A Minimum Principle for Lyapunov Exponents and a Higher–Dimensional Version of a Theorem of Mañé[*]

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We consider compact invariant sets Λ for C^1 maps in arbitrary dimension. We prove that if Λ contains no critical points then there exists an invariant probability measure with a Lyapunov exponent λ which is the *minimum* of all Lyapunov exponents for all invariant measures supported on Λ. We apply this result to prove that Λ is *uniformly expanding* if every invariant probability measure supported on Λ is hyperbolic repelling. This generalizes a well known theorem of Mañé to the higher–dimensional setting.

Key Words: Lyapunov exponents, uniform hyperbolicity, nonuniform hyperbolicity

1. INTRODUCTION AND RESULTS

Hyperbolicity as been a key idea in the modern theory of Dynamical Systems and a basic related problem has been that of proving hyperbolicity of particular systems or general classes under a priori weaker assumptions. In 1985, Mañé [6] proved a remarkable result in the setting of one–dimensional maps to the effect that any compact invariant set not containing any critical points and with all *periodic points hyperbolic repelling* is actually uniformly hyperbolic. The point here is that the periodic points are not assumed to be *uniformly* hyperbolic, and that even if they were, this hyperbolicity does not necessarily extend to the whole invariant set. Generalizations of this result to the higher dimensional setting are quite problematic and there has been no substantial progress to date. Here we solve this problem under the slightly stronger but natural assumptions that all *invariant measures are hyperbolic repelling*, i.e. have positive Lyapunov exponents. As an intermediate result of independent interest we get that if all Lyapunov exponents are positive then they are uniformly positive and the minimum of such exponents is actually realized for some invariant measure.

Throughout the paper, we let M be a compact Riemannian manifold of dimension $d \geq 1$ and let $f : M \to M$ be C^1 map . We say that x is a critical point for f if $\det Df_x = 0$. For a compact invariant set Λ we let $\mathcal{M}(f) = \mathcal{M}(f, \Lambda)$ denote the set of all f–invariant Borel probability measures with support in Λ and $\mathcal{E}(f, \Lambda)$ the subset of all ergodic invariant measures.

1.1. Lyapunov exponents

DEFINITION 1.   We say that λ is a *Lyapunov exponent* for f if there exists a point x and a vector $v \in T_x M$ such that

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \log \|Df_x^n(v)\|.$$

We let $\mathcal{L}(f)$ denote the set of all Lyapunov exponents for f.

Classical results [8, 9] imply that for each ergodic measure $\mu \in \mathcal{M}(f)$ there exists a number $0 < k \leq d$, constants $\lambda^1 < \ldots < \lambda^k$, and a μ–measurable splitting $T_x M = E^1 \oplus \ldots \oplus E^k$ of the tangent bundle over Λ, such that $\lim_{n \to \infty} \frac{1}{n} \log \|Df_x^n(v)\| = \lambda^j$ for μ–almost every x and every non–zero vector $v \in E_x^j \subset T_x M$. For non–ergodic measures the number k, the constants λ^j, and the tangent bundle decomposition may depend on the ergodic component. The constants λ^j are called the Lyapunov exponents associated to the measure μ.

DEFINITION 2.   We let $\mathcal{L}(\mu)$ denote the set of all Lyapunov exponents associated to a given ergodic measure μ

Notice that exceptional points for all invariant measures may still have some have a well defined Lyapunov exponent which may be unrelated to any of the Lyapunov exponents of any ergodic invariant measure. Therefore in general we have

$$\bigcup_{\mu \in \mathcal{E}(f,\Lambda)} \mathcal{L}(\mu) \subsetneq \mathcal{L}(f).$$

1.2. The minimum principle

Lyapunov exponents do not in general vary continuous either with the base point x or with the measure, and so it is hard if not impossible to formulate any general compactness statements about the sets $\mathcal{L}(f)$ of $\cap_{\mu}\mathcal{L}(\mu)$. Here we prove the following

THEOREM 3 (Minimum principle for Lyapunov exponents).
Suppose Λ does not contain any critical points. Then there exists an ergodic measure $\mu \in \mathcal{M}(f,\Lambda)$ such that

$$\inf\{\mathcal{L}(f)\} \in \mathcal{L}(\mu).$$

This can be interpreted as a mild compactness result on $\mathcal{L}(f)$: there exists some invariant probability measure μ with support in Λ with an associated Lyapunov exponent which realizes the infimum over all Lyapunov exponents associated to all invariant measures. It also says that no *freak* exponent can be less than a *proper* exponent associated to some ergodic invariant measure. Exactly the same argument also shows that there exists some (other) measure μ for which

$$\sup\{\mathcal{L}(f)\} \in \mathcal{L}(\mu).$$

We do not show however that $\mathcal{L}(f)$ is closed. We conjecture this to be the case in this setting and believe that the absence of critical points is a necessary condition for such property to hold.

In the application to be given below we shall consider the situation in which all invariant measures have only positive Lyapunov exponents. The minimum principle then immediately implies the following

COROLLARY 4. *Suppose Λ contains no critical points and*

$$\mathcal{L}(f) > 0.$$

Then there exists a constant $\lambda > 0$ such that

$$\mathcal{L}(f) > \lambda.$$

1.3.   Mañé's Theorem in arbitrary dimension

Our next result says that in fact an even stronger statement holds in this case. before stating it we recall some standard notation. The set Λ is said to be *uniformly expanding* if there exist constants $C, \lambda > 0$ such that

$$\|Df_x^n(v)\| \geq Ce^{\lambda n}\|v\| \tag{1}$$

for all $x \in \Lambda$, all $v \in T_x M$ and all $n \geq 1$. A measure μ is *expanding* if $\mathcal{L}(\mu) > 0$. The definition of uniform hyperbolicity implies in particular that all invariant measures supported on Λ are *uniformly* expanding: there exists some $\lambda > 0$ such that $\mathcal{L}(f) \geq \lambda > 0$. The converse however is non–trivial.

THEOREM 5.   *Suppose Λ contains no critical points and*

$$\mathcal{L}(f) > 0.$$

Then Λ is uniformly expanding.

This is a generalization to the higher–dimensional setting of the well known theorem of Mañé [6] which gives the same conclusions in the one–dimensional setting under the weaker assumption that every periodic point is expanding. This has become almost a Folklore Theorem in one–dimensional dynamics for the fundamental role it plays in a huge number of arguments. By comparison, the theory of higher–dimensional non–uniformly expanding maps is just taking off, see [1, 5]. By analogy with the one–dimensional case, perhaps these results might play some significant role in that theory.

We emphasize that the absence of critical points in Λ is a necessary condition. There are many quite generic situations, including for example Collet–Eckmann one–dimensional maps [7], of compact invariant sets Λ satisfying $\mathcal{L}(f) \geq \lambda > 0$ but which contain a critical point and thus clearly cannot be uniformly expanding. Partial results in the direction of our Theorem 5 were obtained in [2, 3] in the context of globally C^1 local diffeomorphisms. Here we provide a generalization of those results by using a quite different argument based on the minimum principle given above. Theorem 5 also implies the following

COROLLARY 6.   *Let $f : M \rightarrow M$ be a $C^{1+\alpha}$ local diffeomorphism of a compact Riemannian manifold M and suppose that*

$$\mathcal{L}(f) > 0.$$

Then there exists a (unique) measure $\mu \in \mathcal{E}(f)$ which is absolutely continuous with respect to the Riemannian volume on M.

The conclusion follows from the uniform hyperbolicity of f by well known and absolutely classical arguments. However we want to emphasize here that the assumptions differ from most kinds of assumptions used to imply the existence of absolutely continuous invariant measures, both in the uniform and non–uniform setting, in at least one important feature. They do not, *a priori*, say anything about the dynamical or hyperbolic properties of a positive measure set of points for the Riemannian volume. They just specify that any invariant measure must have positive Lyapunov exponent. A priori *all* of these measures may be singular with respect to the Riemannian volume. Notice that for this corollary we need the derivative of f to be Hölder continuous, since the proof of the existence of an absolutely continuous invariant measure requires distortion estimates which require more regularity than that needed for the other results.

1.4. Fibred maps

Some straightforward adaptations of the definitions and the arguments allow us to obtain our results in the more general setting of fibred maps. More precisely, we assume as above that $f : M \to M$ is a C^1 map of a compact Riemannian manifold and that Λ is a compact invariant set. We now assume that the tangent bundle $T_\Lambda M$ over Λ admits a continuous decomposition $T_\Lambda M = E^1_\Lambda \oplus E^2_\Lambda$ into Df–invariant subbundles. In particular the angles between the subspaces E^1_x and E^2_x are uniformly bounded below for all $x \in \Lambda$. The (measurable) Oseledets–Ruelle decomposition must be consistent with this continuous decomposition and therefore it makes sense to talk about the sets $\mathcal{L}^1(\mu)$ and $\mathcal{L}^2(\mu)$ of Lyapunov exponents in the directions of E^1 and E^2 respectively. Notice that $\mathcal{L}(\mu) = \mathcal{L}^1(\mu) \cup \mathcal{L}^2(\mu)$. We then have the following

THEOREM 7. *Suppose Λ does not contain any critical points. Then*

$$\inf\{\mathcal{L}^1(f)\} \in \mathcal{L}^1(f).$$

The definition of uniform hyperbolicity given above can also be generalized and we say that the set Λ is *uniformly expanding in the direction of* E^1 if there exist constants $C, \lambda > 0$ such that

$$\|Df^n_x(v)\| \geq Ce^{\lambda n}\|v\| \tag{2}$$

for all $x \in \Lambda$, all $v \in E^1_x$ and all $n \geq 1$. Then we have the following

THEOREM 8. *Suppose Λ contains no critical points and*

$$\mathcal{L}^1(f) > 0.$$

Then Λ is uniformly expanding in the direction of E^1.

We emphasize that we are not assuming here any *partial hyperbolicity*. The results do not depend in anyway on the hyperbolicity properties in the complementary subspace to the one under consideration. Theorems 3 and 5 are special cases of Theorems 7 and 8 corresponding to the case in which the complementary subbundle E_Λ^2 is trivial. The proofs of the more general cases differ from the special cases only in the notation and so, in order to keep this as simple as possible, we shall prove the results explicitly in the special cases.

1.5.   Remarks

Before starting the proof of the Theorems, we make some observations concerning the setup. In particular we want to emphasize the difference between the uniformity statement in Corollary 1 and that in Theorem 5.

An invariant measure μ has at most d distinct Lyapunov exponents and thus the condition $\mathcal{L}(\mu) > 0$ implies that there exists $\lambda^1(\mu) = \inf\{\mathcal{L}(\mu)\}$. By the definition of Lyapunov exponents this implies

$$\liminf_{n\to\infty} \frac{1}{n}\|Df_x^n(v)\| \geq \lambda^1 \tag{3}$$

for μ–almost every x and for every non–zero $v \in T_x M$ (the actual limit exists only for those vectors in the appropriate subspace $E_x^1 \subset T_x M$). In particular (directly from the definition of $\liminf$) there exists, for every $\lambda^1 > \lambda > 0$, a measurable function C_x, non–zero μ–almost everywhere, such that

$$\|Df_x^n(v)\| \geq C_x e^{\lambda n}\|v\| \tag{4}$$

for every non–zero vector $v \in T_x M$ and every $n \geq 1$. Crucially here, the liminf in (3) cannot be assumed to be achieved uniformly and thus the constant C_x is in general not uniformly bounded away from 0.

Thus, the step from $\mathcal{L}(f) > 0$ to uniform expansivity requires two uniformity estimates to be obtained. The first, given by Corollary 1, implies that (4) holds on a set of total probability, (i.e. on a set $\mathcal{B}$ such that $\mu(\mathcal{B}) = 1$ for all measures $\mu \in \mathcal{M}(f)$) with a *uniform bound on the growth rate* λ. However, this still leaves us with a family of *measurable* functions $C_x(\mu)$ which are not, a priori, uniformly bounded below. Theorem 5 says that in the absence of critical points they are indeed uniformly bounded below. Therefore the inequality (4) can be extended to every point of Λ for uniform constants λ and C, as required in the definition (1) of uniform expansion.

After writing this paper we became aware of some previously published papers [4, 10–12] containing related results concerning uniformity of Lyapunov exponents and other quantities in the spirit of our Minimum Principle. The arguments given there are significantly more general and the

proofs significantly more complicated and it is not clear that our result follows immediately from the statements in any of these papers without some additional non–trivial arguments.

2. SETUP AND NOTATION

A key tool in our approach is that of lifting certain quantities to the unit tangent bundle

$$SM = S_\Lambda M = \{(x, v) \in TM : x \in \Lambda, v \in T_x M, \|v\| = 1\}.$$

Notice that SM is a compact, metric, measure space. We let $\pi : SM \to M$ denote the standard projection $\pi(x, v) = x$. We start by defining the map $F : SM \to SM$ by

$$F(x, v) = \left(f(x), \frac{Df_x(v)}{\|Df_x(v)\|} \right).$$

Notice that F is well defined on Λ since Λ contains no critical points and thus $\|Df_x(v)\| \neq 0$. We define iterates of F by

$$F^n(x, v) = \left(f^n(x), \frac{Df_x^n v}{\|Df_x^n v\|} \right).$$

We let $\mathcal{M}(F)$ denote the space of F–invariant probability measures on SM and let

$$\pi^* : \mathcal{M}(F) \to \mathcal{M}(f)$$

denote the standard projection of measures where $\pi^* \mu(A) = \mu(\pi^{-1}(A))$ for any $\mu \in \mathcal{M}(F)$ and Borel set $A \subset \Lambda$. Let $\mathcal{E}(F) \subset \mathcal{M}(F)$ and $\mathcal{E}(f) \subset \mathcal{M}(f)$ denote the subsets of ergodic invariant measures for F and f respectively.

LEMMA 9.   *The projection π^* sends $\mathcal{E}(F)$ to $\mathcal{E}(f)$ and the restriction $\pi^* : \mathcal{E}(F) \to \mathcal{E}(f)$ is surjective.*

Proof.   We show first of all that $\pi^*(\mathcal{E}(F)) \subseteq \mathcal{E}(f)$. Let $\mu \in \mathcal{E}(F)$ and let $\mu^* = \pi^* \mu$. Suppose that $A \subset M$ satisfies $f^{-1}(A) = A$. Then $F^{-1}(\pi^{-1}(A)) = \pi^{-1}(A)$ and by ergodicity we have $\mu(\pi^{-1}(A)) = 0$ or 1. Then by the definition of π^* we have $\mu^*(A) = \pi^* \mu(A) = \mu(\pi^{-1}(A)) = 0$ or 1. Thus μ^* is ergodic.

We now show that $\pi^* : \mathcal{M}(F) \to \mathcal{E}(f)$ is surjective by fixing a measure $\mu^* \in \mathcal{E}(f)$ and finding a measure $\mu \in \mathcal{M}(F)$ with $\pi^* \mu = \mu^*$. By the Birkhoff ergodic theorem we know that the set

$$X = \left\{ x : \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \to \mu^* \right\} \quad \text{satisfies} \quad \mu^*(X) = 1.$$

For such a generic point $x \in X$ and some vector $v \in T_x M$, consider the sequence of measures

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{F^i(x,v)}$$

and let $\mu \in \mathcal{M}(F)$ be the limit of some subsequence μ_{n_k}. We claim that $\pi^* \mu = \mu^*$. To see this, consider a continuous test function $\varphi : M \to \mathbb{R}$. We claim that

$$\int_M \varphi d(\pi^* \mu) = \int_{SM} \varphi \circ \pi d\mu = \int_M \varphi d\mu^*.$$

Since φ is chosen arbitrarily this implies the claim. The first equality follows immediately. To obtain the second we write, for $x \in X$ and $v \in T_x M$ as above,

$$
\begin{aligned}
\int_M \varphi d\mu^* &= \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \\
&= \lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \varphi(f^i(x)) \\
&= \lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} (\varphi \circ \pi)(F^i(x,v)) \\
&= \int_{SM} (\varphi \circ \pi) d\mu.
\end{aligned}
$$

The measure μ is not necessarily ergodic, but we claim that any ergodic component $\tilde{\mu}$ of μ also satisfies $\pi^* \tilde{\mu} = \pi^* \mu = \mu^*$. Indeed consider the set $X \subset M$ of μ^* generic points defined above. Since $\mu^*(X) = 1$ we also have that $\mu(\pi^{-1}(X)) = 1$ and therefore also $\pi^* \tilde{\mu}(X) = \tilde{\mu}(\pi^{-1}(X)) = 1$. Moreover, $\pi^* \tilde{\mu}$ is ergodic by the first part of the statement in the Lemma, and therefore the Dirac averages of almost every point in X converge to $\pi^* \tilde{\mu}$ implying that $\pi^* \tilde{\mu} = \mu^*$. ∎

3. MINIMUM PRINCIPLE

In this section we prove Theorem 3. We define the observable $\phi : SM \to \mathbb{R}$ by

$$\phi(x,v) = \log \|Df_x(v)\|.$$

Notice that ϕ is *continuous* and that we have

LEMMA 10. *For every $(x, v) \in SM$ and every $n \geq 1$ we have*

$$\frac{1}{n} \log \|Df_x^n(v)\| = \frac{1}{n} \sum_{i=0}^{n-1} \phi(F^i(x, v)).$$

In particular

$$\lambda(x, v) = \lim_{n \to \infty} \frac{1}{n} \log \|Df_x^n(v)\| = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(F^i(x, v))$$

whenever such limits exist (and the existence of one limit implies the existence of the other).

Proof. Write first of all

$$\frac{\|Df_x^n(v)\|}{\|v\|} = \frac{\|Df_{f^{n-1}(x)}(Df_x^{n-1}(v))\|}{\|Df_x^{n-1}(v)\|} \frac{\|Df_{f^{n-2}(x)}(Df_x^{n-2}(v))\|}{\|Df_x^{n-2}(v)\|} \cdots$$
$$\cdots \frac{\|Df_{f(x)}(Df_x(v))\|}{\|Df_x(v)\|} \frac{\|Df_x(v)\|}{\|v\|}.$$

Then, taking logs and using the definition of ϕ we have

$$\log \frac{\|Df_x^n(v)\|}{\|v\|} = \sum_{i=0}^{n-1} \log \frac{\|Df_{f^i(x)}(Df_x^i(v))\|}{\|Df_x^i(v)\|}$$
$$= \sum_{i=0}^{n-1} \log \left\| Df_{f^i(x)} \left(\frac{Df_x^i(v)}{\|Df_x^i(v)\|} \right) \right\|$$
$$= \sum_{i=0}^{n-1} \phi(F^i(x, v)).$$

∎

LEMMA 11. *There exists a measure $\hat{\mu} \in \mathcal{E}(F)$ such that*

$$\int_{SM} \phi \, d\hat{\mu} = \inf_{\mu \in \mathcal{M}(F)} \int_{SM} \phi \, d\mu.$$

Proof. By the compactness of $\mathcal{M}(F)$ and continuity of ϕ and of the integral functional $\ell_\phi(\mu) = \int \phi \, d\mu$, it follows that there exists some measure $\bar{\mu}$ for which the the equality in the statement of the Lemma holds. This measure is not necessarily ergodic but we claim that some ergodic component

of $\bar{\mu}$ also satisfies the required equality. Indeed, by the Ergodic Decomposition Theorem [13, page153], there exists a measure τ on $\mathcal{M}(F)$ and a set $\mathcal{E}_{\bar{\mu}}(F) \subseteq \mathcal{E}(F)$ with $\tau(\mathcal{E}_{\bar{\mu}}(F)) = 1$ such that we have $\int_{\mathcal{E}(F)} \nu d\tau = \bar{\mu}$ in the sense that

$$\int_{\mathcal{E}(F)} \left(\int_{SM} \phi d\nu \right) d\tau = \int_{SM} \phi d\bar{\mu}. \tag{5}$$

Now, for any $\nu \in \mathcal{E}_{\bar{\mu}}(F)$ we have

$$\int_{SM} \phi d\nu \geq \inf_{\mu \in \mathcal{M}(F)} \int_{SM} \phi d\mu = \int \phi d\bar{\mu}. \tag{6}$$

If the inequality was strict for a positive τ measure set, we would have

$$\int_{\mathcal{E}(F)} \left(\int_{SM} \phi d\nu \right) d\tau > \int \phi d\bar{\mu}$$

contradicting (5). Therefore there must be a τ full measure (in particular non–empty) set of measures in $\mathcal{E}_{\bar{\mu}}(F)$ for which the equality in (6) holds. ∎

LEMMA 12. $\forall \lambda \in \mathcal{L}(f) \; \exists \mu \in \mathcal{M}(F)$ such that

$$\int \phi d\mu = \lambda.$$

Proof. By assumption, there exists some $(x, v) \in SM$ such that

$$\lambda(x, v) = \lim_{n \to \infty} \frac{1}{n} \log \|Df_x^n(v)\| = \lambda. \tag{7}$$

For such a point (x, v), consider the sequence of measures

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{F^i(x,v)}$$

and let $\mu \in \mathcal{M}(F)$ be the limit of some subsequence μ_{n_k}. For any $n \geq 1$ we have

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(F^i(x, v)) = \int \phi d\mu_n.$$

Therefore, by the definition of weak–star convergence,

$$\lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n-1} \phi(F^i(x, v)) = \lim_{n_k \to \infty} \int \phi d\mu_n = \int \phi d\mu.$$

Moreover, using the existence of the limit in (7), we have

$$\lim_{k \to \infty} \frac{1}{n_k} \log \frac{\|Df_x^n(v)\|}{\|v\|} = \lim_{n \to \infty} \frac{1}{n} \log \frac{\|Df_x^n(v)\|}{\|v\|} = \lambda(x, v) = \lambda.$$

Finally, the definition of ϕ implies

$$\lim_{k \to \infty} \frac{1}{n_k} \log \frac{\|Df_x^n(v)\|}{\|v\|} = \lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n-1} \phi(F^i(x, v)).$$

Substituting this into the two previous equations completes the proof. ∎

LEMMA 13. $\forall\, \mu \in \mathcal{E}(F)\, \exists\, \lambda \in \mathcal{L}(f)$ such that

$$\lambda = \int \phi d\mu.$$

Proof.   By the ergodicity of μ there exists a set $A \subseteq SM$ with $\mu(A) = 1$ such that for all $(x, v) \in A$ there exists a constant

$$\lambda = \lambda(x, v) = \lim_{n \to \infty} \frac{\|Df_x^n(v)\|}{\|v\|} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(F^i(x, v)) = \int \phi d\mu. \quad (8)$$

It remains to show that $\lambda \in \mathcal{L}(f)$, i.e. that there actually exists a measure $\mu^* \in \mathcal{M}(f)$ such that $\lambda \in \mathcal{L}(\mu^*)$, as opposed to the possibility that the limit in (8) exists by complete coincidence for some exceptional point. By Lemma (9) the measure $\mu^* = \pi^*\mu$ is invariant and ergodic for f. Moreover we have $\mu^*(\pi(A)) = 1$. Thus, for μ^* almost every $x \in \pi(A)$ and every $v \in T_xM$ such that $(x, v) \in A$ we have that equation (8) holds and thus λ is one of the Lyapunov exponents associated to the measure μ^* and therefore belongs to $\mathcal{L}(f)$. ∎

Proof (Proof of Theorem 3).   By Lemma 11, there exists a measure $\mu \in \mathcal{E}(F)$ which minimizes the integral $\int \phi d\mu$. By Lemma 13 there exists a Lyapunov exponent $\lambda \in \mathcal{L}(f)$ associated to some measure $\mu^* \in \mathcal{E}(f)$, which satisfies $\lambda = \int \phi d\mu$. Thus it just remains to show that there exist no other Lyapunov exponents $\lambda' < \lambda$. By Lemma 12 this would imply the existence of a measure $\mu' \in \mathcal{M}(F)$ such that $\int \phi d\mu' = \lambda' < \lambda = \int \phi d\mu$ which contradicts the minimality of $\int \phi d\mu$ over all measures in $\mathcal{M}(F)$. ∎

4. UNIFORM EXPANSIVITY

We now prove Theorem 5. We first reformulate the definition of uniform hyperbolicity in the following clearly equivalent form: there exists constants

$\lambda > 0$ and $N > 0$ such that

$$\|Df_x(v)\| \geq e^{\lambda n}\|v\| \quad \forall \, x \in \Lambda \, v \in T_x M, \; n \geq N.$$

Proof (Proof of Theorem 5). By Theorem 3 we can choose a constant λ satisfying

$$\inf\{\mathcal{L}(f)\} = \lambda' > \lambda > 0$$

We assume by contradiction that there exists a sequence of times $n_k \to \infty$, a sequence of points $x_k \in \Lambda$ and a sequence of vectors $v_k \in T_{x_k} M$ such that

$$\|Df_{x_k}^{n_k}(v_k)\| < e^{\lambda n_k}\|v_k\|.$$

Now consider the sequence of measures

$$\mu_{n_k} = \frac{1}{n_k} \sum_{i=0}^{n_k-1} \delta_{F^i(x_k, v_k)}$$

and any (invariant) limit measure μ of this sequence. For simplicity we shall suppose without loss of generality that $\mu_{n_k} \to \mu$. Then we have

$$\frac{1}{n_k} \sum_{i=0}^{n_k-1} \phi(F^i(x_k, v_k)) = \int \phi \, d\mu_{n_k} < \lambda$$

for every n_k. In particular

$$\lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \phi(F^i(x_k, v_k)) = \lim_{k \to \infty} \int \phi \, d\mu_{n_k} = \int \phi \, d\mu \leq \lambda.$$

However, by Lemma 13 this implies $\lambda \in \mathcal{L}(f)$ which contradicts our choice of λ. ∎

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