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Solution of the Problem of the Centre for Cubic Differential System with Three Invariant Straight Lines in Generic Position

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For cubic differential system with three invariant straight lines such that not one pair of the lines is parallel and no more than two lines pass through the same point (in generic position) is proved that a singular point with pure imaginary eigenvalues (a weak focus) is a centre if and only if the first seven Liapunov quantities $V_j,\ j = \overline{1,7}$ vanish.

Key Words: Cubic differential systems, centre-focus problem, invariant algebraic curves, integrability.

1. INTRODUCTION

Consider the real autonomous system of differential equations with polynomial right–hand sides of degree three

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

i.e. $P = P_0 + P_1 + P_2 + P_3, \quad Q = Q_0 + Q_1 + Q_2 + Q_3$, where $P_j,\ Q_j,\ j = \overline{0,3}$ are homogeneous polynomials of degree j with real coefficients and variables. We shall assume that (1) has a singular point (x_0, y_0) with pure imaginary eigenvalues (a weak focus). This point is a centre or a focus. The problem arises of distinguishing between a centre and a focus, i.e. of finding the coefficient conditions under which (x_0, y_0) is, for example, a centre for (1). These conditions are called the conditions for the existence of a centre or the centre conditions and the problem – the problem of the centre.

By an affine change of coordinates and a time rescaling we bring (1) to a cubic system with $P_0 = Q_0 = 0$ and $P_1 = y,\ Q_1 = -x$. In this case, $x_0 = y_0 = 0$ and there exists a function $F(x, y)$ defined in a neighborhood of the origin such that its rate of change along orbits of system (1) is of the form

$$\frac{dF}{dt} = \sum_{j=1}^{\infty} V_j (x^2 + y^2)^{j+1},$$

where V_j are polynomials in the coefficients of (1), called the Liapunov quantities.

It is known that $(0, 0)$ is a centre for (1) if and only if $V_j = 0,\ j = \overline{1, \infty}$, that is when (1) has a first integral of the form $F(x, y) = const$ [11]. Also, it is known that $(0, 0)$ is a centre if and only if (1) has in some neighborhood of the origin an independent of t holomorphic integrating factor $\mu(x, y)$ [1]. The order of the weak focus $(0, 0)$ is r if $V_1 = V_2 = \ldots = V_{r-1} = 0$ but $V_r \neq 0$.

The problem of the centre was completely solved for quadratic system ($P_3 \equiv 0,\ Q_3 \equiv 0$) by Dulac [8] (in this case the order of a weak focus is at most three) and for cubic symmetric system ($P_2 \equiv 0,\ Q_2 \equiv 0$) by K.S.Sibirski [14] (in this case the order of a weak focus is at most five). If the cubic system (1) contains both quadratic and cubic nonlinearities the problem of the centre is solved only in some particular cases (see, for example, [2], [3], [5], [6], [15], [12], [13]).

An algebraic curve $f(x, y) = 0$ is said to be an invariant curve of (1) if there exists a polynomial $K(x, y)$ such that

$$\frac{\partial f}{\partial x} P(x, y) + \frac{\partial f}{\partial y} Q(x, y) \equiv f(x, y) \cdot K(x, y).$$

The polynomial $K(x, y)$ is called the cofactor of the invariant algebraic curve $f(x, y) = 0$.

The quadratic systems and cubic symmetric systems with a singular point of a centre type are Darboux integrable, i.e. these systems have a first integral (an integrating factor) composed of invariant algebraic curves. Hence, the interest arises to study the problem of the centre for polynomial differential systems with invariant algebraic curves. The problem of integrability for polynomial systems with invariant algebraic curves, in particular, with invariant straight lines was considered in the works [1], [4], [10] and other.

To investigation of the problem of the centre for cubic differential systems (1) with invariant straight lines (real or complex) are dedicated the works [5], [6], [7], [15], [16], [18]. In these papers, the problem of the centre was completely solved for cubic systems with at least four invariant straight lines and for some cubic systems with three invariant straight lines. The principal results of these works are gathered in the following two theorems:

THEOREM 1. *Let the cubic system (1) with a weak focus at* (x_0, y_0) *have at least four invariant straight lines. Then*

a) the order of (x_0, y_0) *is at most one (i.e.* (x_0, y_0) *is a centre if and only if* $V_1 = 0$*), if* (x_0, y_0) *does not belong to the invariant straight lines, and*

b) the order of (x_0, y_0) *is at most two (i.e.* (x_0, y_0) *is a centre if and only if* $V_1 = V_2 = 0$*), if* (x_0, y_0) *belongs to the invariant straight lines.*

THEOREM 2. *Let the cubic system (1) with a weak focus at* (x_0, y_0) *have three invariant straight lines. Then the order of a weak focus* (x_0, y_0) *is at most seven, if* (x_0, y_0) *belongs to the union of these invariant straight lines.*

By Theorems 1 and 2 to solve completely the problem of the centre for cubic differential systems with at least three invariant straight lines, it remains to investigate the case of a cubic system (1) with three invariant straight lines in generic position assuming that a singular point with pure imaginary eigenvalues does not belong to these invariant straight lines. This case is considered in the present paper.

2. THE FORM OF A DIFFERENTIAL SYSTEM WITH INVARIANT CURVES

A curve Γ defined by equation $\omega(x, y) = 0$ is called an invariant curve for differential system (1), if

$$\left(\omega_x \cdot P(x,y) + \omega_y \cdot Q(x,y)\right)\Big|_{\Gamma} = 0,$$

where $\omega_x = \partial\omega/\partial x,\ \omega_y = \partial\omega/\partial y$.

For differential systems with invariant curves the following problems arise: a) (direct problem) for a given differential system determine its invariant curves and b) (inverse problem) being given the curves $\omega_1(x, y) = 0, \ldots, \omega_k(x, y) = 0$, find all differential systems for which these curves are invariant. As to inverse problem, the first work concerning this problem is Erugin [9]. In this paper, it is stated that differential systems having invariant the curve $\omega(x, y) = 0$ can be written into the form

$$\dot{x} = F_1(\omega, x, y) - \omega_y \cdot A(x, y), \quad \dot{y} = F_2(\omega, x, y) - \omega_x \cdot A(x, y),$$

where F_1, F_2, A are some functions and $F_1(0,x,y) = F_2(0,x,y) = 0$.

The inverse problem was considered by Amel'kin [1], in the case, when P, Q and ω_j are polynomials. It is shown that differential systems having two invariant algebraic curves $\omega_{1,2}(x,y) = 0$, $\omega_{1x}\omega_{2y} - \omega_{1y}\omega_{2x} \not\equiv 0$ can be written into the form

$$\dot{x} = A_1\omega_1\omega_{2y} - A_2\omega_2\omega_{1y}, \quad \dot{y} = A_2\omega_2\omega_{1x} - A_1\omega_1\omega_{2x},$$

where A_1, A_2 are polynomials.

Christopher and other (see, for example, [4], [10]) examined the inverse problem for polynomial systems in the case of more invariant algebraic curves. Thus, under some generic conditions on curves $\omega_j(x,y) = 0$, $j = \overline{1,k}$, we obtain the following form of a system (1):

$$\dot{x} = \Big(\sum_{j=1}^{k} A_j \frac{\omega_{jy}}{\omega_j} + B\Big)\prod_{j=1}^{k}\omega_j, \quad \dot{y} = -\Big(\sum_{j=1}^{k} A_j \frac{\omega_{jx}}{\omega_j} + D\Big)\prod_{j=1}^{k}\omega_j, \tag{2}$$

where A_j, B, D are polynomials. We shall use the form (2) in the next investigations.

Farther, in this paper, we shall consider the cubic system (1) with real polynomials $P(x,y)$ and $Q(x,y)$, and $\deg(P^2+Q^2) = 6$.

It should be mentioned that a straight line $L \equiv ax + by + c = 0$ is said to be invariant for (1) if and only if there exists a polynomial $K(x,y)$ such that the following identity holds

$$a \cdot P(x,y) + b \cdot Q(x,y) \equiv (ax + by + c) \cdot K(x,y).$$

The polynomial $K(x,y)$ is called the cofactor of the invariant straight line L and $\deg(K) \leq \max\{\deg(P), \deg(Q)\} - 1$. If the cubic system (1) has complex invariant straight lines then, obviously they occur in complex conjugated pairs L and $\overline{L}$. We shall assume that the differential system (1) has exactly three invariant straight lines

$$L_j \equiv a_j x + b_j y + c_j = 0, \quad j = 1,2,3;\ a_j, b_j, c_j \in \mathbf{C} \tag{3}$$

such that not one pair of the lines is parallel and no more than two lines pass through the same point (in generic position), i.e.

$$\Delta_{jl} = \begin{vmatrix} a_j & b_j \\ a_l & b_l \end{vmatrix} \neq 0, \ \ j \neq l, \ \ j,l = 1,2,3; \ \ \Delta_{123} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0. \tag{4}$$

The invariant straight line L_3 can be considered real.

Conditions (4) allow to write the cubic system (1) into the form (2):

$$\begin{aligned}\frac{dx}{dt} &= \Big(\sum_{j=1}^{3} \frac{A_j L_{jy}}{L_j} + p_1\Big) \prod_{j=1}^{3} L_j \equiv P(x,y),\\ \frac{dy}{dt} &= -\Big(\sum_{j=1}^{3} \frac{A_j L_{jx}}{L_j} + q_1\Big) \prod_{j=1}^{3} L_j \equiv Q(x,y),\end{aligned} \tag{5}$$

where $L_{jx} = \partial L_j/\partial x,\ L_{jy} = \partial L_j/\partial y;\ p_1,\ q_1 \in \mathbf{R}$ and $A_j,\ j = 1,2,3$ are linear in x and y. Let $A_j = m_j x + n_j y + s_j,\ j = 1,2,3.$

The straight lines L_1, L_2, L_3 have respectively the cofactors

$$\begin{aligned}K_1(x,y) &= \Delta_{12}L_3A_2 + \Delta_{13}L_2A_3 + (p_1a_1 - q_1b_1)L_2L_3,\\ K_2(x,y) &= \Delta_{23}L_1A_3 + \Delta_{21}L_3A_1 + (p_1a_2 - q_1b_2)L_1L_3,\\ K_3(x,y) &= \Delta_{31}L_2A_1 + \Delta_{32}L_1A_2 + (p_1a_3 - q_1b_3)L_1L_2.\end{aligned} \tag{6}$$

By affine transformations of coordinates and a time rescaling

$$x \to \alpha_1 x + \beta_1 y + \gamma_1, \quad y \to \alpha_2 x + \beta_2 y + \gamma_2, \quad t \to \alpha t \tag{7}$$

the system (5) does not change the form.

Let (x^*, y^*) be a singular point of (5) with pure imaginary eigenvalues. By transformations of the form (7), first we translate (x^*, y^*) at the origin, i.e.

$$P(0,0) = Q(0,0) = 0 \tag{8}$$

and then transform the linear part of $P(x,y)$ to be equal with y, and of $Q(x,y)$ to be equal with $-x$, i.e.

$$P'_x(0,0) = Q'_y(0,0) = 0, \quad P'_y(0,0) = -Q'_x(0,0) = 1. \tag{9}$$

The intersection point of straight lines L_1 and L_2 is a singular point for (5) and has real coordinates. In particular, this point can be $(0,0)$. In this case, $L_{1,2} = x \pm iy$, $i^2 = -1$ and the problem of the centre was solved in [15] (see Theorem 2 of this paper).

By rotating the system of coordinates ($x \to x\cos\varphi - y\sin\varphi, y \to x\sin\varphi + y\cos\varphi$) and rescaling the axes of coordinates ($x \to \alpha x,\ y \to \alpha y$), we obtain

$$L_1 \cap L_2 = (0,1). \tag{10}$$

In this case the invariant straight lines (3) can be written as

$$\begin{aligned}&L_j = a_j x - y + 1,\ L_3 = a_3 x + b_3 y + 1,\\ &a_j \in \mathbf{C},\ j = 1,2;\ a_3, b_3 \in \mathbf{R},\end{aligned} \tag{11}$$

and (4):

$$\begin{aligned}&\Delta_{12} = a_2 - a_1 \neq 0,\ \Delta_{j3} = a_j b_3 + a_3 \neq 0,\ j = 1,2,\\ &\Delta_{123} = b_3 + 1 \neq 0.\end{aligned} \tag{12}$$

The relations (8), (9) and $P(0,1) = Q(0,1) = 0$ put the following conditions on the coefficients of system (5):

$$\begin{aligned}
p_1 &= s_1 + s_2 - b_3 s_3,\\
q_1 &= -a_1 s_1 - a_2 s_2 - a_3 s_3,\\
m_1 &= (a_1^2 s_1 - a_1 a_2 s_1 + a_2 a_3 b_3 s_3 - a_2 b_3 m_3 + a_3^2 s_3 - a_3 m_3 + 1)/\\
&\quad (a_1 - a_2),\\
n_1 &= (a_2 b_3^2 s_3 - a_1 s_1 - a_2 b_3 n_3 + a_2 s_1 + a_2 + a_3 b_3 s_3 - a_3 n_3)/\\
&\quad (a_1 - a_2),\\
m_2 &= (a_1 a_2 s_2 - a_1 a_3 b_3 s_3 + a_1 b_3 m_3 - a_2^2 s_2 - a_3^2 s_3 + a_3 m_3 - 1)/\\
&\quad (a_1 - a_2),\\
n_2 &= (a_1 b_3 n_3 - a_1 b_3^2 s_3 - a_1 s_2 - a_1 + a_2 s_2 - a_3 b_3 s_3 + a_3 n_3)/\\
&\quad (a_1 - a_2).
\end{aligned} \tag{13}$$

3. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A CENTRE

Denote

$$\begin{aligned}
f_1 = {}& a_3(b_3+1)(n_3 - b_3 s_3)^2 - (b_3+1)(n_3 - b_3 s_3)(m_3 - a_3 s_3)\cdot\\
&\cdot (b_3 + a_1 a_2) + (n_3 - b_3 s_3)(m_3 - a_3 s_3)(a_1 a_2 - a_1 a_3 -\\
&- a_2 a_3 + a_3^2) + (b_3+1)(m_3 - a_3 s_3)(1 - a_3 m_3 + a_3^2 s_3) -\\
&- a_3(n_3 - b_3 s_3) + a_1 a_2 (m_3 - a_3 s_3).
\end{aligned} \tag{14}$$

LEMMA 3. *The cubic differential system (5) with conditions (13) and $f_1 = 0$ has a centre at the origin.*

Proof. The system (5) with invariant straight lines $L_1,\ L_2,\ L_3$ and cofactors $K_1,\ K_2,\ K_3$ has a Darboux integrating factor

$$\mu = L_1^{\alpha_1} L_2^{\alpha_2} L_3^{\alpha_3}, \quad \alpha_1,\ \alpha_2,\ \alpha_3 \in \mathbf{C},$$

if and only if the following identity holds

$$\alpha_1 K_1(x,y) + \alpha_2 K_2(x,y) + \alpha_3 K_3(x,y) + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \equiv 0. \tag{15}$$

Let

$$\Delta \equiv a_1\Delta_{23} + a_2a_3 - a_3^2 + b_3^2 + b_3 \neq 0. \tag{16}$$

From (15) by taking into account (6) and (11)–(14), we obtain

$$\begin{aligned}
\alpha_1 = {} & (n_3 - b_3s_3)(a_1\Delta_{23} + a_2a_3 - a_3^2) + a_3(b_3+1)(m_3 - a_3s_3)+\\
& +b_3 - a_1a_2 - 3 - \alpha_2 + b_3\alpha_3;
\end{aligned}$$

$$\begin{aligned}
\alpha_2 = {} & -[(n_3 - b_3s_3)(a_1^2\Delta_{23} + a_1a_2a_3 - a_1a_3^2 + a_3b_3 + a_3)+\\
& +(m_3 - a_3s_3)(b_3+1)(a_1a_3 - b_3) - a_1^2a_2 + a_1b_3 - 2a_1+\\
& +a_2 + a_3 + \Delta_{13}\alpha_3]/\Delta_{12};
\end{aligned}$$

$$\begin{aligned}
\alpha_3 = {} & [-(b_3+1)^2(n_3 - b_3s_3)^2(a_3^2 + a_1^2a_2^2) + 2a_1a_2(b_3+1)(n_3 - b_3s_3)^2\cdot\\
& \cdot(a_1 - a_3)(a_2 - a_3) - (n_3 - b_3s_3)^2(a_1 - a_3)^2(a_2 - a_3)^2 + (b_3+1)\cdot\\
& \cdot(n_3 - b_3s_3)(a_1^2a_2^2 + a_1a_2 - a_1a_3 - a_2a_3 + 2a_3^2) - (a_1 - a_3)(a_2 - a_3)\cdot\\
& \cdot(1 + a_1a_2)(n_3 - b_3s_3) + (b_3+1)^2(a_3^2 + b_3^2)(m_3 - a_3s_3)^2 - (b_3+1)\cdot\\
& \cdot(m_3 - a_3s_3)(a_1a_2a_3 - a_1b_3 - a_2b_3 + 2a_3b_3 + a_3) - \Delta]/\Delta.
\end{aligned}$$

Because in the space of coefficients of (5) the centre variety is closed, the system (5) will have a centre at the origin, even if, the inequality (16) is not satisfied. ▮

Let us consider the cubic systems

$$\begin{aligned}
\dot{x} &= (1 - 2bx)(y + ax^2 - bxy - y^2),\\
\dot{y} &= -[x - 2bx^2 + dxy + by^2 + (1-a)(a-d-2)x^3+\\
& \quad b(a-d)x^2y + (1-a)xy^2 - by^3]
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
\dot{x} &= (1 + gx)(2y + x^2 + 2(b+g)xy - 2y^2)/2,\\
\dot{y} &= (y - 1)(2x - xy + 2gx^2 + 2by^2)/2.
\end{aligned} \tag{18}$$

It should be mentioned that the system (17) has the invariant straight lines

$$1 - 2bx = 0,\ 1 - (b \pm \sqrt{b^2 - a + d + 2}\,)x - y = 0$$

and the system (18) has the invariant straight lines

$$1 + gx = 0, \quad 1 - y = 0, \quad 1 + gx - y = 0.$$

Lemma 4. *For both systems (17) and (18) the origin is a centre.*

Proof. It is easy to check that systems (17) and (18) are reversible (see [19]). Indeed by transformation

$$X = -\frac{x}{1-bx}, \quad Y = \frac{y}{1-bx}$$

the system (17) is reduced to

$$\begin{aligned}\dot{X} &= (1-b^2X^2)(Y+aX^2-Y^2),\\ \dot{Y} &= X[-1-dY+((a-1)(a-d-2)+b^2)X^2+(a-2b^2-1)Y^2-\\ &- ab^2X^2Y+b^2Y^3]\end{aligned}$$

for which OY is an axes of symmetry and by transformation

$$X = \frac{x}{y-2}, \quad Y = \frac{y}{y-2}$$

the system (18) can be reduced to

$$\begin{aligned}&\dot{X} = Y[1-2(b+2g)X+2(1+2bg+2g^2)X^2-Y^2-2gX^3-2bXY^2],\\ &\dot{Y} = (Y^2-1)(X-2gX^2-2bY^2)\end{aligned}$$

for which OX is an axes of symmetry. ∎

Let us consider for (5) the following coefficient conditions:

$$a_1 = (\Delta_1 - a_2a_3 + b_3)/\Delta_{23}; \tag{19}$$

$$m_3 = a_3s_3, \quad n_3 = (b_3^2s_3+1)/b_3; \tag{20}$$

$$m_3 = a_3s_3 - \Delta_{23}/\Delta_2, \quad n_3 = b_3s_3 - (\Delta_1 + b_3 - a_2a_3)/\Delta_2; \tag{21}$$

$$m_3 = a_3s_3 + \Delta_{23}/(\Delta_3\Delta_{123}), \quad n_3 = b_3s_3 + a_2\Delta_{23}/(\Delta_3\Delta_{123}); \tag{22}$$

$$\begin{aligned}m_3 &= a_3s_3 - a_3(a_2^2a_3 - a_2b_3 - a_2\Delta_1 - \Delta_{23})/(2\Delta_3\Delta_{23}\Delta_{123}),\\ n_3 &= b_3s_3 - [a_2b_3(a_2a_3-b_3) - (b_3+\Delta_1)\Delta_{23} - a_2b_3\Delta_1]/\\ &\quad (2\Delta_1\Delta_{23}\Delta_{123}),\end{aligned} \tag{23}$$

where

$$\Delta_1 = a_3^2 + b_3^2, \quad \Delta_2 = \Delta_3 - (b_3+2)\Delta_1, \quad \Delta_3 = a_2^2b_3 + 2a_2a_3 - b_3.$$

LEMMA 5. *The following four series of conditions:* **1)** *(13), (19), (20);* **2)** *(13), (19), (21);* **3)** *(13), (19), (22);* **4)** *(13), (19), (23) are sufficient conditions for the origin to be a centre for system (5).*

Proof. In each of the cases **1), 2)** and **3)** the system (5) has four invariant straight lines. Thus, in conditions **1)** besides the invariant straight lines (11), the system (5) has one more invariant straight line

$$b_3 - a_3x - b_3y = 0;$$

in conditions **2)**:

$$\Delta_2 - (\Delta_1 - a_2a_3 + b_3)\Delta_{23}x + \Delta_{23}^2 y = 0$$

and in conditions **3)**:

$$\Delta_3 + \Delta_1(a_2x - y) = 0.$$

In **1)**, **2)** and **3)** the first two Liapunov quantities vanish. Hence, in each of these cases, the existence of a centre at $(0,0)$ follows from Theorem 1.

In conditions **4)** the system (5) has a Darboux integrating factor

$$\mu(x,y) = \frac{1}{L_1L_2L_3\sqrt{f}},$$

where

$$f = 1 + a_3x + \frac{\gamma_1}{\Delta_{23}}y + \frac{(\gamma_2x + \gamma_3y)^2}{4\Delta_{23}^2\Delta_1},$$
$$\gamma_1 = a_2^2a_3 - a_2a_3^2 - 2a_2b_3 + a_3b_3 - a_3,$$
$$\gamma_2 = a_3(\gamma_1 - b_3\Delta_{23}), \quad \gamma_3 = b_3\gamma_1 + a_3^2\Delta_{23}.$$

∎

4. SOLUTION OF THE PROBLEM OF THE CENTRE

Denote

$$f_2 = a_1\Delta_{23} - \Delta_1 - b_3 + a_2a_3. \tag{24}$$

Remark 6. Let $\Delta_2 = 0$ or $\Delta_3 = 0$. Then the invariant straight lines (11) of system (5) with conditions (13) are real.

Indeed, the straight line $L_3 = a_3x + b_3y + 1$ is real by hypothesis and if $L_1 = a_1x - y + 1$ and $L_2 = a_2x - y + 1$ are complex, then $L_1 = \overline{L_2}$, i.e.

$a_1 = \overline{a_2}$ (see section 2). The case $b_3 = 0,\ a_3 \in R \setminus \{0\}$ is elementary. Let $b_3 \neq 0$. From $\Delta_2 = 0$ we have

$$a_2 = \left(-a_3 \pm (b_3+1)\sqrt{a_3^2 + b_3^2}\right)/b_3 \in \mathbf{R}$$

and from $\Delta_3 = 0$ it follows that

$$a_2 = \left(-a_3 \pm \sqrt{a_3^2 + b_3^2}\right)/b_3 \in \mathbf{R}.$$

Lemma 7. *Let for cubic system (5) the following three series of conditions:* **1)** *(13),* $f_2 = 0, b_3 = 0$*;* **2)** *(13),* $f_2 = 0, \Delta_3 = 0$ *and* **3)** *(13),* $f_2 = 0, \Delta_2 = 0$ *hold. The order of a weak focus* $(0,0)$ *in each of these cases is at most three.*

Proof. We compute the first three Liapunov quantities V_1, V_2, V_3 using the algorithm, described in [17].

1). Let $b_3 = 0$. Express a_1 from $f_2 = 0$: $a_1 = a_3 - a_2$ and substitute in V_1, V_2, V_3. We obtain $V_1 = 0$ and V_2 after cancellation by non–zero terms, looks

$$V_2 = (2n_3 - 1)(2a_3^2 s_3 - 2a_3 m_3 + 1).$$

Assume that $2n_3 - 1 = 0$ and rename the coefficients of system (5), then it is the same as (17) and by Lemma 4 we have a centre at $(0,0)$. Let $2n_3 - 1 \neq 0$. From $2a_3^2 s_3 - 2a_3 m_3 + 1 = 0$ we express m_3 and substitute in V_3:

$$V_3 = a_2(a_2 - a_3).$$

In each of the cases $a_2 = 0$ and $a_2 - a_3 = 0$, the system (5) can be written into the form (18) and we can make use of Lemma 4.

Assume next that $b_3 \neq 0$ and we show that the cases **2)** and **3)** can be reduced to **1)**.

Indeed, from $f_2 = 0$ express a_1:

$$a_1 = (\Delta_1 + b_3 - a_2 a_3)/\Delta_{23}.$$

Denote

$$x_{13} = -\Delta_{23}/\Delta_1, \quad y_{13} = (a_2 a_3 - b_3)/\Delta_1,$$

$$x_{23} = -(b_3+1)/\Delta_{23}, \quad y_{23} = (a_3 - a_2)/\Delta_{23}.$$

Observe that $(x_{j3}, y_{j3}) \in L_j \cap L_3,\ j = 1,2.$

By transformation of coordinates

$$x = y_{13}X + x_{13}Y, \quad y = -x_{13}X + y_{13}Y$$

the invariant straight lines L_1, L_2, L_3 of the system (5) are transformed for the new differential system into the invariant straight lines

$$L_1 = \frac{a_2a_3 - a_2^2 - b_3 - 1}{\Delta_{23}}X - Y + 1, \quad L_3 = a_2X - Y + 1,$$

$$L_2 = \frac{a_2^2a_3 - 2a_2b_3 - a_3}{\Delta_1}X - \frac{\Delta_3}{\Delta_1}Y + 1.$$

In the case of transformation

$$x = y_{23}X + x_{23}Y, \quad y = -x_{23}X + y_{23}Y$$

we have

$$L_1 = \frac{a_3(a_2 - a_3)^2 + a_3\Delta_{123}^2 - 2\Delta_{23}\Delta_{123}}{\Delta_{23}^2}X + \frac{\Delta_2}{\Delta_{23}^2}Y + 1,$$

$$L_2 = \frac{a_2a_3 - a_2^2 - b_3 - 1}{\Delta_{23}}X - Y + 1, \quad L_3 = \frac{\Delta_1 + b_3 - a_2a_3}{\Delta_{23}}X - Y + 1.$$

▮

Theorem 8. *Let the cubic system have three invariant straight lines such that not one pair of the lines is parallel and no more then two lines pass through the same point. Then the order of any weak focus not lying on these lines is at most three.*

Proof. Without loss of generality we can consider that the cubic system is of the form (5) and the conditions (13) hold. In this case, a singular point $(0, 0)$ is a weak focus for (5) and the invariant straight lines L_1, L_2, L_3 are given by formulas (11). Obviously $(0, 0) \notin L_1 \cup L_2 \cup L_3$. As $L_1 \cap L_2 \cap L_3 = \emptyset$ and not one pair of the lines is parallel we are in conditions (12).

For $(0, 0)$ we compute the first three Liapunov quantities V_1, V_2, V_3. The first Liapunov quantity looks: $V_1 = f_1f_2$ (see (14), (24)). If $f_1 = 0$, then the assertion of Theorem 8 follows from Lemma 3. Let $f_2 = 0$. Each of the following cases $b_3 = 0$, $\Delta_2 = 0$ or $\Delta_3 = 0$ was examined in Lemma 7. Therefore, next we shall assume that

$$b_3f_1\Delta_2\Delta_3 \neq 0. \tag{25}$$

From $f_2 = 0$ express a_1 (see (19) and substitute in V_2, V_3. The quantity V_2 cancelled by non–zero terms, looks

$$\begin{aligned} V_2 \;=\; & 2b_3\Delta_{23}\Delta_{123}m_3^2 + m_3(3a_3\Delta_{23} - \Delta_1 - a_2^2b_3 - b_3)- \\ -\; & 2\Delta_{23}\Delta_{123}[s_3\Delta_1(b_3s_3 - 2n_3) + n_3(b_3n_3 + 2a_3m_3)] + \Delta_{23}(3b_3n_3+ \\ +\; & 2n_3 - 1) + s_3[b_3(a_2^2a_3 - 2a_2b_3 - a_3) - \Delta_1(3a_2b_3 + 2a_3)]. \end{aligned}$$

The quantity V_3 is too cumbersome and will be not given here. The coefficient of m_3^2 is non-zero in V_2. Taking into account $V_2 = 0$ we transform V_3 to be linear in m_3. Reduce V_3 by the non-zero factors and calculate the resultant of the polynomials V_2 and V_3 in m_3:

$$\mathrm{Res}(V_2,\, V_3,\, m_3) = 4\Delta_2\Delta_3\Delta_{23}^2\Delta_{123}j_1j_2j_3j_4,$$

where

$$j_1 = b_3n_3 - b_3^2s_3 - 1, \quad j_2 = (n_3 - b_3s_3)\Delta_2 + \Delta_1 + b_3 - a_2a_3,$$
$$j_3 = 2\Delta_1\Delta_{23}\Delta_{123}(n_3 - b_3s_3) + a_2b_3(a_2a_3 - b_3) - (b_3 + \Delta_1)\Delta_{23} - a_2b_3\Delta_1,$$
$$j_4 = \Delta_3\Delta_{123}(n_3 - b_3s_3) - a_2\Delta_{23}.$$

We shall examine separately each of the following cases:
a) $j_1 = 0$; **b)** $j_2 = 0,\ j_1 \neq 0$; **c)** $j_3 = 0,\ j_1j_2 \neq 0$; **d)** $j_4 = 0,\ j_1j_2j_3 \neq 0$.

a) From $j_1 = 0$ express n_3 and substitute in V_2 and V_3:

$$V_2 = (m_3 - a_3s_3)f_3, \quad V_3 = (m_3 - a_3s_3)h_1h_2h_3,$$

where

$$f_3 = 2b_3^2\Delta_{23}\Delta_{123}(m_3 - a_3s_3) - a_3\Delta_{23}(b_3 + 2) - h_3,$$
$$h_1 = \Delta_3 - \Delta_1 - a_3\Delta_{23},$$
$$h_2 = \Delta_3 + b_3(a_2a_3 - b_3),$$
$$h_3 = b_3\Delta_1 + 2a_3\Delta_{23} + b_3^2(a_2^2 + 1).$$

In assumption (25) we have the equality $\{j_1 = 0,\ m_3 - a_3s_3 = 0\} = \{\text{conditions (20)}\}$ and inclusions

$$\{j_1 = f_3 = h_1 = 0\} \subset \{(21)\},\ \{j_1 = f_3 = h_2 = 0\} \subset \{(22)\},$$
$$\{j_1 = f_3 = h_3 = 0\} \subset \{(23)\}.$$

Hence, in this case, the assertion of Theorem 8 follows from Lemma 5.

b) $j_2 = 0$, $j_1 \neq 0$. From $j_2 = 0$ express n_3 and substitute in V_2 and V_3:

$$V_2 = b_3g_1f_4/\Delta_2^2, \quad V_3 = b_3j_1\Delta_{12}\Delta_{23}g_1h_5,$$

where

$$g_1 = \Delta_2(m_3 - a_3s_3) + \Delta_{23},$$
$$f_4 = 2\Delta_2\Delta_{23}\Delta_{123}(m_3 - a_3s_3) - h_1((a_2 - a_3)^2 + \Delta_{123}^2),$$
$$h_4 = a_2^2 - a_2a_3 + b_3 + 1.$$

Taking into account (25), $j_1 \neq 0$, the equality $\{j_2 = g_1 = 0\} = \{(21)\}$ and inclusion $\{j_2 = f_4 = h_4 = 0\} \subset \{(22)\}$ we come to Lemma 5.

c) $j_3 = 0,\ j_1 j_2 \neq 0$. Express n_3 from $j_3 = 0$ and substitute in V_2, V_3 and j_1:

$$V_2 = b_3 g_2 f_5/(2\Delta_1^2\Delta_{23}\Delta_{123}),\ \ V_3 = b_3\Delta_{12}^2\Delta_{23} g_2 h_5/(2\Delta_1^2\Delta_{123}),$$
$$j_1 = a_3 h_5/(2\Delta_1\Delta_{23}\Delta_{123}),$$

where

$$g_2 = 2\Delta_1\Delta_{23}\Delta_{123}(m_3 - a_3 s_3) + a_3(a_2^2 a_3 - a_2 b_3 - a_2\Delta_1 - \Delta_{23}),$$
$$f_5 = 2\Delta_1\Delta_{23}\Delta_{123}(m_3 - a_3 s_3) - h_3, \quad h_5 = b_3^2(a_2^2+1) + 2a_3\Delta_{23} + b_3\Delta_1.$$

Evidently $h_5 \neq 0$. Since $b_3\Delta_{12}\Delta_{23} \neq 0$ (see (25), (12)), then V_2 and V_3 vanish simultaneously if and only if $g_2 = 0$. It remains to observe that $\{j_3 = 0,\, g_2 = 0\} = \{(23)\}$ and to apply Lemma 5.

d) $j_4 = 0,\ j_1 j_2 j_3 \neq 0$. Find n_3 from $j_4 = 0$ and substitute in V_2 and V_3:

$$V_2 = -b_3 f_6 g_3/(\Delta_3^2\Delta_{123}), \quad V_3 = \Delta_3\Delta_{123} j_1 j_2 g_3,$$

where

$$f_6 = 2\Delta_3\Delta_{23}\Delta_{123}(m_3 - a_3 s_3) - h_2(a_2^2+1), \quad g_3 = \Delta_3\Delta_{123}(a_3 s_3 - m_3) + \Delta_{23}.$$

Hence $V_2 = V_3 = 0 \iff g_3 = 0$. The equalities $j_4 = g_3 = 0$ yields to conditions (22) examined in Lemma 5. ▌

From Theorems 1, 2 and 8 stated above follow the complete solution of the problem of the centre for cubic differential systems with at least three invariant straight lines. Thus we have

THEOREM 9. *Any singular point (x_0, y_0) with pure imaginary eigenvalues of a real cubic differential system with at least three invariant straight lines (real, complex, real or complex) is a centre if and only if the first seven Liapunov quantities (focus quantities) vanish at this point.*

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