QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **5**, 275–284 (2004)
ARTICLE NO. 84

Corrections and Complements to "Liouvillian Integration of the Lotka–Volterra System"

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The Lotka–Volterra system of autonomous differential equations consists in three homogeneous polynomial equations of degree 2 in three variables.

This system, or the corresponding vector field $LV(A, B, C)$, depends on three non–zero (complex) parameters and may be written as

$$LV(A, B, C) = x(Cy + z)\partial_x + y(Az + x)\partial_y + z(Bx + y)\partial_z.$$

In fact, $LV(A, B, C)$ can be chosen as a normal form for most of the factored quadratic systems; the study of its first integrals of degree 0 is thus of great mathematical interest.

In the paper into consideration [1], we thus described all possible values of the triple (A, B, C) of non–zero parameters for which $LV(A, B, C)$ has a homogeneous liouvillian first integral of degree 0.

We also discussed the corresponding problem of the liouvillian integration for quadratic factored vector fields that cannot be put in Lotka–Volterra normal form, for instance with some 0 among A, B, C.

There are some errors in the description of these marginal situations that we would like to correct in the present note.

Key Words: Liouvillian integration, Lotka-Volterra system, Darboux polynomials, supply functions.

1. INTRODUCTION

So far as we know, no significant error has been depicted as concerns the main purpose of our work [1], the liouvillian integrability of $LV(A, B, C)$ with $ABC \neq 0$. But there are some marginal errors in the degenerate cases where some parameter among A, B, C is 0.

First, two missing cases of factored systems that are not Lotka–Volterra (even with zeroes) have been noticed by L. Cairo, H. Giacomini and J. Llibre [2]: they must be added to the list (i)–(v) of Section 3.1.

More recently, A. Nowicki [3] discovered a gap in Theorem 12 of [1]; this case has to be added to the list of this theorem as a last one:

25. If $[A, B, C] = [1, n, 0]$ with $n \in \mathbb{N}^\star$, there is an irreducible Darboux polynomial of degree n and cofactor $nx + (n-1)z$; we describe this family in Section 2 of the present note.

Consequently, Proposition 47 of [1]*, concerning strict Darboux polynomials of $LV(A, B, 0)$, has to be completed and its proof corrected* (Section 3).

After the remark of Andrzej Nowicki, I understood what my error was: in a systematic way, when the parameter A (for instance) is 1, I considered this situation as well–known with respect to liouvillian integration, but this is only true if $C \neq 0$. In particular, if $[A, B, C] = [1, B, 0]$ with $B \notin \mathbb{N}^\star$, there is no strict Darboux polynomial whereas y/x is a supply function (even if $B \in \mathbb{N}^\star$ of course).
Consequently, Proposition 49 of [1]*, concerning supply functions without strict Darboux polynomials of $LV(A, B, C)$, has to be completed and its proof corrected too* (Section 4).

2. THE SPORADIC FAMILY OF ANDRZEJ NOWICKI

THEOREM 1. *$LV(1, n, 0), n \in \mathbb{N}^\star$, has a strict irreducible Darboux polynomial of degree n and cofactor $\Lambda = nx + (n-1)z$.*

Proof. Let $A_n(x, y, z) = y^n - zN_{n-1}(x, y)$. From the Darboux relation

$$x\, z\, \partial_x(A_n) \;+\; y\,(x+z)\,\partial_y(A_n) \;+\; z\,(nx+y)\,\partial_z(A_n) \;=\; \Lambda\, A_n,$$

elementary computations show that A_n is the announced homogeneous polynomial with prescribed degree and cofactor if and only if the two–variable homogeneous polynomial $N_{n-1}(x, y)$ of degree $n-1$ follows the *ordinary* differential equation

$$x\, \partial_y N_{n-1} \;+\; N_{n-1} \;=\; y^{n-1}.$$

It is clear that such a N_{n-1} exists and is unique.
As a polynomial in $\mathbb{Q}(x, y)[z]$, A_n has the degree 1 and is irreducible.
As y does not divide N_{n-1}, the coefficients of this one–variable polynomial in z are coprime and A_n is thus irreducible in $\mathbb{Q}[x, y, z]$. ∎

3. THE NEW PROPOSITION 47

Proposition 47 of [1] has to be corrected as follows.

THEOREM 2. *The vector field $LV(A,B,0)$, with $AB \neq 0$ has no strict Darboux polynomial except in the three following cases:*

- $B = 1$ *and* $y - Az$ *is the sought polynomial,*
- $A = -1, B = 2$ *and* $(x+y)^2 - 2yz$ *is the sought polynomial,*
- $A = 1, B = n \in \mathbb{N}^\star$ *and* A_n *of Theorem 1 is the sought polynomial.*

Proof. We have to prove that there is no other possibility. We use the general notations and tools of [1], marginal polynomials and combinatorial determinants.
Let then f be an irreducible strict Darboux polynomial of degree n and cofactor $\lambda x + \mu y + \nu z$. According to Lemma 15, we have $\mu = \gamma_1 = 0$ whereas the two–variable polynomial R is proportional to $x^{\alpha_3}\, y^{\beta_3}$ and the two–variable polynomial P to $y^{\beta_1}\,(y - Az)^{n-\beta_1}$.
The monomial y^n thus appears in P and must appear in R : $\alpha_3 = 0$.
Then, $\beta_3 = \lambda = n$ and $B = n/\gamma_2$ is a rational number with $B \geq 1$.

Now, from the main specific analysis 5.1 with marginal determinants, we have the following facts:

- $D_3 = 0$ but this is not an information : in this special case where $C = 0$, $D_3 \equiv (n - \alpha_2)\,[n - \beta_3]^{\overline{n}} = (n - \alpha_2)\,[n - n]^{\overline{n}}$ is 0.
- $L_1 = n - \lambda = 0$ and we may use further the transfer principle with determinant D_1'.
- $D_2 \equiv L_2\,[1/B - \gamma_2]^{\overline{\gamma_2}}\,[A - \alpha_2]^{\overline{\alpha_2}}\,[1/B + A + 1]^{\overline{n-\gamma_2-\alpha_2-1}}$.

Our analysis then divides in four branches depending on which factor of D_2 vanishes.

$\boxed{L_2 = m + (m - \beta_1)A = 0}$ A is not a positive rational number and from Lemma 15, $\alpha_2 = \beta_1 = 0$, whence $A = -1$. As $L_2 = 0$, we use the transfer principle with D_2': the second factor of D_2' cannot vanish and $1/B \in \mathbb{N}^\star$. As $B \geq 1$, this implies $B = 2$, the second announced possibility.

$\boxed{[1/B - \gamma_2]^{\overline{\gamma_2}} = 0}$ $1/B \in \mathbb{N}^\star$; as $B \geq 1$, $B = 1$. Then $\gamma_2 = m$ and $P \equiv (y - Az)^m$, the well–known first announced possibility.

$\boxed{[A - \alpha_2]^{\overline{\alpha_2}} = 0}$ $A \in \mathbb{N}^\star$. But $D_1' = 0$ means that $2/A \in \mathbb{N}^\star$ as the other factors cannot vanish. Then A is 1 or 2.

- $\boxed{A = 2}$ turns out to be impossible : Proposition 8 shows that M_7 is not a node (the product $-(1+B)$ of the eigenvalues is negative) and there is no way to solve $\rho_7 i_7 + \sigma_7 j_7 = 2m - \alpha_2$ with i_7 and j_7 equal to 0 or 1.

• $\boxed{A=1}$ is the discovery of Andrzej Nowicki : Proposition 8 at point M_4 shows that $i_4 = 0, 1$, whence $\beta_1 = m$ or $\beta_1 = m-1$. $\beta_1 = m$ would mean $B = 1$ (we know this situation). Thus $\beta_1 = m-1$ which implies $\gamma - 1 = 1$ and $B = m$, the degree of the Darboux polynomial.

$\boxed{[1/B + A + 1]^{\overline{m-\gamma_2-\alpha_2-1}} = 0}$ A has to be negative and $\alpha_2 = \beta_1 = 0$. In the transfer $D'_1 = 0$, we have to choose $[2/A + 1]^{\overline{m-2}} = 0$ whence $2/A = -a-1$ with the integer a is in the range $[0..m-2]$.
Then $1/B - 2/(a+1) + 1 = -b$ with b in the integer range $[0..m-\gamma_2-2]$.
As $0 < 1/B \leq 1$, an elementary computation leads to $0 < 1-a-b-ab$, which implies $a = b = 0$ and $B = 1, A = -2$, a subcase of the main possibility.

∎

4. THE NEW PROPOSITION 49

Proposition 49 of [1] has to be corrected in the following way.

Theorem 3. *If $LV(A,B,C)$, where A,B,C may vanish, has no strict Darboux polynomial, and if there exits a supply function, then, up to a natural transformation (circular permutations only!),*

- $[A,B,C] = [-1, 1/2, 0]$ *and* $\dfrac{(y+z)^2}{xy}$ *is the supply function,*
- $[A,B,C] = [1, B, 0]$ *and* $\dfrac{y}{x}$ *is the supply function.*

Proof. A supply function is a homogeneous rational fraction $\dfrac{N}{x^\alpha y^\beta z^\gamma}$ of degree 0 with $[\alpha,\beta,\gamma]$ in $\mathbb{N}^3 \setminus \{[0,0,0]\}$ such that for some non–zero *speed* $\Lambda = \lambda x + \mu y + \nu z$,

$$\begin{aligned} &x(Cy+z)N_x + y(Az+x)N_y + z(Bx+y)N_z \\ &\quad -N((\beta+B\gamma)x + (\gamma+C\alpha)y + (\alpha+A\beta)z) \;=\; \Lambda x^\alpha y^\beta z^\gamma. \end{aligned} \tag{1}$$

If all three exponents are less than or equal to 1, the supply function may be written as $N/(xyz)$ where some of x, y, z may divide N.
This is an effective computation to find all supply functions of this type and we get the two announced cases and also some other situations that we discard because there is a strict Darboux polynomial.

We are thus left with situations *in which some of α,β,γ is greater than or equal to* 2. The supply function can be chosen irreducible, which means that if an exponent of a variable in the denominator is strictly positive, the corresponding variable does not divide the numerator N.

The main tools of our paper have to be used:

- if $\alpha \geq 1$, then there exists a non–zero two variable marginal polynomial P in y and z,
- if $\alpha \geq 2$, then moreover the marginal determinant D_1 vanishes, whereas its factor L_1 cannot vanish except if $ABC + 1 = 0$. As a strict Darboux polynomial of degree 1 exists in this case, we can cancel this possibility and $D_1 = 0$ only because of its other three factors.
- The same is true for β and γ.

Unfortunately, this proof remains a rather long case analysis.

$\boxed{\alpha \geq 2, \beta \geq 2, \gamma \geq 2}$

Marginal two–variable polynomials P, Q, R can be derived from N giving the following set of equations with $\alpha_2, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2$ in $\mathbb{N}$

$$\begin{cases} \gamma + C\alpha &= \gamma_1 = C\alpha_3, \\ \alpha + A\beta &= \alpha_2 = A\beta_1, \\ \beta + B\gamma &= \beta_3 = B\gamma_2 \end{cases} \tag{2}$$

Then A, B, C are non–zero rational numbers and from

$$\alpha_2 + \alpha_3,\ \beta_1 + \beta_3,\ \gamma_1 + \gamma_2 \leq \alpha + \beta + \gamma, \quad \alpha,\ \beta,\ \gamma\ > 0,$$

we deduce $1/A + C \leq 1,\ 1/B + A \leq 1,\ 1/C + B \leq 1.$

If all $\alpha_2, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2$ vanish, then A, B, C are negative rational numbers and their product is -1 : we discard this well–known situation.

Up to circular permutations, it is enough to consider that $\alpha_2\beta_1 \neq 0$ which implies $A = \alpha_2/\beta_1 = (\alpha_2 - \alpha)/\beta = \alpha/(\beta_1 - \beta) > 0$ and $\alpha/\alpha_2 + \beta/\beta_1 = 1$.

$A = 1$ (together with $BC \neq 0$) leads to a strict Darboux polynomial of degree 1: we discard this well–known situation.

Up to a natural transformation, $[A, B, C] \to [1/A, 1/C, 1/B]$, we have to look for $A > 1$; then $B < 0$ hence $\beta_3 = \gamma_2 = 0$ and $B = -\beta/\gamma$.
C cannot be a positive integer; to have $D_1 = 0$, $1/A + C + 1$ has to be an integer in the range $[-(\alpha + \beta + \gamma - \beta_1 - \gamma_1 - 2)..0]$.
Thus $C < 0$, $\gamma_1 = \alpha_3 = 0$, $C = -\gamma/\alpha$ and $(\beta_1 - \beta)/\alpha - \gamma/\alpha + 1$ is an integer in the range $[-(\alpha + \beta + \gamma - \beta_1 - 2)..0]$: $\beta + \gamma = k_1\alpha + \beta_1$ where k_1 is a positive integer.
Now, to have $D_3 = 0$, $1/C + B + 1 = -\alpha/\gamma - \beta/\gamma + 1$ has to be an integer in the range $[-(\alpha + \beta + \gamma - 2)..0]$: $\alpha + \beta = k_3\gamma$ where k_3 is a positive integer.

To have $D_2 = 0$, either A is an integer or $1/B + A + 1$ is an integer in the range $[-(\alpha+\beta+\gamma-\alpha_2)..0]$, which leads to $\alpha+\gamma = k_2\beta+\alpha_2$ where k_2 is a positive integer.
In that last case, adding the three equalities (with k_1, k_2, k_3) gives a contradiction if $k_3 \neq 1$. With $k_3 = 1$, we have $\gamma = \alpha+\beta$, $1/C + B = -1$, $2\beta = (k_1-1)\alpha+\beta_1$ and $2\alpha = (k_2-1)\beta+\alpha_2$.
Whence $2\beta/\beta_1 = (k_1-1)\alpha/\beta_1 + 1$ and $2\alpha/\alpha_2 = (k_2-1)\beta/\alpha_2 + 1$ and, by addition, $k_1 = k_2 = 1$. This leads to $\alpha_2 = 2\alpha, \beta_1 = 2\beta$.
In this case, $1/C + B = -1$, $1/A + C = -1$, $1/B + A = -1$, and there is a strict Darboux polynomial of degree 2 (case **3** of the main list).

We are thus left with the first possibility $A \in \mathbb{N}^{\star\star}$ to have $D_2 = 0$ and we get a candidate situation

$$A \in \mathbb{N}^{\star\star},\ C + 1/A = -k_1\,(k_1 \in \mathbb{N}^\star),\ B + 1/C = -k_3\,(k_3 \in \mathbb{N}^\star).$$

We need an additional argument to eliminate it:

- The denominator does not vanish at M_6 and it is also minimalist at the rational nodes M_4 and M_5. We say than the denominator $x^\alpha y^\beta z^\gamma$ is *minimalist* at some Darboux point if its order is the minimum possible among all orders of the same exposed line. When this is the case the cofactor $\lambda x + \mu y + \nu z$ vanish at that Darboux point. This occurs in particular if the denominator does not vanish at the Darboux point. In the case under consideration, the three points are not on the same line, the cofactor would be 0 and we would have a rational first integral, which we exclude in the present study.

This achieves the search of supply functions without a strict Darboux polynomial when all three α, β, γ are greater than 1: we did not find anything. The other two cases have to be studied with the same attention.

$\boxed{\alpha \geq 2, \beta \geq 2, \gamma < 2, \quad \gamma \text{ may be 0 or 1}}$

If $\gamma = 1$, two–variable marginal polynomials P, Q, R can still be derived from N giving the following set of equations, a special case of (2)

$$\begin{cases} 1 + C\alpha &= \gamma_1 &= C\alpha_3, \\ \alpha + A\beta &= \alpha_2 &= A\beta_1, \\ \beta + B &= \beta_3 &= B\gamma_2 \end{cases}$$

Here too A, B, C are non–zero rational numbers ($B \in \mathbb{Z}$) and

$$1/A + C \leq 1,\ 1/B + A \leq 1,\ 1/C + B \leq 1.$$

As $\gamma < 2$, only equations $D_1 = 0$ and $D_2 = 0$ can be used. For the same reason as before, one among A, B, C at least has to be positive but the situation is no longer symmetrical.

If $B = 1$, we are in a known situation ($ABC \neq 0$) that we discard; thus remaining possibilities for the integer B are $B \geq 2$ and $B \leq -1$. Similarly $1/C$ is also an integer: interesting possibilities are $1/C \geq 2$ and $1/C \leq -1$.

As $1/C+B \leq 1$, there are only three combined possibilities for B and $1/C$.

• $\boxed{B \leq -1 \text{ and } 1/C \leq -1}$ Then $A > 0$, $\gamma_1 = \alpha_3 = 0$, $\beta_3 = \gamma_2 = 0$, $A = \alpha_2/\beta_1 = (\alpha_2 - \alpha)/\beta = \alpha/(\beta_1 - \beta) > 0$ and $\alpha/\alpha_2 + \beta/\beta_1 = 1$.
As $1/A+C+1 \leq 0$ is impossible, $1/A \in \mathbb{N}^\star$ is the only way to have $D_1 = 0$.
As $A + 1/B + 1 \leq 0$ is impossible, $A \in \mathbb{N}^\star$ is the only way to have $D_2 = 0$.
Hence $A = 1$, with $ABC \neq 0$: we discard this well–known situation.

• $\boxed{B \geq 2,\ 1/C \leq -1}$ and $\boxed{B \leq -1,\ 1/C \geq 2}$ are equivalent situations under a natural transformation. Let us choose to study the first one.
To have $D_1 = 0$, $1/A + C + 1 \leq 0$ is impossible and $1/A \in \mathbb{N}^\star$.
Thus, to have $D_2 = 0$, it is impossible that $A + 1/B + 1 \leq 0$ and $A \in \mathbb{N}^\star$.
Hence $A = 1$ again, with $ABC \neq 0$: we discard this situation.

This achieves the study of $\alpha \geq 2$, $\beta \geq 2$ with $\gamma = 1$. Let now γ be 0.

If z divides the numerator N, then adding a non–zero constant to the supply function cancels this fact and we still have the three non–zero marginal polynomials P, Q, R and the following set of equations, a special case of (2)

$$\begin{cases} C\alpha & = \gamma_1 = C\alpha_3, \\ \alpha + A\beta & = \alpha_2 = A\beta_1, \\ \beta & = \beta_3 = B\gamma_2 \end{cases}$$

As $\gamma = 0$ we only have $1/A + C \leq 1$ and $1/B + A \leq 1$.
As $\beta \neq 0$, B is not 0: $B = \beta/\gamma_2$. As $\alpha \neq 0$, $A \neq 0$. As usual, $B = 1, A \neq 0$ has to be discarded. Thus, either $B > 1$ or $0 < B < 1$. On the other hand, $C = \gamma_1/\alpha \geq 0$.

• If $1/B > 1$, A cannot be positive : $A + 1/B$ would be greater than 1. $\alpha_2 = \beta_1 = 0$ and $A = -\alpha/\beta < 0$.
To have $D_2 = 0$, either $1/B$ is an integer in the range $[2..\gamma_2]$ or $-1/B - A$ is an integer k_2 in the range $[1..\alpha + \beta - \gamma_2 - 1]$.

If $-A = k_2 + 1/B > 2$, then $C + 1/A + 1 > 1/2$ and the only possibility to have $D_1 = 0$ is that $C = \gamma_1/\alpha$ is an integer in the range $[2..\gamma_1]$, the case $C = 1$ being, as usual, well–known. We would have $\gamma_1 \geq 2\alpha$, whence $\alpha \leq \beta$ and $1/B = \gamma_2/\beta = -k_2 - A = -k_2 + \alpha/\beta$ and $\gamma_2 - \alpha = k_2\,\beta$.
As $\gamma_2 \leq \alpha + \beta$, this implies $k_2 = 1$ and $1/B \leq 0$, a contradiction.

If $1/B$ is an integer in the range $[2..\gamma_2]$, as $\gamma_2 \leq \alpha + \beta$, $\alpha \geq (1/B\ - 1)\beta$.
If $C = \gamma_1/\alpha$ is an integer in the range $[2..\gamma 1]$, we would have $\gamma_1 \geq 2\alpha$, whence $\alpha \leq \beta$. Then $1/B = 2$, $\alpha = \beta$, $C = 2$ and $A = -1$, a situation to be discarded as $ABC + 1 = 0$.

• Now suppose $B > 1$.
To have $D_2 = 0$, either A is an integer in the range $[2..\alpha_2]$ or $-1/B-A = k_2$, where k_2 is an integer in the range $[1..\alpha + \beta - \alpha_2 - \gamma_2 - 1]$.

If $A > 0$ then A is an integer and we can suppose $A \geq 2$.
To have $D_1 = 0$ the only possibility is that $C = \gamma_1/\alpha$ is an integer in the range $[2..\gamma_1]$, a contradiction with $1/A + C \leq 1$.

If $A < 0$ then $-A = \alpha/\beta = k_2 + 1/B = k_2 + \gamma_2/\beta$ and $\alpha > k_2\,\beta$
To have $D_1 = 0$, either $C = \gamma_1/\alpha$ is an integer in the range $[2..\gamma_1]$ or $-1/A - C = k_1$, where k_1 is an integer in the range $[1..\alpha + \beta - \gamma_1 - 1]$.
$\gamma_1 \geq 2\alpha$ is contradictory with $\gamma_1 \leq \alpha + \beta$ and $C = -1/A - k_1 = -k_1 + \beta/\alpha$ whence $\beta \geq k_1\,\alpha$, a contradiction.

The study of $\alpha \geq 2, \beta \geq 2, \gamma \leq 1$ is achieved : nothing new.

$\boxed{\alpha \geq 2, \beta < 2, \gamma < 2}$

First suppose $\beta = \gamma = 1$. As usual, the two–variable marginal polynomials P, Q, R can be derived from N giving the following set of equations

$$\begin{cases} 1 + C\alpha &= \gamma_1 &= C\alpha_3, \\ \alpha + A &= \alpha_2 &= A\beta_1, \\ 1 + B &= \beta_3 &= B\gamma_2 \end{cases}$$

Because $\alpha\beta\gamma > 0$, none among A, B, C is 0. All of them are rational numbers. If they are negative, their product is -1 and we can discard this usual situation and consider that one among A, B, C is positive.

- Let $C > 0$.

Then $1/C = \alpha_3 - \alpha \in \mathbb{N}^\star$. $C = 1$, $AB \neq 0$ leads to a strict Darboux polynomial. It then suffices to consider the integer $1/C$ greater than 1 and $\alpha_3 = \alpha + 1/C \geq \alpha + 2$.
As $\alpha_3 + \beta_3 \leq \alpha + 2$, the total degree of N, $\beta_3 = 0$, $\alpha_3 = \alpha + 2$, $B = -1$, $C = 1/2$.
$D_1 = 0$ then needs either $1/A \in \mathbb{N}^\star$ or $-1/A - 3/2 \in \mathbb{N}$. Discarding $A = 1$, we are left with $A = -1/(n + 3/2), n \in \mathbb{N}$, which cannot be an integer.

- Let $A > 0$, $C < 0$.

Then $A = \alpha_2/\beta_1 = \alpha_2 - \alpha = \alpha/(\beta_1 - 1)$ and $\alpha/\alpha_2 + 1/\beta_1 = 1$. A is a positive integer and we suppose $A \geq 2$.
From $\alpha_2 + \gamma_2 \leq \alpha + 2$, the total degree of N, we have $A = 2$, $\gamma_2 = \beta_3 = 0$.
Whence $B < 0$, $B = -1$.
To have $D_1 = 0$, $-1/A - C = -1/2 + 1/\alpha = k_1$, with $k_1 \in \mathbb{N}^\star$: impossible.

- Let $B > 0$, $C < 0$, $A < 0$.

Then $\alpha_2 = \beta_1 = 0$, $A = -\alpha$, $\alpha_3 = \gamma_1 = 0$, $C = -1$ and $B = \beta_3/\gamma_2$.
To have $D_1 = 0$, $-1/A - C = 1/\alpha + 1 = k_1 \in \mathbb{N}^\star$: impossible with $\alpha \notin \mathbb{N}^\star$.

Now suppose that $\beta = 1, \gamma = 0$ ($\beta = 0, \gamma = 1$ is an equivalent situation).

If z divides the numerator N, then adding a non–zero constant to the supply function cancels this fact and we still have the three non–zero marginal polynomials P, Q, R and the following set of equations

$$\begin{cases} C\alpha & = \gamma_1 = C\alpha_3, \\ \alpha + A & = \alpha_2 = A\beta_1, \\ 1 & = \beta_3 = B\gamma_2 \end{cases}$$

$B \neq 0$, hence $\beta_3 = 1$, $B = 1/\gamma_2$.
$A = 0$ would give $\alpha = 0$, a contradiction.
$B = 1, A \neq 0$ with an arbitrary C is a well–known case.

Thus $B < 1$, $\gamma_2 \geq 2$ and $\alpha_2 \leq \alpha - 1$ because of the total degree of N.
Then $A = \alpha_2 - \alpha \leq -1$ is a integer and $\alpha_2 = \beta_1 = 0$: $A = -\alpha \leq -2$.

$C < 0$ is impossible. If $C > 0$, then $\alpha_3 = \alpha$ and $C = \gamma_1/\alpha \leq 1 - 1/A \leq 3/2$. $C = 1$ with $B \neq 0$ is well–known and there is no other choice to have $D_1 = 0$ than $-1/A - C = k_1$, an integer in the range $[1..\alpha - \gamma_1]$: C would be negative, a contradiction. Thus $C = 0$: together with $A = -\alpha$, we have $D_1 = 0$.
This case $[A, B, C] = [-\alpha, 1/\gamma_2, 0]$, with integers $2 \leq \gamma_2 \leq \alpha + 1$ deserves special arguments.

• If z does not divide the numerator N, then the marginal polynomial R is not 0 and it is a multiple of the denominator $x^\alpha\, y$.
Adding a well–chosen constant makes z divide the numerator: $N = z\,\overline{N}$.
Then z divides the left–hand side of Equation (1); it must also divide its right–hand side, which is the speed $\lambda\, x + \mu\, y + \nu\, z$ multiplied by $x^\alpha\, y$.
Then $\lambda = \mu = 0$ and, if $\overline{R}$ stands for the evaluation of $\overline{N}$ at $z = 0$, then $\overline{R}$ follows the differential equation

$$x\, y\, \partial_y(\overline{R}) + ((B-1)\, x + y)\, \overline{R} = \nu\, x^\alpha\, y, \quad \nu \neq 0. \tag{3}$$

x and y divide $\overline{R}$. Let then $\overline{R} = y^k\, S$, where y does not divide S.
It follows from Equation (3) that $k = 1$.
Let then $S = x^l\, T$, where x does not divide T.
From Equation (3) again, $l \leq \alpha$ and T follows the differential equation

$$(Bx + y)\, T + x\, y\, \partial_y(T) = \nu\, x^{\alpha - l}. \tag{4}$$

As x does not divide T, the right–side of (4) has to be the non–zero constant ν, a contradiction with the degree of the homogeneous polynomial on the left–hand side of (4).

To finish, suppose that $\beta = 0, \gamma = 0$. If y or z divides the numerator N, then adding a non–zero constant to the supply function cancels this fact and we still have the three non–zero marginal polynomials P, Q, R and the following set of equations

$$\begin{cases} C\alpha &= \gamma_1 &= C\alpha_3, \\ \alpha &= \alpha_2 &= A\beta_1, \\ 0 &= \beta_3 &= B\gamma_2 \end{cases}$$

$\alpha_2 = \alpha \neq 0$. Then $A \neq 0$, $A = \alpha/\beta_1$. C cannot be negative.

If $C = 0$, to have $D_1 = 0$, $1/A$ has to be a positive integer, greater than 1. Then $\beta_1 \geq 2\alpha$, but the total degree of N is α, a contradiction.

If $C \neq 0$, then $\alpha_3 = \alpha$ and $C = \gamma_1/\alpha$. $\beta_1 \geq 1$ and $\beta_1 + \gamma_1 \leq \alpha$ give $\gamma_1 \leq \alpha - 1$: C is not an integer.
To have $D_1 = 0$, $1/A$ has to be a positive integer, greater than 1 as usual. Then, $\beta_1 \geq 2\alpha$, the same contradiction. ▮

ACKNOWLEDGMENTS

Of course, I am very happy to thank Andrzej Nowicki for his discovery that gave me the opportunity to correct a marginal but serious error.

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