QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **5**, 301–336 (2004)
ARTICLE NO. 86

Subharmonic Bifurcations Near Infinity

Marcelo Messias*

Department of Mathematics, Statistics and Computing, Faculty of Sciences and Technology, State University of São Paulo at P.Prudente - FCT/UNESP, Brazil.
E-mail: marcelo@prudente.unesp.br

In this paper are considered periodic perturbations, depending on two parameters, of planar polynomial vector fields having an annulus of large amplitude periodic orbits, which accumulate on a symmetric infinite heteroclinic cycle. Such periodic orbits and the heteroclinic trajectory can be seen only by the global consideration of the polynomial vector fields on the whole plane, and not by their restrictions to any compact region. The global study envolving infinity is performed via the Poincaré Compactification. It is shown that, for certain types of periodic perturbations, one can seek, in a neighborhood of the origin in the parameter plane, curves C^m of subharmonic bifurcations, to which the periodically perturbed system has subharmonics of order m, for sufficiently large integer m. Also, in the quadratic case, it is shown that, as m tends to infinity, the tangent lines of the curves C^m, at the origin, approach the curve C of bifurcation to heteroclinic tangencies, related to the periodic perturbation of the infinite heteroclinic cycle. The results are similar to those stated by Chow, Hale and Mallet–Paret in [4], although the type of systems and perturbations considered there are quite different, since they are restricted to compact regions of the plane.

Key Words: Subharmonic bifurcations, periodic perturbations, polynomial systems.

1. INTRODUCTION

The infinite heteroclinic cycles of planar polynomial vector fields may be either hyperbolic (attractors or repelors) or nonhyperbolic (accumulated by large amplitude periodic orbits), as shown in Figure 1. Stability and bifurcation of these cycles due to autonomous perturbations were considered by many authors, as Sotomayor and Paterlini [13], Dumortier, Roussarie

*This work is part of the author's Phd thesis, developed at IME–USP Brazil, partially supported by the CAPES.

and Rousseau [6] and, more recently, by Gasull, Mañosa and Mañosas [7]. In [10], [11] and [12], the author studied tangencies and transversal heteroclinic bifurcations of such infinite heteroclinic cycles, due to periodic (nonautonomous) perturbations. The present work considers the effect of the periodic perturbations on the annulus of large amplitude periodic orbits which accumulate on these cycles. Also, in the quadratic case, the interaction between the subharmonic bifurcations and heteroclinic tangencies are considered.

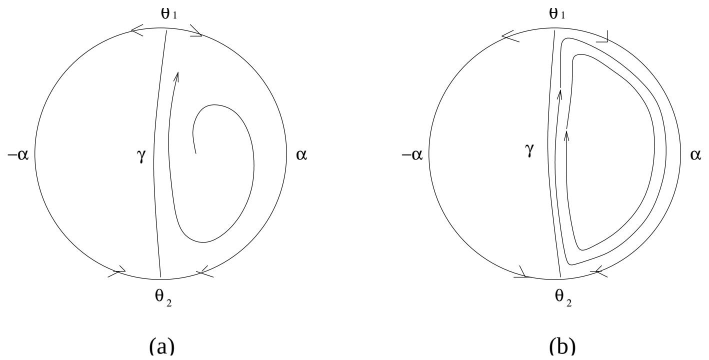

FIG. 1. Hyperbolic and nonhyperbolic symmetric infinite heteroclinic cycles of a planar polynomial vector field on the Poincaré Disk.

It is a fundamental engineering problem to determine the response of a physical system to an applied force, specially when this force is periodic, what may lead to resonances. This subject has a long history in applied science and mathematics and there are several related papers (see, for instance, [2] and references therein). However, the extensive literature there is on the subject is related to periodic perturbations restricted to compact regions of the phase plane. Here a global analysis is carried out, considering perturbations of the large amplitude periodic orbits, which are near infinity and can be seen only by the global consideration of the system in the whole plane. These large amplitude periodic orbits appear in the study of several applied problems, like those related to Astrophysics, fluid dynamics and Lotka–Volterra equations [5].

The results obtained are similar to those stated in [4] and [8], although the type of systems and perturbations considered there are quite different, since they are restricted to compact regions of the plane.

2. BASIC DEFINITIONS AND PRELIMINARIES

In order to formulate the results that will be stated and establish some preliminary notations and results, consider the system

$$\dot{u} = f(u) + \epsilon g(u,t), \tag{1}$$

where $u \in \mathbb{R}^2$, $\epsilon \in \mathbb{R}$ and g is a periodic function of class $\mathcal{C}^k$, $k \geq 2$, in $\mathbb{R}$, with period T, in the variable t.

Suppose that (1) has, for $\epsilon = 0$, an annulus of periodic orbits, in which is contained a periodic orbit Γ, whose period is *in resonance* with the period T of the perturbation function g, that is, there is a natural number m such that the minimum period of Γ is equal to mT. In this case Γ is said to be a *resonant periodic orbit.* Under these assumptions, an important problem is to determine the existence of values of ϵ different from zero, for which the periodically perturbed system (1) has periodic orbits of period mT, close to Γ. Such orbits are said to be *subharmonics of order* m, since their periods are m times the period of $g(u,t)$. In this context, one tries to determine bifurcation curves $\epsilon \mapsto \sigma(\epsilon)$ in the phase plane of (1), with $\sigma(0) \in \Gamma$, such that $\sigma(\epsilon)$ is the initial value of a subharmonic of order m. In other words, one seeks for subharmonics which bifurcate from the resonant orbit Γ, $\sigma(\epsilon)$ being a curve of subharmonic bifurcations.

To study the problem stated above, considering the periodic character of the function g, let us take (1) as an autonomous system defined on the cylinder $\mathbb{R}^2 \times \mathbb{S}^1$, through the change of variable $\phi = t$, $\dot{\phi} = 1$, from which one obtains

$$\begin{aligned} \dot{u} &= f(u) + \epsilon g(u,\phi) \\ \dot{\phi} &= 1, \end{aligned} \tag{2}$$

with $u \in \mathbb{R}^2$ and $\phi \in \mathbb{S}^1$, where $\mathbb{S}^1$ is identified with the interval $[0,T)$, considering $[0,T)$ as the quotient space $\mathbb{R}/TZ$. Due to this change, the periodic orbits of the unperturbed system, (2) with $\epsilon = 0$, become invariant tori, contained in $\mathrm{IR}^2 \times \mathbb{S}^1$.

In this way, as $\dot{\phi} > 0$, one can take a global transversal section through the flow of the system (2) (see Figure 2), given by

$$\Sigma^{\phi_0} = \{(u,\phi) \mid u \in \mathrm{IR}^2 \ \text{ and } \ \phi = \phi_0 \in [0,T)\},$$

on which one can define $P_\epsilon^m : \Sigma^{\phi_0} \to \Sigma^{\phi_0}$, which is the m^{th} iteraction of the Poincaré Map, given by

$$P_\epsilon^m(\xi) \ = \ u(mT, \xi, \epsilon), \tag{3}$$

where $u(t,\xi,\epsilon)$ is the solution of (2) with $u(0,\xi,\epsilon) = \xi \in \Sigma^{\phi_0}$. So, if Γ is a resonant periodic orbit of (1) then, for $\epsilon = 0$, every point $\xi \in \Gamma$ is a

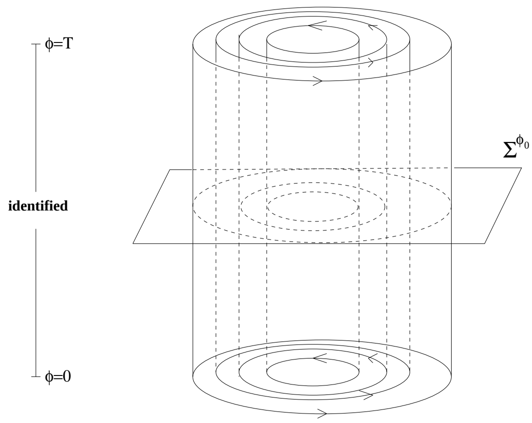

FIG. 2. Invariant tori generated by the periodic perturbations of the annulus of periodic orbits of the system (1) for $\epsilon = 0$.

fixed point of $P_0^m(\xi)$. Indeed, $P_0^m(\xi) = u(mT, \xi, 0) = \xi$. Such fixed points correspond to period m periodic orbits of the system (2), with $\epsilon = 0$, which are subharmonics of order m.

Aiming to verify if these fixed points (and, consequently, these periodic orbits) persist for small enough values of $\epsilon \neq 0$, we shall consider the displacement function

$$\delta(\xi, \epsilon) = P_\epsilon^m(\xi) - \xi = u(mT, \xi, \epsilon) - \xi. \tag{4}$$

So, the zeroes of $\delta(\xi, \epsilon)$, which give the fixed points of P_ϵ^m, correspond to periodic orbits of period mT of the perturbed system (2), which are subharmonics of order m, since $P_\epsilon^j(\xi) \neq \xi$, for $1 < j < m$.

Aiming to study the existence of zeroes of the function δ defined above, in the next two subsections following which is done in [1] and [2] we shall use a nonhomogeneous linear variational equation related to the perturbed system (given by Diliberto's Propostition) in connection with the reduction method of Lyapunov–Schmidt, to obtain some preliminaries results, which will be used in section 3, to prove the statements related to subharmonic bifurcations of large amplitude periodic orbits of polynomial vector fields.

2.1. Variational equations and subharmonic bifurcations

The solution of the *first variational equations*

$$\dot{W} = Df(\varphi_t(\xi))W,$$

related to the system $\dot{u} = f(u)$, over a regular solution $u = \varphi_t(\xi)$, can be obtained with the following Proposition (see [1] or [2]). First, let us recall that the *divergence* and the *curl* of a vector function $f : \mathbb{R}^2 \to \mathbb{R}^2$, with $f(x,y) = (f_1(x,y), f_2(x,y))$, are defined as follows

$$div f(x,y) := \frac{\partial f_1}{\partial x}(x,y) + \frac{\partial f_2}{\partial y}(x,y),$$

$$rot f(x,y) := \frac{\partial f_2}{\partial x}(x,y) - \frac{\partial f_1}{\partial y}(x,y),$$

and the *scalar curvature* of a differentiable curve $t \mapsto (x(t), y(t))$ is

$$\kappa := \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

PROPOSITION 1 (Diliberto's Proposition). *Let φ_t be the flow of the differential equation $\dot{u} = f(u)$, $u \in \mathbb{R}^2$.If $f(u) \neq 0$, then the fundamental matrix $t \to \Phi(t)$, with $\Phi(0) = Id$, solution of the homogeneous variational equation*

$$\dot{W} = Df(\varphi_t(\xi))W,$$

related to $\dot{u} = f(u)$, is such that

$$\Phi(t)f(\xi) = f(\varphi_t(\xi)),$$

$$\Phi(t)f^{\perp}(\xi) = a(t,\xi)f(\varphi_t(\xi)) + b(t,\xi)f^{\perp}(\varphi_t(\xi)),$$

where

$$a(t,\xi) = \int_0^t \Big(2\kappa(s,\xi)\|f(\varphi_s(\xi))\| - rot f(\varphi_s(\xi))\Big) b(s,\xi) ds,$$

$$b(t,\xi) = \frac{\|f(\xi)\|^2}{\|f(\varphi_t(\xi))\|^2}\, e^{\int_0^t div f(\varphi_s(\xi))ds}.$$

Remark 2. The Functions a and b which appear in the integral expressions given above have an important geometric mean: if ξ is on the T–periodic orbit Γ and Σ is transversal to Γ in ξ, then the function b takes the form

$$b(T,\xi) = e^{\displaystyle\int_0^T div f(\varphi_s(\xi))ds},$$

which is exactly the derivative of the Poincaré map related to Γ, on $\xi = \Gamma \cap \Sigma$. On the other side, if Γ belongs to an annulus $\mathcal{A}$ of periodic orbits and

Σ is an orthogonal transversal section to this annulus, then the function $a(T,\xi)$ is proportional to the derivative $T'(\xi)$, of the period function $T(p)$, which, for each $p \in \mathcal{A} \cap \Sigma$, gives the minimum period of the periodic orbit through p, as stated in [1] (or [2], p.329–330).

The remark above will be used later, in conjunction with the following definition:

Definition 3. Let $\mathcal{A}$ be an annulus of periodic orbits of the system $\dot{u} = f(u)$, with $u \in \mathbb{R}^2$. Let $T(p)$ be the period function, which associates to each point $p \in \mathcal{A}$ the minimum period of the orbit through p. If $T'(p) \neq 0$, $\forall p$, then $\mathcal{A}$ is said to be a *regular annulus*.

Consider now a nonhomogeneous equation $\dot{u} = f(u) + \epsilon g(u,t)$, $u \in \mathbb{R}^2$, with the regular solution $\varphi_t(\xi,\epsilon)$.The variational equation given by

$$\dot{W} = Df(\varphi_t(\xi,0))W + g(\varphi_t(\xi,0),t), \tag{5}$$

is called the *second variational equation* related to $\dot{u} = f(u) + \epsilon g(u,t)$, over a regular solution $u = \varphi_t(\xi)$. The solution of (5) can be obtained from Proposition 1, using the variation of constants method, since their homogeneous part coincide. The following lemma holds (the proof is made in [1]):

Lemma 4. *Let φ_t be the flow of the system $\dot{u} = f(u)$, $u \in \mathbb{R}^2$. If $f(u) \neq 0$, then the solution $t \to W(t)$ of the initial value problem*

$$\dot{W} = Df(\varphi_t(\xi))W + g(\varphi_t(\xi),t), \qquad W(0) = 0$$

is given by

$$W(t) = [N(t) + a(t,\xi)M(t)]f(\varphi_t(\xi)) + [b(t,\xi)M(t)]f^{\perp}(\varphi_t(\xi)),$$

with

$$N(t) := \int_0^t \Big[\frac{1}{\|f\|^2} < f, g > - \frac{a(s,\xi)}{b(s,\xi)\|f\|^2} < f^{\perp}, g > \Big](\varphi_s(\xi))ds,$$

$$M(t) := \int_0^t \Big[\frac{1}{b(s,\xi)\|f\|^2} < f^{\perp}, g > \Big](\varphi_s(\xi))ds,$$

where a and b are the functions defined in Proposition 1, and $<,>$ is the usual inner product in $\mathbb{R}^2$.

Suppose that system $\dot{u} = f(u) + \epsilon g(u,t)$ has, for $\epsilon = 0$, a regular annulus of periodic orbits, which contains a periodic orbit Γ, whose period is in resonance with the period T of the function g. In the next subsection it is shown (Theorem 5) that, under these conditions, the existence of zeroes to the displacement function $\delta(\xi,\epsilon) = P_\epsilon^m(\xi) - \xi$, defined in (4), is equivalent to the existence of simple zeroes of the function $M(t)$, that appears in Lemma 4.

2.2. Subharmonic bifurcations on a regular annulus

It can be proved (see [1] p.342) that if ξ belongs to a resonant periodic orbit of $\dot{u} = f(u)$, then $f(\xi) \in Ker[\delta_\xi(\xi,0)]$, where $\delta(\xi,\epsilon) = u(mT,\xi,\epsilon) - \xi$ is defined in (4). As a consequence, it is not possible to use the Implicit Function Theorem to determine the existence of zeroes of the displacement function $\delta(\xi,\epsilon)$, since, even if $\delta(\xi,0) = 0$, $\delta_\xi(\xi,0)$ is not an isomorphism. Then, to solve the problem Lyapunov–Schmidt reduction is used, which consists in considering projections of the δ function on some special spaces and using the Implicit Function Theorem to find zeroes of these projections, as it will be done in the next theorem (the proof follows closely what is done in [1]).

Theorem 5. *Suppose that the system $\dot{u} = f(u) + \epsilon g(u, t+\phi_0)$, given in (2), has, for $\epsilon = 0$, a regular annulus $\mathcal{A}$ of periodic orbits, in the transversal section $\Sigma^{\phi_0} = \{(u,\phi) \mid u \in \mathbb{R}^2$ and $\phi = \phi_0 \in [0,T)\}$ (Figure 2). Suppose that in $\mathcal{A}$ there exists a periodic orbit Γ, whose period is in resonance with the period T of the function g, that is, there is a natural number m such that the minimum period of Γ is mT. Consider both the function*

$$M(s,\phi_0) = \int_0^{mT} e^{-\int_0^t div f(\varphi_{\tau+s}(v))d\tau} < f^\perp(\varphi_{t+s}(v)), g(\varphi_{t+s}(v), t+\phi_0) > dt,$$

where $\varphi_t(v)$ is a parametrization of Γ, and $P_\epsilon^m : \Sigma^{\phi_o} \to \Sigma^{\phi_0}$ is the Poincaré map, defined by $P_\epsilon^m(\xi) = u(mT,\xi,\epsilon)$, where $u(t,\xi,\epsilon)$ is the solution of (2) such that $u(0,\xi,\epsilon) = \xi \in \Sigma^{\phi_0}$. If there exists a point $(\bar{s},\bar{\phi_0})$ such that $M(\bar{s},\bar{\phi_0}) = 0$ and $\frac{\partial M}{\partial s}(\bar{s},\bar{\phi_0}) \neq 0$, then, for $\epsilon \neq 0$ sufficiently small, $P_\epsilon^m(\xi)$ has a fixed point bifurcating from Γ. Such fixed point corresponds to a subharmonic of order m of the perturbed system (2).

Proof. Let φ_t be the flow of the system $\dot{u} = f(u)$ and ψ_t the flow of the orthogonal system $\dot{u} = f^\perp(u)$. Consider a local system of coordinates (r,s), defined on some neighborhood V of the resonant periodic orbit Γ, with V belonging entirely in the annulus $\mathcal{A} \subset \Sigma^{\phi_0}$, through the function $H(r,s) = \psi_r\varphi_s(v)$, where $s \in [0,mT]$, r is near zero, $v \in \Gamma$, being a fixed but arbitrary point. Then one has

$$H_r(r,s) = f^\perp(H(r,s)) \quad \text{and} \quad H_s(r,s) = D\psi_r(\varphi_s(v))f(\varphi_s(v)),$$

which are linearly independent and, consequently, to each $\xi \in V$, there is (r, s) such that $\xi = H(r, s)$. Using theses coordinates, the function $\delta(\xi, \epsilon) = P_\epsilon^m(\xi) - \xi = u(mT, \xi, \epsilon) - \xi$ becomes $\Delta(r, s, \epsilon) = \delta(H(r, s), \epsilon)$.

So, the zeroes of the function $\Delta(r, s, \epsilon)$ correspond to fixed points of the Poincaré map $P_\epsilon^m : \Sigma^{\phi_0} \to \Sigma^{\phi_0}$. Let us apply the Lyapunov–Schmidt reduction in conjunction with the Implicit Function Theorem to find them.

Firstly one has $\Delta(0, s, 0) = \delta(\varphi_s(v), 0) = u(mT, \varphi_s(v), 0) - \varphi_s(v) = 0$. Consider now the projection of Δ on the $f(\varphi_s(v))$ direction, given by

$$\mathcal{P}(r, s, \epsilon) =< \Delta(r, s, \epsilon), f(\varphi_s(v)) > .$$

The partial derivative of $\mathcal{P}$, with relation to r, evaluated on $(0, s, 0)$ is $< \Delta_r(0, s, 0), f(\varphi_s(v)) >$, and $\Delta_r(0, s, 0) = \delta_\xi(\varphi_s(v), 0)f^\perp(\varphi_s(v))$. Also,

$$\begin{aligned} &\delta_\xi(\varphi_s(v), 0)f^\perp(\varphi_s(v)) \\ &= \frac{\partial}{\partial r}\Big(u(mT, \psi_r(\varphi_s(v)), 0) - \psi_r(\varphi_s(v))\Big)|_{r=0} \\ &= \Big(u_\xi(mT, \psi_r(\varphi_s(v)), 0)\dot{\psi}_r(\varphi_s(v)) - \dot{\psi}_r(\varphi_s(v))\Big)|_{r=0} \\ &= u_\xi(mT, \varphi_s(v), 0)f^\perp(\varphi_s(v)) - f^\perp(\varphi_s(v)), \end{aligned}$$

where $u_\xi(mT, \varphi_s(v), 0)$ is the fundamental matrix of the variational equation $\dot{W} = Df(\varphi_s(v))W$, with $W(0) = Id$. So, using Proposition 1 one obtains

$$\begin{aligned} &u_\xi(mT, \varphi_s(v), 0)f^\perp(\varphi_s(v)) = \\ &\quad a(mT, \varphi_s(v))f(\varphi_s(v)) \ + \ b(mT, \varphi_s(v))f^\perp(\varphi_s(v)), \end{aligned}$$

and so one has

$$\begin{aligned} &\delta_\xi(\varphi_s(v), 0)f^\perp(\varphi_s(v)) = \\ &\quad a(mT, \varphi_s(v))f(\varphi_s(v)) + [b(mT, \varphi_s(v)) - 1]f^\perp(\varphi_s(v)). \end{aligned}$$

As the periodic orbit Γ is nonhyperbolic, $b(mT, \varphi_s(v)) = 1$ (see Remark 2 after Proposition 1). So, using the calculations above, it follows that

$$\Delta_r(0, s, 0) = b(s, v)a(mT, \varphi_s(v))f(\varphi_s(v)), \tag{6}$$

from which one obtains

$$< \Delta_r(0, s, 0), f(\varphi_s(v)) >= b(s, v)a(mT, \varphi_s(v))\|f(\varphi_s(v))\|^2, \tag{7}$$

which is different from zero, since $b \neq 0$, $f \neq 0$ on the periodic orbit Γ and $a(mT, \varphi_s(v)) \neq 0$, since by hypothesis Γ is contained in a regular annulus

$\mathcal{A}$, and the function a is proportional to the period function (see Remark 2).

So, as $\mathcal{P}(0,s,0)=0$ and $\mathcal{P}_r(0,s,0)\neq 0$, it follows from the Implicit Function Theorem that, for $\epsilon\neq 0$ small enough, there exists a function $h=h(s,\epsilon)$, with $h(s,0)=0$, such that

$$\mathcal{P}(h(s,\epsilon),s,\epsilon) =< \Delta(h(s,\epsilon),s,\epsilon), f(\varphi_s(v)) >= 0.$$

Define now $\tilde{\Delta}$ as the projection of $\Delta(h(s,\epsilon),s,\epsilon)$ on the $f^{\perp}$ direction, in the following way:

$$\tilde{\Delta}(s,\epsilon) =< \Delta(h(s,\epsilon),s,\epsilon), f^{\perp}(\varphi_s(v)) > .$$

As $\Delta(0,s,0)=0$ and $h(s,0)=0$, then $\tilde{\Delta}(s,0)=0$. So, taylor expanding $\tilde{\Delta}$ in ϵ, around $\epsilon=0$, one obtains

$$\tilde{\Delta}(s,\epsilon)=\epsilon[\tilde{\Delta}_\epsilon(s,0)+O(\epsilon)]=\epsilon d(s,\epsilon),$$

from which follows that the simple zeros of $d(s,\epsilon)=\tilde{\Delta}_\epsilon(s,0)+O(\epsilon)$ correspond to the zeroes of $\Delta(r,s,\epsilon)$, for $\epsilon\neq 0$ sufficiently small. Indeed, if $d(\bar{s},0)=0$ and $\frac{\partial d}{\partial s}(\bar{s},0)\neq 0$ then, by the implicit function theorem there exists a function $s=s(\epsilon)$ such that $d(s(\epsilon),\epsilon)=0$, for sufficiently small ϵ. Therefore, diminishing ϵ if necessary, one has, from the calculations above, $\Delta(h(s(\epsilon),\epsilon),s(\epsilon),\epsilon)=0$.

To finish the proof, it is necessary to show that these simple zeroes correspond to the zeroes of the expression $M(s,\phi_0)$ given in the theorem. To see that, it is sufficient to derive $\tilde{\Delta}$ with respect to ϵ, evaluating in $\epsilon=0$, from which one obtains

$$\tilde{\Delta}_\epsilon(s,0) =< \Delta_r(h(s,0),s,0)h_\epsilon(s,0)+\Delta_\epsilon(h(s,0),s,0)), f^{\perp}(\varphi_s(v)) > .$$

Since $h(s,0)=0$, using the expression Δ_r given in (6), it follows that

$$\tilde{\Delta}_\epsilon(s,0) =< \Delta_\epsilon(0,s,0), f^{\perp}(\varphi_s(v)) > .$$

On the other hand, from the definition of $\Delta(r,s,\epsilon)$ one has $\Delta_\epsilon(r,s,0)=u_\epsilon(mT,r,s,0)$, where $u_\epsilon(t,r,s,0)$ is the solution of the variational equation $\dot{W}=Df(\varphi_t(\xi))W+g(\varphi_t(\xi,0),t+\phi_0)$, which, from Lemma 4 is given by

$$W(t)=[N(t)+a(t,\xi)M(t)]f(\varphi_t(\xi))+[b(t,\xi)M(t)]f^{\perp}(\varphi_t(\xi)).$$

In (r,s) coordinates one has $\xi=H(r,s)=\psi_r\varphi_s(v)$. As we are interested in the computation of $\Delta_\epsilon(0,s,0)=u_\epsilon(mT,0,s,0)$, taking $r=0$, from which

$\xi = \varphi_s(v)$, and considering $t = mT$ in the expression of $M(t)$ given in Lemma 4, one obtains the following expression for $M(mT)$:

$$\frac{1}{\|f(\varphi_s(v))\|^2}\int_0^{mT} e^{-\int_0^t divf(\varphi_{\tau+s}(v))d\tau} < f^{\perp}(\varphi_{t+s}(v)), g(\varphi_{t+s}(v), t+\phi_0) > dt.$$

Therefore,

$$\tilde{\Delta}_\epsilon(s,0) =< \Delta_\epsilon(0,s,0), f^{\perp}(\varphi_s(v)) >= b(mT,v)\|f^{\perp}(\varphi_s(v))\|^2 M(mT),$$

and, as $b(mT,v) = 1$, for Γ is nonhyperbolic,

$$\begin{aligned}
&\tilde{\Delta}_\epsilon(s,0)\\
&= \int_0^{mT} e^{-\int_0^t divf(\varphi_{\tau+s}(v))d\tau} < f^{\perp}(\varphi_{t+s}(v)), g(\varphi_{t+s}(v), t+\phi_0) > dt\\
&= M(s,\phi_0),
\end{aligned}$$

and the result follows from the Implicit Function Theorem. ∎

The theorem above will be used in the next section in the analysis of the subharmonic bifurcations near infinity for two parameter families of planar polynomial vector fields.

3. SUBHARMONIC BIFURCATIONS OF POLYNOMIAL VECTOR FIELDS WITH AN INVARIANT STRAIGHT LINE

Let X be a planar polynomial vector field of degree n, with an invariant straight line, free from singularities, connecting two hyperbolic saddle points at infinity. Without loss of generality, one can consider such straight line as the y axis and the system of ordinary differential equations determined by the vector field takes the form

$$\begin{aligned}
\dot{x} &= xP(x,y)\\
\dot{y} &= Q(x,y),
\end{aligned}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials of degree $n-1$ and n, respectively, with $Q(0,y) > 0, \forall y$. Note that the degree of $Q(0,y)$ must be even, otherwise $Q(0,y)$ has at least one real root and, consequently, the system has at least one critical point in the y axis.

For technical reasons, we perform a translation in the system above, in order to let the straight line $\{x = 1\}$ invariant, from which one obtains, after renaming the coefficients and variables, the equivalent system

$$\begin{aligned}
\dot{x} &= (x-1)P(x,y)\\
\dot{y} &= Q(x,y),
\end{aligned} \tag{8}$$

with $Q(1,y) > 0, \ \forall y$.

Suppose that the system (8) has only two saddles at infinity. Then, the compactification of the system on the Poincaré disk has an infinite heteroclinic cycle, composed by $\gamma(s)$ (which is the compactification of the solution contained in the line $\{x = 1\}$), θ_1 and θ_2 (which are the saddles at infinity) and $\alpha(s)$ (which is the trajectory arc of the compactfied system belonging to the border of the Poincaré disk) (see Figure 1 at the introduction).

Suppose now that the singular heteroclinic cycle $\gamma(s) \cup \theta_1 \cup \alpha(s) \cup \theta_2$ is accumulated by periodic orbits of the compactfied system (Figure 1(b)). Such periodic orbits correspond to *large amplitude periodic orbits* of the planar polynomial vector field. Then, consider the periodic perturbation of (8) given by

$$\begin{array}{rcl} \dot{x} & = & \tilde{P}(x,y) + \mu + Af(t) \\ \dot{y} & = & Q(x,y) + Ag(t), \end{array} \tag{9}$$

where $\tilde{P}(x,y) = (x-1)P(x,y)$, f and g are C^k functions, $k \geq 2$, periodic with period $T = 2\pi$, with μ and A being real small parameters. It is a periodic perturbation of amplitude A, combined with an autonomous perturbation, given by the parameter μ. The autonomous perturbation is transversal to the annulus of periodic orbits and, consequently, "breaks" these orbits, while the parameter A controls the amplitude of the nonautonomous (periodic) part of the perturbation.

In this section the effect of the periodic perturbation above on the annulus of large amplitude periodic orbits will be analyzed. The method used can also be applied to the analysis of other types of two parameter perturbations, as, for example, the type considered in [14],

$$\begin{array}{rcl} \dot{x} & = & \tilde{P}(x,y) + \mu_1 f(\omega t) \\ \dot{y} & = & Q(x,y) + \mu_2 g(\omega t), \end{array}$$

where μ_1 and μ_2 are real parameters.

In order to study the system on the whole plane let us consider the Poincaré compactification of (9) through the change to polar coordinates

$$x = \frac{\cos\theta}{\rho} \qquad\qquad y = \frac{\sin\theta}{\rho}$$

in order that $\{\rho = 0\}$ correspond to the equator, from which one obtains, after the multiplication by ρ^{n-1}, which is equivalent to the introduction of a new time s via $\frac{dt}{ds} = \rho^{n-1}$ [10]:

$$\begin{array}{rcl} \dot{\theta} & = & \sum_{i=1}^{n} \rho^i A_{n-i}(\theta) + \bar{f}(\theta, \rho, \mu, A, t(s)) \\ \dot{\rho} & = & -\sum_{i=0}^{n} \rho^{i+1} R_{n-i}(\theta) + \bar{g}(\theta, \rho, \mu, A, t(s)), \end{array} \tag{10}$$

where

$$\begin{aligned} A_k(\theta) &= -P_k(\cos\theta,\sin\theta)\sin\theta + Q_k(\cos\theta,\sin\theta)\cos\theta \\ R_k(\theta) &= P_k(\cos\theta,\sin\theta)\cos\theta + Q_k(\cos\theta,\sin\theta)\sin\theta, \end{aligned}$$

for $k = 0, 1, \dots, n$, with $\theta \in [0, 2\pi)$ and $\rho \in [0, \infty)$,

$$\begin{aligned} \bar{f}(\theta,\rho,\mu,A,s) &= \rho^n \left[Ag(t(s))\cos\theta - (\mu + Af(t(s)))\sin\theta\right], \\ \bar{g}(\theta,\rho,\mu,A,s) &= -\rho^{n+1}\left[(\mu + Af(t(s))\cos\theta + Ag(t(s))\sin\theta\right]. \end{aligned}$$

Observe that the functions $\bar{f}$ and $\bar{g}$ depend explicitly on the time s, which is the time of the compactfied system, on the Poincaré disk (the system is nonautonomous). Also, $\{\rho = 0\}$ correspond to the points on the border of the disk, which represent the points at infinity.

The question now is: will the large amplitude periodic orbits, which exist for $\mu = A = 0$, persist for $\mu \neq 0$ and $A \neq 0$ sufficiently small?

Following the ideas presented in the previous section, taking $\phi = t(s)$ such that $\dot{\phi} = \frac{dt}{ds} = \rho^{n-1}$, one can transform (10) into an autonomous system, defined on the solid torus $\mathrm{D}^2 \times \mathbb{S}^1$, given by

$$\begin{aligned} \dot{\theta} &= \sum_{i=1}^{n} \rho^i A_{n-i}(\theta) + \bar{f}(\theta,\rho,\mu,A,\phi) \\ \dot{\rho} &= -\sum_{i=0}^{n} \rho^{i+1} R_{n-i}(\theta) + \bar{g}(\theta,\rho,\mu,A,\phi) \\ \dot{\phi} &= \rho^{n-1}, \end{aligned} \tag{11}$$

where $\phi \in \mathbb{S}^1$, which is identified with the interval $[0, 2\pi)$, considering it as the quotient space $\mathbb{R}/2\pi\mathbb{Z}$. A representation of the system (11), with $\mu = A = 0$, on the solid torus, is presented in Figure 3. The large amplitude periodic orbits of the planar system become, on the extended phase space $\mathrm{D}^2 \times \mathbb{S}^1$, invariant tori.

One can now apply the results stated in the previous section. In order to do that, consider (11) in the form

$$\dot{u} = F(u) + \mu G_1(u) + A G_2(u,\phi) \tag{12}$$
$$\dot{\phi} = \rho^{n-1} \tag{13}$$

where $u = (\theta, \rho)$, $\phi \in [0, 2\pi)$, and

$$\begin{aligned} F(\theta,\rho) &= \Big(\sum_{i=1}^{n} \rho^i A_{n-i}(\theta), -\sum_{i=0}^{n} \rho^{i+1} R_{n-i}(\theta)\Big), \\ G_1(\theta,\rho) &= (-\rho^n sin\theta, -\rho^{n+1} cos\theta), \\ G_2(\theta,\rho) &= (\rho^n(g(\phi)cos\theta - f(\phi)sin\theta), -\rho^{n+1}(f(\phi)cos\theta + g(\phi)sin\theta)). \end{aligned}$$

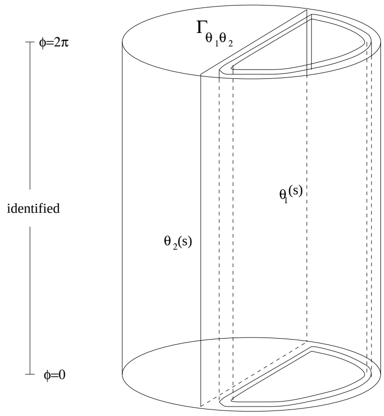

FIG. 3. Invariant tori generated by the periodic orbits of the compactfied polynomial vector field (11) with $\mu = A = 0$.

As $\dot{\phi} > 0$ and the system is 2π–periodic in ϕ, one can consider the global cross section to the flow of the system, given by

$$\Sigma^{\phi_0} = \{(\theta, \rho, \phi) \mid (\theta, \rho) \in [0, 2\pi) \times [0, +\infty) \quad \text{and} \quad \phi = \phi_0 \in [0, 2\pi)\},$$

on which one can define the m^{th} iterated of the Poincaré return map, related to (11), $P^m_{\mu,A} : \Sigma^{\phi_0} \longrightarrow \Sigma^{\phi_0}$, given by

$$P^m_{\mu,A}(u_0) = u(mT, u_0, \phi_0, \mu, A),$$

where $u(s, u_0, \phi_0, \mu, A)$ is the solution of

$$\dot{u} = F(u) + \mu G_1(u) + AG_2(u, \phi(s, \phi_0)), \tag{14}$$

with $u(0, u_0, \phi_0, \mu, A) = u_0$, and $\phi(s, \phi_0)$ is the solution of $\dot{\phi} = \rho^{n-1}$ (equation (13)), with $\phi(0) = \phi_0$. Thus, for $s = 0$, $u_0 \in \Sigma^{\phi_0}$. This way, fixed points of $P^m_{\mu,A}$ correspond to order m subharmonics of (11), since $P^j_{\mu,A}(u_0) \neq u_0$ for $1 < j < m$.

Remember that we are assuming that the system $\dot{u} = F(u)$, associated to the compactfied system (11) has, for $\mu = A = 0$, an annulus $\mathcal{A}$ of periodic orbits, which accumulate on the infinite heteroclinic cycle $\theta_1 \cup \gamma(s) \cup \theta_2 \cup \alpha(s)$. We are interested in determining which of these periodic orbits will persist for the periodically perturbed system (14), for μ and A sufficiently small.

We prove now two technical lemmas, which will be used to prove Theorem 8, the main result of this section.

Lemma 6. *The annulus $\mathcal{A} \subset \Sigma^{\phi_0}$ of periodic orbits of (14), with $\mu = A = 0$, which accumulate on the infinite heteroclinic cycle $\gamma(s) \cup \theta_1 \cup \alpha(s) \cup \theta_2$, is regular in a sufficiently small neighborhood of the cycle.*

Proof. Consider the Poincaré cross section, transversal to the flow of the system (14), with $\mu = A = 0$, given by $\Sigma = \{(\theta, \rho) \mid \theta = \pi \text{ and } \rho \in [0, \bar{\rho})\}$,with $\bar{\rho}$ sufficiently small (see Figure 4).

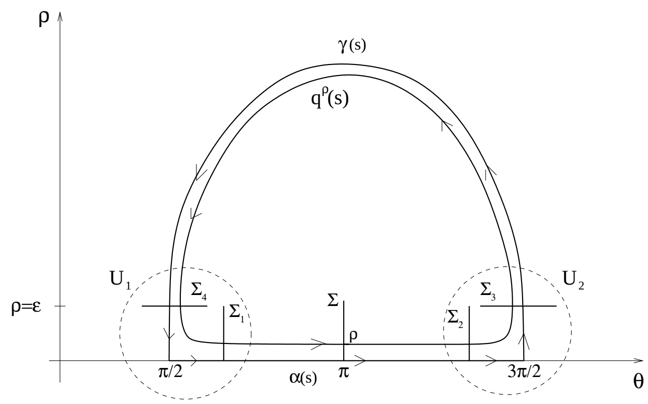

FIG. 4.

Let $q^\rho(s)$ be the periodic orbit contained in the annulus $\mathcal{A}$, such that $q^\rho(0) = (\rho, \pi) \in \Sigma$, with $\rho \in (0, \bar{\rho})$. Let $T(\rho)$ be the period of $q^\rho(s)$. We shall prove that $T'(\rho) \neq 0$, for all $\rho \in (0, \bar{\rho})$ (diminishing $\bar{\rho}$, if necessary).

In order to accomplish that, consider sections Σ_i, $i = 1, 2, 3, 4$, transversal to $\mathcal{A}$, in the following way (Figure 4):

$$
\begin{aligned}
\Sigma_1 &= \{(\theta, \rho) \mid \theta = \frac{\pi}{2} + \epsilon \quad \text{and} \quad \rho \in [0, \epsilon)\},\\
\Sigma_2 &= \{(\theta, \rho) \mid \theta = \frac{3\pi}{2} - \epsilon \quad \text{and} \quad \rho \in [0, \epsilon)\},\\
\Sigma_3 &= \{(\theta, \rho) \mid \theta \in (\frac{3\pi}{2} - \epsilon, \frac{3\pi}{2} + \epsilon) \quad \text{and} \quad \rho = \epsilon\},\\
\Sigma_4 &= \{(\theta, \rho) \mid \theta \in (\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon) \quad \text{and} \quad \rho = \epsilon\},
\end{aligned}
$$

with $0 < \epsilon < \bar{\rho}$.

As the periodic orbits $q^\rho(s)$ accumulate on the cycle $\gamma(s) \cup \theta_1 \cup \alpha(s) \cup \theta_2$, for $\bar{\rho}$ sufficiently small, each orbit $q^\rho(s)$, with $\rho \in (0, \bar{\rho})$, crosses the transversal sections Σ_i, $i = 1, 2, 3, 4$, defined above. Then one can consider the period function

$$T(\rho) = T_1(\rho) + T_2(\rho) + T_3(\rho) + T_4(\rho),$$

where $T_i(\rho)$ is the time taken to $q^\rho(s)$ to go from Σ_i to Σ_{i+1} (identifying Σ_5 with Σ_1, to close the cycle).

From the continuous dependence theorem with respect to the initial conditions, it follows that $T_1(\rho)$ and $T_3(\rho)$ are analytic functions of ρ, such that

$$\lim_{\rho\to 0^+} T_1(\rho) = T_1(0) < +\infty \quad \text{and} \quad \lim_{\rho\to 0^+} T_3(\rho) = T_3(0) < +\infty. \tag{15}$$

On the other hand, the times T_2 and T_4 are the transition times of $q^\rho(s)$ in the neighborhood of the saddle points $\theta_1 = (\theta_1, 0)$ and $\theta_2 = (\theta_2, 0)$ of the system (11) in the equator, being $\theta_1 = \pi/2$ and $\theta_2 = 3\pi/2$.By hypothesis, such points are hyperbolic and the jacobian matrix of the compactfied system, evaluated in these points, is given by

$$\begin{pmatrix} A_n^{'}(\theta_i) & A_{n-1}(\theta_i) \\ 0 & -R_n(\theta_i) \end{pmatrix},$$

with $i = 1, 2$. So, by a theorem due to Hartman (see [9], pg. 258), there exist neighborhoods U_1 and U_2 of θ_1 and θ_2, in which the system is conjugated to

$$\begin{matrix} \dot\theta = A_n'(\theta_1)(\theta - \pi/2) \\ \dot\rho = -R_n(\theta_1)\rho, \end{matrix} \quad \text{and} \quad \begin{matrix} \dot\theta = A_n'(\theta_2)(\theta - 3\pi/2) \\ \dot\rho = -R_n(\theta_2)\rho, \end{matrix}$$

where $-R_n(\theta_1) < 0 < A_n'(\theta_1)$ and $A_n'(\theta_2) < 0 < -R_n(\theta_2)$ are the eigenvalues of the system on the saddle points θ_1 and θ_2, respectively.

Taking ϵ fixed, sufficiently small, so that $\Sigma_1, \Sigma_4 \subset U_1$ and $\Sigma_2, \Sigma_3 \subset U_2$ (Figure 4), one has, from the second equations in the linear systems above, that the transition time close to the saddle points, as a function of ρ, is given by

$$T_2(\rho) = -\frac{1}{R_n(\theta_2)} ln\Big(\frac{\epsilon}{\rho_2(\rho)}\Big) > 0 \quad \text{and} \quad T_4(\rho) = -\frac{1}{R_n(\theta_1)} ln\Big(\frac{\rho_1(\rho)}{\epsilon}\Big) > 0, \tag{16}$$

with $\rho_1(\rho)$ being the point where $q^\rho(s)$ crosses the transversal section Σ_1, $\rho_2(\rho)$ is the point where it crosses Σ_2, being ρ_1 and ρ_2 differentiable functions of ρ, with

$$\lim_{\rho\to 0^+} \rho_1(\rho) = \lim_{\rho\to 0^+} \rho_2(\rho) = 0.$$

It follows from the expression (16) that

$$\lim_{\rho\to 0^+} T_2(\rho) = +\infty \quad \text{and} \quad \lim_{\rho\to 0^+} T_4(\rho) = +\infty. \tag{17}$$

Also, differentiating T_2 and T_4 with respect to ρ one has

$$T_2'(\rho) = \frac{\rho_2'(\rho)}{R_n(\theta_2)\rho_2(\rho)} < 0 \quad \text{and} \quad T_4'(\rho) = -\frac{\rho_1'(\rho)}{R_n(\theta_1)\epsilon\rho_1(\rho)} < 0,$$

from which follows that

$$\lim_{\rho \to 0^+} T_2'(\rho) = -\infty \quad \text{and} \quad \lim_{\rho \to 0^+} T_4'(\rho) = -\infty. \tag{18}$$

Finally, from (15), (17) and (18) follows that, for sufficiently small ρ, $T'(\rho) < 0$ and $\lim_{\rho \to 0^+} T(\rho) = +\infty$, which ends the proof. ▮

Using the lemma above, one can make a sketch of the graphic of period function $T(\rho)$, which is shown in Figure 5. Still using this lemma, one can prove the following result.

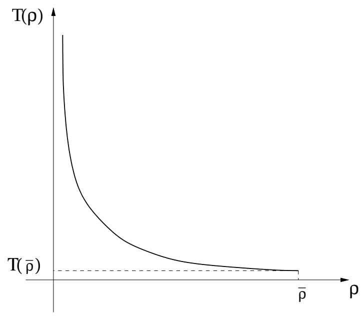

FIG. 5. Sketch of the period function graphic.

LEMMA 7. *If $\bar{\rho}$ is sufficiently small, there exists a sequence $\{\rho_{m_i}\}_{i \in N} \subset (0, \bar{\rho})$, with $\lim_{i \to +\infty} \rho_{m_i} = 0$, to which corresponds a family $\{q^{\rho_{m_i}}(s)\}_{i \in N}$, of periodic orbits of the system (14), with $\mu = A = 0$, belonging to the annulus $\mathcal{A}$, such that the period $T(\rho^{m_i})$ of $q^{\rho_{m_i}}(s)$ satisfies, for each i, the following resonance condition with the period 2π of the perturbation function $G_2(u, \phi(s))$ in 14*

$$T(\rho_{m_i}) = 2\pi m_i, \tag{19}$$

where m_i are natural numbers such that $lim_{i \to +\infty} m_i = +\infty$.

Proof. The proof follows immediately from the previous lemma, since $T(\rho)$ is a continuous function which tends to infinity as $\rho \to 0$, assuming all the values in the interval $[T(\bar{\rho}), +\infty)$. Also, as $\lim_{i \to +\infty} \rho_{m_i} = 0$, and

$T(\rho_{m_i}) \to +\infty$ as $\rho_{m_i} \to 0$, from the resonance relation (19) it follows that $m_i \to \infty$, as $i \to +\infty$. ▌

Using these lemmas and the results stated in the previous section (Theorem 5), we shall prove the following, which is the main result of the paper.

THEOREM 8. *Consider the system* $\dot{u} = F(u)+\mu G_1(u)+AG_2(u, \phi(s, \phi_0))$, *given in (14). Suppose that, for* $\mu = A = 0$, *the system has a singular cycle* $\gamma(s) \cup \theta_1 \cup \alpha(s) \cup \theta_2$, *accumulated by an annulus* $\mathcal{A}$ *of periodic orbits. Consider the functions*

$$S_m = \int_0^{mT} e^{-\int_0^s divF(q^{\rho_m}(\tau))d\tau} < F^{\perp}(q^{\rho_m}(s)), G_1(q^{\rho_m}(s)) > ds,$$
$$M_m(\phi_0) = \int_0^{mT} e^{-\int_0^s divF(q^{\rho_m}(\tau))d\tau} < F^{\perp}(q^{\rho_m}(s)), G_2(q^{\rho_m}(s), \phi(s, \phi_0)) > ds,$$

where $q^{\rho_m}(s)$ *is a resonant periodic orbit belonging to* $\mathcal{A}$ *and* F, G_1 *and* G_2 *are the functions on the system (14),* $F^{\perp}$ *being the orthogonal of* F.

If S_m *is different from zero and* M_m *has only two nondegenerated critical points, a maximum and a minimum, given by* ϕ_{max} *and* ϕ_{min}, *such that* $M''(\phi_{max}) \neq 0$ *and* $M''(\phi_{min}) \neq 0$, *with* $M(\phi_{min}) < 0 < M(\phi_{max})$, *then the following results hold: there exist, for sufficiently small* μ, *differentiable curves* $A_1^m(\mu)$ *and* $A_2^m(\mu)$, *with* $A_1^m(0) = A_2^m(0) = 0$, *which divide the parameter space* (μ, A) *in two disjoint regions* R_1 *and* R_2 *(Figure 6) such that:* ***a)*** *for* (μ, A) *in* R_1, *the perturbed system has two subharmonics of order* m, *which bifurcate from the resonant periodic orbit* $q^{\rho_m}(s)$; ***b)*** *for* (μ, A) *in* R_2, *there are no subharmonics bifurcating from* $q^{\rho_m}(s)$; ***c)*** *for* (μ, A) *on the border of the regions* R_1 *and* R_2 *(in the curves* $A_1^m(\mu)$ *and* $A_2^m(\mu)$*), there exists only one subharmonic of order* m, *that bifurcates from* $q^{\rho_m}(s)$.

Remark 9. The functions S_m and $M_m(\phi_0)$ which appear in the theorem above correspond to the functions $M(s, \phi_0)$, with $s = 0$, in Theorem 5. Also, it can be proved that $\frac{\partial M}{\partial s} = \frac{\partial M}{\partial \phi_0}$ ([15], pg. 503). S_m does not depend on ϕ_0, for it is related to the autonomous part of the perturbation, while $M_m(\phi_0)$ is related to the nonautonomous (periodic) part.

Proof. Consider the Poincaré map $P^m_{\mu,A}(u) : \Sigma^{\phi_0} \to \Sigma^{\phi_0}$, associated to (14), given by

$$P^m_{\mu,A}(u) = u(mT, u, \phi_0, \mu, A),$$

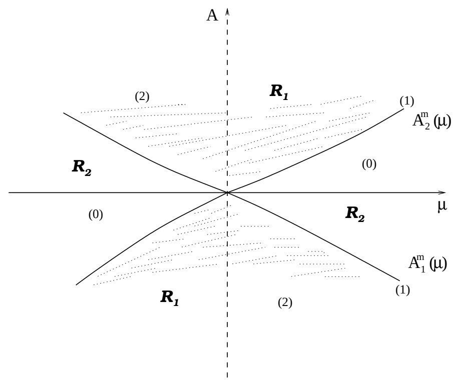

FIG. 6.

where u is the solution of $\dot{u} = F(u) + \mu G_1(u) + AG_2(u, \phi(s, \phi_0))$, with $u(0, u, \phi_0, \mu, A) = u \in \Sigma^{\phi_0}$, and

$$\Sigma^{\phi_0} = \{(u, \phi) \mid u \in \mathbb{R}^2 \text{ and } \phi = \phi_0 \in [0, 2\pi)\}.$$

Taylor expanding $P^m_{\mu,A}$, with respect to the parameters μ and A, near $\mu = A = 0$, one obtains

$$\begin{aligned} P^m_{\mu,A}(u) &= u(mT, u, \phi_0, 0, 0) + u_\mu(mT, u, \phi_0, 0, 0)\mu \\ &\quad + u_A(mT, u, \phi_0, 0, 0)A + R(\phi_0, \mu, A), \end{aligned}$$

where $R(\phi_0, \mu, A) = a(\phi_0, \mu, A)\mu^2 + b(\phi_0, \mu, A)\mu A + c(\phi_0, \mu, A)A^2 + h.o.t$, and a, b, c are differentiable functions such that $a(\phi_0, 0, 0) = b(\phi_0, 0, 0) = c(\phi_0, 0, 0) = 0$

Suppose that u belongs to one of the resonant periodic orbits of the unperturbed systems, say $q^{\rho_m}(s)$, given by Lemma 7. Then, one has $u(mT, u, \phi_0, 0, 0) = u$, from which follows that

$$P^m_{\mu,A}(u) - u = u_\mu(mT, u, \phi_0, 0, 0)\mu + u_A(mT, u, \phi_0, 0, 0)A + R(\phi_0, \mu, A).$$

Taking

$$\Delta(\phi_0, \mu, A) := u_\mu(mT, u, \phi_0, 0, 0)\mu + u_A(mT, u, \phi_0, 0, 0)A + R(\phi_0, \mu, A),$$

it follows that the fixed points of $P^m_{\mu,A}$ are given by the zeroes of Δ.

From Theorem 5 and Lemma 6, which guarantee that the annulus $\mathcal{A}$ of (14) is regular, it follows that, for μ and A sufficiently small, the zeroes of the function Δ depend only on their projection over the direction

$f^{\perp}(q^{\rho_m}(s))$, that is, they depend only on the components of u_μ and u_A on the direction of $f^{\perp}(q^{\rho_m}(s))$, where u_μ and u_A are the solutions of the second variational equation associated to (14), obtained from differentiating the system with respect to μ and A, respectively.

Therefore, using Lemma 4, one has

$$< \Delta(\phi_0, \mu, A), f^{\perp}(q^{\rho_m}(s)) >:= S_m\mu + M_m(\phi_0)A + R(\phi_0, \mu, A),$$

where

$$S_m = \int_0^{mT} e^{-\int_0^s divF(q^{\rho_m}(\tau))d\tau} < F^{\perp}(q^{\rho_m}(s)), G_1(q^{\rho_m}(s)) > ds,$$
$$M_m(\phi_0) = \int_0^{mT} e^{-\int_0^s divF(q^{\rho_m}(\tau))d\tau} < F^{\perp}(q^{\rho_m}(s)), G_2(q^{\rho_m}(s), \phi(s, \phi_0)) > ds,$$

which are the projections of u_μ and u_A, respectively, on the direction of $f^{\perp}(q^{\rho_m}(s))$.

Factoring μ in the expression above, one obtains

$$\begin{aligned} &< \Delta(\phi_0, \mu, A), f^{\perp}(q^{\rho_m}(s)) >:= \\ &\quad \mu[S_m + M_m(\phi_0)\eta + \tilde{R}(\phi_0, \mu, \eta)] := \mu\tilde{\Delta}(\phi_0, \mu, \eta), \end{aligned}$$

where $\eta = A/\mu$,

$$\tilde{R}(\phi_0, \mu, \eta) = a(\phi_0, \mu, \eta\mu)\mu + b(\phi_0, \mu, \eta\mu)\mu\eta + c(\phi_0, \mu, \eta\mu)\mu\eta^2 = O(\mu)$$

and, in this way, for $\mu \neq 0$ sufficiently small, the zeroes of the projection of the function $\Delta(\phi_0, \mu, A)$ on the direction of $f^{\perp}(q^{\rho_m}(s))$ coincide with the zeroes of the function

$$\tilde{\Delta}(\phi_0, \mu, \eta) = S_m + M_m(\phi_0)\eta + O(\mu). \tag{20}$$

Using now the hypotheses that $S_m \neq 0$ and $M_m(\phi_0)$ have a non degenerated maximum and a minimum, with $M_m(\phi_{min}) < 0 < M_m(\phi_{max})$, and supposing, without loss of generality, that $S_m > 0$, one can determine the regions R_1 and R_2 described in the theorem from the function $\tilde{\Delta}$. Indeed, $\tilde{\Delta}(\phi_0, \mu, \eta) = 0$ if and only if

$$S_m + M_m(\phi_0)\eta + O(\mu) = 0. \tag{21}$$

On the other hand,

$$M_m(\phi_{min}) \;\leq\; M_m(\phi_0) \;\leq\; M_m(\phi_{max}).$$

From the relations above, isolating $M(\phi_0)$ in (21) and being $\eta = A/\mu$, it follows that

$$M_m(\phi_{min}) \ \le\ -\frac{S_m}{\eta} - O(\mu^2) \ \le\ M_m(\phi_{max}),$$

from which one has the relations

$$\eta \ge -\frac{S_m}{M_m(\phi_{min})} - O(\mu) \quad \text{or} \quad \eta \ge -\frac{S_m}{M_m(\phi_{max})} - O(\mu) \quad \text{if} \quad \eta > 0,$$

and

$$\eta \ \le\ -\frac{S_m}{M_m(\phi_{min})} - O(\mu) \quad \text{or} \quad \eta \le -\frac{S_m}{M_m(\phi_{max})} - O(\mu) \quad \text{if} \quad \eta < 0,$$

The inequalities $\eta \ \ge\ -\frac{S_m}{M_m(\phi_{max})} - O(\mu)$ and $\eta \ \le\ -\frac{S_m}{M_m(\phi_{min})} - O(\mu)$ are always satisfied, due to the hypotheses. From the other inequalities, one has (changing back the change of variables $\eta = A/\mu$), for $\mu > 0$,

$$A \ge -\frac{S_m}{M_m(\phi_{min})}\mu - O(\mu^2 \quad \text{and} \quad A \le -\frac{S_m}{M_m(\phi_{max})}\mu - O(\mu^2).$$

Also, if $\mu < 0$ one has

$$A \le -\frac{S_m}{M_m(\phi_{min})}\mu - O(\mu^2) \quad \text{and} \quad A \ \ge\ -\frac{S_m}{M_m(\phi_{max})}\mu - O(\mu^2).$$

These four inequalities determine the regions R_1 and R_2 (Figure 6), in the parameter plane, in which there exist one subharmonic of order m for (14), which bifurcates from the resonant orbit $q^{\rho_m}(s)$. Such regions are delimited by the curves $A_1^m(\mu)$ and $A_2^m(\mu)$, with μ small enough, given by

$$A_1^m(\mu) = -\frac{S_m}{M_m(\phi_{max})}\mu - O(\mu^2), \quad A_2^m(\mu) = -\frac{S_m}{M_m(\phi_{min})}\mu - O(\mu^2),$$

and depend only on the values of S_m, $M_m(\phi_{min})$ and $M_m(\phi_{max})$.

Let us show, using the hypotheses $M''(\phi_{max}) < 0$ and $M''(\phi_{min}) > 0$, that, if (μ, A) belongs to the interior of the regions R_1 and R_2, there exist two subharmonics of order m bifurcating from the resonant periodic orbit $q^{\rho_m}(s)$ and, if (μ, A) is in the curves $A_1^m(\mu)$ or $A_2^m(\mu)$,there exists only one subharmonic of order m.

Taylor expanding $M(\phi_0)$ with respect to ϕ_0 around $\phi_0 = \phi_{max}$ and putting it in the expression of $\tilde{\Delta}(\phi_0, \mu, \eta)$ in (20), one obtains

$$\tilde{\Delta}(\phi_0, \mu, \eta) = S_m + M_m(\phi_{max})\eta + \beta\eta(\phi_0 - \phi_{max})^2 + O((\phi_0 - \phi_{max})^3)\eta + O(\mu),$$

with $\beta = M''(\phi_{max})/2 < 0$. So, if $(\mu, A) \in A_1^m(\mu)$, then

$$\eta = -S_m/M_m(\phi_{max}) - O(\mu),$$

which, substituted in the last expression, implies that $\tilde{\Delta}(\phi_0, \mu, \eta) = 0$ if, and only if,

$$\beta\eta(\phi_0 - \phi_{max})^2 + O((\phi_0 - \phi_{max})^3)\eta = 0.$$

As, for μ sufficiently small, $\beta\eta \neq 0$, it follows that the expression above is equal to zero if, and only if, $\phi_0 - \phi_{max} = 0$. Thus, if $(\mu, A) \in A_1^m(\mu)$, $\phi_0 = \phi_{max}$ is the unique zero of $\tilde{\Delta}(\phi_0)$, which correspond to a subharmonic of order m, bifurcating from $q^{\rho_m}(s)$.

In the same way, by Taylor expanding $M(\phi_0)$ around ϕ_{min}, one can conclude that, if (μ, A) is in $A_2^m(\mu)$, then $\phi_0 = \phi_{min}$ is the only zero of $\tilde{\Delta}(\phi_0)$.

Suppose now that (μ, A) belongs to the interior of the region R_2. The equality

$$\begin{aligned}\tilde{\Delta}(\phi_0, \mu, \eta) &= S_m + M_m(\phi_{max})\eta + \beta\eta(\phi_0 - \phi_{max})^2 \\ &+ O((\phi_0 - \phi_{max})^3)\eta + O(\mu) = 0,\end{aligned}$$

is equivalent to

$$\begin{aligned}\beta\eta{\phi_0}^2 - 2\beta\eta\phi_0\phi_{max} + \beta\eta{\phi_{max}}^2 + S_m + M_m(\phi_{max})\eta \\ +O((\phi_0 - \phi_{max})^3)\eta + O(\mu) = 0.\end{aligned}$$

The equation above, as a function of ϕ_0, has two real roots if, and only if, its discriminant is strictly positive, which, after a straight forward calculations leads to

$$-4\beta\eta[S_m + M_m(\phi_{max})\eta + O((\phi_0 - \phi_{max})^3)\eta + O(\mu)] > 0.$$

As, for small enough μ, $\beta\eta > 0$, the inequality above holds if, and only if,

$$\eta < -S_m/M_m(\phi_{max}) - O((\phi_0 - \phi_{max})^3)\eta - O(\mu).$$

Taking $(\phi_0 - \phi_{max})$ of order $\mu^{2/3}$, as $\eta = A/\mu$, it follows that $O((\phi_0 - \phi_{max})^3)\eta = O(\mu)$. Then, the last inequality takes the form

$$\eta < \frac{-S_m}{M_m(\phi_{max})} - O(\mu)$$

which implies that (μ, A) belongs to the interior of the region R_2, with $A > 0$,for μ sufficiently small.

In the same way, after expanding $\tilde{\Delta}$ around $\phi_0 = \phi_{min}$, one can conclude that the equation $\tilde{\Delta}(\phi_0, \mu, \eta) = 0$ has two real roots if (μ, A) belongs to the interior of the region R_2, with $A < 0$.

These two real roots of $\tilde{\Delta} = 0$ correspond to the $2m$ fixed points of the Poincaré map $P^m_{\mu,A}$, which correspond to the two subharmonics of order m of the perturbed system (14), which bifurcate from the resonant orbit $q^{\rho_m}(s)$. The theorem is proved. ▮

The hypotheses on the function $M(\phi_0)$ in the previous theorem are directly related to the properties of the periodic perturbation function $G_2(u, \phi(s, \phi_0))$ of the system (14), as it can be easily seen.

Looking at the Figure 6, let us see the scenery described in Theorem 8. Fixing a value of the parameter $\mu > 0$, taking A varying in the vertical direction, from $A = 0$, one has: initially, for A closer to zero, the system does not have subharmonics bifurcating from $q^{\rho_m}(s)$; when the parameter A crosses the value $A = A^m_1(\mu)$, one subharmonic of order m arises; finally, for $A > A^m_1(\mu)$, there exist two subharmonics of order m arising from the resonant orbit $q^{\rho_m}(s)$. These subharmonics correspond to fixed points of the Poincaré map $P^m_{\mu,A}$, with μ fixed, sufficiently small. This scenery is commonly found in the study of one parameter family of difeomorfisms, which is given, in this case, by $P^m_{\mu,A}$, considering the parameter μ fixed. As the value of A crosses the curve $A^m_1(\mu)$, a saddle node bifurcation occurs, which gives rise to the subharmonics.

In the next section, the quadratic case will be considered, since it appears in several applied equations, as Lotka–Volterra model, Blasius equation of the fluid flow and Endem–Fowler equations of Astrophysics ([5], [14]).

4. QUADRATIC CASE

Consider the family of quadratic polynomial vector fields

$$\begin{aligned} \dot{x} &= xy - y \\ \dot{y} &= \alpha + mx + ny + ax^2 + bxy + cy^2. \end{aligned} \tag{22}$$

and suppose that the system satisfies the hypotheses (H1)–(H3) below:

(H1) $(b + n)^2 - 4(\alpha + m + a)c < 0$ and $m^2 - 4a\alpha > 0$;
(H2) $0 < c < 1$;
(H3) $b^2 - 4a(c - 1) < 0$.

Then, as it is proved in [10], (22) has only two hyperbolic saddles at infinity, θ_1 and θ_2, connected by a heteroclinic trajectory $\gamma(s)$, which is a reparametrization of the solution contained in the straight line $\{x = 1\}$, forming the infinite heteroclinic cycles $\gamma(s) \cup \theta_1 \cup \pm\alpha(s) \cup \theta_2$ on the Poincaré disk (see Figures 1 and 7). Also, the system has two critical points on the x axis, which are foci or centers.

Lemma 10. *The system (22) has two centers on the x axis if, and only if, $n = b = 0$.*

Proof. System (22) has two critical points in the x axis, given by

$$E_{1,2} = \frac{-m \pm \sqrt{m^2 - 4a\alpha}}{2a},$$

and the trace of the jacobian matrix evaluated in these points is equals to zero if, and only if, $n = b = 0$, which is a necessary condition for E_1 and E_2 to be centers.

Considering the change of variables $(x, y) \to (x, -y)$ it follows that, if $b = n = 0$, the system is symmetric with respect to the x axis and, therefore, E_1 and E_2 are centers. ∎

The phase plane of the compactification of the system (22) on the Poincaré disk, with $n = b = 0$, is as shown in Figure 7.

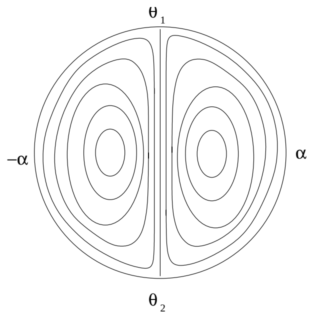

FIG. 7. Phase plane of the system (22) with $n = b = 0$.

Consider now the following periodic perturbation of system (22):

$$\begin{aligned} \dot{x} &= xy - y + \mu + A(a_1 \cos t + a_2 \sin t) \\ \dot{y} &= \alpha + mx + ax^2 + cy^2 + A(b_1 \cos t + b_2 \sin t), \end{aligned} \tag{23}$$

where μ and A are real parameters.

In order to use the results stated in the previous section to consider subharmonic bifurcations to the system above, it must be observed first that, although the annulus of periodic orbits tend, on the one hand, to the critical point E_2 and, on the other, to the infinite heteroclinic cycle $\gamma(s) \cup \theta_1 \cup +\alpha(s) \cup \theta_2$, it is not possible to guarantee that it is regular. In fact, it is still an open problem to determine the number of critical points of the period function associated to an annulus of periodic orbits of a polynomial vector field in the case in which it is unbounded (see, for example, [3]). So, as it was done previously, it will be analyzed only the subharmonic bifurcations that occur with the large amplitude periodic orbits, which is the part of the annulus that is regular, as proved in Lemma 6.

The compactification of (23), in polar coordinates (θ, ρ) is given by

$$\begin{aligned}
\dot{\theta} &= a\cos^3\theta + (c-1)\sin^2\theta\cos\theta + \rho(m\cos^2\theta + \sin^2\theta) + \rho^2\alpha\cos\theta \\
&\quad -\mu\rho^2\sin\theta + A\rho^2[(b_1\cos s + b_2\sin s)\cos\theta - (a_1\cos s + a_2\sin s)\sin\theta] \\
\dot{\rho} &= -\rho[c\sin^3\theta + (a+1)\cos^2\theta\sin\theta + \rho(m-1)\sin\theta\cos\theta + \rho^2\alpha\sin\theta] \\
&\quad -\mu\rho^3\cos\theta - A\rho^3[(a_1\cos s + a_2\sin s)\cos\theta + (b_1\cos s + b_2\sin s)\sin\theta]
\end{aligned}$$

Taking $F(\theta, \rho) = (F_1(\theta, \rho), F_2(\theta, \rho))$, where

$$F_1(\theta, \rho) = a\cos^3\theta + (c-1)\sin^2\theta\cos\theta + \rho(m\cos^2\theta + \sin^2\theta) + \rho^2\alpha\cos\theta,$$

$$F_2(\theta, \rho) = -\rho[c\sin^3\theta + (a+1)\cos^2\theta\sin\theta + \rho(m-1)\sin\theta\cos\theta + \rho^2\alpha\sin\theta],$$

$$G_1(\theta, \rho) = (-\rho^2\sin\theta, -\rho^3\cos\theta),$$

and

$$G_2(\theta, \rho) = (G_2^1(\theta, \rho), G_2^2(\theta, \rho)),$$

with

$$G_2^1 = \rho^2[(b_1\cos s + b_2\sin s)\cos\theta - (a_1\cos s + a_2\sin s)\sin\theta]$$

$$G_2^2 = -\rho^3[(a_1\cos s + a_2\sin s)\cos\theta + (b_1\cos s + b_2\sin s)\sin\theta],$$

one can write the system in the following way

$$\dot{u} = F(u) + \mu G_1(u) + AG_2(u, \phi(s, \phi_0)), \tag{24}$$

where $u = (\theta, \rho)$, and $\phi(s, \phi_0)$ is the solution of $\dot{\phi} = \rho$, with $\phi(0) = \phi_0$.

One can now apply the results stated in the previous theorems. First it will be shown that the hypotheses of Theorem 8 hold for the periodic perturbation above, that is, the functions S_m and $M_m(\phi_0)$ associated with (24), defined in the theorem, satisfy the conditions $S_m \neq 0$ and $M_m(\phi_0)$ has only one maximum point ϕ_{max}, and a minimum, ϕ_{min}, non degenerated ($M''(\phi_{max}) \neq 0$ and $M''(\phi_{min}) \neq 0$).

Lemma 11. *The functions S_m and $M_m(\phi_0)$, defined in Theorem 8, related to the system (24), are given by*

$$S_m = -\int_0^{mT} e^{-\int_0^s \sigma(\tau)d\tau} \rho^3 [a\cos^2\theta + c\sin^2\theta + \rho m\cos\theta + \rho^2\alpha] ds,$$

and

$$M_m(\phi_0) = -\int_0^{mT} e^{-\int_0^s \sigma(\tau))d\tau} \rho(s)^3 K(\theta, \rho, \phi_0, s) ds,$$

where $K(\theta, \rho, \phi_0, s)$ is equals to

$$\begin{aligned}&[a_1\cos(\phi(s,\phi_0)) + a_2\sin(\phi(s,\phi_0))](a\cos^2\theta + c\sin^2\theta + \rho m\cos\theta + \rho^2\alpha)\\ &\qquad + [b_1\cos(\phi(s,\phi_0)) + b_2\sin(\phi(s,\phi_0))](\rho\sin\theta - \rho^2\sin\theta\cos\theta),\end{aligned}$$

where $(\theta, \rho) = (\theta(s), \rho(s))$ are the coordinates of the resonant periodic orbit $q^{\rho_m}(s)$ of (24), and

$$\sigma(s) = \sum_{i=0}^{2} \rho(s)^i A'_{n-i}(\theta(s)) - \sum_{i=0}^{2} (i+1)\rho(s)^i R_{n-i}(\theta(s)),$$

with $A_k(\theta)$ and $R_k(\theta)$ defined above.

Proof. Consider the polynomial vector field

$$\begin{aligned}\dot{x} &= P(x,y) + \epsilon f(s)\\ \dot{y} &= Q(x,y) + \epsilon g(s),\end{aligned}$$

and its compactification in (θ, ρ) coordinates, given by

$$\begin{aligned}\dot{\theta} &= \sum_{i=1}^{n} \rho^i A_{n-i}(\theta) + \bar{f}(\theta, \rho, \epsilon, \phi)\\ \dot{\rho} &= -\sum_{i=0}^{n} \rho^{i+1} R_{n-i}(\theta) + \bar{g}(\theta, \rho, \epsilon, \phi)\\ \dot{\phi} &= \rho,\end{aligned}$$

where

$$\begin{aligned}A_k(\theta) &= -P_k(\cos\theta, \sin\theta)\sin\theta + Q_k(\cos\theta, \sin\theta)\cos\theta\\ R_k(\theta) &= P_k(\cos\theta, \sin\theta)\cos\theta + Q_k(\cos\theta, \sin\theta)\sin\theta,\end{aligned}$$

for $k = 0, 1, 2$, with $\theta \in [0, 2\pi)$ and $\rho \in [0, \infty)$,

$$\begin{aligned} \bar{f}(\theta, \rho, \epsilon, \phi) &= \rho^n[\epsilon g(\phi)\cos\theta - \epsilon f(\phi)\sin\theta], \\ \bar{g}(\theta, \rho, \epsilon, \phi) &= -\rho^{n+1}[\epsilon f(\phi)\cos\theta + \epsilon g(\phi)\sin\theta]. \end{aligned}$$

Let us calculate, for this system, the terms of the integrand in $M(\phi_0)$, given in Theorem 8. In order to do that, let us take

$$F(\theta, \rho) = \left(\sum_{i=1}^{n} \rho^i A_{n-i}(\theta), -\sum_{i=0}^{n} \rho^{i+1} R_{n-i}(\theta)\right).$$

Then, neglecting the arguments of the functions, to simplify, and taking $\sin\theta = s$ and $\cos\theta = c$, one obtains:

$$\begin{aligned} &< F^{\perp}, (\bar{f}, \bar{g}) >= \\ &c\bar{f}\sum_{i=0}^{n} \rho^{i+1} P_{n-i} + s\bar{f}\sum_{i=0}^{n} \rho^{i+1} Q_{n-i} - s\bar{g}\sum_{i=0}^{n} \rho^{i} P_{n-i} + c\bar{g}\sum_{i=0}^{n} \rho^{i} Q_{n-i}. \end{aligned}$$

Substituting $\bar{f}$ and $\bar{g}$ on the expression above, one obtains, after a straightforward calculation

$$\begin{aligned} < F^{\perp}, (\bar{f}, \bar{g}) > &= g\rho^{n+1}\textstyle\sum_{i=0}^{n} \rho^i P_{n-i} - f\rho^{n+1}\sum_{i=0}^{n} \rho^i Q_{n-i} \\ &= \rho^{n+1} < \Big(-\textstyle\sum_{i=0}^{n} \rho^i Q_{n-i}, \sum_{i=0}^{n} \rho^i P_{n-i}\Big), (f, g) >, \end{aligned}$$

where P_k and Q_k are the homogeneous polynomials of degree k, components of P and Q, which are the polynomial that determine the unperturbed system on the plane, ($P = \sum_{k=0}^{n} P_k$ and $Q = \sum_{k=0}^{n} Q_k$), f and g are the perturbation functions of the polynomial vector field (P, Q), and $<, >$ is the usual inner product in $\mathbb{R}^2$.

For the system (23), whose compactification is given by (24), there exist two pairs (f, g) of perturbation functions: $(f, g) = (1, 0)$, related to the parameter μ (autonomous part), and

$$(f, g) = (a_1 \cos s + a_2 \sin s, b_1 \cos s + b_2 \sin s),$$

related to the parameter A (nonautonomous part).

Putting these expressions and the quadratic polynomials given in (23) on the equations $< F^{\perp}, (\bar{f}, \bar{g}) >$ obtained above, and considering the divergence of the unperturbed system, given by

$$divF(q^{\rho_m}(s)) = \sigma(s) = \sum_{i=0}^{n} \rho(s)^i A'_{n-i}(\theta(s)) - \sum_{i=0}^{n} (i+1)\rho(s)^i R_{n-i}(\theta(s)),$$

one can obtain the expressions S_m and $M_m(\phi_0)$ stated in the lemma. ∎

LEMMA 12. *For ρ_m sufficiently small one has $S_m < 0$, for*

$$\int_0^{mT} e^{-\int_0^s \sigma(\tau)d\tau}\rho(s)^3[a\cos^2\theta(s)+c\sin^2\theta(s)+\rho(s)m\cos\theta(s)+\rho(s)^2\alpha]ds>0.$$

Proof. Consider the periodic orbit $q^{\rho_m}(s) = (\theta(s), \rho(s))$. For ρ_m sufficiently small such periodic orbit is closer to the infinite heteroclinic cycle $\theta_1 \cup \gamma(s) \cup \theta_2 \cup \alpha(s)$ and we can consider the cross section Σ_1 and Σ_2, transversal to this orbit, defined by

$$\begin{aligned}
\Sigma_1 &= \{(\theta,\rho) \mid \theta \in (\frac{3\pi}{2} - \epsilon, \frac{3\pi}{2} + \epsilon), \rho = \epsilon\},\\
\Sigma_2 &= \{(\theta,\rho) \mid \theta \in (\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon), \rho = \epsilon\},
\end{aligned}$$

with $\epsilon > 0$ sufficiently small (see Figure 8).

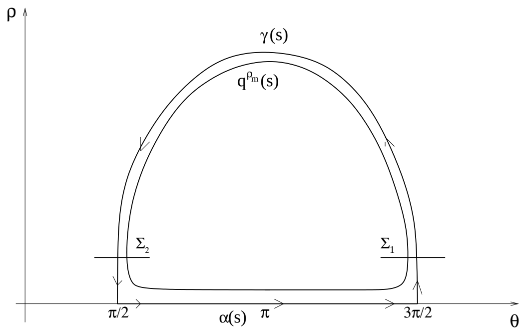

FIG. 8.

This way, taking $q^{\rho_m}(0) \in \Sigma_1$, one can divide the integral in two parts:

$$\begin{aligned}
&\int_0^{mT} e^{-\int_0^s \sigma(\tau)d\tau}\rho(s)^3[a\cos^2\theta(s) + c\sin^2\theta(s) + \rho(s)m\cos\theta(s) + \rho(s)^2\alpha]ds = \\
&\quad \int_0^{T_1} e^{-\int_0^s \sigma(\tau))d\tau}\rho(s)^3[a\cos^2\theta(s) + c\sin^2\theta(s) + \rho(s)m\cos\theta(s) + \rho(s)^2\alpha]ds + \\
&\quad \int_{T_1}^{mT} e^{-\int_0^s \sigma(\tau))d\tau}\rho(s)^3[a\cos^2\theta(s) + c\sin^2\theta(s) + \rho(s)m\cos\theta(s) + \rho(s)^2\alpha]ds \\
&= S_1 + S_2,
\end{aligned}$$

where T_1 is the time taken to the orbit $q^{\rho_m}(s)$ to go from Σ_1 to Σ_2.

Let us prove that $S_1 > 0$ and that S_2 is closer to zero, for ρ_m sufficiently small.

Considering the cross sections defined above (Figure 8) and that $q^{\rho_m}(s)$ stay ϵ–close to the singular cycle $\gamma(s) \cup \theta_1 \cup \alpha(s) \cup \theta_2$, for $s \in [T_1, mT]$ one has $0 < \rho(s) \leq \epsilon << 1$, from which follows that $\rho(s)^3 < \epsilon$. On the other hand, being $\rho(s)$ bounded, the term

$$\sigma(s) = \sum_{i=0}^{n} \rho(s)^i A'_{n-i}(\theta(s)) - \sum_{i=0}^{n} (i+1)\rho(s)^i R_{n-i}(\theta(s)),$$

is bounded, and the term $\exp\{-\int_0^s \sigma(\tau)d\tau\}$ is also bounded for $s \in [T_1, mT]$, say by a constant $K_1 < \infty$. Also, it is clear that $[a\cos^2\theta(s) + c\sin^2\theta(s) + \rho(s)m\cos\theta(s) + \rho(s)^2\alpha] \leq K_2$, for some constant $K_2 < \infty$. Then one has

$$\int_{T_1}^{mT} e^{-\int_0^s \sigma(\tau))d\tau} \rho(s)^3[a\cos^2\theta(s) + c\sin^2\theta(s) + \rho(s)m\cos\theta(s) + \rho(s)^2\alpha]ds < K_1K_2(mT - T_1)\epsilon. \tag{25}$$

In order to prove that $S_1 > 0$, we will take into account the fact that, for ρ_m sufficiently small and $s \in [0, T_1]$, one has, by the continuous dependence theorem, that $q^{\rho_m}(s)$ stay close to the portion $\gamma(s)$ between Σ_1 and Σ_2 (see Figure 8). Considering the relations

$$x\rho = \cos\theta, \qquad y\rho = \sin\theta \qquad \text{and} \qquad \frac{ds}{dt} = \frac{1}{\rho^{n-1}} \tag{26}$$

and changing the variables in the integral above, one obtains S_1 given by

$$\int_0^{T_1} e^{-\int_0^s \sigma(\tau)d\tau} \rho(s)^3[a\cos^2\theta(s) + c\sin^2\theta(s) + \rho(s)m\cos\theta(s) + \rho(s)^2\alpha]ds =$$
$$\int_{t_1}^{t_2} e^{-\int_0^s \tilde{\sigma}(\tau))d\tau} \|(x(t), y(t))\|^{-2}[ax(t)^2 + cy(t)^2 + mx(t) + \alpha]dt =$$
$$\int_{t_1}^{t_2} e^{-\int_0^s \tilde{\sigma}(\tau))d\tau} \|(x(t), y(t))\|^{-2} Q(x(t), y(t))dt,$$

where $q(t) = (x(t), y(t))$, with $t \in [t_1, t_2]$, is the portion of the periodic orbit on the plane, in (x, y) coordinates, which is a reparametrization of the portion of $q^{\rho_m}(s)$, with $s \in [0, T_1]$.

As the change of coordinates (26) is a difeomorfism for $(x, y) \neq (0, 0)$, and $s \in [0, T_1]$, $q^{\rho_m}(s)$ stay ϵ–close to $\gamma(s)$, on the Poincaré disk, it follows from the continuous dependence theorem that for $t \in [t_1, t_2]$ the periodic orbit $q(t)$ stay close to the solution $\varphi(t) = (1, \varphi_2(t))$ of the polynomial system, on the plane, contained in the invariant line $\{x = 1\}$ (corresponding to $\gamma(s)$ on the disk). Then, the coordinate $x(t)$ of the periodic orbit $q(t)$ stay close to 1, for $t \in [t_1, t_2]$. As, by hypothesis, $Q(1, y) > 0, \quad \forall y$, one has, by continuity, that the integrand of S_1 is the product of positive functions and, therefore, the integral is strictly positive, and the lemma is proved. ▌

LEMMA 13. *The function $M_m(\phi_0)$, associated with (24), has two non degenerate critical points on the interval $[0, 2\pi)$, being one of them a minimum and the other a maximum.*

Proof. As stated in Lemma 11, the expression of $M_m(\phi_0)$ is given by

$$M_m(\phi_0) = \int_0^{mT} e^{-\int_0^s \sigma(\tau)d\tau} \rho(s)^3 K(\theta, \rho, \phi_0) ds,$$

where

$$\sigma(s) = \sum_{i=0}^{n} \rho^i A'_{n-i}(\theta) - \sum_{i=0}^{n} (i+1)\rho^i R_{n-i}(\theta),$$

which does not depend on ϕ_0, and

$$\begin{aligned} K(\theta, \rho, \phi_0) = \\ &[a_1 \cos(\phi(s, \phi_0)) + a_2 \sin(\phi(s, \phi_0))](a \cos^2\theta + c \sin^2\theta + \rho m \cos\theta + \rho^2 \alpha) \\ &+ [b_1 \cos(\phi(s, \phi_0)) + b_2 \sin(\phi(s, \phi_0))](\rho \sin\theta - \rho^2 \sin\theta \cos\theta), \end{aligned}$$

where $(\theta, \rho) = (\theta(s), \rho(s))$ are the coordinates of the resonant orbit $q^{\rho_m}(s)$ of (24).

Calculating the first two derivatives of $M_m(\phi_0)$ with respect to ϕ_0 it follows that $M''_m(\phi_0) = -\zeta(\phi_0)M_m(\phi_0)$, for every $\phi_0 \in [0, 2\pi)$. Therefore $M_m(\phi_0)$ has nondegenerate critical points, a maximum and a minimum, on the interval $[0, 2\pi)$, with $M(\phi_{min}) < 0 < M(\phi_{max})$. ▌

Using Lemmas 12 and 13 one can conclude, using Theorem 8, that the bifurcation diagram of the system (24) is similar to that shown in Figure 6.

5. ASYMPTOTIC BEHAVIOR OF THE SUBHARMONIC BIFURCATIONS

The subharmonics of order m, detected by the theorem of the previous section, for the quadratic system, correspond only to those that bifurcate from the resonant orbit $q^{\rho_m}(s)$, when the parameters μ and A vary on a neighborhood of the origin, in the parameter space.

Although, by Lemma 7, for $\bar{\rho}$ sufficiently small, there exists a countable sequence $\rho_{m_i} \subset (0, \bar{\rho})$, with $i \in \mathbb{N}$, such that, to each element of this sequence correspods a resonant periodic orbit $q^{\rho_{m_i}}(s)$, of the unperturbed system (24), contained in the regular annulus, that accumulate on the infinite heteroclinic cycle. Also, $\lim_{i\to+\infty}\rho_{m_i} = 0$, from which one can conclude that the resonant periodic orbits $q^{\rho_{m_i}}(s)$ tend, as a set, to the singular cycle, as $i \to +\infty$.

Based on these considerations in this section it will be done an analysis of the asymptotic behavior of the subharmonics of order m_i, which bifurcate from the resonant one $q^{\rho_{m_i}}(s)$, as the sequence ρ_{m_i} tends to zero.

The aim is to prove that the tangent line, at the origin, to the bifurcation curves, $A_1^{m_i}(\mu)$ and $A_2^{m_i}(\mu)$, associated to the resonant orbit $q^{\rho_{m_i}}(s)$, tends to the heteroclinic tangency curves, , as $i \to +\infty$.

$$A_M(\mu) = -\frac{S}{M(\phi_{max})}\mu - O(\mu^2) \quad \text{and} \quad A_m(\mu) = -\frac{S}{M(\phi_{min})}\mu - O(\mu^2)$$

obtained by this author in [10]. In other words, it shall be proved the following convergences

$$S_{m_i} \to S, \quad M_{m_i}(\phi_{max}) \to M(\phi_{max}) \quad \text{and} \quad M_{m_i}(\phi_{min}) \to M(\phi_{min}),$$

as $i \to +\infty$.

Since, for each m_i there exist two subharmonics of order m_i, bifurcating from the resonant orbit $q^{\rho_{m_i}}(s)$, as the parameters μ, A cross the curves $A_1^{m_i}(\mu)$ e $A_2^{m_i}(\mu)$, then, if $m_i \to +\infty$, the total number of subharmonics of the system tends to infinity. On the other hand, as the resonant periodic orbits $q^{\rho_{m_i}}(s)$ accumulate on the infinite heteroclinic cycle $\gamma(s)\cup\theta_1\cup\alpha(s)\cup\theta_2$, the subharmonics of order m_i bifurcating from them accumulate on the heteroclinic tangencies. In the Figure 9 a sketch of this convergence is shown, based on the considerations above.

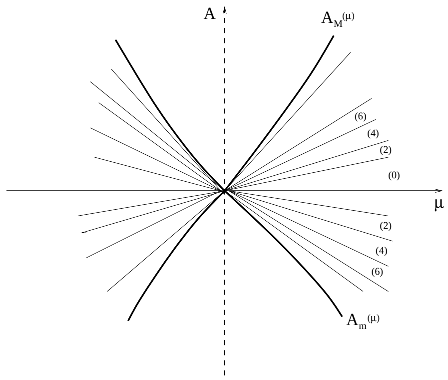

FIG. 9.

Consider then the following periodic perturbation of the quadratic system with two centers and an invariant straight line

$$\begin{aligned}\dot{x} &= xy - y + \mu + A(a_1 \cos t + a_2 \sin t)\\ \dot{y} &= \alpha + mx + ax^2 + cy^2 + A(b_1 \cos t + b_2 \sin t),\end{aligned}$$

with its compactification on the Poincaré disk

$$\dot{u} = F(u) + \mu G_1(u) + AG_2(u, \phi(s, \phi_0),$$

given in (24).

The subharmonic bifurcation curves for this system, related to a resonant periodic orbit $q^{\rho_{m_i}}(s)$, are given by

$$A_1^{m_i}(\mu) = -\frac{S_{m_i}}{M_{m_i}(\phi_{max})}\mu - O(\mu^2)$$

and

$$A_2^{m_i}(\mu) = -\frac{S_{m_i}}{M_{m_i}(\phi_{min})}\mu - O(\mu^2),$$

where

$$\begin{aligned}S_{m_i} &= \int_0^{m_iT} e^{-\int_0^s \sigma(\tau)d\tau}\rho_{m_i}(s)^3[a\cos^2\theta_{m_i}(s) + c\sin^2\theta_{m_i}(s)\\ &\qquad + \rho_{m_i}(s)m\cos\theta_{m_i}(s) + \rho_{m_i}(s)^2\alpha]ds\\ M_{m_i}(\phi_{max}) &= \int_0^{m_iT} e^{-\int_0^s \sigma(\tau)d\tau}\rho_{m_i}(s)^3 F(\theta_{m_i}(s), \rho_{m_i}(s), \phi_{max})ds,\\ M_{m_i}(\phi_{min}) &= \int_0^{m_iT} e^{-\int_0^s \sigma(\tau)d\tau}\rho_{m_i}(s)^3 F(\theta_{m_i}(s), \rho_{m_i}(s), \phi_{min})ds,\end{aligned}$$

with

$$\sigma(s) = \sum_{i=0}^{2}\rho_{m_i}(s)^i A'_{n-i}(\theta_{m_i}(s)) - \sum_{i=0}^{2}(i+1)\rho_{m_i}(s)^i R_{n-i}(\theta_{m_i}(s)),$$

and

$$\begin{aligned}&F(\theta_{m_i}(s), \rho_{m_i}(s), \phi_0) =\\ &-(a_1\cos(\omega s + \phi_0) + a_2\sin(\omega s + \phi_0))\sum_{i=0}^{2}\rho_{m_i}(s)^i Q_{n-i}(\cos\theta_{m_i}(s), \sin\theta_{m_i}(s))\\ &+(b_1\cos(\omega s + \phi_0) + b_2\sin(\omega s + \phi_0))\sum_{i=0}^{2}\rho_{m_i}(s)^i P_{n-i}(\cos\theta_{m_i}(s), \sin\theta_{m_i}(s)),\end{aligned}$$

where $(\theta_{m_i}(s), \rho_{m_i}(s))$ are the coordinates of the resonant orbit $q^{\rho_{m_i}}(s)$, P_k and Q_k the homogeneous polynomials of degree k such that $P(x,y) = \sum_{k=0}^{2} P_k(x,y)$ and $Q(x,y) = \sum_{k=0}^{2} Q_k(x,y)$, where $P(x,y) = xy - y$ and $Q(x,y) = \alpha + mx + ax^2 + cy^2$ are the polynomials which determine the quadratic system in the plane.

From the study developed in [10], it follows that the heteroclinic tangencies, related to the periodic perturbation of the heteroclinic connection $\gamma(s)$ of the system (24), are given, in the parameter space, by

$$A_M(\mu) = -\frac{S}{M(\phi_{max})}\mu - O(\mu^2) \qquad \text{and} \qquad A_m(\mu) = -\frac{S}{M(\phi_{min})}\mu - O(\mu^2),$$

where

$$\begin{aligned} S &= \int_{-\infty}^{+\infty} e^{-\int_0^s \sigma(\tau)d\tau} \rho_0(s)^3 [a\cos^2\theta_0(s) + c\sin^2\theta_0(s) \\ &\qquad\qquad + \rho m \cos\theta_0(s) + \rho^2(s)\alpha] ds, \\ M(\phi_{min}) &= \int_{-\infty}^{+\infty} e^{-\int_0^s \sigma(\tau)d\tau} \rho_0(s)^3 F(\theta_0(s), \rho_0(s), \phi_{min}) ds, \\ M(\phi_{max}) &= \int_{-\infty}^{+\infty} e^{-\int_0^s \sigma(\tau)d\tau} \rho_0(s)^3 F(\theta_0(s), \rho_0(s), \phi_{max}) ds, \end{aligned}$$

with

$$\sigma(s) = \sum_{i=0}^{2} \rho_0(s)^i A'_{n-i}(\theta_0(s)) - \sum_{i=0}^{2} (i+1)\rho_0(s)^i R_{n-i}(\theta_0(s)),$$

and

$$\begin{aligned} &F(\theta_0(s), \rho_0(s), \phi_0) = \\ &-(a_1 cos(\omega s + \phi_0) + a_2 sin(\omega s + \phi_0)) \sum_{i=0}^{n} \rho_0(s)^i Q_{n-i}(cos(\theta_0(s)), sin(\theta_0(s))) \\ &+(b_1 cos(\omega s + \phi_0) + b_2 sin(\omega s + \phi_0)) \sum_{i=0}^{n} \rho_0(s)^i P_{n-i}(cos(\theta_0(s)), sin(\theta_0(s))), \end{aligned}$$

$(\theta_0(s), \rho_0(s))$ being the coordinate of the heteroclinic trajectory $\gamma(s)$, which connects the saddle points θ_1 and θ_2, of the compactified unperturbed polynomial system, and P_k and Q_k are the homogeneous polynomials of degree k such that $P(x,y) = \sum_{k=0}^{2} P_k(x,y)$ and $Q(x,y) = \sum_{k=0}^{2} Q_k(x,y)$, being P and Q the quadratic polynomials which determine the planar system.

It is clear the similarity in the expressions of the integral formulas S and S_m, $M(\phi_0)$ and $M_m(\phi_0)$. Let us prove then, firstly, that $S_{m_i} \to S$, as $i \to +\infty$.

LEMMA 14. *$S_{m_i} \to S$ as $i \to +\infty$.*

Proof. In order to establish the convergence, let us show that, given $\epsilon > 0$, there exists $i \in \mathbb{N}$ such that $|S - S_{m_i}| < \epsilon$.

As $\rho_{m_i} \to 0$, for $i \to +\infty$, it follows that the correspondent resonant periodic orbit, $q^{\rho_{m_i}}(s)$ such that $q^{\rho_{m_i}}(0) = \rho_{m_i}$, tends, as a set, to the singular cycle $\gamma(s) \cup \theta_1 \cup \alpha \cup \theta_2$ (see Figure 8), as $i \to +\infty$.

Then, one can take Poincaré sections, transversal to $q^{\rho_{m_i}}(s)$ and to $\gamma(s)$, given by

$$\begin{aligned}
\Sigma_1 &= \{(\theta,\rho) \mid \theta \in (\frac{3\pi}{2} - \epsilon, \frac{3\pi}{2} + \epsilon) \quad \text{and} \quad \rho = \epsilon\}, \\
\Sigma_2 &= \{(\theta,\rho) \mid \theta \in (\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon) \quad \text{and} \quad \rho = \epsilon\},
\end{aligned}$$

with $\epsilon > 0$ sufficiently small (see Figure 8).

As, for $i \to +\infty$, $q^{\rho_{m_i}}(s)$ tends to the singular cycle $\gamma(s) \cup \theta_1 \cup \alpha \cup \theta_2$, for i sufficiently large one has from the continuous dependence theorem, that there exists $\tau > 0$ such that $q^{\rho_{m_i}}(s)$ stays $\epsilon-$close to $\gamma(s)$, for every $s \in [-\tau, \tau]$, with

$$q^{\rho_{m_i}}(-\tau), \gamma(-\tau) \in \Sigma_1 \qquad \text{and} \qquad q^{\rho_{m_i}}(\tau), \gamma(\tau) \in \Sigma_2.$$

Consider then the integral S in the form

$$\begin{aligned}
&\int_{-\infty}^{-\tau} e^{-\int_0^s \sigma(\tau)d\tau}\rho_0(s)^3[acos^2\theta_0(s) + csin^2\theta_0(s) + \rho mcos\theta_0(s) + \rho^2(s)\alpha]ds \\
&+ \int_{-\tau}^{\tau} e^{-\int_0^s \sigma(\tau)d\tau}\rho_0(s)^3[acos^2\theta_0(s) + csin^2\theta_0(s) + \rho mcos\theta_0(s) + \rho^2(s)\alpha]ds \\
&+ \int_{\tau}^{+\infty} e^{-\int_0^s \sigma(\tau)d\tau}\rho_0(s)^3[acos^2\theta_0(s) + csin^2\theta_0(s) + \rho mcos\theta_0(s) + \rho^2(s)\alpha]ds.
\end{aligned}$$

Using now the periodicity of $q^{\rho_{m_i}}(s)$, one can write the integral S_{m_i} in the following way

$$\begin{aligned}
&\int_{-m_iT/2}^{m_iT/2} e^{-\int_0^s \sigma(\tau)d\tau}\rho_{m_i}(s)^3[acos^2\theta_{m_i}(s) + csin^2\theta_{m_i}(s) \\
&+\rho_{m_i}(s)mcos\theta_{m_i}(s) + \rho_{m_i}^2(s)\alpha]ds
\end{aligned}$$

and thus, as $m_i \to +\infty$, for $i \to +\infty$ (Lemma 7), one obtains

$$\begin{aligned}
&S_{m_i} = \\
&\int_{-m_iT/2}^{-\tau} e^{-\int_0^s \sigma(\tau)d\tau}\rho_{m_i}^3[a\cos^2\theta_{m_i} + c\sin^2\theta_{m_i} + \rho_{m_i}m\cos\theta_{m_i} + \rho_{m_i}^2\alpha]ds \\
&+ \int_{-\tau}^{\tau} e^{-\int_0^s \sigma(\tau)d\tau}\rho_{m_i}^3[a\cos^2\theta_{m_i} + c\sin^2\theta_{m_i} + \rho_{m_i}mcos\theta_{m_i} + \rho_{m_i}^2\alpha]ds \\
&+ \int_{\tau}^{m_iT/2} e^{-\int_0^s \sigma(\tau)d\tau}\rho_{m_i}^3[a\cos^2\theta_{m_i} + c\sin^2\theta_{m_i} + \rho_{m_i}m\cos\theta_{m_i} + \rho_{m_i}^2\alpha]ds,
\end{aligned}$$

where $\theta_{m_i} = \theta_{m_i}(s)$ and $\rho_{m_i} = \rho_{m_i}(s)$ are the coordinates of the resonant orbit $q^{\rho_{m_i}}(s)$.

In order to simplify the calculations, let us call $F(q^{\rho_{m_i}}(s))$ and $G(\gamma(s))$ the integrands of S_{m_i} and S, respectively. This way, from the considerations above follows that

$$
\begin{aligned}
|S - S_{m_i}| = \Big| & \int_{-\infty}^{-\tau} G(\gamma(s))ds + \int_{-\tau}^{\tau} G(\gamma(s))ds + \int_{\tau}^{+\infty} G(\gamma(s))ds \\
& - \int_{-m_iT/2}^{-\tau} F(q^{\rho_{m_i}}(s))ds - \int_{-\tau}^{\tau} F(q^{\rho_{m_i}}(s))ds - \int_{\tau}^{m_iT/2} F(q^{\rho_{m_i}}(s))ds \Big| \\
\leq & \Big| \int_{-\infty}^{-\tau} G(\gamma(s))ds \Big| + \Big| \int_{-m_iT/2}^{-\tau} F(q^{\rho_{m_i}}(s))ds \Big| \\
& + \Big| \int_{-\tau}^{\tau} G(\gamma(s))ds \Big| + \Big| \int_{-\tau}^{\tau} F(q^{\rho_{m_i}}(s))ds \Big| \\
& + \Big| \int_{\tau}^{+\infty} G(\gamma(s))ds \Big| + \Big| \int_{\tau}^{m_iT/2} F(q^{\rho_{m_i}}(s))ds \Big|.
\end{aligned}
$$

Let us consider each one of the integrals above, for $i \to +\infty$ (and, consequently, $\rho_{m_i} \to 0$).

As, for i sufficiently large and $s \in [-\tau, \tau]$, one has that $\gamma(s)$ and $q^{\rho_{m_i}}(s)$ stay ϵ–close (continuous dependence theorem) and, being G and F continuous functions of $\gamma(s)$ and $q^{\rho_{m_i}}(s)$, respectively, with

$$\lim_{i \to +\infty} F(q^{\rho_{m_i}}(s)) = G(\gamma(s)),$$

one has that, for i sufficiently large,

$$
\begin{aligned}
& \Big| \int_{-\tau}^{\tau} G(\gamma(s))ds - \int_{-\tau}^{\tau} F(q^{\rho_{m_i}}(s))ds \Big| \\
= & \Big| \int_{-\tau}^{\tau} [G(\gamma(s)) - F(q^{\rho_{m_i}}(s))]ds \Big| < \epsilon.
\end{aligned}
$$

Consider now $\gamma(s) = (\theta_0(s), \rho_0(s))$ and $q^{\rho_{m_i}}(s) = (\theta_{m_i}(s), \rho_{m_i}(s))$. Based on the construction of the transversal sections Σ_1 and Σ_2 (Figure 8) and considering that $q^{\rho_{m_i}}(s)$ tends to the singular cycle $\gamma(s) \cup \theta_1 \cup \alpha(s) \cup \theta_2$, it follows that, for i sufficiently large and $s \in [-\infty, -\tau)$ or $s \in [\tau, +\infty)$, the following inequalities hold

$$|\theta_{m_i}(s)| < \epsilon \quad \text{and} \quad |\rho_{m_i}(s)| < \epsilon.$$

From these considerations, one has that

$$\Big| \int_{-m_iT/2}^{-\tau} F(q^{\rho_{m_i}}(s))ds \Big| \leq \int_{-m_iT/2}^{-\tau} \Big| F(q^{\rho_{m_i}}(s)) \Big| ds \; < \epsilon_1 M_1$$

and also

$$\Big| \int_{\tau}^{m_iT/2} F(q^{\rho_{m_i}}(s))ds \Big| \leq \int_{\tau}^{m_iT/2} \Big| F(q^{\rho_{m_i}}(s)) \Big| ds \; < \epsilon_2 M_2,$$

where

$$M_1 = \sup_{s\in[-m_iT/2,-\tau]} \Big| F(q^{\rho_{m_i}}(s)) \Big| \quad \text{and} \quad M_2 = \sup_{s\in[\tau,m_iT/2]} \Big| F(q^{\rho_{m_i}}(s)) \Big|.$$

As $m_i \to +\infty$ and the improper integral above is convergent (see [2] for a proof), for $i \to +\infty$ one can take τ sufficiently large (say $\tau = \frac{m_iT}{2} - \epsilon$), so that

$$\Big| \int_{-\infty}^{-\tau} G(\gamma(s))ds \Big| < \epsilon/4 \quad \text{and} \quad \Big| \int_{\tau}^{\infty} G(\gamma(s))ds \Big| < \epsilon/4.$$

Taking finally $\epsilon_i,\ i = 1, 2$ in the apropriate form in the majorities above, one can conclude that

$$|S - S_{m_i}| \; < \; \epsilon,$$

for every given ϵ, from which the converge of the integrals follows, for $i \to +\infty$. ∎

Lemma 15. *$M_{m_i}^{(j)}(\phi_{max}) \to M^{(j)}(\phi_{max})$, as $i \to +\infty$, for $j = 0, 1, 2$, where (j) denotes the j^{th} derivative of M with respect to ϕ_0.*

Proof. The proof is analogous to that of Lemma 14. ∎

From the proof of lemmas above follows that the heteroclinic tangencies of the infinite heteroclinic cycle of the compactfied vector field are limits of subharmonic bifurcations, which occur on the annulus of large amplitude periodic orbits. Also, as $m_i \to +\infty$ for $i \to +\infty$ and for each m_i there exist two subharmonics of order m_i which bifurcate from the resonant periodic orbit $q^{\rho_{m_i}}(s)$, one can conclude that there exists an infinite number of subharmonics in the neighborhood of the heteroclinic tangencies.

Similar results were stated in [4], although for other classes of vector fields and periodic perturbations, restricted to compact regions of the plane.

REFERENCES

1. C. Chicone, *Bifurcation of nonlinear oscillations and frequency entrainment near resonance*, SIAM J. Math. Anal. **23** (1992), no. 6, 1577–1608.
2. C. Chicone, *Ordinary differential equations with applications*, Texts in Appl. Math. **34**, Springer–Verlag, New York, 1999.
3. C. Chicone and F. Dumortier, *Finiteness for critical periods of planar analytic vector fields*, Nonlinear Analysis, Theory, Methods & Applications **20** (1993), no. 4, 315–335.
4. S-N. Chow, J. Hale and J. Mallet–Paret, *An example of bifurcation to homoclinic orbits*, J. Differential Equations **37** (1980), 351–373.
5. W. A. Coppel, *A survey of quadratic systems*, J. Differential Equations **2** (1996), 293–304.
6. F. Dumortier, R.Roussarie and C.Rousseau, *Hilbert's 16th problem for quadratic vector fields*, J. Differential Equations **110** (1994), 86–133.
7. A.Gasull, V.Mañosa and F.Mañosas, *Stability of certain planar unbounded polycycles*, J. Math. Anal. Appl. **269** (2002), 332–351.
8. J. Hale and P.Z. Táboas, *Interaction of Dumping and Forcing in a Second Order Equation*, Nonlinear Analisys, Theory, Methods & Applications **2** (1978), no. 1, 77–84.
9. P.Hartman, *Ordinary differential equations*, John Wiley and Sons, New York, 1964.
10. M.Messias, *Perturbações periódicas de ciclos heteroclínicos infinitos de campos vetoriais polinomiais planares*, PhD Thesis, IME–USP, Brazil, 2000.
11. M.Messias, *Periodic perturbations of quadratic planar polynomial vector fields*, Ann. Braz. Acad. Scienc. **74** (2002), no. 2, 193–198.
12. M.Messias and M.Meneguette Jr., *On the existence of infinite heteroclinic cycles in polynomial systems and its dynamic consequences*, Proc. of the ICNAAM – Intern. Conf. Numer. Anal. and Appl. Math., WILEY–VCH Verlag, Weinheim (2004), 261–264.
13. J. Sotomayor and R. Paterlini, *Bifurcations of polynomial vector fields in the plane*, Canadian Mathematical Society, Conference Proceedings **8** (1987), 665–685.
14. P.Z. Táboas, *Periodic solutions of a forced Lotka–Volterra equation*, J. Math. Analysis and Appl.**124** (1987), 82–97.
15. S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*, Texts in Appl. Math. 2, Springer–Verlag, New York, 1990.