$ to system (35), we have

$$\begin{aligned} \dot{u} &= \tilde{b}_1 s + uQ_3'(s,1) + u^2 Q_4'(s,1) + \cdots = \tilde{b}_1 s - lu + \cdots, \\ \dot{s} &= s[P_2'(s,1) - Q_3'(s,1)] + su[P_4'(s,1)) - Q_4'(s,1)] + \cdots = (l+1)s + \cdots. \end{aligned}$$

Evidently $u = s = 0$ is a saddle. Hence the local phase portrait of system (35) at $x = u = 0$ is topologically Figure 28(h). Consequently the local phase portrait of system (17) at $x = y = 0$ is Figure 3(57).

To sum up, we know from the above analysis that there are no new pictures for the local phase portrait of system (17) at $x = y = 0$ in this section.

Now the analysis of the singular point $x = y = 0$ of system (17) is finished. The pictures for its local phase portrait are exactly those in Figure 3. Thus the proof of Theorem D is finished.

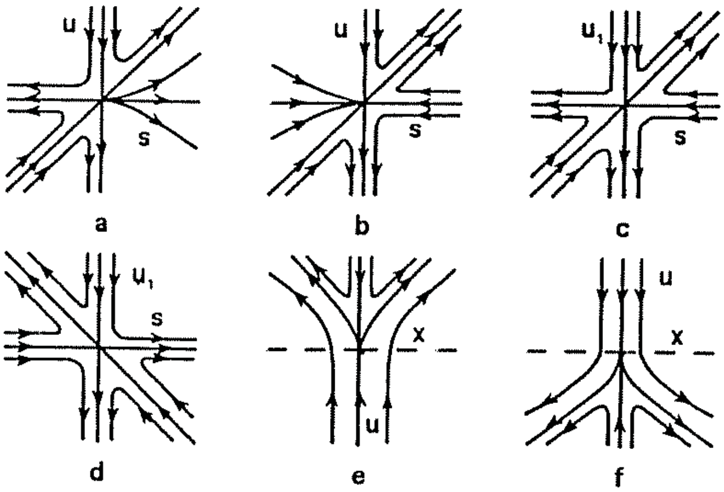

FIG. 29. The local phase portraits obtained in §9.2.3.

10. APPENDIX: THREE FACTS USED IN THE MAIN TEXT

This section contains three facts used in the main body of this paper. In §10.1, we prove two lemmas which are used in §1. In §10.2, we study a special kind of 1-degree singular points. The material here is used in §7.2 and in the discussion of the case that $\bar{a}_2 = 0$ and $Q(0,0) \neq 0$ in §7.3.5.

10.1. Two lemmas used in §1

LEMMA 25. *C^1-equivalence is strictly finer than qualitative equivalence.*

Proof. It suffices to find two singular points which are qualitatively equivalent, but not C^1-equivalent. To do this, we consider the following two systems

$$\dot{x} = -dx^3 + ex^2y + (c-1)xy^2, \quad \dot{y} = cy^3 + exy^2 - dx^2y, \tag{43}$$

and

$$\dot{\bar{x}} = \bar{x}, \qquad \dot{\bar{y}} = -\bar{y}, \tag{44}$$

where $d > 0, c > 0$ and e are real constants. Let p_1 and p_2 denote the singular points $x = y = 0$ and $\bar{x} = \bar{y} = 0$ respectively. It is easy to check that these two singular points are saddles. Their separatrices coincide with the coordinate axes. So they are qualitatively equivalent, and this lemma is proved by showing that they are not C^1-equivalent for some suitable values of the parameters d, c and e.

We first observe that if p_1 and p_2 are C^1-equivalent through the diffeomorphism $\Psi : (x, y) \mapsto (\bar{x}, \bar{y})$, and Γ is any smooth closed curve around p_1, then the number of contact points of system (43) with Γ is equal to the number of contact points of system (44) with $\Psi(\Gamma)$. Actually Ψ maps contact points to contact points.

Consider the circle $\Gamma_r : x^2 + y^2 = r^2$. The contact points of system (43) with Γ_r are just the intersection points of Γ_r with the curve

$$x \cdot [-dx^3 + ex^2y + (c-1)xy^2] + y \cdot [cy^3 + exy^2 - dx^2y] = 0. \tag{45}$$

Let $y = xu$, then (45) is changed to

$$cu^4 + eu^3 + (c-d-1)u^2 + eu - d = 0. \tag{46}$$

Evidently the number of contact points of system (43) with Γ_r is not less than twice the number of different real solutions of equation (46).

Let

$$e = \frac{1}{2} + d - c. \tag{47}$$

Then $u = 1$ is a root of (46). Inserting (47) to (46), we have

$$cu(u^2+1)(u-1) + d(u^2+1)(u-1) + \frac{1}{2}u(u-1)^2 = 0. \tag{48}$$

From (48), we know that for some values of $c > 0$ and $d > 0$, (46) has a root $u = u_0$ such that $0 < u_0 < \frac{1}{2}$ (e.g., one can choose $d > 0, c > 0$ sufficiently small such that the values of the polynomial in (48) at $u = 0$ and $u = \frac{1}{2}$ have different signs). Since $c \cdot d > 0$, (46) has at least one negative root. Therefore we have shown that for some values of the parameters $c > 0, d > 0$ and e, system (43) has at least six contact points with Γ_r. In the following, we prove that p_1 and p_2 cannot be C^1-equivalent in this case.

Assume this is not the case, the map $\Psi : (x, y) \mapsto (\bar{x}, \bar{y}) = \Psi(x, y)$ realizes the C^1-equivalence between p_1 and p_2. Without loss of generality, we assume that Ψ keeps orientation of orbits. Since Ψ sends orbits of system (43) to orbits of system (44), and separatrices to separatrices, Ψ must have the following form:

$$\Psi(x, y) = (\alpha x + o(r)), \beta y + o(r)) \quad \text{as } r = \sqrt{x^2 + y^2} \to 0,$$

where $\alpha \cdot \beta \neq 0$. Hence $\Psi(\Gamma_r)$ is described by

$$(\frac{\bar{x} + o(\bar{r})}{\alpha})^2 + (\frac{\bar{y} + o(\bar{r})}{\beta})^2 = r^2 \text{ as } \bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2} \to 0 \text{ (or equivalently, } r \to 0).$$

The contact points $(\bar{x}, \bar{y})$ of system (44) with $\Psi(\Gamma_r)$ must satisfy

$$\bar{x} \cdot (\frac{2\bar{x} + o(\bar{r})}{\alpha^2}) - \bar{y} \cdot (\frac{2\bar{y} + o(\bar{r})}{\beta^2}) = 0 \quad \text{as } \bar{r} \to 0. \tag{49}$$

Let $\bar{x} = \bar{r}\cos\bar{\theta}$, $\bar{y} = \bar{r}\sin\bar{\theta}$. Applying the Implicit Function Theorem to (49) at $\bar{r} = 0$, we obtain exactly four functions

$$\theta_i(\bar{r}) = \theta_{0i} + o(\bar{r}) \quad \text{as } \bar{r} \to 0,$$

where $i = 1, 2, 3, 4$, $0 < \theta_{0i} < 2\pi$ satisfies

$$\frac{\cos^2\theta_{0i}}{\alpha^2} - \frac{\sin^2\theta_{0i}}{\beta^2} = 0.$$

Thus we have proved that there are exactly 4 contact points of system (44) with $\Psi(\Gamma_r)$. This contradicts the fact that the number of contact points of system (43) with Γ_r must be equal to the number of contact points of system (44) with $\Psi(\Gamma_r)$. So p_1 and p_2 cannot be C^1-equivalent. The proof is finished. ∎

Lemma 26. *The center-focus class is non-empty in the set of singular points of degree m if and only if m is odd.*

Proof. Consider the singular point $x = y = 0$ of system (2). We first show that it cannot be a center or a focus if m is even. To do this, let us consider its local phase portrait on the unit circle (see §2.4). If $x = y = 0$ is a center or a focus, the unit circle should be invariant, and all its regular orbits (i.e., those which are not singular points) must have the same orientation (i.e., either clockwise or counter-clockwise). Now consider the characteristic polynomial $D(x, y)$ of this singular point. It is either identically zero, or homogeneous of odd degree. In the first case, the unit circle is not invariant. In the second case, there is a real linear homogeneous polynomial $\alpha x + \beta y$ whose multiplicity is odd in the factorization of $D(x, y)$. This linear factor corresponds to a singular point on the unit circle around which the two regular orbits have different orientations, namely, one is clockwise, and the other is counter-clockwise. So in both cases, $x = y = 0$ cannot be a center or a focus.

If m is odd, we show that there are singular points which are centers or foci. To do this, we consider the following system

$$\dot{x} = P_m(x, y), \qquad \dot{y} = Q_m(x, y).$$

Here, as before, $P_m(x, y), Q_m(x, y)$ are homogeneous of degree m. The polynomial $D(x, y) = xQ_m(x, y) - yP_m(x, y)$ is of even degree. So one can find P_m and Q_m such that $D(x,y)$ has no real linear factors. In this case, $x=y=0$ is a center or a focus. ▮

10.2. Local phase portraits of a special kind of singular points of degree 1

In this subsection, we will study the local phase portrait of the system

$$\begin{aligned}\dot{x} &= a_1y + a_3x^2 + a_2x^3 + y[P_1(x,y) + P_2(x,y) + \cdots],\\ \dot{y} &= b_1xy + b_3x^2y + y^2\left[b_2 + Q_1(x,y) + Q_2(x,y) + \cdots\right]\end{aligned} \tag{50}$$

at $x=y=0$, where P_i and Q_i are homogeneous polynomials of degree i, $a_1, a_2, a_3, b_1, b_2, b_3$ are real constants. We assume a_1 is positive, $a_3 \neq 0$ in the following analysis. In this case we can take $a_3 > 0$ by using the rescaling $(x,y) \mapsto (-x,-y)$ in system (50) which keeps a_1 unchanged and changes a_3 to $-a_3$.

To start, we apply the blowing-up $y = xu$ to system (50). Then we have

$$\begin{aligned}\dot{x} =& a_1xu + a_3x^2 + a_2x^3 + x^2u[P_1(1,u) + xP_2(1,u) + \cdots],\\ \dot{u} =& -a_1u^2 + (b_1-a_3)xu + (b_3-a_2)x^2u\\ &+ xu^2\left[b_2 - P_1(1,u) + x[Q_1(1,u) - P_2(1,u)] + \cdots\right].\end{aligned} \tag{51}$$

Now we apply the following two changes of variables

$$u = xu_1,\ d\tau = xdt; \qquad x = uv,\ d\tau = udt$$

to (51), and we obtain the following two systems

$$\begin{aligned}\dot{x} &= a_1xu_1 + a_3x + a_2x^2 + x^2u_1\left[P_1(1,xu_1) + xP_2(1,xu_1) + \cdots\right],\\ \dot{u}_1 &= -2a_1u_1^2 + (b_1-2a_3)u_1 + (b_3-2a_2)xu_1 + xu_1^2[\cdots],\end{aligned} \tag{52}$$

and

$$\dot{u} = -a_1u + \cdots, \qquad \dot{v} = 2a_1v + \cdots,$$

where $[\cdots]$ is some analytic function. The singular point $u=v=0$ is a saddle. On the u_1-axis system (52) has only singular points with at least one non-zero eigenvalue. Thus, by Theorem A, it is easy to get all possible pictures for the phase portrait of system (52) on the u_1-axis. They are shown in Figure 30. According to the theory of blowing-ups (§2), we obtain Figure 31 as the corresponding pictures for the phase portrait of system (51) on the u-axis, where $x=u=0$ is the unique singular point. Finally Figure 32 shows all the possibilities of the local phase portrait of system (50) at $x=y=0$. The details are given in the following analysis.

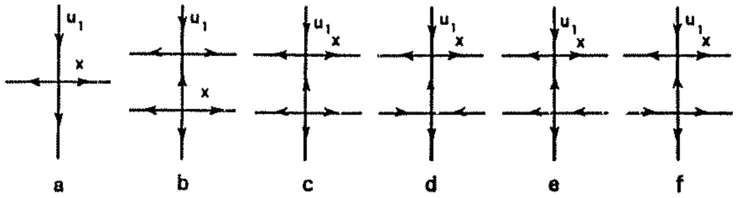

FIG. 30. The phase portrait of system (52) on the u_1-axis.

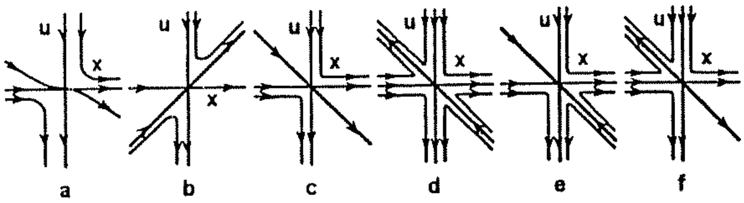

FIG. 31. The phase portrait of system (51) on the u-axis.

If $b_1 = 2a_3$, system (52) has only one singular point on the u_1-axis. We have Figure 30(a) for the phase portrait of system (52) on the u_1-axis. Correspondingly, we obtain Figure 31(a) for the local phase portrait of system (51) at $x = u = 0$, and Figure 32(a) for the local phase portrait of system (50) at $x = y = 0$.

If $b_1 > 2a_3$, system (52) has two singular points on the u_1-axis: $(0, 0)$ and $(0, \frac{b_1-2a_3}{2a_1})$, where the matrix of the linear part is $\begin{pmatrix} a_3 & 0 \\ 0 & b_1-2a_3 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{2}b_1 & 0 \\ * & 2a_3-b_1 \end{pmatrix}$ respectively. Figure 30(b) is the phase portrait of system (52) on the u_1-axis. We obtain Figures 31(b) and 32(b) for the local phase portraits of $x = u = 0$ and $x = y = 0$ respectively.

Now assume $b_1 < 2a_3$. There are 2 singular points on the u_1-axis: $x = u_1 = 0$ and $x = u_1 - \frac{b_1-2a_3}{2a_1} = 0$. The point $x = u_1 = 0$ is a saddle. The point $x = u_1 - \frac{b_1-2a_3}{2a_1} = 0$ has four possibilities. If $b_1 > 0$ we obtain Figures 30(c), 31(c) and 32(a). If $b_1 < 0$, we obtain Figures 30(d), 31(d) and 32(c). If $b_1 = 0$, the point $x = u_1 - \frac{b_1-2a_3}{2a_1}$ can be a saddle, a node or a saddle-node. But new pictures are produced only when it is a saddle-node, and the new pictures which we will obtain are Figures 30(e) and 30(f). Correspondingly we obtain Figures 31(e) and 31(f) for the local phase portrait of system (51) at $x = u = 0$; and Figures 32(d) and 32(e) for the local phase portrait of system (50) at $x = y = 0$.

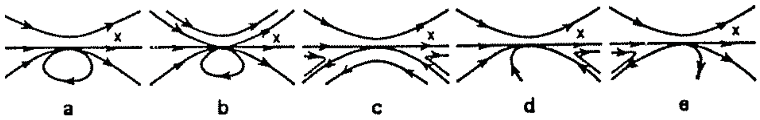

FIG. 32. The local phase portrait of system (50) at $x = y = 0$.

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