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# Some Central Forces-Stability

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In this paper we deal with attractive central forces, more precisely with the system

$$\ddot{x} = -xf(x,y), \quad \ddot{y} = -yf(x,y), \quad f(0,0) > 0 \quad f \in \mathcal{C}^\omega.$$

We characterize the stability of the origin whenever the system admits a first integral of the following kind

$$V(x,y,\dot{x},\dot{y}) = u(\dot{x},\dot{y}) + \Pi(x,y),$$

with $u(\dot{x},\dot{y}) = a\dot{x}^2 + b\dot{x}\dot{y} + c\dot{y}^2$ undefinite and make repairs to some optimisms we have committed in [3].

*Key Words:* Non-Linearly Coupled Oscilators, Liapounov Stability.

## 1. INTRODUCTION

Let us consider the following system of ODEs.

$$\ddot{x} = -xf(x,y), \quad \ddot{y} = -yf(x,y), \tag{1}$$

$f : \Omega = \overset{\circ}{\Omega} \subset \mathbf{R}^2 \to \mathbf{R}$, $0 \in \Omega$, $f \in \mathcal{C}^\omega$, $f(0,0) > 0$.

This can be related to a point on a plane, under the influence of a central force. Remark that (1) admits the first integral $s = x\dot{y} - y\dot{x}$, called the areal integral and that (up to a scale time changing) we can suppose $f(0,0) = 1$.

Suppose there exist $a,\ b,\ c \in \mathbf{R}$, and a function $\Pi : D \to \mathbf{R}, D = \overset{\circ}{D} \subset \Omega,\ 0 \in D$, such that

$$V(x, y, \dot{x}, \dot{y}) = a\dot{x}^2 + b\dot{x}\dot{y} + c\dot{y}^2 + \Pi(x, y) \tag{2}$$

is a first integral of (1) and that $u(\dot{x}, \dot{y}) = a\dot{x}^2 + b\dot{x}\dot{y} + c\dot{y}^2$ is an undefinite quadratic form.

In this case, $f$ has a particular structure, namely there exists a function

$$g(u) : I = \overset{\circ}{I} \to \mathbf{R},\ 0 \in I,\ g \in \mathcal{C}^\omega$$

such that $f(x, y) = g(u)$, with $u = ax^2 + bxy + cy^2$.

Indeed the condition

$$\dot{V} = 2a\dot{x}\ddot{x} + b\ddot{x}\dot{y} + b\dot{x}\ddot{y} + 2c\dot{y}\ddot{y} + \Pi_x\dot{x} + \Pi_y\dot{y} = 0.$$

with (2) and (1) gives

$$\Pi_x = (2ax + by)f, \quad \Pi_y = (2cy + bx)f.$$

So the condition $\Pi_{xy} = \Pi_{yx}$ gives

$$(2ax + by)f_y = (2cy + bx)f_x.$$

The general analytic solution of this last equation is

$$f(x, y) = g(u),$$

with $u = ax^2 + bxy + cy^2$, the function $g \in \mathcal{C}^\omega$, $g(0) = 1$ being arbitrary.

The system (1) then becomes

$$\ddot{x} = -xg(u), \quad \ddot{y} = -yg(u), \qquad \text{with } u = ax^2 + bxy + cy^2. \tag{3}$$

The quadratic form $u$ plays an important role.

Our principal contribution is

Theorem 1. *For an indefinite $u$ and under the hypothesis of analiticity, the origin is stable if and only if $g$ (so $f$) is constant.*

An easy calculus shows that

$$\Pi(x, y) = \int_0^{ax^2+bxy+cy^2} g(\xi)\mathrm{d}\xi = \int_0^u g(\xi)\mathrm{d}\xi.$$

Up to a linear change of variables, we can suppose that $u = xy$ and the system (3) becomes

$$\ddot{x} = -xg(u), \quad \ddot{y} = -yg(u), \quad u = xy, \tag{4}$$

called the *main system*, in the sequel, which admits the following first integral

$$\dot{x}\dot{y} + \int_0^u g(\xi)\mathrm{d}\xi = K. \tag{5}$$

We have: $\dot{u} = x\dot{y} + y\dot{x}$, $\ddot{u} = x\ddot{y} + y\ddot{x} + 2\dot{x}\dot{y}$, thus

$$\ddot{u} = -2ug(u) - 2\int_0^u g(\xi)\mathrm{d}\xi + 2K. \tag{6}$$

called the *auxiliary equation*, in the sequel.

Therefore, we obtain a family of differential equations, with a parameter $K$. Such family admits the following first integral

$$\frac{\dot{u}^2}{2} + 2u\left[\int_0^u g(\xi)\mathrm{d}\xi - K\right] = \frac{\dot{u}^2}{2} + W(u). \tag{7}$$

Given $K$, all solutions to equation (6), for each small enough $|K|$, are periodic. Indeed, if we put

$$G(u) = ug(u) + \int_0^u g(\xi)\mathrm{d}\xi,$$

it is easy to verify that, to each small enough $|K|$, there corresponds a unique number $\overline{u}$, such that $G(\overline{u}) = K$.

The equation (6) becomes: $\ddot{u} = -2[G(u) - G(\overline{u})]$. For $u = \overline{u} + U$, we have:

$$\ddot{U} = -2[G(\overline{u} + U) - G(\overline{u})] = -2U\frac{G(\overline{u} + U) - G(\overline{u})}{U} = -UF(\overline{u}, U),$$

where $F(\overline{u}, 0) = \lim_{U\to 0} F(\overline{u}, U) = 2G'(\overline{u}) = 4g(\overline{u}) + 2\overline{u}g'(\overline{u}) > 0$, for small enough $|\overline{u}|$. A simple calculus shows that $F \in \mathcal{C}^\omega$ ($g \in \mathcal{C}^\omega$). Therefore, for each small enough $|\overline{u}|$, the solutions of

$$\ddot{U} = -UF(\overline{u}, U), \tag{8}$$

are periodic ($U = 0$ is a center).

## 2. RELATION BETWEEN THE SOLUTIONS OF THE MAIN SYSTEM AND OF THE AUXILIARY EQUATION

The deduction of equation (6) leads us to the following result: if $(x(t), y(t))$ is a solution of (4), then $u(t) = x(y)y(t)$ is a solution of (6), where $K$ is determined by (5). Vice-versa: given $K$ and a solution $u(t)$ of (6), what are the solutions of (4) such that $x(t)y(t) = u(t)$?

For each $K$, the solutions of (6) are periodic. Therefore, the problem can be solved in an instant $t_1$, for which $u(t)$ is maximum, i.e. $u(t_1) = u_{\max}$, $\dot{u}(t_1) = 0$.

Furthermore, since (4) and (6) are autonomous, we can suppose $t_1 = 0$. Before the solution of the problem, let us make the following considerations:

Given $K$, $u(t, u_0, K)$ stands for the solution of (6) with initial values $u_0 = u_{\max} \geq 0$, $\dot{u}_0 = 0$, and $\tau(u_0, K)$ stands for its period. Under this solution, (4) yields the following Hill system family:

$$\ddot{x} = -xg(u(t, u_0, K)), \quad \ddot{y} = -yg(u(t, u_0, K)). \tag{9}$$

Note that the above stated problem is equivalent to determine the solutions of (9), $x(t, u_0, K)$, $y(t, u_0, K)$, with initial values $x_0, y_0, \dot{x}_0, \dot{y}_0$, which are also solutions of (4).

In fact, under a straight verification, the solutions of (4) and (9) will coincide if and only if

$$x(t, u_0, K)y(t, u_0, K) = u(t, u_0, K),$$

which means:

$$\begin{aligned} &x_0 y_0 = u_0, \quad x_0\dot{y}_0 + y_0\dot{x}_0 = 0, \\ &\dot{x}_0\dot{y}_0 + \int_0^{u_0} g(\xi)\mathrm{d}\xi = K \end{aligned} \tag{10}$$

The values $x_0, y_0, \dot{x}_0, \dot{y}_0$ which satisfy (10) will be called consonant (compatible) with the fixed values $u_0 \geq 0$, $\dot{u}_0 = 0$ and $K$, and the correspondent solutions will be called consonant with the solution $u(t, u_0, K)$.

Since $u_0 \geq 0$, by the second relation in (10), we will have $\dot{x}_0\dot{y}_0 \leq 0$ and by the third $\int_0^{u_0} g(\xi)\mathrm{d}\xi \geq K$.

The trivial cases, $u_0 = 0$ and $K \leq 0$ or $u_0 > 0$ and $\int_0^{u_0} g(\xi)\mathrm{d}\xi = K$, correspond to solutions of (4) with null areal velocity, i.e., solutions contained in straight lines of the plane $xy$ through the origin.

We can consider only the case in which the areal velocity $s$ is strictly positive and $x_0 > 0$, $y_0 > 0$. Then, $\dot{x}_0 < 0$ and $\dot{y}_0 > 0$.

From (10), we have

$$\dot{x}_0 = -x_0\sqrt{\frac{\int_0^{u_0} g(\xi)\mathrm{d}\xi - K}{u_0}}\,, \qquad \dot{y}_0 = y_0\sqrt{\frac{\int_0^{u_0} g(\xi)\mathrm{d}\xi - K}{u_0}}. \tag{11}$$

We therefore see that: given $K$, $u_0 > 0$, $\int_0^{u_0} g(\xi)\mathrm{d}\xi > K$ and supposing $s > 0$, through each point $x_0 > 0$, $y_0 > 0$, such that $x_0 y_0 = u_0$, there is only one velocity $\dot{x}_0, \dot{y}_0$ and so only one consonant solution.

## 3. FLOQUET THEORY

Let $y_1(t, u_0, K)$ and $y_2(t, u_0, K)$ be the fundamental solutions of each one of the equations of system (9), i.e.,

$$y_1(0, u_0, K) = 1,\ \dot{y}_1(0, u_0, K) = 0,\ y_2(0, u_0, K) = 0,\ \dot{y}_2(0, u_0, K) = 1.$$

So the consonants solutions can be written

$$\begin{cases} x(t, u_0, K) = x_0\left[y_1(t, u_0, K) - \sqrt{\dfrac{\int_0^{u_0} g(\xi)\mathrm{d}\xi - K}{u_0}}\, y_2(t, u_0, K)\right], \\ y(t, u_0, K) = y_0\left[y_1(t, u_0, K) + \sqrt{\dfrac{\int_0^{u_0} g(\xi)\mathrm{d}\xi - K}{u_0}}\, y_2(t, u_0, K)\right], \end{cases} \tag{12}$$

$\forall x_0 > 0,\ y_0 > 0,\ x_0 y_0 = u_0$.

In an analogous way of the deduction of (11), in the instant $\tau(u_0, K)$ we will have the following relations:

$$\begin{cases} \dot{x}(\tau(u_0, K), u_0, K) = -x(\tau(u_0, K), u_0, K)\sqrt{\dfrac{\int_0^{u_0} g(\xi)\mathrm{d}\xi - K}{u_0}} \\ \dot{y}(\tau(u_0, K), u_0, K) = y(\tau(u_0, K), u_0, K)\sqrt{\dfrac{\int_0^{u_0} g(\xi)\mathrm{d}\xi - K}{u_0}}. \end{cases} \tag{13}$$

From the first relations in (13) and (12), we have

$$\dot{y}_1(\tau(u_0, K), u_0, K) - \sqrt{\frac{\int_0^{u_0} g(\xi)\mathrm{d}\xi - K}{u_0}}\,\dot{y}_2(\tau(u_0, K), u_0, K)$$
$$= -\left[y_1(\tau(u_0, K), u_0, K) - \sqrt{\frac{\int_0^{u_0} g(\xi)\mathrm{d}\xi - K}{u_0}}\, y_2(\tau(u_0, K), u_0, K)\right]\sqrt{\frac{\int_0^{u_0} g(\xi)\mathrm{d}\xi - K}{u_0}}$$

and since $\dot{y}_2(\tau(u_0,K),u_0,K) = y_1(\tau(u_0,K),u_0,K)$ ([1] even case), it results

$$\dot{y}_1(\tau(u_0,K),u_0,K) = y_2(\tau(u_0,K),u_0,K)\frac{\int_0^{u_0} g(\xi)\mathrm{d}\xi - K}{u_0}.$$

There are only two possibilities:

• Either $\dot{y}_1(\tau(u_0,K),u_0,K) = y_2(\tau(u_0,K),u_0,K) = 0$ and all consonants solutions $x(t,u_0,K)$, $y(t,u_0,K)$ are periodic and their corresponding periods are $2\tau(u_0,K)$.
• Or [1]

$$\frac{\dot{y}_1(\tau(u_0,K),u_0,K)}{y_2(\tau(u_0,K),u_0,K)} = \frac{y_1^2(\tau(u_0,K),u_0,K)-1}{y_2^2(\tau(u_0,K),u_0,K)} = \frac{\int_0^{u_0} g(\xi)\mathrm{d}\xi - K}{u_0} > 0$$

what means $y_1^2(\tau(u_0,K),u_0,K) > 1$.

In this case we will prove that $\forall x_0 > 0$, $y_0 > 0$, $x_0y_0 = u_0$, all the solutions $x(t,u_0,K)$ (respec. $y(t,u_0,K)$) will be unbounded in the future if
$y_2(\tau(u_0,K),u_0,K) > 0$ (respec. $y_2(\tau(u_0,K),u_0,K) < 0$).

In fact, from Floquet theory in the even case, the characteristic equation of any of the relations (9) is

$$\rho^2 - 2y_1(\tau(u_0,K),u_0,K)\rho + 1 = 0$$

and its roots $\rho_1$ and $\rho_2$ are: $\rho_1 < -1$ and $0 > \rho_2 > -1$.

It's easy to see that:

if $y_2(\tau(u_0,K),u_0,K) > 0$ (respec. $y_2(\tau(u_0,K),u_0,K) < 0$) the consonants solutions $x(t,u_0,K)$, $y(t,u_0,K)$ are respectively the eigen-vectors associated to the eigen-values $\rho_1$ and $\rho_2$ (respec. $\rho_2$ and $\rho_1$). In the first case

$$\begin{cases} x(t+\tau(u_0,K),u_0,K) = \rho_1 x(t,u_0,K) \\ y(t+\tau(u_0,K),u_0,K) = \rho_2 y(t,u_0,K), \end{cases}$$

$$\begin{cases} x(t+2\tau(u_0,K),u_0,K) = \rho_1^2 x(t,u_0,K) \\ y(t+2\tau(u_0,K),u_0,K) = \rho_2^2 y(t,u_0,K), \end{cases}$$

and therefore $x(t,u_0,K)$ is unbounded in the future. In the second case $y(t,u_0,K)$ is unbounded in the future.

## 4. FLOW AFTER TWO PERIODS

Let us give a more geometrical proof of the same results obtained in 3.

We proved in 2 that: given $K$, $u_0 > 0$, $\dot{u}_0 = 0$, $\int_0^{u_0} g(\xi)\mathrm{d}\xi > K$, and supposing $s > 0$ through each point of the hyperbola branch $x_0 y_0 = u_0$ in the first quadrant, there is a unique consonant solution $x(t, u_0, K)$, $y(t, u_0, K)$ such that $x(0, u_0, K) = x_0$, $y(0, u_0, K) = y_0$.

The flow of this solution in the instant $2\tau(u_0, K)$ is a diffeomorphism of the hyperbola branch and its first coordinate, given by $x(2\tau(u_0, K), u_0, K) = \varphi(x_0)$ is a diffeomorphism of the half-straight line $x > 0$, $y = 0$ and then it is monotonous. (The function $\psi$ defined by $y(2\tau(u_0, K), u_0, K) = \psi(y_0)$ is a diffeomorphism of the half-straight line $x = 0$, $y > 0$).

We therefore have two possibilities:

- Either $y_1^2(\tau(u_0, K), u_0, K) = 1$, and then all consonants solutions will be periodic and $\varphi$ is the identity.
- Or $y_1^2(\tau(u_0, K), u_0, K) > 1$, and no consonant solution will be periodic. Then $\varphi$ does not admit a fixed point ($\varphi(x_0) \neq x_0$, $\forall x_0$). Supposing $\varphi(x_0) > x_0$ (respec. $\varphi(x_0) < x_0$) we can conclude that $x(t, u_0, K)$ (respec. $y(t, u_0, K)$) is unbounded in the future.

## 5. STABILITY OF THE ORIGIN OF THE MAIN SYSTEM

Let us consider in the plane $u, K$ the region

$$u > 0 \quad \text{and} \quad \int_0^u g(\xi)\mathrm{d}\xi > K. \tag{14}$$

From the explanation we can conclude that:

- If in (14) there is a vanishing sequence $(u_n, K_n)$ (tending to the origin), in which $y_1^2(\tau(u_n, K_n), u_n, K_n) > 1$ then there will be a sequence $(x_n, y_n, \dot{x}_n, \dot{y}_n)$ tending to the origin of (4), in which all the correspondent solutions will be unbounded. As a result, the origin (4) will be unstable.
- Otherwise, the restriction of $y_1^2(\tau(u, K), u, K)$ to the intersection of the region (14) with a convenient neighbourhood of the origin will be equal to 1. In this case there will be a neighbourhood of the origin of (4) in which all the solutions are periodic.

in this way, we showed the following

THEOREM 2. *A necessary and sufficient condition for the stability of the origin in (4) is that the restriction of $y_1^2(\tau(u, K), u, K)$ to the intersection of the region (14) with a convenient neighbourhood of the origin be identical to 1.*

## 6. CONCLUSION

Let us consider the system

$$\begin{cases} \ddot{U} = -UF(\overline{u}, U) \\ \ddot{y} = -yg(\overline{u} + U) \end{cases}$$

in which the first equation is (8) and, as we have seen,

$$F(\overline{u}, 0) = 4g(\overline{u}) + 2\overline{u}g'(\overline{u}).$$

In the the region (14) we have shown $y_1^2(\tau(u, K), u, K) \geq 1$.

From [2], we know that

$$4g(\overline{u}) = n^2[4g(\overline{u}) + 2\overline{u}g'(\overline{u})],$$

has a natural solution $n$, for every number $\overline{u}$ small enough, otherwise $y_1^2(\tau(u, K), u, K) < 1$ for every number $u$ in a convenient neighbourhood of $\overline{u}$. It happens if and only if $\overline{u}g'(\overline{u}) = 0$.

Under the hypothesis of analiticity of $g$, if locally $y_1^2(\tau(u, K), u, K) = 1$ in the region (14), then $y_1^2(\tau(u, K), u, K) = 1$ in a neighbourhood of $u = 0$, $K = 0$. Therefore $g'(\overline{u}) = 0$ for every small enough $|\overline{u}|$, what happens if and only if $g$ is locally constant.

This way, we showed the following

Theorem 3. *Under the hypothesis of analiticity and undefiniteness of $u$, a necessary and sufficient condition for the stability of the origin of (4), is that $g$ be locally constant.*

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