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Nonholonomic Systems and the Geometry of Constraints

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In a recent paper [9] we analyze conservation of volume for a series of examples of mechanical systems with linear, affine and non linear constraints aiming to make evident some geometric aspects related with them. Here, we only consider examples with linear constraints (defined by a constant rank distribution), in which we have conservation of volume. Conservation of volume means, equivalently, that the orthogonal distribution (the metric is defined by the kinetic energy) is minimal (see [15]) and so, if it is integrable, the corresponding foliation has minimal leaves. Properties of the falling penny and of the vertical disc rolling on a horizontal plane without slipping are very special. A dynamically symmetric sphere that rolls without slipping on a given surface $S \subset \mathbb{R}^3$ conserves volume, and the orthogonal distribution is integrable if, and only if, S is *parallel* to a surface with a fixed constant mean curvature. Semi–simple Lie groups endowed with suitable metrics have foliations with minimal leaves. Geometric questions related with the kinematics of the rolling motion of two surfaces are also considered.

Key Words: Non-holonomic systems, Liouville theorem, minimal leaves, rolling motion

1. NOTATION

All the data we will consider in this paper are of class C^∞. Let us start with a Riemannian manifold (M, g) where M has dimension n and represents the configuration space; g is the metric tensor with associated Levi–Civita connection denoted by ∇. The map $K = (1/2)g : TM \to \mathbb{R}$ is the kinetic energy and $\mathcal{D} \subset TM$ denotes a vector sub–bundle of the tangent bundle $\tau_{TM} : TM \to M$. $\mathcal{D}$ can be seen, equivalently, as a distribution of plane fields with constant rank m on M and represents the constraint for the velocities; to say that the mechanical system is non holonomic means that $\mathcal{D}$ is non integrable. Denote by $\mathcal{D}^\perp$ the orthogonal distribution to $\mathcal{D}$ with respect to g that is a distribution of constant rank $(n-m)$ (see [10, 7, 12, 14, 9, 13]).

2. THE TOTAL SECOND FUNDAMENTAL FORM OF A DISTRIBUTION $\tilde{\mathcal{D}}$

This notion was introduced in [12] (see also [5] and [14]) and we recall it for the sake of completeness. Let $\tilde{\mathcal{D}}$ be a distribution (integrable or not) defined on M; the total second fundamental form of $\tilde{\mathcal{D}}$ is a bilinear vector bundle morphism $B_{\tilde{\mathcal{D}}} : TM \times_M \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}^\perp$ defined in the following way: fix $x \in M$, consider $(\bar{X}, \bar{Y}) \in T_xM \times \tilde{\mathcal{D}}_x$, choose local extensions X, Y of $\bar{X}, \bar{Y}$ with $Y \in \tilde{\mathcal{D}}$ and consider $B_{\tilde{\mathcal{D}}}(\bar{X}, \bar{Y})$ as the orthogonal projection $P^\perp_{\tilde{\mathcal{D}}}[(\nabla_X Y)_x]$; the value of $B_{\tilde{\mathcal{D}}}(\bar{X}, \bar{Y})$ does not depend on the local extensions X, Y of $\bar{X}, \bar{Y}$.

Remark 1. At this point it is important to emphazise ([12]) that when $\tilde{\mathcal{D}}$ is integrable, the restriction of $B_{\tilde{\mathcal{D}}}$ to $\tilde{\mathcal{D}} \times_M \tilde{\mathcal{D}}$ gives, precisely, the classical second fundamental form of the leaves of the foliation defined by $\tilde{\mathcal{D}}$. In particular when the trace of the second fundamental form of a leaf is equal to zero, one concludes that the leaf is minimal (see [15]).

The meaning of minimality just mentioned in the remark above is the following: if Ω is a sufficiently small domain contained in a leaf of $\tilde{\mathcal{D}}$ (which is an immersed submanifold of M) and has a regular boundary $\partial\Omega$, then the volume of Ω in the induced metric of $\tilde{\mathcal{D}} \subset M$ is less or equal to the volume of any other submanifold of M with the same boundary.

3. D'ALEMBERT PRINCIPLE:EQUATIONS OF MOTION AND THE CONDITION FOR INVARIANCE OF A VOLUME ON $\mathcal{D}$

The non holonomic systems we want to consider aim to the determination of motions $q(t) \in M$ such that $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t, under the condition that the difference between the acceleration $\frac{D\dot{q}}{dt}$ and the external force $(-\mathrm{grad}V)(q(t))$ has to be orthogonal to $\mathcal{D}_{q(t)}$; in other words, this means that D'Alembert principle (see [12, 14]) holds. That leads to the second order ODE

$$\frac{D\dot{q}}{dt} + P_{\mathcal{D}}(\mathrm{grad}V(q)) = B_{\mathcal{D}}(\dot{q}, \dot{q}) \tag{1}$$

where we need to consider the initial conditions $q(0) \in M$ and $\dot{q}(0) \in \mathcal{D}_{q(0)}$.

The ODE system (1) defines a vector field on the manifold $\mathcal{D} \subset TM$; under a certain condition, that vector field leaves invariant a local volume on $\mathcal{D} \cap TU$ where $U(q^1, ..., q^n)$ is a local system of coordinates on M (see [12] for more details).

Theorem 2 (Kupka and Oliva). *There exists a local volume form ω defined on TU such that its restriction to $\mathcal{D} \cap TU$ is invariant under the flow of the vector field defined by (1) if, and only if, the trace of the restriction of $B_{\mathcal{D}^\perp}$ to $\mathcal{D}^\perp \times_M \mathcal{D}^\perp$ vanishes.*

To obtain a (global) volume form on $\mathcal{D}$ one needs to assume that $\mathcal{D}$ is orientable as submanifold of TM.

Corollary 3. *If the flow defined by system (1) conserves the local volume ω considered in Theorem 2 and $\mathcal{D}^\perp$ is integrable, then the leaves of $\mathcal{D}^\perp$ are immersed minimal submanifolds of M.*

The proof follows from Theorem 2 and Remark 1.

4. THE HOMOGENEOUS FALLING PENNY AND THE VERTICAL ROLLING DISC

The so called falling penny is a homogeneous disc rolling without slipping on a horizontal plane. The configuration space $M = \mathbb{R}^2 \times SO(3)$ is 5–dimensional and has local coordinates $(x, y, \phi, \theta, \psi)$, the mass of the penny is $m > 0$ and its radius is R (see [3]). The constraint $\mathcal{D}$ is defined, locally, by the zeros of the linear differential forms $dx + R\,d\psi \cos(\phi)$ and $dy + R\,d\psi \sin(\phi)$; so it is a non integrable rank 3 distribution. After writting the kinetic energy we see in [9] that the volume mentioned in Theorem 2 is not conserved. On the other hand applying Frobenius theorem we see that the rank 2 distribution $\mathcal{D}^\perp$ is integrable; thus no conclusions one can take about its minimality.

Now we analize the vertical disc (not necessarily homogeneous) rolling without slipping on a horizontal plane. The configuration manifold is $M = \mathbb{R}^2 \times S^1 \times S^1$ so it is 4–dimensional with local coordinates (x, y, ϕ, ψ) and the constraint distribution is written as in the example related with the falling penny. The kinetic energy is also presented in [9], defining the metric. In this case we have $dim\mathcal{D}\ dim\mathcal{D}^\perp = 2$ and there is conservation of volume. On the other hand the rank 2 distribution $\mathcal{D}^\perp$ is integrable. So, by Corollary 3, the leaves of the foliation defined by $\mathcal{D}^\perp$ are 2–dimensional manifolds and so they are minimal surfaces immersed in the 4–dimensional manifold M.

5. SEMI–SIMPLE LIE GROUPS UNDER CARTAN DECOMPOSITION

Let us start by recalling the following definitions and results for semi-simple Lie algebras (we follow here the notation of [8]; see also [5] and [9]):

1. A Lie algebra g is called *semisimple* if the Killing form

$$\kappa(X,Y)trace(adX adY)$$

on $g \times g$ is non–degenerate. An analytical Lie group is *semisimple* if its Lie algebra is semisimple.

2. Let g be a Lie algebra. Then $\theta \in Aut(g)$ is an involution if $\theta^2 = 1$.

3. If g is a real semisimple Lie algebra, then an involution θ on g is called a *Cartan involution* if the symmetric bilinear form

$$\kappa_\theta(X,Y) = -\kappa(X,\theta Y)$$

is positive definite, where κ is the Killing form of g.

4. Every real semisimple Lie algebra has a Cartan involution. Moreover any two Cartan involutions are conjugate via $Int(g)$.

5. Any Cartan involution yields a Cartan Decomposition $g = t \oplus p$, with

$$\begin{aligned} t &= \{X \in g \mid \theta(X) = X\}, \\ p &= \{X \in g \mid \theta(X) = -X\}, \end{aligned}$$

where t is a maximal compactly embedded subalgebra of g (for more details see [8]).

6. The following properties hold:

(i) $[t,t] \subset t, \quad [t,p] \subset p, \quad [p,p] \subset t,$

(ii) $\kappa_\theta(t,p) = \kappa(t,p) = 0,$

(iii) $\kappa|_t$ is negative definite, $\kappa|_p$ is positive definite.

On a semisimple analytical Lie group G with Lie algebra g, let us consider the left invariant distribution defined by $\mathcal{D}_e = p$ and the left invariant metric associated with an arbitrary metric on g such that p and t are orthogonal, for instance, $\langle X,Y\rangle = \kappa_\theta(X,Y)$ for all $X,Y \in g$.

Theorem 4. *Let G be an analytic semi–simple Lie group considered as the configuration space of a mechanical system where the kinetic energy and the constraint are given by the invariant metric and distribution, respectively, introduced above. Then the left invariant distribution $\mathcal{D}^{\perp}$ such that ${\mathcal{D}^{\perp}}_e = t$ is integrable and $B_{\mathcal{D}^{\perp}}|_{\mathcal{D}^{\perp}\times\mathcal{D}^{\perp}}$ vanishes. So, in particular, the leaves of $\mathcal{D}^{\perp}$ are minimal submanifolds of G.*

As concrete examples, we mention the so called pseudo–rigid bodies (see [14]), whose configuration space is $SL(n)$; for any $X \in sl(n)$, $\theta(X) = -X^{\dagger}$, $X^{\dagger}$ being the transpose of X. Another example is $SO(n,1)$ (see [5]) whose Lie algebra is $so(n,1) = \{X \in gl(n+1) | X^{\dagger} I_{\bar{n}} + I_{\bar{n}} X^{\dagger} = 0\}$ where $I_{\bar{n}} = diag(-I_n, 1)$.

Remark 5. A totally geodesic distribution is a distribution invariant under the geodesic spray of the Levi–Civita connection. So, if $\mathcal{D}^{\perp}$ is totally geodesic we have $B^s_{\mathcal{D}^{\perp}}|_{\mathcal{D}^{\perp}\times\mathcal{D}^{\perp}} = 0$ and then the GMA flow conserves volume. Semi–simple Lie groups with the Cartan decomposition satisfy this latter property.

6. SPHERE ROLLING WITHOUT SLIPPING ON A HORIZONTAL PLANE

The configuration space in this section is the 5–dimensional manifold $\mathbb{R}^2 \times SO(3)$; we deal with the classical local coordinates $(x, y, \phi, \theta, \psi)$. As in [1, 7, 9], let us consider I_1, I_2, I_3 as the moments of inertia corresponding to the distribution of mass, and $|a| > 0$ being the distance between the center of mass and the geometric center of the sphere of radius δ. The total mass is $m > 0$ and the non integrable constraint distribution $\mathcal{D}$ is defined by the zeros of the two differential forms

$$dx - \delta[-d\psi \sin(\theta)\cos(\phi) + d\theta \sin(\phi)]$$

and

$$dy - \delta[-d\psi \sin(\theta)\sin(\phi) - d\theta \cos(\phi)].$$

The kinetic energy is obtained from formula (4) of [9].

We quote from [9] the two following facts:

(i) the Routh sphere, that is ($I_1 = I_2 \neq I_3$) and $|a| > 0$, does not conserve volume;

(ii) the Chaplygin sphere, that is ($I_1 = I_2 \neq I_3$) and $|a| = 0$, does conserve volume.

On the other hand, using Frobenius theorem one concludes that, in the case of Chaplygin sphere, the rank 2 distribution $\mathcal{D}^{\perp}$ is integrable. Since homogeneous spheres are of Chaplygin type the same is true for homogeneous spheres.

7. A DYNAMICALLY SYMMETRIC SPHERE ROLLING WITHOUT SLIPPING ON A REGULAR SURFACE OF $\mathbb{R}^3$

THEOREM 6. *Let us consider a rigid sphere of radius $\delta > 0$ and $k = \sqrt{I/m}$ its radius of gyration; m is is the mass and I is the moment of inertia of the sphere around any axis passing through its center (we assume that the mass distribution has spherical symmetry). The sphere rolls without slipping on a fixed surface $\mathcal{S} \subset \mathbb{R}^3$. Then:*

1. We have conservation of volume.
2. The distribution $\mathcal{D}^\perp$ is integrable if, and only if, the surface $\mathcal{C}$ parallel to S defined by the all possible positions of the center of mass of the sphere, has the absolute value of the constant mean curvature equal to $H(\mathcal{C}) = \frac{\delta}{2k^2}$.

Proof. Since our computation is local we may assume that our regular surface is a smooth graph, say $(x, y) \to \Phi(x, y)$. If the normal vector is given by $(\partial_x\Phi, \partial_y\Phi, -1)$, the absolute value of the mean curvature $H(\mathcal{C})$ of that graph surface can be determined (see [17]) by the following formula:

$$\frac{\left|[1 + (\partial_x\Phi)^2]\partial_{yy}\Phi + [1 + (\partial_y\Phi)^2]\partial_{xx}\Phi - 2(\partial_x\Phi)(\partial_y\Phi)(\partial_{xy}\Phi)\right|}{\left|-2[1 + (\partial_x\Phi)^2 + (\partial_y\Phi)^2]^{3/2}\right|}$$

On the other hand, by the Frobenius theorem the computation relative to the condition for the integrability of $\mathcal{D}^\perp$ (when we deal with a general smooth surface and use a local graph representation) is equivalent to

$$[1 + (\partial_x\Phi)^2]\partial_{yy}\Phi + [1 + (\partial_y\Phi)^2]\partial_{xx}\Phi - 2(\partial_x\Phi)(\partial_y\Phi)(\partial_{xy}\Phi)$$

$+ (\delta/k^2)[1 + (\partial_x\Phi)^2 + (\partial_y\Phi)^2]^{3/2} = 0$ or, necessarily, $H(\mathcal{C}) = \frac{\delta}{2k^2}$. ∎

Special cases for $\mathcal{C}$ implying the integrability of $\mathcal{D}^\perp$ are: spheres, circular cylinders, Delaunay surfaces, etc., provided that the absolute value of the constant mean curvature of $\mathcal{C}$ be equal to $H(\mathcal{C}) = \frac{\delta}{2k^2}$. It is clear that we need to choose $\mathcal{S}$ in a such way that the corresponding $\mathcal{C}$ has the desired value for $H(\mathcal{C})$, after taking into account the radius δ of the rolling sphere.

For example, for a homogeneous sphere of radius δ we have $k^2 = (2/5)\delta^2$ and so $H(\mathcal{C})\ \frac{5}{4\delta}$. Let us choose as $\mathcal{S}$ a spherical surface of radius R. One can see that we have integrability of the distribution $\mathcal{D}^\perp$ if and only if the rolling sphere moves inside $\mathcal{S}$ with a radius $\delta = (5/9)R$. If the moving sphere rolls on $\mathcal{S}$ but outside it, then $\mathcal{D}^\perp$ is never integrable. One can then state the following corollary.

COROLLARY 7. *If a homogeneous sphere of radius δ rolls without slipping inside another sphere of radius R then the corresponding rank two distribution $\mathcal{D}^{\perp}$ orthogonal to the constraint distribution $\mathcal{D}$ is integrable and minimal if, and only if, $\delta = (5/9)R$. The geometrical consequence is that the compact manifold $S^2 \times SO(3)$ (which represents the configuration space of the mechanical problem) admits a foliation with dimension two immersed minimal leaves.*

Another example that we leave to the reader is the case of a small sphere rolling internally or externally on the surface of a circular cylinder.

It can also be considered the case of a sphere of radius δ rolling on a surface S (to be determined) such that its *parallel* surface $\mathcal{C}$ is a suitable Delaunay surface.

From Corollary 3 and when there is integrability of $\mathcal{D}^{\perp}$ it follows that the leaves of this foliation are minimal surfaces of $\mathcal{S} \times SO(3)$.

8. ROLLING SURFACES

In this section we do not need to assume D'Alembert principle because we will deal only with geometric questions related to the kinematics of the problem to be studied.

In a previous section we considered the case of a dynamically symmetric sphere rolling without slipping on a regular surface $\mathcal{S} \subset \mathbb{R}^3$. The configuration space in this problem is identified with the 5–dimensional manifold $\mathcal{S} \times SO(3)$ and the fact that the sphere rolls without slipping corresponds to the consideration of a constant rank 3 distribution as constraints for the velocities. If we want to consider rolling without slipping or twisting the constraint has to be given by a constant rank 2 non integrable distribution. This model gives rise to an example of a so called system of Cartan type (see [4] where the authors investigate the geometry of the so called rigid integral curves of rank 2 distributions on manifolds; see also the recent book [13] by R. Montgomery).

Following [4] let Σ_1 and Σ_2 be oriented surfaces embedded in Euclidean space and let F_1 (resp. F_2) denote the corresponding oriented orthonormal frame bundle of Σ_1 (resp. Σ_2). Each of F_1 and F_2 are principal $SO(2)$–bundles. It is well known that there are canonical 1–forms α_1, α_2 and $\alpha_{21}(= -\alpha_{12})$ on F_1 and correspondingly β_1, β_2 and $\beta_{21}(= -\beta_{12})$ on F_2 (see [17, 6]) satisfying on F_1

$$\begin{aligned} d\alpha_1 &= \alpha_{21} \wedge \alpha_2 \\ d\alpha_2 &= -\alpha_{21} \wedge \alpha_1 \\ d\alpha_{21} &= A\alpha_1 \wedge \alpha_2 \end{aligned}$$

and analogous equations valid on F_2 with β instead of α and B instead of A, where A (respectively, B) is the Gauss curvature of Σ_1 (respectively, Σ_2).

Let $SO(2)$ acting diagonally on $F_1 \times F_2$:

$$(g, (\bar{p}_1, \bar{p}_2)) \in SO(2) \times (F_1 \times F_2) \to (g\bar{p}_1, g\bar{p}_2) \in F_1 \times F_2$$

that is a proper and free action. Set $M = (F_1 \times F_2)/SO(2)$ the 5–manifold representing the configuration space. If p_1 and p_2 are the projections of $\bar{p}_1$ and $\bar{p}_2$, an element of M has a natural interpretation as a triple (p_1, p_2, i) where $i : T_{p_1}\Sigma_1 \to T_{p_2}\Sigma_2$ is an oriented isometry.

Let us take a curve $\gamma : [a, b] \to M$, $\gamma(t) = (u_1(t), u_2(t), i(t))$, where $u_i(t) : [a, b] \to \Sigma_i$, $i = 1, 2$, are smooth curves satisfying $i(t)(\dot{u}_1(t)) = \dot{u}_2(t)$ for all $a \le t \le b$; that last condition corresponds to the property *rolling without slipping*. If one wants to prevent twisting one considers $e_1, f_1 : [a, b] \to T\Sigma_1$ as any parallel orthonormal frame field along the curve u_1, such that the corresponding frame field $e_2, f_2 : [a, b] \to T\Sigma_2$ defined by $e_2(t) = i(t)(e_1(t))$, $f_2(t) = i(t)(f_1(t))$, is also paralell along u_2.

In usual formulations of the problem, one starts with $\Sigma_1 \subset \mathbb{R}^3$ as a stationary surface and one can think on a rigid body in $\mathbb{R}^3$ whose boundary surface Σ_2 is into tangential contact with Σ_1. So Σ_1 and Σ_2 are tangent one to the other in an initial point P_0. Let us consider two smooth curves $u_i : [0, l_i] \to \Sigma_i$, $i = 1, 2$, and assume they are parametrized by the corresponding arc lenghts s, with $u_i'(s) := t_i(s) \neq 0$, $s \in [0, l_i]$, $i = 1, 2$. So, by hypothesis $u_1(0) = u_2(0) = P_0$ and $u_1'(0) = u_2'(0)$. We have well defined the moving frames $E_i = (t_i, \eta_i, N_i)$ with $\eta_i = N_i \times t_i$, $i = 1, 2$ (here we are ommiting the argument s, $s \in [0, l_i]$, for simplicity of notation). Let us see how these two curves allow us to define a rolling of Σ_2 on Σ_1 (thanks to [2] and also to a private communication from C.E. Harle). The component N_i comes from the unitary normal field that gives the orientation of Σ_i, so $(t_i(s), \eta_i(s))$ is an oriented frame field tangent to Σ_i at $u_i(s)$, $i = 1, 2$. For a curve $u_i(s)$ contained in Σ_i we have the well known Darboux relations

$$\begin{aligned}
\frac{dt_i}{ds} &= k_g^{(i)}\eta_i + h^{(i)}N_i \\
\frac{d\eta_i}{ds} &= -k_g^{(i)}t_i + \tau^{(i)}N_i \\
\frac{dN_i}{ds} &= -h^{(i)}t_i - \tau^{(i)}\eta_i,
\end{aligned}$$

where $k_g^{(i)} = k_g^{(i)}(s)$ is the geodesic curvature of $u_i = u_i(s)$ on Σ_i, $i = 1, 2$ (see [6]). Recall that u_i is a geodesic of Σ_i if, and only if, $k_g^{(i)}(s) = 0$ for all s, $i = 1, 2$.

For each value of the parameter s we consider the map $I(s) \in SO(3)$ defined by $I(s)(t_1) = t_2$, $I(s)(\eta_1) = \eta_2$, $I(s)(N_1) = N_2$; that corresponds to rolling without slipping. By derivative with respect to s we obtain

$$I'(s)I^{-1}(s)t_2 = I'(s)t_1 = \frac{dt_2}{ds} - I(s)\frac{dt_1}{ds} = (k_g^{(2)} - k_g^{(1)})\eta_2 + (h^{(2)} - h^{(1)})N_2$$

and analogous expressions for $I'(s)I^{-1}(s)\eta_2$ and $I'(s)I^{-1}(s)N_2$. Since $I'(s)I^{-1}(s)$ is skew symmetric, there exists a vector $\omega(s) \in \mathbb{R}^3$, called the direction of the instantaneous axis of rotation, such that $I'(s)I^{-1}(s) = \omega(s)\times$ (see [14]) and so (ommiting s variable, for simplicity) we have:

$$\begin{aligned}
\omega \times t_2 &= (k_g^{(2)} - k_g^{(1)})\eta_2 + (h^{(2)} - h^{(1)})N_2 \\
\omega \times \eta_2 &= -(k_g^{(2)} - k_g^{(1)})t_2 + (\tau^{(2)} - \tau^{(1)})N_2 \\
\omega \times N_2 &= -(h^{(2)} - h^{(1)})t_2 - (\tau^{(2)} - \tau^{(1)})\eta_2,
\end{aligned}$$

therefore

$$\omega = (\tau^{(2)} - \tau^{(1)})t_2 - (h^{(2)} - h^{(1)})\eta_2 + (k_g^{(2)} - k_g^{(1)})N_2.$$

A special rolling can be considered if we impose ω orthogonal to N_2 for all values of the parameter s, that is, if and only if, $k_g^{(2)}(s) = k_g^{(1)}(s)$ for all s. As we will see, that kind of rolling turns out to be without slipping or twisting and implies the following about the curves: curve u_1 determines curve u_2 and conversely. In particular if u_1 is a geodesic of Σ_1, then u_2 is a geodesic of Σ_2 and conversely.

As we mentioned above, the rolling without slipping defined in [4] means that for the smooth curve $\gamma(s) = (u_1(s), u_2(s), i(s))$, the oriented isometry $i(s)$ has the property that $i(s)(t_1(s)) = t_2(s)$ and so $i(s)(\eta_1(s)) = \eta_2(s)$, $s \in [a, b]$. On the other hand to avoid twisting one starts with a parallel oriented orthonormal frame $(e_1, f_1) = (e_1(s), f_1(s))$ along $u_1 = u_1(s)$. Then both covariant derivatives $\frac{De_1}{ds}$ and $\frac{Df_1}{ds}$ vanish; so since $e_1 = At_1 + B\eta_1$ and $f_1 = \bar{A}t_1 + \bar{B}\eta_1$ one obtains by covariant derivative (see for instance [14]):

$$\begin{aligned}
&\frac{dA}{ds}t_1 + \frac{dB}{ds}\eta_1 + A\frac{Dt_1}{ds} + B\frac{D\eta_1}{ds} = 0, \\
&\frac{d\bar{A}}{ds}t_1 + \frac{d\bar{B}}{ds}\eta_1 + \bar{A}\frac{Dt_1}{ds} + \bar{B}\frac{D\eta_1}{ds} = 0,
\end{aligned}$$

where one can assume $A(s) = \bar{B}(s) = \cos(\theta(s))$ and $B(s) = -\bar{A}(s) = \sin(\theta(s))$. On the other hand, from the well known relations above and the definition of covariant derivative it follows that $\frac{Dt_1}{ds} = k_g^{(1)}\eta_1$ and $\frac{D\eta_1}{ds} = -k_g^{(1)}t_1$. Then $\frac{dA}{ds} = Bk_g^{(1)}$; $\frac{d\bar{B}}{ds} = -\bar{A}k_g^{(1)}$; also $\frac{dB}{ds} = -Ak_g^{(1)}$ and $\frac{d\bar{A}}{ds} =$

$\bar{B}k_g^{(1)}$. Then $\frac{dA}{ds} = -\sin(\theta(s))\dot{\theta}(s) = Bk_g^{(1)}(k_g^{(1)})\sin(\theta(s))$; analogously $\frac{dB}{ds} = \cos(\theta(s))\dot{\theta}(s) = -Ak_g^{(1)} - (k_g^{(1)})\cos(\theta(s))$. Thus $k_g^{(1)} = -\dot{\theta}$. But the definition given by Bryant and Hsu of rolling without slipping or twisting implies also that $i(s)t_1(s) = t_2(s)$ and that the two vectors $i(s)e_1 = e_2$ and $i(s)f_1 = f_2$ form a parallel frame, that is, $\frac{De_2}{ds}$ and $\frac{Df_2}{ds}$ vanish. Now, arguing as before one sees that $k_g^{(2)} = -\dot{\theta}$. From the above one concludes that $k_g^{(1)} = k_g^{(2)}$. That shows the equivalence between the special rolling mentioned above and the rolling without slipping or twisting defined in [4].

Another statement equivalent to both definitions can also be obtained with the following equalities: $i(s)(t_1(s)) = t_2(s)$ and $i(s)\frac{Dt_1}{ds} = \frac{Dt_2}{ds}$. It follows from the first one and from the fact that $i(s)$ is an oriented isometry that we also have $i(s)(\eta_1(s)) = \eta_2(s)$, because (t_2, η_2) is a positive frame. Since we observe that $i(s)\frac{Dt_1}{ds} = k_g^{(1)}i(s)\eta_1 = k_g^{(1)}\eta_2$ and that $\frac{Dt_2}{ds} = k_g^{(2)}\eta_2$ the equivalence is now an easy matter.

Theorem 8. *Let Σ_1 and Σ_2 be oriented surfaces embedded in Euclidean space and $u_1(s)$, $u_2(s)$ be two smooth contact curves on Σ_1 and Σ_2, respectively, parametrized by arc lenght with $u'_i(s) \neq 0$, $i = 1, 2$, for all values of s, and obtained by the rolling without slipping of Σ_1 along Σ_2. Then the rolling is without slipping or twisting if and only if $u_1(s)$ and $u_2(s)$ have the same geodesic curvatures.*

Bryant and Hsu described completely the canonical constant rank 2 distribution $\mathcal{D}$ on M which has the property that the curves on M tangent to $\mathcal{D}$ (the so called $\mathcal{D}$–curves) represent the possible ways of rolling without slipping or twisting and they showed that the distributions $\mathcal{D}_1 = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ and $\mathcal{D}_1 + [\mathcal{D}_1, \mathcal{D}_1]$ have constant rank 3 and maximum rank 5, respectively, on the open set characterized by $A \neq B$, where, as defined above, A (resp.B) is the Gauss curvature of Σ_1 (resp. Σ_2) that is $\mathcal{D}$ is said to be a distribution of Cartan type in that open set. In particular $\mathcal{D}$ and $\mathcal{D}_1$ are bracket generating (see [4] and [13]).

For a sake of completeness, we recall some arguments of [4]; let $\tilde{\mathcal{D}}$ be the distribution of constant rank 3, defined on $F_1 \times F_2$ by the Pfaff equations

$$\begin{aligned} \alpha_1 - \beta_1 &= 0 \\ \alpha_2 - \beta_2 &= 0 \\ \alpha_{21} - \beta_{21} &= 0. \end{aligned}$$

In fact, the 1–forms appearing in these equations are the pull–back, by the canonical projections, of the global canonical 1–forms on F_1 and F_2. That distribution $\tilde{\mathcal{D}}$ is invariant under the action of $SO(2)$ on $F_1 \times F_2$ and contains the tangent vectors to the fibers of the quotient map $F_1 \times$

$F_2 \mapsto M = (F_1 \times F_2)/SO(2)$. So, by *push–down* of $\tilde{\mathcal{D}}$, one obtains, on the manifold M, a constant rank 2 distribution $\mathcal{D}$. The motions of rolling without slipping or twisting correspond, precisely, to the $\mathcal{D}$–curves, that is the constant rank 2 distribution $\mathcal{D}$ describes the nonholonomic constraints of the considered rolling system.

Now, on the manifold M, one defines three 1–forms θ^i, $i = 1, 2, 3$, by the equalities:

$$\begin{aligned} \alpha_1 - \beta_1 &= \theta^1 \\ \alpha_2 - \beta_2 &= \theta^2 \\ \alpha_{21} - \beta_{21} &= \theta^3; \end{aligned}$$

to these 1–forms we add two new 1–forms ω^1 and ω^2 defined by the relations $2\omega^1(\alpha_1 + \beta_1)$ and $2\omega^2 = (\alpha_2 + \beta_2)$. It can be proved the following:

LEMMA 9. *The five 1–forms θ^1, θ^2, θ^3, ω^1 and ω^2 are linearly independent at each point of the configuration space M. Moreover,*

$$\begin{aligned} d\theta^1 &= \theta^3 \wedge \omega^2 + \frac{1}{2}(\alpha_{21} + \beta_{21}) \wedge \theta^2 \\ d\theta^2 &= -\theta^3 \wedge \omega^1 - \frac{1}{2}(\alpha_{21} + \beta_{21}) \wedge \theta^1 \\ d\theta^3 &= \frac{1}{2}(A - B)\omega^1 \wedge \omega^2 + \frac{1}{2}(A - B)\theta^1 \wedge \theta^2 + \frac{1}{2}(A + B)[\theta^1 \wedge \omega^2 + \omega^1 \wedge \theta^2]. \end{aligned}$$

LEMMA 10. *On the open set in M where $A \neq B$, the constant rank 2 distribution $\mathcal{D}$ corresponding to motions without slipping or twisting is of Cartan type.*

Proof. Let $(X_1, X_2, X_3, Y_1, Y_2)$ be the dual frame corresponding to the sequence

$$(\theta^1, \theta^2, \theta^3, \omega^1, \omega^2).$$

The distribution $\mathcal{D}$ is framed by (Y_1, Y_2) and $\mathcal{D}_1 = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ is framed by $(Y_1, Y_2, [Y_1, Y_2])$. In fact, the basic formula

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

and the previous lemma imply that $\theta^1([Y_1, Y_2]) = \theta^2([Y_1, Y_2]) = 0$ and $\theta^3([Y_1, Y_2])\frac{1}{2}(A - B)$. So, $\mathcal{D}_1$ has constant rank 3 on the open set in M where $A \neq B$. To finish the proof one has to show that the distribution $\mathcal{D}_2 = \mathcal{D}_1 + [\mathcal{D}_1, \mathcal{D}_1]$ has maximum rank 5; for this one considers the sequence

$$(Y_1, Y_2, [Y_1, Y_2], [Y_1, [Y_1, Y_2]], [Y_2, [Y_1, Y_2]]),$$

show that the last two vector fields do not belong to $\mathcal{D}_1$ and that all the five vector fields are linearly independent at each point of the open set in M where $A \neq B$. ∎

We quote from [4] the following result:

THEOREM 11 (Bryant and Hsu). *Consider the rolling without slipping or twisting of an oriented surface Σ_1 along another oriented surface Σ_2, both embedded in Euclidean space. On the configuration space M, the corresponding rank 2 distribution $\mathcal{D}$ is of Cartan type provided that we restrict ourselves to the open set in M characterized by $A \neq B$ where A (resp B) is the Gauss curvature of Σ_1 (resp Σ_2). One can find a unique singular curve through each point in each direction tangent to the distribution $\mathcal{D}$ and that family of curves describes the motion of rolling Σ_2 along Σ_1 in a such way that each of the contact curves u_i traces out a geodesic in Σ_i, $i = 1, 2$.*

For a characterization of singular curves (also called non–regular in [4]) see [4, 13, 12, 14]. A nice presentation of the subject, in the context of subriemannian geometry, can also be seen in [11].

Observe that not all motions defining a rolling without slipping or twisting (of an oriented surface Σ_1 along another surface Σ_2, both embedded in an Euclidean space) give rise to two geodesics as contact curves. In fact, Sharpe (see [16]) proved, in a quite general context, that given an arbitrary smooth curve u_2 on Σ_2, there is a unique motion defining a rolling of Σ_1 along Σ_2 without slipping or twisting that has u_2 as contact curve. A simple explicit example can be obtained by considering a circumference $\mathcal{C}$ of radius r on the surface of a sphere of radius R, with $R > r$, and take a straight cone of basis $\mathcal{C}$ and vertex P, the cone being tangent to the sphere along $\mathcal{C}$; let Q be a point of $\mathcal{C}$ and one can think that the segment PQ lies on a horizontal plane. Now, if the rigid body (cone + sphere) starts to roll over the plane with the point P fixed, during this rolling motion the circle $\mathcal{C}$ rolls over another circle $\mathcal{C}_1$ (on the plane) of center P and radius equal to PQ. An elementary computation shows that $\mathcal{C}$ as a curve on the sphere and $\mathcal{C}_1$ as a curve on the plane have the same geodesic curvatures, both equal to $\frac{(R^2-r^2)^{\frac{1}{2}}}{rR}$. So, by Theorem 8 that rolling motion of the sphere along the horizontal plane is without slipping or twisting but the contact curves are not geodesics.

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