QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **6**, 9–29 (2005)
ARTICLE NO. 90

Local Analytic Models for Families of Hyperbolic Vector Fields

Patrick Bonckaert

and

Koen Neirynck

Hasselt University, Agoralaan, gebouw D, B-3590 Diepenbeek, Belgium
E-mail: patrick.bonckaert@uhasselt.be

We study local analytic models for hyperbolic singularities of analytic families of real analytic vector fields. We emphasize on those cases where the normal form theorems of Poincaré and Siegel do not apply, for example for a family of saddles. We also consider the preservation of possible symmetries.

Key Words: normal form, families of analytic vector fields, symmetry

1. INTRODUCTION

We look for analytic models near hyperbolic singularities of families of real analytic vector fields X_ε. The interesting case deals with saddles, since for sources or sinks we have the results of Poincaré [1, 2]. For families we cannot use the Siegel theorem since the condition on the small divisors is fragile. Even on the formal level (i.e. power series) the number of resonances between the eigenvalues is infinite for a family: for instance in the case of a planar saddle this comes to the density of the rationals in $\mathbf{R}$. One option is to use a C^k ($k < \infty$) normal form for the family [5]. Here we want to remain within the analytic category, and have to allow a less simplified form.

A first standard simplification is to use stable and unstable manifolds, and to 'straighten' them, i.e. to write the vector field such that these are linear subspaces. The fact that these invariant manifolds are analytic and depend analytically on the parameter will also follow from the results in this paper. The normal form we aim at will be moreover be 'as flat as desired' along these invariant manifolds if there are no low order resonances for X_0.

This approach can already be found in [3, 11, 12] and we extend the results in [12], on which our methods are inspired. Even though in this paper we confine ourselves to the case of a family of vector fields, we can prove similar results for a family of diffeomorphisms [7]. We shall also prove that possible symmetries are preserved in our local analytic model and by the changes of variables.

2. SETTINGS AND PRELIMINARIES

As we only aim local conjugacies near the singularity, we will restrict to analytic functions being convergent power series on a polydisk

$$\mathbf{D}(a,R) := B(a_1,R_1)\times\cdots\times B(a_n,R_n)$$

where $a=(a_1,\cdots,a_n)\in\mathbf{C}^n$ and $R=(R_1,\cdots,R_n)\in(\mathbf{R}^+\setminus\{0\})^n$. We need a few facts from local analytic function theory [4]. We say that a series $\sum_{m\in\mathbf{N}^n}a_m(z)$ converges normally on a poly-disk $\mathbf{D}(a,R)$ if $\sum_{m\in\mathbf{N}^n}\sup_{z\in K}|a_m(z)|$ converges on every compact set $K\subset\mathbf{D}(a,R)$. If f is analytic on the poly-disk $\mathbf{D}(a,R)$, we have

$$f(z)=\sum_{m\in\mathbf{N}^n}\frac{\partial^{|m|}f}{\partial z^m}(a)\frac{(z-a)^m}{m!},\ z\in\mathbf{D}(a,R),$$

with normal convergence. This normal convergence implies that $\sum_{m\in\mathbf{N}^n}a_m(z)$ exists and is independent of the order of summation and that the sum is analytic if all a_m are analytic.

In what follows we want to work with functions that are analytic in a variable $z\in\mathbf{C}^p$ and a parameter $\varepsilon\in\mathbf{C}^q$. Equipping $\mathbf{C}^{p+q}$ with the maximum-norm, the cartesian product of a poly-disk in $\mathbf{C}^p$ with a poly-disk in $\mathbf{C}^q$ is a poly-disk in $\mathbf{C}^{p+q}$. Introducing $\mathbf{e}:=(1,1,\cdots,1)\in\mathbf{C}^n$ (with $n\geq 1$) this choice of norm gives us that $B(a,R)=\mathbf{D}(a,R\mathbf{e})\subset\mathbf{C}^n$. So by the normal convergence, we have for each analytic function $f(z,\varepsilon)$ on $\mathbf{D}(a,R\mathbf{e})\times\mathbf{D}(b,r)$ in $\mathbf{C}^p\times\mathbf{C}^q$ that $f(z,\varepsilon)=\sum_{m\in\mathbf{N}^p}f_m(\varepsilon)(z-a)^m$, for each $z\in\mathbf{D}(a,R)$ and $\varepsilon\in\mathbf{D}(b,r)$, with normal convergence and each $f_m(\varepsilon)$ is analytic in $\mathbf{D}(b,r)$. Conversely if the series $\sum_{m\in\mathbf{N}^p}f_m(\varepsilon)(z-a)^m$ converges normally on $\mathbf{D}(a,R)\times\mathbf{D}(b,r)$ and each $f_m(\varepsilon)$ is analytic on $\mathbf{D}(b,r)$, then by uniform convergence of the series on each compact subset the function defined by the sum $f(z,\varepsilon):=\sum_{m\in\mathbf{N}^p}f_m(\varepsilon)(z-a)^m$ is analytic on $\mathbf{D}(a,R)\times\mathbf{D}(b,r)$ as it is clear that $f_m(\varepsilon)(z-a)^m$ is analytic on $\mathbf{D}(a,R)\times\mathbf{D}(b,r)$ and we have normal convergence. We will need the following consequence of this:

Proposition 1. *Let $f_m(\varepsilon)$ be an analytic function on $\mathbf{D}(b,r)$ for each $m\in\mathbf{N}^p$ and $g(z)=\sum_{m\in\mathbf{N}^p}g_m(z-a)^m$ is an analytic function on $\mathbf{D}(a,R)$,* *with g_m real and positive, such that*

$$|f_m(\varepsilon)| \le g_m, \forall m \in \mathbf{N}^p,$$

then the function $f : \mathbf{C}^{p+q} \to \mathbf{C}$ with

$$f(z,\varepsilon) := \sum_{m \in \mathbf{N}^p} f_m(\varepsilon)(z-a)^m$$

is analytic on $\mathbf{D}(a,R) \times \mathbf{D}(b,r)$.

Consider a p-parameter family of n-dimensional real vector fields X_ε with a singularity of hyperbolic type. We assume that the vector field can be written as a convergent power series in its variables and the parameter ε such that if we extend this vector field to $\mathbf{C}^n$, i.e. we replace each real variable by a complex one, we obtain a complex power series that converges on a poly-disk $\mathbf{D}(0,R) \times \mathbf{D}(0,r) \subset \mathbf{C}^n \times \mathbf{C}^p$. So we consider a real analytic family. We also assume that all eigenvalues of the linear part at the singularity have multiplicity 1 for $\varepsilon = 0$. Using the Implicit Function Theorem, we may assume that the singular point is the origin for all ε near zero and that X_ε is given by

$$X_\varepsilon : \dot{x} = A_\varepsilon x + f_\varepsilon(x), \tag{1}$$

where $f_\varepsilon(x) = O(|x|^2)$ is an analytic function of (x,ε) on a poly-disk $\mathbf{D}(0,R)$ and A_ε is in Jordan Normal Form, in such a way that the eigenvalues of A_ε with negative real part are labeled from 1 upto s and those with positive real part from $s+1$ upto n.

In order to calculate the formal normal form it is convenient to have a diagonal linear part at the origin, therefore we will use complex coordinates. Using the matrix

$$Q = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},$$

we obtain the change of coordinates $z = Px$ where P is a complex $n \times n$ matrix.

Applying this change of coordinates (1) is transformed into

$$Y_\varepsilon : \dot{z} = B_\varepsilon z + F_\varepsilon(z) \tag{2}$$

where

$$\begin{aligned} B_\varepsilon \;=\; & \mathsf{diag}(\nu_1(\varepsilon), \cdots, \nu_a(\varepsilon), \alpha_1(\varepsilon) + i\beta_1(\varepsilon), \alpha_1(\varepsilon) - i\beta_1(\varepsilon), \cdots, \\ & \alpha_b(\varepsilon) + i\beta_b(\varepsilon), \alpha_b(\varepsilon) - i\beta_b(\varepsilon), \mu_1(\varepsilon), \cdots, \mu_c(\varepsilon), \\ & \gamma_1(\varepsilon) + i\delta_1(\varepsilon), \gamma_1(\varepsilon) - i\delta_1(\varepsilon), \cdots, \gamma_d(\varepsilon) + i\delta_d(\varepsilon), \gamma_d(\varepsilon) - i\delta_d(\varepsilon)). \end{aligned}$$

As f_ε is a real analytic function with a complex extension converging on a poly-disk $\mathbf{D}(0,R)$, we have that F_ε is an analytic function of (z,ε) where z has the following properties:

- if $\lambda_j(\varepsilon)$ is a real eigenvalue of A_ε, then $\overline{z_j}=z_j=x_j$, in other words z_j is a real variable,
- if $\lambda_j(\varepsilon)$ and $\lambda_{j+1}(\varepsilon)$ form a pair of complex conjugate eigenvalues of A_ε, then $\overline{z_j}=z_{j+1}$. So $x_j=\frac{z_j+z_{j+1}}{2}$ and $x_{j+1}=\frac{z_j-z_{j+1}}{2i}$.

Therefore F_ε will be analytic for

- $|z_j|=|x_j|<R_j$, if z_j is real,
- $|z_j+z_{j+1}|=2|x_j|<2R_j$ and $|z_j-z_{j+1}|<2R_{j+1}$, if $\overline{z_j}=z_{j+1}$.

This immediately gives the following properties of F_ε:

- if z_j is real, then $\overline{F_{\varepsilon,j}(z)}=F_{\varepsilon,j}(z)$,
- if $\overline{z_j}=z_{j+1}$, then $\overline{F_{\varepsilon,j}(z)}=F_{\varepsilon,j+1}(z)$.

3. SPECTRAL CONDITIONS AND RESULTS

In Siegel's Theorem the eigenvalues are assumed to be a fortiori non-resonant. Here we want to relax the notion of non-resonance a bit. Therefore we consider a complex $n\times n$ matrix A with $\mathsf{Spec}(A)=\{\lambda_1,\cdots,\lambda_n\}$ where $\lambda_1,\cdots,\lambda_s$ have negative real part and $\lambda_{s+1},\cdots,\lambda_n$ have positive real part. Following [12] we consider (for any integer $\ell\geq 1$):

$$\mathcal{S}_{\ell,n,s} := \left\{ m\in\mathbf{N}^n | \sum_{j=1}^{s} m_j<\ell \text{ or } \sum_{j=s+1}^{n} m_j<\ell \right\} \tag{3}$$

$$\mathcal{T}_{\ell,n,s} := \left\{ m\in\mathbf{N}^n | \sum_{j=1}^{s} m_j\geq\ell \text{ and } \sum_{j=s+1}^{n} m_j\geq\ell \right\} \tag{4}$$

so $\mathbf{N}^n=\mathcal{S}_{\ell,n,s}\cup\mathcal{T}_{\ell,n,s}$ and $\mathcal{S}_{\ell,n,s}\cap\mathcal{T}_{\ell,n,s}=\emptyset$. For any formal power series $F(x)=\sum_{m\in\mathbf{N}^n}F_m x^m$ we have

$$\begin{aligned} F(x) &= \sum_{m\in\mathcal{S}_{\ell,n,s}} F_m x^m + \sum_{m\in\mathcal{T}_{\ell,n,s}} F_m x^m \\ &=: [F(x)]^{\mathcal{S}_{\ell,n,s}} + [F(x)]^{\mathcal{T}_{\ell,n,s}}. \end{aligned}$$

We recall that $\mathsf{Spec}(A)$ is a resonant set if there exists a $m \in \mathbf{N}^n$ with $|m| \geq 2$ and $k = 1, \cdots, n$ such that

$$\sum_{j=0}^{n} m_j \lambda_j = \lambda_k. \tag{5}$$

In what follows we will fix an integer ℓ (which one wants to take as large as possible in applications) and demand that no element of $\mathcal{S}_{\ell,n,s}$ is a solution of (5). In such a case we will say that $\mathcal{S}_{\ell,n,s}$ **causes no resonances** in $\mathsf{Spec}(A)$. It is obvious that if $\mathsf{Spec}(A)$ is non-resonant, then $\mathcal{S}_{\ell,n,s}$ will cause no resonance in $\mathsf{Spec}(A)$. To fix the ideas we give an example of a hyperbolic singularity which is resonant but no element of $\mathcal{S}_{\ell,n,s}$ (for given ℓ and n) satisfies (5). Take $\ell = 19$, $n = 3$, $s = 1$ and eigenvalues -11, $9+i$ and $9-i$, then the first resonance equation becomes

$$-11m_1 + (9+i)m_2 + (9-i)m_3 = -11.$$

We consider all solutions of this equation and take the solution with the smallest stable and unstable 'length'. In this case $m = (19, 11, 11)$, i.e. $m_1 = 19$ and $m_2 + m_3 = 22$. Thus, even though the system is resonant, $\mathcal{S}_{19,3,1}$ causes no resonances in $\{-11, 9+i, 9-i\}$.

In the statement of the results of this paper we shall use the following norms:

$$\begin{aligned} |y| &= \max_{1\leq j\leq n} |y_j|, \forall y \in \mathbf{C}^n \\ \|F\|_r &= \max_{|x|\leq r} |F(x)| \end{aligned}$$

for any continuous function F on $\mathbf{D}(0, r\mathbf{e})$.

THEOREM 2. *Consider a fixed integer $\ell \geq 1$ and an n-dimensional real vector field*

$$X_\varepsilon : \dot{x} = A_\varepsilon x + f_\varepsilon(x) \tag{6}$$

such that A_ε is a real $n \times n$ matrix in Jordan Normal Form, $f_\varepsilon(x) = O(|x|^2)$ for $x \to 0$ where f_ε is a real analytic function of x and ε such that $f_\varepsilon^\diamond$, the complex extension of $f_\varepsilon(x)$, is analytic on a poly-disk $\mathbf{D}(0,R) \times \mathbf{D}(0,r)$ and $\mathcal{S}_{\ell,n,s}$ causes no resonances in $\mathsf{Spec}(A_0)$, where s denotes the number of eigenvalues of A_0 that have a negative real part. Then there exists positive constants r_0, r_1, K_0, K_1, ρ and a change of coordinates

$$x = y + \phi_\varepsilon(y) \tag{7}$$

which is real analytic in (y, ε) *such that* $\phi_\varepsilon^\diamond$ *is analytic on* $\mathbf{D}(0, r_1\mathbf{e}) \times \mathbf{D}(0, \rho\mathbf{e})$*, such that* $\|\phi_\varepsilon\|_q \leq K_0 q^2$ *for* $q < r_0$ *and* $\varepsilon \in \mathbf{D}(0, \rho\mathbf{e})$*, and (7) conjugates (6) to*

$$Y_\varepsilon : \dot{y} = A_\varepsilon y + g_\varepsilon(y), \tag{8}$$

where $g_\varepsilon(y)$ *is real analytic in* (y, ε)*,* $[g_\varepsilon(y)]^{\mathcal{S}_{\ell,n,s}} = 0$ *and*

$$|g_\varepsilon(y)| \leq K_1 |(y_1, \cdots, y_s)|^\ell |(y_{s+1}, \cdots, y_n)|^\ell$$

for $y \in \mathbf{D}(0, r_1\mathbf{e})$*.*

From the properties of g_ε in (8) and the fact that the transformation given by (7) is analytic in the variable and the parameter gives us the following result as a corollary of Theorem 2.

Corollary 3. *Under the conditions of Theorem 2 we have that the stable and unstable manifold of* X_ε *at the origin are real analytic manifolds depending in a real analytic way on the parameter* ε*.*

If the original family admits symmetry, then we have the following result.

Theorem 4. *If - under the conditions of Theorem 2 - the family of vector fields* X_ε *admits an analytic family of symmetries* S_ε *(i.e.* S_ε *is an analytic family of linear maps such that* $(S_\varepsilon)_* X_\varepsilon = X_\varepsilon$*), then the transformation given by (7) commutes with* S_ε *and the resulting family of vector fields given by (8) admits the same family of symmetries.*

Remark 5. If we replace the family X_ε in Theorem 4 is reversible instead of symmetric with respect to S_ε, i.e. $(S_\varepsilon)_* X_\varepsilon = -X_\varepsilon$, then the result of Theorem 4 remains valid provided that $\ell \leq 3$. For instance this implies that the local stable and unstable manifold can straightened by means of an analytic change of variables which commutes with the symmetry.

4. ABSENCE OF SMALL DIVISORS

In this section we want to show that if $\mathcal{S}_{\ell,n,s}$ causes no resonances in $\mathsf{Spec}(A_0)$, then there exists a constant $\rho > 0$ such that for all $\varepsilon \in B(0, \rho) \subset \mathbf{C}^p$ we have that $\mathcal{S}_{\ell,n,s}$ causes no resonances in $\mathsf{Spec}(A_\varepsilon)$.

We consider the kth resonance equation (for $k = 1, \cdots, n$):

$$\begin{aligned}\sum_{j=1}^{a} r_j \nu_j(0) + \sum_{j=1}^{b} s_j(\alpha_j(0) + i\beta_j(0)) + \sum_{j=1}^{b} \tilde{s}_j(\alpha_j(0) - i\beta_j(0)) \ + \\ \sum_{j=1}^{c} t_j \mu_j(0) + \sum_{j=1}^{d} u_j(\gamma_j(0) + i\delta_j(0)) + \sum_{j=1}^{d} \tilde{u}_j(\gamma_j(0) - i\delta_j(0)) \ = \ \lambda_k(0)\end{aligned} \tag{9}$$

where $\lambda_j(\varepsilon)$ is the jth component of $\Lambda_\varepsilon = (\lambda_1(\varepsilon), \cdots, \lambda_n(\varepsilon))$. Looking more closely at this equation, one should note that there are actually two equations to consider: one coming from the real parts and one coming from the imaginary parts.

PROPOSITION 6. *Let A_ε be as in (6). Then the eigenvalues of A_0 are resonant iff the eigenvalues of $\tilde{A}_0$ are resonant, where $\tilde{A}_\varepsilon$ is the $(a+b+c+d)\times(a+b+c+d)$ matrix defined by*

$$\tilde{A}_\varepsilon = \begin{pmatrix} A_\varepsilon^{(1)} & 0 & 0 & 0 \\ 0 & \tilde{A}_\varepsilon^{(2)} & 0 & 0 \\ 0 & 0 & A_\varepsilon^{(3)} & 0 \\ 0 & 0 & 0 & \tilde{A}_\varepsilon^{(4)} \end{pmatrix}$$

where $A_\varepsilon^{(1)}$ and $A_\varepsilon^{(3)}$ are defined in (1) and $A_\varepsilon^{(2)} = diag(\alpha_1(\varepsilon), \cdots, \alpha_b(\varepsilon))$ and $A_\varepsilon^{(4)} = diag(\gamma_1(\varepsilon), \cdots, \gamma_d(\varepsilon))$.

Proof. The eigenvalues of A_0 form a resonant set iff (5) has a solution. Looking at the real and the imaginary part of this equation, we obtain the following two equations

$$\begin{aligned} \sum_{j=1}^{a} r_j\nu_j(0) + \sum_{j=1}^{b}(s_j+\tilde{s}_j)\alpha_j(0) + \sum_{j=1}^{c} t_j\mu_j(0) & \\ + \sum_{j=1}^{d}(u_j+\tilde{u}_j)\gamma j(0) &= \Re(\lambda_k(0)), \qquad (10) \\ \sum_{j=1}^{b}(s_j-\tilde{s}_j)\beta_j(0) + \sum_{j=1}^{d}(u_j-\tilde{u}_j)\delta_j(0) &= \Im(\lambda_k(0)). \qquad (11) \end{aligned}$$

If $\Im(\lambda_k(0)) = 0$, a solution of (11) is given by taking $\tilde{s}_j = s_j$ for $j = 1, \cdots, b$ and $\tilde{u}_j = u_j$ for $j = 1, \cdots, d$. If $\Im(\lambda_k(0)) \neq 0$, then we have to look at the sign of $\Re(\lambda_k(0))$. In the positive case there is a $q \in \{1, \cdots, d\}$ such that we take $\tilde{u}_q(0) = \tilde{u}_q(0) \pm 1$ (the $\pm$ is determined by the sign of $\Im(\lambda_k(0))$). Taking $\tilde{s}_j = s_j$ for $j = 1, \cdots, b$ and $\tilde{u}_j = u_j$ for $j \neq q$, we find a solution of (11). In the negative case there is a $q \in \{1, \cdots, d\}$ such that we take $\tilde{s}_q(0) = \tilde{s}_q(0) \pm 1$ (the $\pm$ is determined by the sign of $\Im(\lambda_k(0))$). Taking $\tilde{s}_j = s_j$ for $j \neq q$ and $\tilde{u}_j = u_j$ for $j = 1, \cdots, d$, we find a solution of (11). In all of these cases (10) is reduced to

$$\sum_{j=1}^{a} r_j\nu_j(0) + \sum_{j=1}^{b} s_j(2\alpha_j(0)) + \sum_{j=1}^{c} t_j\mu_j(0) + \sum_{j=1}^{d} u_j(2\gamma_j(0)) = \Re(\lambda_k(0)).$$

This latter equation is equivalent with saying that there is resonance between the eigenvalues of

$$\hat{A} = \begin{pmatrix} A_0^{(1)} & 0 & 0 & 0 \\ 0 & 2\tilde{A}_0^{(2)} & 0 & 0 \\ 0 & 0 & A_0^{(3)} & 0 \\ 0 & 0 & 0 & 2\tilde{A}_0^{(4)} \end{pmatrix}.$$

In a similar way one proves that the eigenvalues of $\tilde{A}_0$ are resonant iff the eigenvalues of $\hat{A}$ are resonant. ▮

From the proof of Proposition 6 we obtain the following result.

Corollary 7. *Let A_ε be as in (6). Then $\mathcal{S}_{\ell,n,s}$ causes no resonances in* $\mathsf{Spec}(A_0)$ *iff $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ causes no resonances in* $\mathsf{Spec}(\tilde{A}_0)$*, where $\tilde{A}_\varepsilon$ is defined in Proposition 6 and*

$$\begin{aligned} \tilde{n} &= a+b+c+d, \\ \tilde{s} &= a+b, \end{aligned}$$

$$\frac{\ell}{2} - \max_{1\le k\le n} \frac{\tilde{\lambda}_k(0)}{2} \le \tilde{\ell} \le \frac{\ell}{2} - \min_{1\le k\le n} \frac{\tilde{\lambda}_k(0)}{2}.$$

The exact value of $\tilde{\ell}$ depends on $\mathsf{Spec}(\tilde{A}_0)$.

In order to fix the ideas we give some examples of this situation:

- Consider $\mathsf{Spec}(A_0) = \{-3, 5+i, 5-i\}$, then $\mathcal{S}_{6,3,1}$ causes no resonances in $\mathsf{Spec}(A_0)$ (and 6 is the maximal value of ℓ causing no resonances). As $\mathsf{Spec}(\tilde{A}_0) = \{-3, 5\}$, we have that $\mathcal{S}_{3,2,1}$ causes no resonances in $\mathsf{Spec}(\tilde{A}_0)$, hence $\tilde{\ell} = 3 = \frac{\ell}{2}$.
- Consider $\mathsf{Spec}(A_0) = \{-2, 5+i, 5-i\}$, then $\mathcal{S}_{2,3,1}$ causes no resonances in $\mathsf{Spec}(A_0)$. As $\mathsf{Spec}(\tilde{A}_0) = \{-2, 5\}$, we have that $\mathcal{S}_{2,2,1}$ causes no resonances in $\mathsf{Spec}(\tilde{A}_0)$, hence $\tilde{\ell} = 2 = \frac{\ell}{2} - \frac{(-2)}{2}$.

To facilitate the notations, we use the constants $\tilde{\ell}$, $\tilde{n}$ and $\tilde{s}$ defined in Corollary 7, this way we can write that $\tilde{A}_\varepsilon$ is an $\tilde{n}\times\tilde{n}$ matrix and that there are $\tilde{s}$ stable directions. Also we introduce $\tilde{\Lambda}_\varepsilon$ as the $\tilde{n}$-tuple of eigenvalues of $\tilde{A}_\varepsilon$.

Now we look at the kth resonance equations on the eigenvalues of $\tilde{A}_0$:

$$\sum_{j=1}^{a} r_j\nu_j(0) + \sum_{j=1}^{b} s_j\alpha_j(0) + \sum_{j=1}^{c} t_j\mu_j(0) + \sum_{j=1}^{d} u_j\gamma_j(0) = \tilde{\lambda}_k(0). \quad (12)$$

As we assume that the eigenvalues are non-resonant, (12) has no non-trivial solutions in $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$. We can interpret this non-resonance in the following geometrical way. Consider the $\tilde{n}$-tuple

$$(r_1, \cdots, r_a, s_1, \cdots, s_b, t_1, \cdots, t_c, u_1, \cdots, u_d)$$

as a point on the grid $\mathbf{Z}^{\tilde{n}}$, then the non-resonance of the eigenvalues of $\tilde{A}_0$ means that the hyperplane H with equation given by (12) contains only one of the "grid points" in $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$. This point is the intersection of H with the x_k-axis (the kth axis in $\mathbf{R}^{\tilde{n}}$) and it has coordinates $e_k := (0, \cdots, 0, 1, 0, \cdots, 0)$ with a 1 on the kth position. The hyperplane H will intersect the x_j-axis (for $j \neq k$) in the point $\frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} e_j$. For each point P of $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ (with $|P| \geq 2$) we consider the hyperplanes through the points P and e_k. These hyperplanes will intersect each axis in a point of the form $\left(\frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} + \eta_{H',P}\right) e_j$ where $\eta_{H',P} \in \mathbf{R} \setminus \{0\}$ depends on the hyperplane H' and the point P. As we are working in $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ we know that $\min_{P \in \mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}} |\eta_{H',P}| > 0$, so there exists a $\theta > 0$ such that $\theta = \min_{P \in \mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}} |\eta_{H',P}| > 0$. Let us denote the hyperplane that gives this θ by $\hat{H}$, the intersection of this hyperplane with the axis will give us the "closest" resonance. This way we have obtained a bound for the ratio of the eigenvalues of $\tilde{A}_\varepsilon$:

$$\frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} - \theta < \frac{\tilde{\lambda}_k(\varepsilon)}{\tilde{\lambda}_j(\varepsilon)} < \frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} + \theta, \tag{13}$$

for $j = 1, \cdots, \tilde{n}$. The region U of $\mathbf{R}^{\tilde{n}}$ defined by the bounds

$$\frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} - \theta < \frac{x_k}{x_j} < \frac{\tilde{\lambda}_k(0)}{\tilde{\lambda}_j(0)} + \theta,$$

for $j = 1, \cdots, \tilde{n}$, is an open subset of $\mathbf{R}^{\tilde{n}}$ containing $\tilde{\Lambda}_0$. We know that for each $k = 1, \cdots, \tilde{n}$, $\varepsilon \mapsto \tilde{\lambda}_k(\varepsilon)$ is a continuous map that is either strictly positive either strictly negative in a neighbourhood of the origin. As U is open, the continuity of the mappings $\varepsilon \mapsto \frac{\tilde{\lambda}_k(\varepsilon)}{\tilde{\lambda}_j(\varepsilon)}$ gives us the existence of a $\rho_k > 0$ such that (13) is fulfilled for all $\varepsilon \in B(0, \rho_k)$. Taking ρ as minimum of all ρ_k (as there only a finite number of ρ_k, we have that $\rho > 0$), we have that $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ causes no resonances on the eigenvalues of $\tilde{A}_\varepsilon$ for all $\varepsilon \in B(0, \rho) = \mathbf{D}(0, \rho\mathbf{e})$. By virtue of Proposition 6 we have that $\mathcal{S}_{\ell,n,s}$ causes no resonances on the eigenvalues of A_ε if $\varepsilon \in B(0, \rho) = \mathbf{D}(0, \rho\mathbf{e})$.

Proposition 8. *If $\mathcal{S}_{\ell,n,s}$ causes no resonances in $\mathsf{Spec}(A_0)$, then there exists a positive constant κ such that $\forall m \in \mathcal{S}_{\ell,n,s}$ and $\forall \varepsilon \in B(0, \rho) =$*

$\mathbf{D}(0,\rho\mathbf{e})$ *(where ρ was determined in the previous argumentation):*

$$|\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon)| \geq \kappa |m| \tag{14}$$

where $1 \leq j \leq n$ and $\lambda_j(\varepsilon)$ denotes the jth eigenvalue of A_ε.

As we have that

$$\begin{array}{rcl} |\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon)| & \geq & |\Re(\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon))| \\ & \geq & \left|\left\langle \tilde{\Lambda}_\varepsilon, \tilde{m}\right\rangle - \Re(\lambda_j(\varepsilon))\right| \end{array}$$

where $\tilde{m} \in \mathcal{S}_{\ell,\tilde{n},\tilde{s}}$ is related to m as follows:

$$\begin{array}{ll} \tilde{m}_j = m_j & \text{for } 1 \leq j \leq a \\ \tilde{m}_{a+j} = m_{a+2j-1} + m_{a+2j} & \text{for } 1 \leq j \leq b \\ \tilde{m}_{a+b+j} = m_{a+2b+j} & \text{for } 1 \leq j \leq c \\ \tilde{m}_{a+b+c+j} = m_{a+2b+c+2j-1} + m_{a+2b+c+2j} & \text{for } 1 \leq j \leq d, \end{array}$$

and

$$|\tilde{m}| = |m|.$$

Proposition 8 will be a consequence of

Proposition 9. *There exists a positive constant K such that for the eigenvalues of $\tilde{A}_\varepsilon$ we have that $\forall \varepsilon \in B(0,\rho) = \mathbf{D}(0,\rho\mathbf{e})$:*

$$\left|\left\langle \tilde{\Lambda}_\varepsilon, m\right\rangle - \tilde{\lambda}_j(\varepsilon)\right| \geq K|m| \tag{15}$$

for all $m \in \mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ and $j = 1, \cdots, \tilde{n}$.

To prove Proposition 9 we need another result. To make the proof a bit clearer, we will assume that the eigenvalues of $\tilde{A}_0$ meet

$$\tilde{\lambda}_1(0) \leq \cdots \leq \tilde{\lambda}_{\tilde{s}}(0) < 0 < \tilde{\lambda}_{\tilde{s}+1}(0) \leq \cdots \leq \tilde{\lambda}_{\tilde{n}}(0).$$

This can be achieved by a permutation of the basis vectors, so it won't effect the result given in (15).

Before stating and proving the lemma, we need to introduce the following notations

$$q_0(\varepsilon) := \begin{cases} \max\limits_{1\leq j\leq \tilde{s}} \tilde{\lambda}_j(\varepsilon) & \text{if } \min\limits_{1\leq j\leq \tilde{s}} |\tilde{\lambda}_j(\varepsilon)| < \min\limits_{\tilde{s}+1\leq j\leq \tilde{n}} |\tilde{\lambda}_j(\varepsilon)| \\ \min\limits_{\tilde{s}+1\leq j\leq \tilde{n}} \tilde{\lambda}_j(\varepsilon) & \text{if } \min\limits_{1\leq j\leq \tilde{s}} |\tilde{\lambda}_j(\varepsilon)| > \min\limits_{\tilde{s}+1\leq j\leq \tilde{n}} |\tilde{\lambda}_j(\varepsilon)| \end{cases}$$

$$
\begin{aligned}
q_{+}(\varepsilon) &:= \begin{cases} \min\limits_{1\le j\le \tilde{s}} \tilde{\lambda}_j(\varepsilon) & \text{if } q_0(\varepsilon) < 0 \\ \max\limits_{\tilde{s}+1\le j\le \tilde{n}} \tilde{\lambda}_j(\varepsilon) & \text{if } q_0(\varepsilon) > 0 \end{cases} \\
q_{-}(\varepsilon) &:= \begin{cases} \max\limits_{\tilde{s}+1\le j\le \tilde{n}} \tilde{\lambda}_j(\varepsilon) & \text{if } q_+(\varepsilon) < 0 \\ \min\limits_{1\le j\le \tilde{s}} \tilde{\lambda}_j(\varepsilon) & \text{if } q_+(\varepsilon) > 0 \end{cases} \\
\lceil x \rceil &:= \min\{k \in \mathbf{Z} | x \le k\}.
\end{aligned}
$$

We remark that q_0, q_+ and q_- are always continuous functions of ε but not necessarily analytic functions of ε.

For each $m \in \mathcal{S}_{\ell,n,s}$ we use the following notations which denotes the splitting with respect to the stable and the unstable directions:

$$
\begin{aligned}
M_s &:= (m_1, \cdots, m_s) \\
M_u &:= (m_{s+1}, \cdots, m_n).
\end{aligned}
$$

Lemma 10. *As $\mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ causes no resonances in $\mathsf{Spec}(\tilde{A}_\varepsilon)$, we have for all $m \in \mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ satisfying*

$$
|m| \ge \left\lceil \frac{q_+(\varepsilon)}{q_0(\varepsilon)} - (\tilde{\ell}-1)\frac{q_-(\varepsilon)}{q_0(\varepsilon)} + (\tilde{\ell}-1) \right\rceil := \Xi(\varepsilon, \tilde{\ell}) \tag{16}
$$

the following inequality

$$
\left| \left\langle \tilde{\Lambda}_\varepsilon, m \right\rangle - \tilde{\lambda}_j \right| \ge |(\tilde{\ell}-1)q_-(\varepsilon) + (|m| - \tilde{\ell} + 1)q_0(\varepsilon) - q_+(\varepsilon)| \tag{17}
$$

for all $j = 1, \cdots, \tilde{n}$.

Proof. First we establish the inequality for those m for which $|m|$ is "sufficiently" large, afterwards we show that these $|m|$ are bounded below by $\Xi(\varepsilon, \tilde{\ell})$.

First we consider the case where $|M_{\tilde{s}}| < \tilde{\ell}$. For $|M_{\tilde{u}}|$ sufficiently large $\left\langle \tilde{\Lambda}_\varepsilon, m \right\rangle - \tilde{\lambda}_j(\varepsilon)$ will be positive. So taking $|M_{\tilde{s}}| = \tilde{\ell} - 1$, we have

$$
\begin{aligned}
\left\langle \tilde{\Lambda}^{\tilde{s}}_\varepsilon, M_{\tilde{s}} \right\rangle &\ge (\tilde{\ell}-1)\tilde{\lambda}_1(\varepsilon), \\
\left\langle \tilde{\Lambda}^{\tilde{u}}_\varepsilon, M_{\tilde{u}} \right\rangle &\ge \tilde{\lambda}_{\tilde{s}+1}(\varepsilon)(|m| - \tilde{\ell} + 1), \\
-\tilde{\lambda}_j(\varepsilon) &\ge -\tilde{\lambda}_{\tilde{n}}(\varepsilon),
\end{aligned}
$$

where

$$
\tilde{\Lambda}^{\tilde{s}}_\varepsilon := (\tilde{\lambda}_1(\varepsilon), \cdots, \tilde{\lambda}_{\tilde{s}}(\varepsilon)),
$$

$$\tilde{\Lambda}_\varepsilon^{\tilde{u}} \;:=\; (\tilde{\lambda}_{\tilde{s}+1}(\varepsilon), \cdots, \tilde{\lambda}_{\tilde{n}}(\varepsilon)).$$

So we can conclude

$$\left\langle \tilde{\Lambda}, m \right\rangle - \tilde{\lambda}_j(\varepsilon) \geq (\tilde{\ell}-1)\tilde{\lambda}_1(\varepsilon) + \tilde{\lambda}_{s+1}(\varepsilon)(|m| - \tilde{\ell} + 1) - \tilde{\lambda}_{\tilde{n}}(\varepsilon) > 0. \quad (18)$$

Second we consider the case where $|M_{\tilde{u}}| < \tilde{\ell}$. For $|M_{\tilde{s}}|$ sufficiently large $\left\langle \tilde{\Lambda}_\varepsilon, m \right\rangle - \tilde{\lambda}_j(\varepsilon)$ will be negative. So taking $|M_{\tilde{u}}| = \tilde{\ell} - 1$, we have

$$\begin{aligned} \left\langle \tilde{\Lambda}_\varepsilon^{\tilde{s}}, M_{\tilde{s}} \right\rangle &\leq (|m| - \tilde{\ell} + 1)\tilde{\lambda}_{\tilde{s}}(\varepsilon), \\ \left\langle \tilde{\Lambda}_\varepsilon^{\tilde{u}}, M_{\tilde{u}} \right\rangle &\leq (\tilde{\ell} - 1)\tilde{\lambda}_{\tilde{n}}(\varepsilon), \\ -\tilde{\lambda}_j(\varepsilon) &\leq -\tilde{\lambda}_1(\varepsilon). \end{aligned}$$

So we can conclude

$$\left\langle \tilde{\Lambda}, m \right\rangle - \tilde{\lambda}_j(\varepsilon) \leq (\tilde{\ell}-1)\tilde{\lambda}_{\tilde{n}}(\varepsilon) + \tilde{\lambda}_s(\varepsilon)(|m| - \tilde{\ell} + 1) - \tilde{\lambda}_1(\varepsilon) < 0. \quad (19)$$

Combining (18) and (19) we find the inequality stated in (17).

The right-hand side of (17) will be increasing after the unique zero of this function. A short calculation will give us the lower bound $\Xi(\varepsilon, \tilde{\ell})$ as stated in (16). ▮

Proof (of proposition 9). From (17) we deduce that there exists a constant K_* for all $m \in \mathcal{S}_{\tilde{\ell},\tilde{n},\tilde{s}}$ with $|m| \geq \Xi(\varepsilon, \tilde{\ell})$ such that $\left| \left\langle \tilde{\Lambda}_\varepsilon, m \right\rangle - \tilde{\lambda}_j(\varepsilon) \right| \geq K_*|m|$. Starting from the right-hand side of (17) we have that

$$\begin{aligned} |(\tilde{\ell}-1)q_-(\varepsilon) + (|m| - \tilde{\ell} + 1)q_0(\varepsilon) - q_+(\varepsilon)| &\geq \\ |q_0(\varepsilon)|.|m| - |(\tilde{\ell}-1)q_-(\varepsilon) + (1-\tilde{\ell})q_0(\varepsilon) - q_+(\varepsilon)|. \end{aligned}$$

The latter expression will be positive for $|m| \geq \Xi_1(\varepsilon, \tilde{\ell})$. Take $|m| \geq \xi_{\varepsilon,\tilde{\ell}} := \max\{\Xi(\varepsilon, \tilde{\ell}), \Xi_1(\varepsilon, \tilde{\ell})\}$, then

$$\begin{aligned} |q_0(\varepsilon)| - \tfrac{|(\tilde{\ell}-1)q_-(\varepsilon) + (1-\tilde{\ell})q_0(\varepsilon) - q_+(\varepsilon)|}{|m|} &\geq \\ K_* := \inf_{\varepsilon \in B(0,\rho)} \left(|q_0(\varepsilon)| - \frac{|(\tilde{\ell}-1)q_-(\varepsilon) + (1-\tilde{\ell})q_0(\varepsilon) - q_+(\varepsilon)|}{\xi_{\varepsilon,\tilde{\ell}}} \right), \end{aligned}$$

hence

$$|(\tilde{\ell}-1)q_-(\varepsilon) + (|m| - \tilde{\ell} + 1)q_0(\varepsilon) - q_+(\varepsilon)| \geq K_*|m|$$

and

$$K_* \leq \inf_{\varepsilon \in B(0,\rho)} |q_0(\varepsilon)|.$$

For each r with $1 \leq |m| < \Xi(\varepsilon, \tilde{\ell})$ we can find a constant $K_m > 0$ such that $\left|\left\langle \tilde{\Lambda}_\varepsilon, m \right\rangle - \tilde{\lambda}_j(\varepsilon)\right| \geq K_m |m|$: we just take

$$K_m := \inf_{\varepsilon \in B(0,\rho)} \frac{\left|\left\langle \tilde{\Lambda}_\varepsilon, m \right\rangle - \tilde{\lambda}_j(\varepsilon)\right|}{|m|}.$$

Defining

$$K := \min \left(\{K_r \mid |r| < \Xi(\varepsilon, \tilde{\ell})\} \cup \{K_*\} \right),$$

we have the wanted constant. K will be strictly positive as it is the minimum of a finite set of strictly positive numbers. ▮

5. PROOF OF THEOREM 2

The proof of Theorem 2 consists of 3 parts: first we determine two equations that will give us (7), second we show that there exists a formal solution and finally we show that the formal solution converges, i.e. there exists an analytic solution.

5.1. Determining the change of coordinates

In this subsection we want to establish an equation which will allow us to determine the transformation (7) we are seeking. From now on we will work in the complexified setting given by (2), this will make it easier to determine a formal solution. This means that we will need to "complexify" the function ϕ_ε given by (7).

So we have a vector field given by

$$\dot{z} = B_\varepsilon z + F_\varepsilon(z)$$

where $z = Px$ with P defined previously, which we want to transform into a vector field

$$\dot{w} = B_\varepsilon w + G_\varepsilon(w)$$

where $P^{-1}w \in \mathbf{R}^n$ (as we wish to return to a real vector field at the end) and $[G_\varepsilon(w)]^{\mathcal{S}_{\ell,n,s}} = 0$, by a transformation

$$z = w + \varphi_\varepsilon(w).$$

In order to return to a real vector field we will have

$$\phi_\varepsilon(w) = P^{-1} \cdot \varphi_\varepsilon(Pw) \tag{20}$$

and

$$G_\varepsilon(w) = P^{-1} \cdot g_\varepsilon(Pw)$$

where P is the matrix defined previously, that gives the change of basis. Performing this transformation we find the following two equalities

$$\begin{aligned} \dot{z} &= (I_n + D_w\varphi_\varepsilon(w))(B_\varepsilon w + G_\varepsilon(w)) \\ \dot{z} &= B_\varepsilon(w + \varphi_\varepsilon(w)) + F_\varepsilon(w + \varphi_\varepsilon(w)) \end{aligned}$$

If we introduce the operator L_{B_ε}

$$L_{B_\varepsilon}\varphi_\varepsilon(w) = D_w\varphi_\varepsilon(w)B_\varepsilon w - B_\varepsilon\varphi_\varepsilon(w) \tag{21}$$

then these equations can be combined to obtain

$$L_{B_\varepsilon}\varphi_\varepsilon(w) = F_\varepsilon(w + \varphi_\varepsilon(w)) - G_\varepsilon(w) - D_w\varphi_\varepsilon(w)G_\varepsilon(w) \tag{22}$$

We split (22) up into two separate equations. This splitting up will be done with respect to $\mathcal{S}_{\ell,n,s}$ and $\mathcal{T}_{\ell,n,s}$. Thus we will solve

$$L_{B_\varepsilon}\varphi_\varepsilon(w) = [F_\varepsilon(w + \varphi_\varepsilon(w))]^{\mathcal{S}_{\ell,n,s}} \tag{23}$$

$$[F_\varepsilon(w + \varphi_\varepsilon(w))]^{\mathcal{T}_{\ell,n,s}} = (I_n + D_w\varphi_\varepsilon(w))G_\varepsilon(w) \tag{24}$$

If we can solve (23), then we can determine $G_\varepsilon(w)$ directly as $(I_n + D_w\varphi_\varepsilon(w))$ is invertible in a sufficiently small neighbourhood of the origin. We know that the formal expansion of φ_ε starts with terms of degree 2 in w, so multiplying $[F_\varepsilon(w + \varphi_\varepsilon(w))]^{\mathcal{T}_{\ell,n,s}}$ with $(I_n + D_w\varphi_\varepsilon(w))^{-1}$ will only increase the degree of each term in w, hence

$$\left[(I_n + D_w\varphi_\varepsilon(w))^{-1}\,[F_\varepsilon(w) + \varphi_\varepsilon(w))]^{\mathcal{T}_{\ell,n,s}}\right]^{\mathcal{S}_{\ell,n,s}} = 0,$$

so $[G_\varepsilon(w)]^{\mathcal{S}_{\ell,n,s}} = 0$.

Also from (24) we immediately have that $|G_\varepsilon(w)| \le K_1|W_s|^\ell|W_u|^\ell$, where $W_s = (w_1, \cdots, w_s)$ and $W_u = (w_{s+1}, \cdots, w_n)$. Hence by virtue of (20) we have the same bounds for $g_\varepsilon(y)$.

In the next subsections we will solve (23) and show that the solution has all properties as stated in Theorem 2.

5.2. Formal solution of (23)

A direct calculation shows that

$$L_{B_{\varepsilon,j}}(vw^m) = v(\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon))w^m,$$

$1 \leq j \leq n$, for any $m \in \mathbf{N}^n$ and any $v \in \mathbf{C}^n$. This means that if we want to have a formal solution $\varphi_\varepsilon(w) = \sum_{|m|\geq 2} a_m(\varepsilon)w^m$, then (23) becomes

$$\sum_{|m|\geq 2} a_{m,j}(\varepsilon)(\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon))w^m = \left[\sum_{|m|\geq 2} F_{m,j}(\varepsilon)\left(w + \sum_{|k|\geq 2} a_k(\varepsilon)w^k\right)^m\right]^{\mathcal{S}_{\ell,n,s}} \tag{25}$$

where $1 \leq j \leq n$ and

$$F_\varepsilon(w) = \sum_{|m|\geq 2} F_m(\varepsilon)w^m.$$

We now show how (25) can be solved formally. First we take the coefficient of w^M for $M \in \mathbf{N}^n$ with $|M| = 2$, then (25) gives

$$a_{M,j}(\varepsilon) = \frac{F_{M,j}(\varepsilon)}{\langle \Lambda_\varepsilon, M\rangle - \lambda_j(\varepsilon)}$$

thus $a_{M,j}(\varepsilon)$ is an analytic function. We now proceed by induction, so assume that $a_{m,j}$ is an analytic function of ε for all $m \in \mathbf{N}^n$ with $2 \leq |m| \leq N-1$. Now take a $m \in \mathbf{N}^n$ with $|m| = N$. Taking the coefficients of w^m in (25) we find

$$a_{m,j}(\varepsilon) = \sigma_m \frac{F_{m,j}(\varepsilon) + \sum\limits_{\substack{r\in\mathbf{N}^n \\ |r|\leq N-1}} P_r^m((a_k(\varepsilon))|_{|k|\leq N-1})F_{r,j}(\varepsilon)}{\langle \Lambda_\varepsilon, m\rangle - \lambda_j(\varepsilon)} \tag{26}$$

where σ_m is defined by

$$\sigma_m := \begin{cases} 1 & \text{if } \ m \in \mathcal{S}_{\ell,n,s} \\ 0 & \text{if } \ m \in \mathcal{T}_{\ell,n,s} \end{cases}$$

and where P_r^m is a polynomial with positive integer coefficients. This result can be proved by induction.

As we know that all $F_{r,j}$ are analytic on the same poly-disk and the denominator is non-zero, the induction hypothesis will give us that $a_{M,j}(\varepsilon)$ is an analytic function of ε in a poly-disk independent of M and j.

5.3. Convergence of the formal solution

We now want to prove that this formal solution converges, i.e. we have an analytic solution in w. For this we will use the classical technique of *majorants* [6, 10, 12]. Using this technique in combination with Proposition 1 will give us that $\varphi_\varepsilon(w)$ is analytic in (w,ε). Given two formal power series $f(z)=\sum_{m\in\mathbf{N}^n} f_m z^m$ and $g(z)=\sum_{m\in\mathbf{N}^n} g_m z^m$, one says that g is a *majorant* of f if we have that $|f_m|\leq g_m$, $\forall m\in\mathbf{N}^n$. One should note that in the latter definition the coefficients of $g(z)$ must be real and positive whilst the coefficients of $f(z)$ may be complex.

Given $m\in\mathcal{S}_{\ell,n,s}$ with $|m|\geq 2$, we have that

$$\nu(m) := \inf_{\varepsilon\in B(0,\rho)} \min_{1\leq k\leq n} |\langle \Lambda_\varepsilon, m\rangle - \lambda_k(\varepsilon)|$$

is bounded away from zero by virtue of Proposition 8. If we use the notation

$$\tilde{c}_m := \sup_{\varepsilon\in B(0,\tilde{\rho})} \max_{1\leq k\leq n} |F_{m,k}(\varepsilon)|,$$

for a fixed $\tilde{\rho}$ with $0<\tilde{\rho}<\rho$, then we can define

$$\tilde{F}(w) = \sum_{|m|\geq 2} \tilde{c}_m w^m \mathbf{e}$$

so $\tilde{F}$ is analytic in w, $\tilde{F}_1=\cdots=\tilde{F}_n$ and $\tilde{F}_j$ is a majorant of $F_{\varepsilon,j}$ for $j=1,\cdots,n$.

Let $\tilde{\varphi}(w)=\sum_{|m|\geq 2}\tilde{a}_m w^m$ be the solution of

$$\sum_{\substack{|m|\geq 2\\ m\in\mathcal{S}_{\ell,n,s}}} \nu(m)\tilde{a}_m w^m = \left[\tilde{F}(w+\tilde{\varphi}(w))\right]^{\mathcal{S}_{\ell,n,s}}. \tag{27}$$

As the coefficients on the right-hand side of (23) are majorised by the coefficients on the right-hand side of (27) and the moduli of the coefficients on the left-hand side of (23) are majorising the coefficients on the left-hand side of (27), hence by division and (26) we obtain that $\tilde{\varphi}$ is a majorant of φ_ε for all $\varepsilon\in B(0,\tilde{\rho})$, in other words

$$|a_{m,j}(\varepsilon)| \leq \tilde{a}_{m,j}, j=1,\cdots,n \tag{28}$$

we also have that $\tilde{\varphi}_1=\cdots=\tilde{\varphi}_n$.

We would like to reduce the of question of convergence to a 1-dimensional problem. Therefore we will need another majorant. We define

$$c_k := \sum_{|m|=k} \tilde{c}_m$$

and

$$\hat{F}(Z) := \sum_{k\geq 2} c_k Z^k, Z \in \mathbf{C}$$

then $\hat{F}(Z)$ equals $\tilde{F}_j(Z\mathbf{e})$ for each $j = 1, \cdots, n$, so $\hat{F}(Z)$ is obviously a majorant for each component of $\tilde{F}(Z\mathbf{e})$. As $\hat{F}(Z)\mathbf{e} = \tilde{F}(Z\mathbf{e})$, $\hat{F}$ is analytic iff $|Z| \leq R_j$ for all $j = 1, \cdots, n$. Hence $\hat{F}(Z)$ is analytic on $B(0, \hat{R}) = \mathbf{D}(0, \hat{R}\mathbf{e})$ where $\hat{R} = \min_{1\leq j\leq n} R_j$. In the same line of arguments we introduce

$$\nu_k := \min_{\substack{|m|=k \\ m\in\mathcal{S}_{\ell,n,s}}} \nu(m)$$

then by Proposition 8 we know there exists a constant $\kappa > 0$ for which we have

$$\nu_k \geq \kappa k.$$

We can look at the solution $\hat{\varphi}(Z) = \sum_{k\geq 2} \hat{a}_k Z^k$ of

$$\sum_{k\geq 2} \kappa k \hat{a}_k Z^k = \hat{F}(Z + \hat{\varphi}(Z)). \tag{29}$$

As before we obtain that $\hat{\varphi}(Z)$ is a majorant of each component of $\tilde{\varphi}(Z\mathbf{e})$, i.e.

$$\tilde{a}_{m,j} \leq \hat{a}_k$$

for all $m \in \mathcal{S}_{\ell,n,s}$ with $|m| = k$ and $1 \leq j \leq n$.

As $k \geq 2$, it is obvious that $\sum_{k\geq 2} \kappa k \hat{a}_k Z^k$ is a majorant for $\sum_{k\geq 2} \kappa \hat{a}_k Z^k$. We know that $\hat{F}$ is analytic on $B(0, \hat{R})$, so we have that

$$\overline{\lim_{k\to\infty}} \sqrt[k]{c_k} = \frac{1}{\hat{R}}. \tag{30}$$

Take a small but fixed $\delta > 0$, then (30) implies that there exists a $K \in \mathbf{N}$ such that for all $k \geq K$ we have

$$\sqrt[k]{c_k} \leq \frac{1+\delta}{\hat{R}},$$

whence

$$c_k \leq \left(\frac{1+\delta}{\hat{R}}\right)^k, \forall k \geq K.$$

For $2 \le k \le K-1$ we obviously have

$$c_k \le c_k \left(\frac{\hat{R}}{1+\delta}\right)^k \left(\frac{1+\delta}{\hat{R}}\right)^k.$$

Defining $\check{R} = \frac{\hat{R}}{1+\delta}$ and

$$\check{c} := \max\left(\left\{ c_k \left(\frac{\hat{R}}{1+\delta}\right)^k \middle| \, 2 \le k \le K-1 \right\} \cup \{1\}\right),$$

we have that

$$c_k \le \check{c}\left(\frac{1}{\check{R}}\right)^k, \forall k \ge 2.$$

As $\check{F}(Z) := \check{c}\sum_{k\ge 2}\left(\frac{Z}{\check{R}}\right)^k$ is a geometrical series, we know that $\check{F}(Z)$ is analytic on $B(0,\check{R})$ and $\check{F}$ is a majorant of $\hat{F}$.

Let $\Phi(Z) = \sum_{k\ge 2} \check{a}_k Z^k$ be the solution of

$$\kappa\Phi(Z) = \check{F}(Z+\Phi(Z)), \tag{31}$$

then Φ will be a majorant of $\hat{\varphi}$.

As $\check{F}$ is given by a geometrical series (31) becomes

$$\begin{aligned} \kappa\Phi(z) &= \check{c}\left(\frac{Z+\Phi(Z)}{\check{R}}\right)^2 \sum_{k\ge 2}\left(\frac{Z+\Phi(Z)}{\check{R}}\right)^{k-2} \\ &= \frac{\check{c}}{\check{R}^2}\frac{(Z+\Phi(Z))^2}{1-\frac{Z+\Phi(Z)}{\check{R}}}, \end{aligned}$$

which gives the following quadratic equation in $\Phi(Z)$:

$$\left(\check{c}+\kappa\check{R}\right)\Phi(Z)^2 + \left(\left(2\check{c}+\kappa\check{R}\right)Z - \kappa\check{R}^2\right)\Phi(Z) + \check{c}Z^2 = 0. \tag{32}$$

The discriminant of (32) is given by

$$D(Z) = \left(\left(2\check{c}+\kappa\check{R}\right)Z - \kappa\check{R}^2\right)^2 - 4\left(\check{c}+\kappa\check{R}\right)\check{c}Z^2.$$

Now Φ is given by

$$\Phi(Z) = \frac{\check{R}^2\kappa - \left(2\check{c}+\kappa\check{R}\right)Z - \sqrt{D(Z)}}{2\left(\check{c}+\kappa\check{R}\right)},$$

where we take the solution with $-\sqrt{D(Z)}$ as we need the solution without constant and linear terms in its formal series expansion. Now it is clear that Φ is analytic in $B(0,\check{R})$. From the series expansion it is clear that $\|\Phi\|_r \leq K_0 r^2$ for any $r < r_0 := \check{R}$, and by virtue of the majorisation we have the same bound for φ_ε. Writing down everything in its real components we obtain the properties stated in Theorem 2.

6. SYMMETRIC CASE

Consider an analytic family of linear maps $S_\varepsilon : \mathbf{R}^n \to \mathbf{R}^n$, then S_ε is a *symmetry* of the family of vector fields X_ε if $(S_\varepsilon)_* X_\varepsilon = X_\varepsilon$. Consider a family of real vector fields X_ε with a symmetry S_ε. First of all we put $DX_\varepsilon(0)$ in its Jordan Normal Form A_ε, this can be done with a suitable matrix M_ε such that $A_\varepsilon = M_\varepsilon^{-1} DX_\varepsilon(0) M_\varepsilon$, hence the vector field becomes $\tilde{X}_\varepsilon = M_\varepsilon^{-1} \cdot X_\varepsilon \circ M_\varepsilon$. It is well-known, see for instance [2], that under a linear change of coordinates the symmetry S_ε of X_ε is transformed into the symmetry $\tilde{S}_\varepsilon = M_\varepsilon^{-1} \cdot S_\varepsilon \cdot M_\varepsilon$.

So from now on we will assume that $DX_\varepsilon(0) = A_\varepsilon$ is already in its Jordan Normal Form. As S_ε is a symmetry of the family of real vector fields X_ε, we have that $T_\varepsilon := P^{-1} S_\varepsilon P$ is a symmetry of the complexified system where P was defined previously. Given the fact that T_ε is a symmetry of $\dot{z} = B_\varepsilon z + F_\varepsilon(z)$, we necessarily have that T_ε commutes with B_ε and F_ε. As B_ε is diagonal and all its eigenvalues are non-zero and have multiplicity 1, T_ε will be diagonal as well. Therefore T_ε cannot "mix up" stable directions with unstable directions. This gives us that T_ε also commutes with $[\cdot]^{\mathcal{S}_{\ell,n,s}}$ and $[\cdot]^{\mathcal{T}_{\ell,n,s}}$.

6.1. ϕ_ε commutes with S_ε

First we show that T_ε commutes with φ_ε. To obtain this result we need to look at (23). We know that φ_ε is the unique analytic non-zero solution of (23), so if we prove that $T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon$ is also a solution of (23) then by unicity we have that $\varphi_\varepsilon = T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon$ or in other words $T_\varepsilon \circ \varphi_\varepsilon = \varphi_\varepsilon \circ T_\varepsilon$.

Let us define $\psi_\varepsilon := T_\varepsilon^{-1} \circ \psi_\varepsilon \circ T_\varepsilon$, then ψ_ε is a solution of (23) iff

$$D\psi_\varepsilon(w) B_\varepsilon w - B_\varepsilon \psi_\varepsilon(w) = [F_\varepsilon(w + \psi_\varepsilon(w))]^{\mathcal{S}_{\ell,n,s}}$$

or equivalently

$$\begin{aligned} D\left(T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon\right)(w) B_\varepsilon w \;&-\; B_\varepsilon \left(T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon\right)(w) \\ &=\; \left[F_\varepsilon(w + (T_\varepsilon^{-1} \circ \varphi_\varepsilon \circ T_\varepsilon)(w)\right]^{\mathcal{S}_{\ell,n,s}}. \quad (33) \end{aligned}$$

As T_ε commutes with B_ε, F_ε and $[\cdot]^{\mathcal{S}_{\ell,n,s}}$, (33) is equivalent with

$$T_\varepsilon^{-1} \cdot D\varphi_\varepsilon(T_\varepsilon w)T_\varepsilon B_\varepsilon w - T_\varepsilon^{-1} B_\varepsilon \varphi_\varepsilon(T_\varepsilon w) = T_\varepsilon^{-1} \left[F_\varepsilon(T_\varepsilon w + \varphi_\varepsilon(T_\varepsilon w))\right]^{\mathcal{S}_{\ell,n,s}} . \tag{34}$$

As T_ε is invertible we can put $T_\varepsilon w =: z$ for all $z \in \mathbf{C}^n$. Hence (34) is equivalent with

$$D\varphi_\varepsilon(z)B_\varepsilon z - B_\varepsilon \varphi_\varepsilon(z) = \left[F_\varepsilon(z + \varphi_\varepsilon(z))\right]^{\mathcal{S}_{\ell,n,s}} . \tag{35}$$

Obviously (35) is equivalent with the demand that φ_ε is a solution of (23). So if φ_ε is a solution of (23) also ψ_ε will be a solution of (23) and vice versa. As we have proved that (23) has a unique solution, necessarily $\psi_\varepsilon = \varphi_\varepsilon$.

From this it is straightforward to prove that ϕ_ε and S_ε commute.

6.2. g_ε commutes with S_ε

To obtain the desired commutation result we need (24) and prove that G_ε commutes with T_ε. We use the same line of arguments as in the previous section. So define $\Gamma_\varepsilon := T_\varepsilon^{-1} \circ G_\varepsilon \circ T_\varepsilon$, then Γ_ε is a solution of (24) iff

$$\left[F_\varepsilon(w + \varphi_\varepsilon(w))\right]^{\mathcal{T}_{\ell,n,s}} = (I_n + D\varphi_\varepsilon(w))\Gamma_\varepsilon(w)$$

or equivalently

$$\left[F_\varepsilon(w + \varphi_\varepsilon(w))\right]^{\mathcal{T}_{\ell,n,s}} = (I_n + D\varphi_\varepsilon(w))(T_\varepsilon^{-1} \circ G_\varepsilon \circ T_\varepsilon)(w). \tag{36}$$

As T_ε commutes with φ_ε we have by differentiating both sides of $\varphi_\varepsilon \circ T_\varepsilon(w) = T_\varepsilon \circ \varphi_\varepsilon(w)$ that $D\varphi_\varepsilon(T_\varepsilon w)T_\varepsilon w = T_\varepsilon D\varphi_\varepsilon(w)$, which is equivalent with

$$T_\varepsilon^{-1} D\varphi_\varepsilon(T_\varepsilon w)T_\varepsilon w = D\varphi_\varepsilon(w). \tag{37}$$

Applying (37) on (36) gives us

$$\left[F_\varepsilon(w + \varphi_\varepsilon(w))\right]^{\mathcal{T}_{\ell,n,s}} = (I_n + T_\varepsilon^{-1} D\varphi_\varepsilon(T_\varepsilon w)T_\varepsilon)(T_\varepsilon^{-1} G_\varepsilon(T_\varepsilon w)). \tag{38}$$

As T_ε commutes with B_ε, φ_ε, F_ε and $[\cdot]^{\mathcal{T}_{\ell,n,s}}$, (38) is equivalent with

$$T_\varepsilon^{-1} \left[F_\varepsilon(T_\varepsilon w + \varphi_\varepsilon(T_\varepsilon w))\right]^{\mathcal{T}_{\ell,n,s}} = T_\varepsilon^{-1}(I_n + D\varphi_\varepsilon(T_\varepsilon w))G_\varepsilon(T_\varepsilon w). \tag{39}$$

Putting $T_\varepsilon w =: z$ for all $z \in \mathbf{C}^n$ we obtain that (39) is equivalent with

$$\left[F_\varepsilon(z + \varphi_\varepsilon(z))\right]^{\mathcal{T}_{\ell,n,s}} = (I_n + D\varphi_\varepsilon(w))G_\varepsilon(w). \tag{40}$$

Obviously (40) is equivalent with the demand that G_ε is a solution of (24). This means that Γ_ε is a solution of (24) iff G_ε is a solution of (24). As (24) has a unique solution, necessarily $\Gamma_\varepsilon = G_\varepsilon$, hence $T_\varepsilon \circ G_\varepsilon = G_\varepsilon \circ T_\varepsilon$. As we did before one derives that this latter equality is equivalent with $S_\varepsilon \circ g_\varepsilon = g_\varepsilon \circ S_\varepsilon$.

This means we have proved Theorem 4.

REFERENCES

1. N.N. Bruslinskaja, Finiteness theorem for families of vector fields in the neighbourhood of a Poincaré-type singularity, *Funct. Anal. Appl.* **5** (1971), 177-181.
2. S. Chow, C. Li & D. Wang, *Normal Forms and Bifurcations of Planar Vector Fields* (1994), Cambridge University Press.
3. A.J. Homburg, Periodic attractors, strange attractors and hyperbolic dynamics near homoclinic orbits to saddle-focus equilibria, *Nonlinearity* **15** (2002), 1029-1050.
4. L. Hörmander, *An Introduction to Complex Analysis in Several Variables* (1973), North-Holland Publishing Company.
5. Y.S. Il'yashenko & S.Y. Yakovenko, Finitely-smooth normal forms of local families of diffeomorphisms and vector fields, *Russian Mathematical Surveys* **46:1** (1991), 1-43.
6. J. Moser, The Analytic Invariants of an Area-Preserving Mapping Near a Hyperbolic Fixed Point, *Communications on Pure and Applied Mathematics* **9** (1956), 673-692.
7. K. Neirynck, *Local equivalence and conjugacy of families of vector fields and diffeomorphisms*, *PhD thesis*, Hasselt University, 2005.
8. H. Poincaré, *Sur les proprietés des fonctions définies par les équa-tions aux différences partielles. Thése.* (1879), Gauthier-Villars: Paris.
9. C.L. Siegel, Über die normalform analytischer differential gleichungen in der nahe einer gleichgewichtslösung, *Math. Phys. kl. IIa* **5** (1952), 21-36.
10. L. Stolovitch, Sur un théorème de Dulac, *Ann. Inst. Fourier (Grenoble)* **44:5** (1994), 1397-1433.
11. W. Tucker, The Lorenz attractor exists, *C.R. Acad. Sci. Paris Sér. I Math.* **328:12** (1999), 1197-1202.
12. W. Tucker, Robust normal forms for saddles of analytic vector fields, *Nonlinearity* **17** (2004), 1965-1983.