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On Complex Vector Fields Having Simply–Connected Orbits

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In this paper we study the classification polynomial vector fields with isolated singularities on $\mathbf{C}^2$ under the hypothesis that the non–singular orbits are simply–connected. Also we regard the case these orbits are *cylinders*. Regardless the natural relations with the study of complete vector fields on $\mathbf{C}^2$ which is carried out in [9], we give examples where the vector field is not complete. Our techniques are based on the geometry of the corresponding projective foliation.

Key Words: Simply–connected orbit, parabolic Riemann surface, holomorphic foliation.

1. INTRODUCTION

Let $X = (P, Q)\colon \mathbf{C}^2 \to \mathbf{C}^2$ be a polynomial vector field. For our purposes we may assume that $\gcd(P, Q) = 1$ and X has isolated singularities on $\mathbf{C}^2$. As it is well–known there exists a unique (singular holomorphic) foliation $\mathcal{F}_X$ on $\mathbf{CP}(2)$, with finite singular set $\operatorname{sing}(\mathcal{F}_{\mathrm{X}})$, and whose leaves $L \subset \mathbf{CP}(2)$ satisfy: $L^* := L \cap \mathbf{C}^2$ is an orbit of X for all L leaf of $\mathcal{F}_X$, except possibly for the case $L \subset \mathbf{P}^1_\infty = \mathbf{CP}(2) \setminus \mathbf{C}^2$. Motivated by well–known results on real foliations [3] as well as the study of complete vector fields [9] we consider the case the orbits of X are *planes* (isomorphic to $\mathbf{C}$) or *cylinders* (isomorphic to $\mathbf{C}^*$). One question we consider is related to the following example. By means of well–known constructions we may consider proper domains $U \subset \mathbf{C}^2$, $U \neq \mathbf{C}^2$, which are biholomorphic to $\mathbf{C}^2$. We may also proceed such construction in a way that for any horizontal line $L \subset \mathbf{C}^2$ the intersection $L \cap U$ is conformally equivalent to the disk $\mathbf{D} = \{z \in \mathbf{C}, |z| < 1\}$. Thus we obtain a foliation $\mathcal{F}_o$ on $\mathbf{C}^2$ whose leaves are disks. A first question is: *It is possible to construct such an* algebraic *foliation by disks on* $\mathbf{C}^2$*?* We prove that such a construction is not possible if we admit only generalized curves as singularities (appearing necessarily at $\mathbf{P}^1_\infty$) (cf. Theorem A). In what follows we give some examples illustrating our basic situations.

Example 1. *Foliations by Planes.*

(i) We begin with an example of a *non–complete* vector field with simply–connected orbits. Take $X = (1+xy)\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y}$. Clearly X is non–singular and the line $\{y=0\}$ is an orbit diffeomorphic to $\mathbf{C}$. We show that all the other orbits are diffeomorphic to $\mathbf{C}$; in fact, if L^* is the X–orbit corresponding to the fiber $\{ye^{xy}=c\}$, other than the orbit $\{y=0\}$, then we define $y(z) = y_o e^z$ and $x(z) = \frac{k-z}{y_o}e^{-z}$, for some suitable constant $k \in \mathbf{C}$. The map $\varphi(z) = (x(z), y(z)), z \in \mathbf{C}$ thus obtained is a parametrization of L^*, and therefore the orbit L^* is diffeomorphic to $\mathbf{C}$. On the other hand since since we have $\dot{y} = y^2$ on the flow equation for X, it follows that X is not complete on $\mathbf{C}^2$. Finally, we remark that $\mathcal{F}$ is holomorphically conjugated to a product foliation $\Gamma : \mathbf{C}\times\{y\}$, $y \in \mathbf{C}$, on $\mathbf{C}^2$. In fact, this conjugacy is given by the entire automorphism $\Psi \in \mathrm{Aut}(\mathbf{C}^2)$, $\Psi(\mathrm{x},\mathrm{y}) = (-\mathrm{x}e^{\mathrm{xy}}, \mathrm{y}e^{-\mathrm{xy}})$ and the vector field $\Psi_* X = e^{xy}$. *X is complete, nevertheless, it is not polynomial.* Later on we will return to the problem of trivializing the foliations $\mathcal{F}_X$.

(ii) We consider a polynomial *Poincaré–Dulac normal form* $\tilde{X} = (mx - y^m)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, $m \in \mathbf{N}$. Flow integration shows that $\tilde{X}$ has orbits diffeomorphic to $\mathbf{C}$ in $\mathbf{C}^2\setminus\{y=0\}$, but the one contained in $\{y=0\}$ is diffeomorphic to $\mathbf{C}^*$. $\tilde{X}$ has a Liouvillian first integral, $\tilde{F}(x,y) = ye^{x/y^m}$. Given any polynomial map $\Phi\colon \mathbf{C}^2 \to \mathbf{C}^2$, such that $\Phi\big|_{\mathbf{C}^2\setminus\{y=0\}}$ is an automorphism, and $\Phi(x,0) = 0, \forall x \in \mathbf{C}$, the vector field $X = \Phi_*\tilde{X}$ is polynomial and has simply–connected orbits (planes) on $\mathbf{C}^2$, provided that it is non–singular over the line $(y=0)$; for instance we can take $\Phi(x,y) = (1+xy^k, y)$, $k \in \mathbf{N}$, giving the vector field $X = \big((m-k)xy^k + m - y^m\big)\frac{\partial}{\partial x} + y^{k+1}\frac{\partial}{\partial y}$, which has the first integral $F(x,y) = ye^{\frac{1+xy^k}{y^m}}$.

(iii) Let $X(x,y) = \frac{\partial}{\partial x} + yb(x)\frac{\partial}{\partial y}$, where $b(x)$ is a non–zero polynomial on x. The 1–form $\omega = dy - yb(x)dx$ defines $\mathcal{F}_X$ on $\mathbf{C}^2$; we have $\omega = yd(\log y - B(x)) = \frac{d(ye^{B(x)})}{ye^{B(x)}}$ where $B(x) = \int_1^x b(z)dz \in \mathbf{C}[x]$. The orbits of X are planes parameterized by $\mathbf{C} \ni x \mapsto (x, y = C\cdot\exp B(x))$, where $C \in \mathbf{C}$. The line $\mathbf{P}^1_\infty = \mathbf{CP}(2)\backslash\mathbf{C}^2$ is invariant, and $\mathcal{F}_X$ is given in the coordinates (u,v) as above by: $u^{2+r}dv + v(-u^{r+1} + \tilde{b}(u))du$, where $r = \deg b(x)$ and $\tilde{b}(x) = x^r\cdot b(\frac{1}{x})$ satisfies $\tilde{b}(0) \neq 0$. Moreover $\mathrm{sing}(\mathcal{F}) = \{(\mathrm{u}=0, \mathrm{v}=0)\}$, which is non–dicritical; $\mathcal{F}_X$ has the entire first integral $F(x,y) = ye^{b(x)}$. As above we can use polynomial maps $\Phi \in \mathrm{Aut}(\mathbf{C}^2\setminus(\mathrm{y}=0))$ to generate other examples of the form $\Phi_* X$. For instance if we take $\Phi(x,y) = (xy^k + \alpha(y), y)$, then we will have the first integral $\tilde{F} = ye^{b(xy^k+\alpha(y))}$. We can complete the above example with:

Claim 2. Let $\mathcal{F}^*$ be an algebraic *Bernoulli foliation* on $\mathbf{C}^2$ say, $\mathcal{F}^* : \omega = p(x)dy - (y^2a(x) + yb(x))dx = 0$. Assume that every leaf of $\mathcal{F}^*$ on $\mathbf{C}^2$ is simply–connected. Then $\mathcal{F}^*$ is given by $dy - yb(x)dx = 0$.

Proof. As we will see (Lemma 33) $\operatorname{sing}(\mathcal{F}^*) = \phi$, it follows that $p(x)$ is constant and we can assume that $p(x) = 1$. Thus $\omega = dy - (y^2a(x) + yb(x))dx$. Let $B(x) = \int_1^x b(z)dz$. Then, $\frac{1}{y^2 \exp B(x)}.\omega = d\big(\frac{-1}{y \exp B(x)} + A(x)\big)$ where $A(x) = \int_1^x a(z)dz$. Thus the leaves of $\mathcal{F}$ are given by $y = \frac{\exp B(x)}{C - A(x)}$ where $C \in \mathbf{C}$. The function $A(x)$ is polynomial, therefore if it is not constant there exists at least one value of $x_C \in \mathbf{C}$ such that $A(x_C) - C = 0$, $\forall C \in \mathbf{C}$. This shows that the leaves of $\mathcal{F}^*$ are simply–connected if only if $a(x) = 0$ and therefore $\omega = dy - yb(x)dx$; in particular these leaves are planes. ▌

(iv) Let $X = \frac{\partial}{\partial x}$; the orbits of X are planes, $\mathbf{P}^1_\infty$ is $\mathcal{F}_X$–invariant and in the coordinates $(u = \frac{1}{x}, v = \frac{y}{x})$ $\mathcal{F}_X$ is given by $udv - vdu = 0$; therefore $q_o = (u = 0, v = 0)$ is a *dicritical singularity* (cf. §2.5).

(v) We take the polynomial vector field $X = \frac{\partial}{\partial x} + (a(x)y + b(x))\frac{\partial}{\partial y}$. Straightforward flow integration shows that X is complete and its flow maps ϕ_t are injective for all $t \in \mathbf{C}$, therefore X has simply–connected orbits diffeomorphic to $\mathbf{C}$; X is a particular case of a *Riccati differential equation* on $\mathbf{C}^2$ (see also Example 42). If $\omega = dy - (a(x)y + b(x))dx$ and $\eta = a(x)dx$, then (see [23]) $d(\omega/e^{\int \eta}) = 0$ so that we have the entire first integral $F = \frac{\omega}{e^{\int \eta}}$.

Example 3. *Foliations by cylinders.*
Now we give examples where X has orbits diffeomorphic to cylinders (i.e., diffeomorphic to $\mathbf{C}^*$).

(i) Once again we begin with an example of a *non–complete* vector field; the *Euler equation* $\{\dot{x} = x^2, \dot{y} = x + y\}$ gives an example with orbits diffeomorphic to cylinders. Indeed, integrating this equation we obtain $x = \frac{1}{c-t}$ and $\dot{y} - y + \frac{1}{t-c} = 0$. The solutions are therefore defined over the sets $\{t \in \mathbf{C}, t \neq c\} \simeq \mathbf{C}^*$. Though the orbits are cylinders, this equation is not in the list of Theorem **B** below; the reason is the existence of a *wild* singularity at $\mathbf{P}^1_\infty$ (see §6 (iv)).

(ii) Take $X = x\frac{\partial}{\partial x} + (a(x)y + b(x))\frac{\partial}{\partial y}$ polynomial; straightforward computations show that X is complete with periodic flow on $\mathbf{C}^2$. So the orbits are diffeomorphic to $\mathbf{C}^*$. Finally, if we take $\omega = i_X(dx \wedge dy) =$

$xdy - (a(x)y + b(x))dx$, and $\eta = \frac{(1+a(x))}{x}dx$, then $d\omega = \eta \wedge \omega$, the form η is therefore a closed rational 1–form with $d\omega = \eta \wedge \omega$, so that $d(\frac{\omega}{e^{\int \eta}}) = 0$ and we have a holomorphic first integral $F = \int \frac{\omega}{e^{\int \eta}}$ on $\mathbf{C}^2$. Notice that $(\eta)_\infty \cap \mathbf{C}^2 = (x = 0)$, which has only one irreducible component and $(\eta)_\infty \cap \mathbf{P}^1_\infty$ is a *dicritical singularity* (definition in §2.5). Compare this example with Theorem **B**(i).

(iii) $X = x\frac{\partial}{\partial x} + yb(x)\frac{\partial}{\partial y}$ has generic orbit diffeomorphic to $\mathbf{C}^*$; indeed X is complete with a periodic flow. As in Claim 2 above we can prove a kind of converse of this result:

Claim 4. Let $\mathcal{F}^*$ be a rational pull–back on $\mathbf{C}^2$ of an algebraic *Bernoulli foliation* $\mathcal{B} : p(x)dy - (y^2a(x) + yb(x))dx = 0$. Assume that the generic leaf of $\mathcal{F}^*$ is diffeomorphic to $\mathbf{C}^*$. Then there exists an affine change of coordinates in $\mathbf{C}^2$ such that $\mathcal{B}$ is given by one of the forms:

(a) $dy - (y^2 + yb(x))dx = 0$,
(b) $xdy - (y^2 + yb(x))dx = 0$,
(c) $xdy - yb(x)dx = 0$.

Remark 5. According to the above examples we may have non–complete polynomial vector fields with simply–connected orbits or with cylindrical orbits. This shows that the techniques used in [9] (e.g. the results of Borel and Kizuka, and the results of M. Suzuki for holomorphic flows, cf. [9]) do not apply to the present situation, though some of the geometrical ideas still persist. Our approach is therefore different, we use the theory of Nishino, Suzuki and H. Saito on meromorphic functions on $\mathbf{C}^2$ and use the solvability of the *holonomy* of the line at infinity in order to classify the foliation, as well as the theorems of Darboux and Zaidenberg–Lin. *To put in evidence this geometrical ideas is one of our goals and this is why we state our results for $\mathbf{C}^2$ instead of an affine algebraic manifold.* We refer to [5], [4], [6] and [7] for the notion of holonomy group, singular projetive holonomy group, resolution of singularities, dicritical singularity, saddle–node singularity and further information.

Definition 6. A singularity $p \in \text{sing}(\mathcal{F})$ is a *generalized curve* if $\pi^{-1}(p)$ contains no saddle–node singularity for $\tilde{\mathcal{F}}$ ([5]). We shall say that p is an *extended generalized curve* if $\pi^{-1}(p)$ exhibits no saddle–node at corners of D and no saddle–nodes with strong manifold transverse to D. In other words, the only saddle–nodes appear outside corners and have strong manifold contained in D.

The following Theorems **A** and **B** are the main results proved in this paper.

Theorem 7 (Theorem **A**). *Let X be a polynomial vector field on $\mathbf{C}^2$ such that each non–singular orbit is simply–connected. We have the following possibilities for $\mathcal{F}_X$:*

1. $\mathcal{F}_X$ is given by $dy = 0$ in some suitable affine chart $(x, y) \in \mathbf{C}^2$.

2. $\mathrm{sing}(\mathcal{F}_\mathrm{X}) \subset \mathbf{P}^1_\infty$ *is non–dicritical and the line $\mathbf{P}^1_\infty$ is invariant. If* $\mathrm{sing}(\mathcal{F}_\mathrm{X}) \cap \mathbf{P}^1_\infty$ *is a single extended generalized curve singularity then, after an affine change of coordinates, $\mathcal{F}_X$ has one of the forms:*

(i) $d\big(ye^{P(x,y)}\big) = 0$,

(ii) $d\big(ye^{\frac{1+y^k P(x,y)}{y^m}}\big) = 0$, $k, m \in \mathbf{N}$, *or*

(iii) $(W, P)^*(dy - (a(x)y + b(x))dx) = 0$ *for some polynomials* $a, b \in \mathbf{C}[x]$, $W, P \in \mathbf{C}[x, y]$, *where W is primitive. In particular, the simply–connected orbits of X are diffeomorphic to* $\mathbf{C}$.

Next we study the case where the generic orbit of X (see §2.2) is diffeomorphic to $\mathbf{C}^*$ (a cylinder). A result of M.Suzuki [27] states that if X is complete then it admits a meromorphic first integral in $\mathbf{C}^2$. We have a similar result for (not necessarily complete) polynomial vector fields. The list of possibilities is considerably larger:

Theorem 8 (Theorem **B**). *Let X be a polynomial vector field on $\mathbf{C}^2$ with generic orbit diffeomorphic to $\mathbf{C}^*$. We have the following possibilities for $\mathcal{F}_X$:*

1. $\mathcal{F}_X$ has a rational first integral f and in suitable affine coordinates we have:

(i)The origin is a dicritical singularity $\mathbf{C}^2$ *and* $f = \frac{x^p}{y^q}$, $f = \frac{x^p}{(ax^k - by^\ell)^q}$; *or* $f = \frac{(ax^k + by^\ell)^p}{(cx^k + dy^\ell)^q}$, *where* $p, q \in \mathbf{N}, k > 1, \ell > 1$, $a, b, c, d \in \mathbf{C}^*$.

(ii)There exists no dicritical singularity in $\mathbf{C}^2$, *and* $f = x^m[x^\ell y + P_\ell(x)]^n$, $m, n \in \mathbf{Z} - \{0\}, P_\ell \in \mathbf{C}[x]$ *of degree* $\leq \ell - 1$.

2. $\mathcal{F}$ has a meromorphic (but no rational) first integral F on $\mathbf{C}^2$. In this case $\mathbf{P}^1_\infty$ is invariant, and we have the following possibilities after an entire change of coordinates:

(i) X has no dicritical singularity on $\mathbf{C}^2$, and therefore there exists a primitive holomorphic first integral $F \colon \mathbf{C}^2 \to \overline{\mathbf{C}}$ *of the form* $F = x^m[x^\ell y + P_\ell(x)]^n) = 0$, $m, n \in \mathbf{Z} - \{0\}, P_\ell \in \mathbf{C}[x]$ *of degree* $\leq \ell - 1$.

(ii)The origin is a dicritical singularity of X. If it is a simple singularity, that is $DX(0,0)$ is non–singular, then there exists an entire automorphism $\psi \in \mathrm{Aut}(\mathbf{C}^2)$, *which linearizes X, that is,* $\psi_* X = nx\frac{\partial}{\partial x} + my\frac{\partial}{\partial y}$, $n, m \in \mathbf{N}$, *and therefore there exists only one singularity in* $\mathbf{C}^2$.

3. $\mathbf{P}^1_\infty$ *is invariant and contains no dicritical singularity. If each* $p \in \operatorname{sing}(\mathcal{F}) \cap \mathbf{P}^1_\infty$ *is a generalized curve, then* X *admits a Liouvillian first integral and we have the following possibilities:*

(i) $\mathcal{F}_X$ *is given by a closed rational and in suitable affine coordinates we have:*

(a) $\frac{dy}{y} + d\big(\frac{P(x,y)}{y^m}\big) = 0$;

(b) $\frac{dx}{x} + \lambda\frac{dy}{y} + d\big(\frac{P(x,y)}{x^n y^m}\big) = 0$,

(c) $\frac{dx}{x} + \lambda\frac{dx}{x-a} + d\big(\frac{P(x,y)}{x^n (x-a)^m}\big) = 0$*; where* $a, \lambda \in \mathbf{C}^*, n, m \in \mathbf{N}, P \in \mathbf{C}[x,y]$.

(ii)$\mathcal{F}_X$ *is a rational pull–back of a Bernoulli foliation of one of the forms:*

(d) $dy - (y^2 + yb(x))dx = 0$,

(e) $xdy - (y^2 + yb(x))dx = 0$,

(f) $xdy - yb(x)dx = 0$.

The proof of Theorems **A** and **B** can be (roughly) outlined as follows. First we remark that due to Riemman–Koebe's Uniformization Theorem for Riemann surfaces if the line at infinity $\mathbf{P}^1_\infty = \mathbf{CP}(2) \setminus \mathbf{C}^2$ is not $\mathcal{F}_X$–invariant then the leaves of $\mathcal{F}_X$ are contained in rational algebraic curves and therefore there exists some rational first integral. Using the results of Zaidenberg–Lin ([13]) and T. Nishino ([18], [19]) we can finish the description of this case. This same situation occurs in Theorem **A** if $\mathbf{P}^1_\infty$ contains some dicritical singularity; in the case of Theorem **B** we conclude the existence of a meromorphic first integral. Thus we may consider only the case $\mathbf{P}^1_\infty$ is $\mathcal{F}_X$–invariant and has no dicritical singularities. In this situation, we shall introduce the *singular projective holonomy groups* associated to a resolution of singularities of $\mathcal{F}_X\big|_{\mathbf{P}^1_\infty}$ (as in [6], [23]). Using the fact that the leaves contain no cycles (Theorem **A**) or arguments involving the existence of invariant transverse measures (cf. [7]) (Theorem **B**) and the density of hyperbolic fixed points for non–solvable subgroups of germs of one variable complex diffeomorphisms ([1], [29]) we conclude that these enriched holonomy groups must be solvable. Now [22], [23] apply in order to give us a Liouvillian first integral for $\mathcal{F}_X$ on $\mathbf{CP}(2)$. Theorems **A** and **B** follow from an analysis of the possibilities for such a first integral, which comes either from a closed rational 1–form or from a Bernoulli (suitable pull–back of a Riccati) differential equation on $\mathbf{C}^2$.

The structure of the paper is as follows: In §2 we introduce the main tools from Complex Analysis, Riemann Surfaces and Holomorphic Dynamics we shall use throughout the paper. §3 is dedicated to the preparation of the proof of Theorem **A** which is completed in §4. In §5 we give the proof of Theorem **B** and related facts. Finally, in §6 we give some remarks concerning the statements of Theorems A and B as well as some conjecture concerning the general case.

This paper is conceived to be accessible also to non–specialists on singular holomorphic foliations, thus we have included in §2 some useful material on the subject.

2. MAIN TOOLS

2.1. Algebraic curves and algebraic foliations

We study relations between the topology of an algebraic curve and its simplest polynomial expression. First we recall the following result of Zaidenberg–Lin.

THEOREM 9 (Zaidenberg–Lin, [13]). *Given a simply–connected algebraic affine curve $C \subset \mathbf{C}^2$, $C = \{P = 0\}, P \in \mathbf{C}[x, y]$; then there exists an algebraic automorphism $T\colon \mathbf{C}^2 \to \mathbf{C}^2$, such that either $P \circ T(x, y) = y^k p(x)$, or $P \circ T(x, y) = p(x) \prod_{j=1}^{r}(a_j x^k + b_j y^\ell), k, \ell \in \mathbf{N}, < k, \ell >= 1, p(x) \in \mathbf{C}[x]$. In particular $C \simeq \mathbf{C}$ (conformal equivalence) and P is quasi–homogeneous.*

A consequence of this result is that C has at most one singular point in $\mathbf{C}^2$. By an *affine change of coordinates on* $\mathbf{C}^2$ we mean a polynomial automorphism of $\mathbf{C}^2$, and by an *analytical change of coordinates* we mean an entire automorphism of $\mathbf{C}^2$. The following result is known as Stein Factorization Theorem:

THEOREM 10 (Stein, [10]). *Let R be a rational function on $\mathbf{CP}(2)$. Then there exists a rational function R_o on $\mathbf{CP}(2)$ such that:*

(i) The fibers $R_o^{-1}(\lambda), \lambda \in \mathbf{C} \cup \{\infty\}$ are connected.
(ii) R is constant along the fibers of R_o.
(iii) $R = T(R_o)$ for some rational map $T\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$.

Such a function R_o is called primitive function. Moreover we have:

(iv) If S is a rational function such that S is constant along the fibers of R_o then $S = A(R_o)$ for some rational map $A\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$. In particular S is primitive if, and only if, $A \in \mathbf{P}SL(2, \mathbf{C})$.

We shall call a *pencil* on $\mathbf{CP}(2)$, a family of algebraic curves $\{\lambda P - \mu Q = 0\}$, where P and Q are polynomials on $\mathbf{C}^2$, parameterized by $[\lambda\,;\,\mu] \in \mathbf{CP}(1)$. If $\mathcal{F}$ is an algebraic foliation on $\mathbf{CP}(2)$, with a rational first integral, then we have a natural associated pencil, (P, Q) where $R = P/Q$ is a rational first integral for $\mathcal{F}$. Using the results above and a theorem of Bertini we can prove that:

LEMMA 11 ([12]). *Let $\mathcal{F}$ be a pencil as above. There exists a pair (P_1, Q_1) with P_1, Q_1 polynomials on $\mathbf{C}^2$, such that:*

(i) P_1 and Q_1 have no non–constant common factors, and $R_1 = P_1/Q_1$ is also a first integral for $\mathcal{F}$.

(ii) The polynomial $\lambda P + \mu Q$ is irreducible for almost all $(\lambda, \mu) \in \mathbf{C}^2$.

(iii) $P/Q = \varphi(P_1/Q_1)$ for some rational map $\varphi : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$.

Such a pair will be called primitive and given any other primitive pair (P_2, Q_2) we have $P_1/Q_1 = \sigma(P_2/Q_2)$ for some Möebius map σ.

THEOREM 12 (Darboux, [12]). *Let X be a polynomial vector field on $\mathbf{C}^2$. Then X admits a rational first integral if, and only if, X has infinitely many algebraic orbits.*

THEOREM 13 (Dimca–Saito, [8]). *Let X be a polynomial vector field with isolated singularities on $\mathbf{C}^2$. Assume that for the 1–form $\omega = i_X(dV)$ (where $dV = dx \wedge dy$ is volume element), there exists a closed polynomial 1–form η on $\mathbf{C}^2$ with $d\omega = \eta \wedge \omega$. Then there exist polynomials $W, P \in \mathbf{C}[x, y]$ with W primitive, such that: $\omega = (W, P)^*(dy - (a(x)y + b(x))dx)$, for some $a(x), b(x) \in \mathbf{C}[x]$, $\eta = a(W)dW$.*

2.2. Potential Theory and foliations

We get some notions from [24] and [26]. A Riemann surface R is *hyperbolic* if given a point $p \in R$ it admits a (finite) Green function with pole at p. The basic example is the disk $\mathbf{D}$. Riemann–Koebe's Uniformization Theorem assures that hyperbolic Riemann surfaces have holomorphic universal covering equivalent to $\mathbf{D}$. A non–hyperbolic and non–compact Riemann surface is said to be *parabolic*. A (holomorphic singular one–dimensional) foliation $\mathcal{F}$ on a Stein surface M^2 is *parabolic* is all its leaves are parabolic Riemann surfaces [26].

THEOREM 14 (M. Suzuki, [26]). *Let $\mathcal{F}$ be a foliation on a Stein surface M^2. Assume that the set $\mathcal{P}(\mathcal{F}) = \{p \in M \setminus \mathrm{sing}(\mathcal{F}), \mathrm{L_p\,is\,parabolic}\}$ has positive logarithmic capacity. Then $\mathcal{F}$ is parabolic. If moreover the leaves of $\mathcal{F}$ are properly embedded then $\mathcal{F}$ admits a meromorphic first integral on M.*

DEFINITION 15. ([24]) Let $\mathcal{F}$ be a foliation on a Stein surface M^2 and R a Riemann surface. We say that the *generic leaf* of $\mathcal{F}$ is diffeomorphic to R if the set $\{p \in M, L_p \not\simeq R\}$ has zero transverse logarithmic capacity, where $\simeq$ means conformal equivalence. If X is a holomorphic vector field on M we say that the *generic orbit* of X is diffeomorphic to R if it is true for the generic leaf of the corresponding foliation $\mathcal{F}$ on M.

As a consequence of this result M.Suzuki has achieved the following conclusion for holomorphic flows on Stein surfaces:

THEOREM 16 ([27]). *Let X be a complete holomorphic vector field on a Stein surface M^2. The generic orbit of X is diffeomorphic to exactly one of the following Riemann surfaces:* $\mathbf{C}$, $\mathbf{C}^*$. *If the generic orbit of X is diffeomorphic to $\mathbf{C}^*$ then X admits a meromorphic first integral.*

Remark 17. Applying Theorem 16 above we conclude that if Z is a complete holomorphic vector field on $\mathbf{C}^2$ then the generic orbit of Z is diffeomorphic to exactly one of the following Riemann surfaces $\mathbf{C}, \mathbf{C}^*$. On the other hand, given X a non–complete polynomial vector field on $\mathbf{C}^2$, it is not clear whether there exists a well–defined generic conformal type for its orbits.

2.3. Meromorphic functions of type $\mathbf{C}$ and $\mathbf{C}^*$

Let now f be a *meromorphic function* on $\mathbf{C}^2$. We denote by $\sigma(f) \subset \mathbf{C}^2$ the set of indeterminacy points of f. Then we assume $\sigma(f)$ is a codimension ≥ 2 analytic subset, and since for any open subset $V \subset \overline{\mathbf{C}}$ the set $f^{-1}(V) \subset Dom(f) := \mathbf{C}^2 \setminus \sigma(f)$ is an open Stein subset [19], we conclude that Nishino's results above hold for the restriction $f\big|_{Dom(f)}$. Thus we may introduce the following terminology [28]: We say that f *is of type* $\mathbf{C}$, *(respectively of type* $\mathbf{C}^*$*)* if every prime surface of $f\big|_{Dom(f)}$ is analytically isomorphic to $\mathbf{C}$ respectively if almost every prime surface of $f\big|_{Dom(f)}$ is analytically isomorphic to $\mathbf{C}^*$. We conclude from above that f is of type $\mathbf{C}$ if, and only if, the set $\{c \in \mathbf{C}, (S_c = \{(x,y) \in Dom(f), f(x,y) - c = 0\}) \simeq \mathbf{C}\}$ has positive logarithmic capacity.

THEOREM 18 ([28]). *Let f be a meromorphic function of type $\mathbf{C}$ on $\mathbf{C}^2$. Then $Dom(f) = \mathbf{C}^2$ and there is an analytical change of coordinates on $\mathbf{C}^2$ such that $f = x$.*

THEOREM 19 (H. Saito and M. Suzuki, [21], [28]). *Let f be a primitive meromorphic function of type $\mathbf{C}^*$ on $\mathbf{C}^2$. There exists an analytical change of coordinates on $\mathbf{C}^2$ such that f writes $f = \xi(f_o)$ where $\xi(z)$ is a degree one rational function, and $f_o = x^m[x^\ell y + P_\ell(x)]^n$, $m, n \in \mathbf{Z} - \{0\}, \ell \in \mathbf{N}, P_\ell(x) \in \mathbf{C}[x]$ has degree $\leq \ell - 1$. In case $\ell = 0$ we define $P_\ell(x) = 0$.*

Combining Theorems 12 and 14 we obtain:

PROPOSITION 20. *Let $R\colon \mathbf{C}^2 \to \overline{\mathbf{C}}$ be a primitive rational function without indeterminacy points (i.e., $Dom(R) = \mathbf{C}^2$). If R is of type $\mathbf{C}^*$ then there exists an affine change of coordinates such that $R(x,y) = x^m[x^\ell y + P_\ell(x)]^n$, $m, n \in \mathbf{Z} - \{0\}, \ell \in \mathbf{N}, P_\ell(x) \in \mathbf{C}[x]$ has degree $\leq \ell - 1$.*

2.4. Fixed points of subgroups of $\mathrm{Diff}(\mathbf{C}, 0)$

We shall denote by $\mathrm{Diff}(\mathbf{C}, 0)$ the group of germs of complex diffeomorphisms f fixing $0 \in \mathbf{C}$, say $f(z) = \lambda z + \sum\limits_{j=1}^{+\infty} a_j z^j, \lambda \neq 0$. Let $G \subset \mathrm{Diff}(\mathbf{C}, 0)$ be a finitely generated subgroup.

Theorem 21 ([1, 17, 29]). *Suppose G is non–solvable.*

(i) The basin of attraction of (the pseudo–orbits of) G is an open neighborhood of the origin $0 \in \Omega$.

(ii) Either G has dense pseudo–orbits in some neighborhood V of the origin or there exists an invariant germ of analytic curve Γ (equivalent to $\mathrm{Im}(z^k) = 0$ for some $k \in \mathbf{N}$) where G has dense pseudo–orbits and also G has dense pseudo–orbits in each component of $V \setminus \Gamma$.

(iii) There exists a neighborhood $0 \in V \subset \Omega$, where G has a dense set of hyperbolic fixed points.

Thus, according to this result, a subgroup $G \subset \mathrm{Diff}(\mathbf{C}, 0)$ having discrete pseudo–orbits outside the origin or without dense fixed points close to the origin must be solvable.

3. SIMPLY–CONNECTED ORBITS OF POLYNOMIAL VECTOR FIELDS

Throughout the next sections of this paper X will denote a polynomial vector field on $\mathbf{C}^2$ with isolated singularities. We shall also denote by $\mathcal{F}_X$ the corresponding foliation on $\mathbf{CP}(2)$. For simplicity, by an *orbit* we shall mean a non–singular orbit, that is, an orbit of $X\big|_{\mathbf{C}^2 \setminus \mathrm{sing(X)}}$. By *for almost every* will mean *for all except for a zero logarithmic capacity subset.* Let $L^* \subset \mathbf{C}^2$ be a simply–connected orbit of X and denote by $L \supset L^*$ the leaf of $\mathcal{F}$ that contains L^*. If $\mathbf{P}^1_\infty$ is $\mathcal{F}$-invariant then we have $L = L^*$. We will take a regard at some properties of the orbits of a polynomial vector field X on $\mathbf{C}^2$, from the dynamical point of view. We introduce the complex projective plane $\mathbf{CP}(2)$ where $\mathbf{CP}(2) \setminus \mathbf{C}^2 = \mathbf{P}^1_\infty \simeq \mathbf{CP}(1)$.

Using the Maximum Modulus Principle one can easily prove the following well–known fact:

Lemma 22. *Let $\mathcal{F}$ be a foliation on $\mathbf{CP}(2)$ and L be a leaf of $\mathcal{F}$. Then, $\overline{L} \cap \mathbf{P}^1_\infty \neq \emptyset$.*

This first remark allows us to focus, in a certain sense, our argumentation to what occurs in a neighborhood of $\mathbf{P}^1_\infty$. We shall assume that $\mathbf{P}^1_\infty$ is $\mathcal{F}$–invariant where $\mathcal{F}$ is the foliation on $\mathbf{CP}(2)$ given on $\mathbf{C}^2$ by the orbits of X.

Example 23. We take $P(x,y) = p(x)\prod_{j=1}^{r}(a_j x^k - b_j y^\ell), k, \ell \in \mathbf{N}, < k, \ell >= 1, a_j, b_j \in \mathbf{C}^*$ or $P(x,y) = p(x)y^k, k$ where $p(x) \in \mathbf{C}[x]$. Then P is a quasi–homogeneous polynomial and $C := (P = 0)$ is an affine algebraic curve. If C is irreducible and if we have a simply–connected orbit L^* of a polynomial vector field X on $\mathbf{C}^2$ such that $L^* \subset \mathbf{C}$ then we have $L^* = C$ so that C is non–singular on $\mathbf{C}^2$. This implies that we have the following list of possibilities for $P(x,y)$: $ax + by$, $ax + by^k, a, b \in \mathbf{C}, a \neq 0, k \in \mathbf{N}$. We may therefore perform a polynomial change of coordinates on $\mathbf{C}^2$ and obtain $P(x,y) = y$. Summarizing we have:

Lemma 24. *Let X be a polynomial vector field on $\mathbf{C}^2$ having an algebraic simply–connected orbit, $L^* \subset \mathbf{C}^2$. We may find affine coordinates $(x,y) \in \mathbf{C}^2$ such that $L^* = (y = 0)$.*

Next we prove the following lemma.

Lemma 25. *Let X be a polynomial vector field on $\mathbf{C}^2$ having a simply–connected orbit, $L^* \subset \mathbf{C}^2$. Then we have two possible cases:*

(i) $L^* \simeq \mathbf{D} \Rightarrow \mathbf{P}^1_\infty$ is $\mathcal{F}$–invariant.
(ii) $L^* \simeq \mathbf{C} \Rightarrow$ either $\mathbf{P}^1_\infty$ is $\mathcal{F}$–invariant or $L = L^*, \overline{L} \simeq \overline{\mathbf{C}}$ and $\overline{L} \subset \mathbf{CP}(2)$ is a rational curve.

Proof. Assume that $\mathbf{P}^1_\infty$ is not $\mathcal{F}$–invariant. We first claim that $L \cap \mathbf{P}^1_\infty \neq \infty$ or $\emptyset \neq \overline{L} \setminus L \subset \text{sing}(\mathcal{F}) \cap \mathbf{P}^1_\infty$, in this last case $\sharp(\overline{L} \cap \mathbf{P}^1_\infty) = 1$ and $\overline{L}$ is a rational curve on $\mathbf{CP}(2)$. In order to prove it we begin by recalling that according to Lemma 22 above we have $\overline{L} \cap \mathbf{P}^1_\infty \neq \emptyset$. Take therefore $p \in \overline{L^*} \cap \mathbf{P}^1_\infty$. If $p \notin \text{sing}(\mathcal{F})$ then by the Flow–Box Theorem we have $L \cap \mathbf{P}^1_\infty \neq \emptyset$. Assume now that $\overline{L} \setminus L \subset \text{sing}(\mathcal{F}) \cap \mathbf{P}^1_\infty$ and take $p \in \overline{L} \setminus L$. Then by Remmert–Stein Theorem [11] $\overline{L}$ is an analytic curve in $\mathbf{CP}(2)$. By a Theorem of Chow [11] $\overline{L}$ is an algebraic curve. Now, $\overline{L} = L^* \cup (\overline{L} \cap \mathbf{P}^1_\infty)$ where $D = \overline{L} \cap \mathbf{P}^1_\infty$ is finite. Thus we have neither $L^* \simeq \mathbf{D}$ nor $\sharp D \geq 2$. This gives $L^* \simeq \mathbf{C}$ and $\overline{L} \simeq \overline{\mathbf{C}}$. Therefore $\overline{L}$ is rational (and by Bezout's Theorem it is a straight line provided that it is smooth at the point $\overline{L} \cap \mathbf{P}^1_\infty$).

This proves (ii). Notice that this already implies the following: $\mathbf{P}^1_\infty$ non–invariant $\Rightarrow L^*$ is not equivalent to $\mathbf{D}$ by Zaidenberg–Lin Theorem. ▌

Proposition 26 ([8]). *If L^* is simply–connected and L contains a separatrix $\Gamma \cap \mathbf{P}^1_\infty \neq \emptyset$ then $L^* \simeq \mathbf{C}$ and $\overline{L}$ is a rational curve.*

Corollary 27. *If X has infinitely many simply–connected (non–singular) orbits and $\mathbf{P}^1_\infty$ is not $\mathcal{F}_X$–invariant then X admits a rational first integral.*

Proof. Indeed, according to Lemma 25(ii) X has infinitely many algebraic orbits. By Darboux's Theorem (Theorem 12) X has a rational first integral. ▌

Corollary 28. *If almost every orbit of X is simply–connected and* $\operatorname{sing}(\mathcal{F}_{\mathrm{X}}) \cap \mathbf{P}^1_\infty$ *contains some dicritical singularity then X admits a rational first integral.*

Proof. Let $p \in \operatorname{sing}(\mathcal{F}_{\mathrm{X}}) \cap \mathbf{P}^1_\infty$ be a dicritical singularity. Given an orbit L^* containing a separatrix through p we have that $\overline{L} \simeq \mathbf{C} \cup \{\infty\} = \overline{\mathbf{C}}$, and therefore $\overline{L}$ is a rational curve. On the other hand p is dicritical so there is a *sector* of rational algebraic curves $\overline{L}$ as above. Darboux's Theorem gives a rational first integral for X. ∎

Proposition 29. *Assume that X has infinitely many simply–connected orbits and admits a rational first integral. Then there are affine coordinates $(x, y) \in \mathbf{C}^2$ such that X writes $X = \lambda \frac{\partial}{\partial x}$, $\lambda \in \mathbf{C}^*$.*

Proof. According to Stein Factorization Theorem (Theorem 10) we may take a primitive rational first integral say, $R = \frac{P}{Q}$, with $P, Q \in \mathbf{C}[x, y]$ relatively prime.

Claim 30. X has no dicritical singularity in $\mathbf{C}^2$.

Indeed, if there exists $(x_o, y_o) \in \mathbf{C}^2$ such that X has a dicritical singularity at (x_o, y_o) then $P(x_o, y_o) = Q(x_o, y_o) = 0$ and as a consequence, *all* the orbits L^* of X on $\mathbf{C}^2$ satisfy $(x_o, y_o) \in \overline{L^*} \subset \mathbf{C}^2$. This implies in particular that for those $L^* \simeq \mathbf{C}$ we have $\overline{L^*} \simeq \mathbf{C} \cup \{(x_o, y_o)\} \simeq \overline{\mathbf{C}}$, which is not possible for $\overline{\mathbf{C}}$ is compact and $\mathbf{C}^2$ contains no compact curve. This proves the claim.

The claim implies that $(P = 0) \cap (Q = 0) = \emptyset$. Now, applying Lemma 24 we may choose affine coordinates so that $P - Q = x$: indeed, the orbits of X in $\mathbf{C}^2$ are given by $\{\lambda P + \mu Q = 0\}$, $\lambda, \mu \in \mathbf{C}$. We may therefore assume that the leaf $P - Q = 0$ is generic and diffeomorphic to $\mathbf{C}$ (by a Theorem of Bertini [12] the set of singular fibers of a rational map is finite). Thus we have $\frac{P}{Q} = \frac{x+Q}{Q}$ and the leaves in $\mathbf{C}^2$ are given by $\lambda(x + Q) + \mu Q = 0$ and generically by $x + \nu Q = 0, \nu \in \mathbf{C}$. This gives the first integral $\frac{x}{Q}$. Since $\mathbf{C}^2$ contains no dicritical singularities we may write $Q(x, y) = 1 + xq(x, y), q \in \mathbf{C}[x, y]$ and the first integral $\frac{x}{1+xq(x,y)}$ · Now we consider a generic level say, $C = \{x(1 - q(x, y)) - 1 = 0\}$. Then C is smooth. Let $C_o = \{xy - 1 = 0\} \subset \mathbf{C}^2$. Then $C = \sigma^{-1}(C_o)$ for $\sigma \colon \mathbf{C}^2 \to \mathbf{C}^2$, $\sigma(x, y) = (x, 1 - q(x, y))$. Thus $\sigma\big|_C \colon C \to C_o$ is a branched finite covering. Clearly $C_o \simeq \mathbf{C}^* \in C_o$ (indeed, $\mathbf{C}^* \ni x \mapsto (x, \frac{1}{x})$ is a diffeomorphism).

Claim 31. If C is simply–connected then $q(x, y)$ is constant.

proof. By hypothesis $C \subset \mathbf{C}^2$ is simply–connected so that by Zaidenberg–Lin Theorem (Theorem 9) we have $C \simeq \mathbf{C}$. Thus we have a finite branched covering $\sigma_1 \colon \mathbf{C} \to \mathbf{C}^*$ given by $\sigma\big|_C \colon C \to C_o$. By Picard's Theorem σ_1 is rational (for it is finite map) so that it must be polynomial. But this implies that $\sigma_1(\mathbf{C}) = \mathbf{C}$ by the Fundamental Theorem of Algebra, except in the case σ_1 is constant. This implies that $1 - q(x,y)$ is constant and therefore $q(x,y)$ is constant. The claim is proved.

We have now $\frac{x}{1+xq(x,y)} = \frac{x}{1+\lambda x}$ for some $\lambda \in \mathbf{C}$ and the proposition is proved. ▮

PROPOSITION 32. *Assume that almost every orbit of X is simply–connected, $\mathbf{P}^1_\infty$ is $\mathcal{F}_X$–invariant,* $\operatorname{sing}(\mathcal{F}_X) \cap \mathbf{P}^1_\infty$ *is non–dicritical and contains only generalized curves. Then each singular projective holonomy group of $\mathcal{F}_X\big|_{\mathbf{P}^1_\infty}$ is solvable without dense fixed points.*

Proof. It is enough to observe that every simply–connected leaf contains only trivial cycles and therefore any singular projective holonomy group has only trivial fixed points over such a leaf. The conclusion follows from Theorem 21. ▮

4. PROOF OF THEOREM A

LEMMA 33. *If the orbits of X are simply–connected and $\mathbf{P}^1_\infty$ is $\mathcal{F}_X$–invariant then X is non–singular in $\mathbf{C}^2$.*

Proof. Suppose by contradiction that there exists $q_o \in \mathbf{C}^2 \cap \operatorname{sing}(X)$. According to [5] there exists a separatrix Γ for $\mathcal{F}$, $q_o \in \Gamma$. That is, Γ is a germ of analytic curve containing q_o such that $\Gamma \backslash q_o$ is contained in some orbit L^* of X. Fixed this orbit we denote as usual L the corresponding leaf of $\mathcal{F}_X$. We claim that L is contained in a rational curve. This rational curve intersects the line at the infinity at another singularity and therefore we will achieve a contradiction. Now, if $L \not\subset \mathbf{C}^2$ then according to Proposition 26 L is contained in a rational curve thus we may assume that $L = L^*$. We have therefore $L \simeq \mathbf{C}$ or $\mathbf{D}$. It is clear that $\mathcal{F}$ has no other singularity with a separatrix *contained* in L. Therefore using the techniques of [8] one can show, even in the case $L \simeq \mathbf{D}$, that $\overline{L} = L \cup \{q_o\} \simeq \overline{\mathbf{C}}$ and therefore L is contained in some rational curve on $\mathbf{CP}(2)$ as claimed (see Proposition 26 and [8]). Now, this implies $\overline{L} \cap \mathbf{P}^1_\infty \neq \phi$ by Bezout's Theorem, therefore $L = \overline{L} \backslash (\operatorname{sing}(\mathcal{F}) \cap \overline{\mathrm{L}}) \subset \overline{\mathrm{L}} \backslash \{\mathrm{q_o}, \mathrm{q_1}\}$ for some point $q_1 \neq q_o$. This contradicts the fact that $L \simeq \mathbf{C}$ and $\overline{L} \simeq \overline{\mathbf{C}}$. Hence X has no singularities in $\mathbf{C}^2$. ▮

LEMMA 34. *Suppose the generic non–singular orbits of X are simply–connected and X has no rational first integral. Then $\mathbf{P}^1_\infty$ is invariant,* $\operatorname{sing}(\mathcal{F}) \cap \mathbf{P}^1_\infty$ *is nondicritical and if $\tilde{q} \in \operatorname{sing}(\tilde{\mathcal{F}}) \cap \mathrm{D}$ is a saddle–node singularity then $\tilde{p}$ has its strong manifold contained in some component of the resolution divisor D of $\mathcal{F}\big|_{\mathbf{P}^1_\infty}$. In particular if $p \in \operatorname{sing}(\mathcal{F}) \cap \mathbf{P}^1_\infty$ is an extended generalized curve then it is indeed a generalized curve.*

Proof. From Corollaries 27 and 28 it only remains to prove that if there exists $\tilde{p} \in D$, which is a saddle–node singularity, this strong manifold is contained in some component of D. If the strong manifold of the saddle–node $\tilde{p}$ is transverse to the divisor D, then it is contained in some leaf $L \subset \mathbf{C}^2$ of $\mathcal{F}$. Now, since this leaf is simply–connected it follows that the holonomy of Γ (associated to this singularity $\tilde{p} \in \operatorname{sing}(\tilde{\mathcal{F}})$) is trivial. Therefore we have a saddle–node $\tilde{p}$ whose strong manifold has trivial holonomy. This is an absurd because according to [14] the holonomy of a strong manifold is of the form $h(z) = z + akz^{k+1} + \dots$, $a \in \mathbf{C}^*$, $k \in \mathbf{N}$. Thus, this type of singularity does not appear. ▮

LEMMA 35. *Let X be as in Lemma 34 above. There exists at most one singularity of $\mathcal{F}$ (lying over $\mathbf{P}^1_\infty$) which exhibits separatrices different from $\mathbf{P}^1_\infty$. Indeed there exists at most one separatrix transverse to $\mathbf{P}^1_\infty$.*

Proof. In fact, if there are $q_1 \neq q_o$, $\{q_o, q_1\} \subset \mathbf{P}^1_\infty$, admitting non–trivial separatrices, then these separatrices are *rational* algebraic curves and therefore either they coincide (what is not possible because as we have seen each separatrix intersects $\mathbf{P}^1_\infty$ only once) or they must intersect in an affine singularity of $\mathcal{F}$ which gives another contradiction. This proves that at most one singularity of $\mathcal{F}$ on $\mathbf{P}^1_\infty$ exhibits separatrices transverse to $\mathbf{P}^1_\infty$. Now we prove that the number of these separatrices is at most one. In fact if there are two of these separatrices, say Γ_1 and Γ_2 through a singularity q_o, then they are contained on parallel affine lines say, L_1 and L_2. Now, through the singularity q_o pass three lines $L_1, L_2, \mathbf{P}^1_\infty$ which are invariant by the foliation $\mathcal{F}$ and by performing a blow–up centered at q_o we obtain a an element $\gamma \in \pi_1(\mathbf{P}^1)$, where $\mathbf{P}^1 \simeq \mathbf{CP}(1)$ is the new projective line of the exceptional divisor of the blow–up, which comes from the commutator of the non–trivial elements γ_1 and $\gamma_2 \in \pi_1(\mathbf{P}^1 \backslash \operatorname{sing}(\widetilde{\mathcal{F}}))$ defined by the lines L_1, L_2. Now we make an important remark: Since the holonomy of $\mathbf{P}^1 \backslash \operatorname{sing}(\widetilde{\mathcal{F}})$ is solvable, and since $\pi_1(\mathbf{P}^1 \backslash \operatorname{sing}(\widetilde{\mathcal{F}}))$ is clearly non–solvable [1] it follows that the holonomy homomorphism $\pi\colon \pi_1(\mathbf{P}^1 \backslash \operatorname{sing}(\widetilde{\mathcal{F}})) \to \operatorname{Hol}(\widetilde{\mathcal{F}}, \mathbf{P}^1 \backslash \operatorname{sing}(\widetilde{\mathcal{F}}))$ is non–injective (already at level of commutators). Therefore we can find an element $[\gamma]$ belonging to the kernel of π. This element lifts into an element $[\widetilde{\gamma}]$ on any leaf $\tilde{\mathcal{L}}$ of $\widetilde{\mathcal{F}}$ (a *cycle*), sufficiently close to $\mathbf{P}^1$ (recall that since $\mathcal{F}$ is non–dicritical, $\mathbf{P}^1$ is $\widetilde{\mathcal{F}}$–invariant). Now we claim that $\tilde{\mathcal{L}}$ is not simply–connected. In fact, the element $[\widetilde{\gamma}] \in \pi_1(\widetilde{\mathcal{L}})$ cannot be (homotopic to) $[0]$ in a neighborhood of the divisor $\mathbf{P}^1$, therefore (since L_1, L_2 are invariant) the homotopy must take place in $\mathbf{C}^2 \backslash (L_1 \cup L_2)$, which is also impossible. This gives a contradiction. ▮

[1] In fact, $\pi_1(\mathbf{P}^1 \backslash \{p_1, \dots, p_{r+1}\})$ is generated by the loops γ_j around the points p_j, and there exists one sole relation $[\gamma_1] * \dots * [\gamma_{r+1}] = [0]$.

LEMMA 36. *Let $\mathcal{F}$ be as in Theorem* **A** *without rational first integral. Then, there exists a projective change of coordinates $T\colon \mathbf{CP}(2) \to \mathbf{CP}(2)$, $T(x,y) = (X,Y)$, such that in the new affine chart (X,Y) we have (i) $\mathbf{P}^1_\infty = (Y=0)$, (ii) If it exists, the sole separatrix C of $\mathcal{F}$ transverse to $\mathbf{P}^1_\infty$ is given by $(X=0)$*

Proof. We may assume that $\mathcal{F}$ exhibits exactly two invariant curves, $\mathbf{P}^1_\infty$ and a rational curve C. Now, given the line $\mathbf{P}^1_\infty$ and the rational curve C, using Zaidenberg–Lin Theorem (Theorem 9) we can choose new affine coordinates (u,v) such that in these coordinates we have $C = O_v, \mathbf{P}^1_\infty$ is still the line at the infinity and we have $L_1 \cap \mathbf{P}^1_\infty = (u=0, v=\infty)$. ▮

LEMMA 37. *Let X be as in Theorem* **A** *without rational first integral. Assume that $\mathcal{F}$, is given by a closed rational 1–form Ω on $\mathbf{CP}(2)$. There exists an affine change of coordinates such that $\mathcal{F}$ has one of the forms:*

(a)$\frac{dy}{y} + dP = 0$,
(b)$\frac{dy}{y} + d\big((1+y^kP(x,y))/y^m\big) = 0$, $k,m \in \mathbf{N}, P(x,y) \in \mathbf{C}[x,y]$.

Proof. First we notice that $(\Omega)_\infty \cap \mathbf{C}^2 \neq \emptyset$ (otherwise $\mathcal{F}$ has a polynomial first integral on $\mathbf{C}^2$). According to Lemma 36 above we can choose an affine chart $(x,y) \in \mathbf{C}^2$, so that we have $(\Omega)_\infty \cap \mathbf{C}^2 = (y=0)$. Now we recall according to the so called Integration Lemma (see [22]), a closed meromorphic 1–form Ω on $\mathbf{CP}(n)$ can be written as $\Omega\big|_{\mathbf{C}^n} = \sum\limits_1^r \lambda_j \frac{df_j}{f_j} + d\big(g/\prod\limits_1^r f_j^{n_j-1}\big)$, with polar divisor $(\Omega)_\infty \cap \mathbf{C}^n = \bigcup\{f_j=0\}$, λ_j is the residue of Ω along $\{f_j=0\}$, g is a polynomial as well as f_j, n_j is the order of $\{f_j=0\}$ as a pole of Ω. Therefore in our case we can write $\Omega = a\frac{dy}{y} + dR(x,y)$ for some rational function $R(x,y)$. We can assume that $a=1$ (if $a=0$ then $\mathcal{F}$ has a rational first integral). Therefore the leaves of $\mathcal{F}$ are the level curves of the function $F(x,y) = y\exp R(x,y)$. Now the fact that Ω has its eventual poles over $(y=0)$, and $\mathcal{F}$ has no affine singularities implies that either $R = P \in \mathbf{C}[x,y]$ or $R(x,y) = (1+y^kP(x,y))/y^m$ for some polynomial $P(x,y)$ and some $k,m \in \mathbf{N}$. This finishes the proof of the lemma. ▮

Proof (Proof of Theorem **A**). Let X be as in Theorem **A**. According to Lemma 34 (see also Corollaries 27 and 28) if $\mathbf{P}^1_\infty$ is not $\mathcal{F}$–invariant or $\mathrm{sing}(\mathcal{F}_{\mathrm{X}}) \cap \mathbf{P}^1_\infty$ contains some dicritical singularity, then X admits a rational first integral and in this case Proposition 29 implies that X is of the form $X = \lambda\frac{\partial}{\partial x}$ for some $\lambda \in \mathbf{C}^*$ and some choice of the affine coordinates. Assume now that X admits no rational first integral, but is given by some closed rational 1–form Ω. Then Lemma 37 implies that we may write $\Omega = \frac{dy}{y} + dP$ or $\Omega = \frac{dy}{y} + d(\frac{1+y^kP(x,y)}{y^m})$ for some choice of the affine coordinates. Finally we may assume that $\mathbf{P}^1_\infty$ is $\mathcal{F}_X$–invariant, $\operatorname{sing}(\mathcal{F}_{\mathrm{X}}) \cap \mathbf{P}^1_\infty$ is non–dicritical and contains (by hypothesis) only extended generalized curves. According to Lemma 34 the singularities $p \in \mathbf{P}^1_\infty \cap \operatorname{sing}(\mathcal{F}_{\mathrm{X}})$ are generalized curves. By Proposition 32 the singular projective holonomy groups appearing in the resolution of singularities of $\mathcal{F}_X\big|_{\mathbf{P}^1_\infty}$ are solvable without dense fixed points. The fact that all these groups are well–defined and solvable implies the following [23]: Given the dual 1–form $\omega = i_X(dx \wedge dy)$ we can find a closed rational 1–form η such that $d\omega = \eta \wedge \omega$, and η has polar divisor of order one, as in [22],[23]. By its turn, the existence of such η (and the hypothesis on $\operatorname{sing}(\mathcal{F}_{\mathrm{X}}) \cap \mathbf{P}^1_\infty$) imply that $\mathcal{F}_X$ is either given by a closed rational 1–form of it is a rational pull–back of a Bernoulli foliation $p(x)dy - (y^2 a(x) + y b(x))dx = 0$. Now, if $(\eta)_\infty \cap \mathbf{C}^2 = \emptyset$ then according to Dimca–Saito Theorem 13 we have (c) in Theorem **A** and we are done.

Lemma 38. *η is polynomial on $\mathbf{C}^2$.*

Proof. Suppose by contradiction that $(\eta)_\infty \cap \mathbf{C}^2 \neq \emptyset$, according to Lemma 36 we have $(\eta)_\infty = \mathbf{P}^1_\infty \cup L_1$ for some projective line L_1 transverse to $\mathbf{P}^1_\infty$ and with $L_1 \cap \mathbf{P}^1_\infty = \{p_o\} \subset \operatorname{sing}(\mathcal{F}_{\mathrm{X}})$. Choose a small bidisc U centered at p_o with local coordinates (u, v) such that $L_1 \cap U : (u = 0)$, $\mathbf{P}^1_\infty \cap U : (v = 0)$ and a small transverse disc $\Sigma : (v = v_o)$. Let $\gamma_o \subset \mathbf{P}^1_\infty$ be a loop $|v| = |v_o| \neq 0$. Since the holonomy of the leaf $L_\infty := \mathbf{P}^1_\infty \setminus \operatorname{sing}(\mathcal{F}_{\mathrm{X}}) = \mathbf{P}^1_\infty \setminus \{\mathrm{p_o}\} \simeq \mathbf{C}$ is trivial we conclude that given any $u \in \Sigma, u \neq 0$ the loop γ_o lifts to a loop $\gamma_u \subset L_u$ in the leaf $L_u \subset \mathbf{C}^2$ that contains the point (u, v_o). Since $\eta\big|_{L_u}$ is closed holomorphic and since $L_u \simeq \mathbf{C}$ necessarily we have $\int_{\gamma_u} \eta = \int_{\gamma_u} \eta\big|_{L_u} = 0$. On the other hand, clearly for any simple loop $\gamma \subset \mathbf{C}^2 \backslash L_1$ around L_1 we have $\int_\gamma \eta = \pm 2\pi\sqrt{-1}\lambda$ where $\lambda \neq 0$ is the residue of η along L_1. This gives a contradiction. ∎

Theorem **A** is now proved. ∎

5. POLYNOMIAL VECTOR FIELDS HAVING CYLINDRICAL ORBITS

In this section we prove Theorem **B**.

Lemma 39. *Suppose X has generic orbit diffeomorphic to $\mathbf{C}^*$ and $\mathbf{P}^1_\infty$ is not $\mathcal{F}_X$–invariant. Given any generic orbit L^* of X the closure $\overline{L} \subset \mathbf{CP}(2)$ is an algebraic curve.*

Proof. Since L^* is generic we may assume that there exists $p \in \overline{L^*} \cap \mathbf{P}^1_\infty$ such that $p \notin \operatorname{sing}(\mathcal{F}_{\mathrm{X}})$. Thus we may trivialize $\mathcal{F}_X$ locally around p and take a small transverse disk $p \in \Sigma \subset \mathbf{P}^1_\infty$. If $\sharp(L \cap \Sigma) = \infty$ then we have a contradiction for $L^* \simeq \mathbf{C}^*$. Thus $\sharp(L \cap \Sigma) < \infty$ and by the Transversal Uniformity Lemma [3] L^* is closed in $\mathbf{C}^2$ and $\overline{L} \backslash L \subset \operatorname{sing}(\mathcal{F}_{\mathrm{X}})$. This implies by Remmert–Stein Theorem and Chow Theorem [11] that $\overline{L}$ is algebraic of dimension one. It remains to study the case $\emptyset \neq (\overline{L^*} \cap \mathbf{P}^1_\infty) \setminus \operatorname{sing}(\mathcal{F}_{\mathrm{X}})$. In this case once again by the theorems of Remmert–Stein and Chow $\overline{L}$ is an algebraic curve. ▌

Remark 40. This same result holds if we assume that $L^* \simeq R \backslash F$, where R is a simply–connected Riemann surface, and $F \subset R$ is finite (always assuming that $\mathbf{P}^1_\infty$ is transverse to $\mathcal{F}$). On the other hand this result does not extend to the case $\mathbf{P}^1_\infty$ is $\mathcal{F}$–invariant: take $X = x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}, \lambda \in \mathbf{C} \backslash \mathbf{R}$. The orbits $L^* \not\subset (x.y = 0)$ are diffeomorphic to $\mathbf{C}$ and are not algebraic, nevertheless $\mathbf{P}^1_\infty$ is $\mathcal{F}$–invariant.

PROPOSITION 41. *Assume that X has generic orbit diffeomorphic to $\mathbf{C}^*$ and $\mathbf{P}^1_\infty$ is not $\mathcal{F}_X$–invariant. Then X admits a rational first integral. In particular, almost every orbit of X is diffeomorphic to $\mathbf{C}^*$.*

Proof. First we remark that by Lemma 39 and Darboux's Theorem (Theorem 12) there exists a rational first integral. Choose a primitive rational first integral $f: \mathbf{C}^2 \to \mathbf{CP}(1)$. Then, by hypothesis, for an infinite set of values $c \in \mathbf{C}$ the fiber $f_c = f^{-1}(c) \subset \mathbf{C}^2$ satisfies $f_c \simeq \mathbf{C}^*$. On the other hand f_c is an algebraic curve, whose degree does not depend on c and which (generically for c) meets $\mathbf{P}^1_\infty$ (which is not $\mathcal{F}$–invariant) at transverse intersection points. Thus Bezout's Theorem shows that this number of intersection points is fixed (for generic c). Moreover, using the fact that the f_c have (for generic c) a fixed conformal type (g, n) we conclude that, except for a zero logarithmic capacity set of exceptional values c, the affine curves $f_c \cap \mathbf{C}^2$ are diffeomorphic. This implies that almost every orbit of X is diffeomorphic to $\mathbf{C}^*$. ▌

Next we give an example of a foliation on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ with generic orbit diffeomorphic to the disk $\mathbf{D}$, however this is not the case for the corresponding foliation on $\mathbf{CP}(2)$.

EXAMPLE 42. Generic Riccati differential equations.
Let $X = (p(x), a(x)y^2 + b(x)y + c(x))$ with generic (polynomial) coefficients. X defines a *Riccati differential equation*; if $c \neq 0$ them mostly there exist no algebraic invariant curve other than the invariant vertical lines given by $\{p(x) = 0\}$. We know that $\mathcal{F}$ comes from a foliation $\overline{\mathcal{F}}$ on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ which is (outside the invariant vertical lines) a suspension of a group $G \subset \mathbf{P}SL(2, \mathbf{C})$ given by some representation $\pi_1(\overline{\mathbf{C}} \setminus \{p(x) = 0\}) \to \mathbf{P}SL(2, \mathbf{C})$. Mostly we have (for degree $p \geq 3$) the group G is free and without dense fixed points. Thus from one hand Picard's Theorem implies that the leaves on $(\overline{\mathbf{C}} \times \overline{\mathbf{C}}) \setminus \{p(x) = 0\}$ are covered by $\mathbf{D}$ and from other hand (using

the fact that the restriction of the first projection $(\overline{x}, \overline{y}) \mapsto \overline{x}$ to these leaves is a covering map) we conclude that the referred leaves are actually diffeomorphic to $\mathbf{D}$.

Finally, we remark that by Painleve's Theorem [20] any leaf L which is not a vertical line admits meromorphic parameterizations and this shows that L crosses a finite number of times the line $(y = \infty)$. Using now the canonical birational morphism $\sigma\colon \overline{\mathbf{C}} \times \overline{\mathbf{C}} \to \mathbf{CP}(2)$ we obtain that the foliation $\mathcal{F}$ induced by X on $\mathbf{CP}(2)$ has leaves L with the property that: $L^* \simeq \mathbf{D} \setminus D$ with $\sharp D = \infty$, and $\overline{L}$ is not algebraic, provided that L does not come from an invariant line.

Now we return to the situation of Theorem **B**.

Lemma 43. *Let X be given with generic orbit diffeomorphic to $\mathbf{C}^*$ and having a rational first integral f (which we may assume to be primitive). Then we have two possibilities after some affine change of coordinates:*

(i) The origin of $\mathbf{C}^2$ is a dicritical singularity and we have either $f = \frac{x^p}{y^q}$, $f = \frac{x^p}{(ax^k - by^\ell)^q}$; or $f = \frac{(ax^k+by^\ell)^p}{(cx^k+dy^\ell)^q}$, $p, q \in \mathbf{N}, k > 1, \ell > 1, a, b, c, d \in \mathbf{C}^*$.

(ii) There exists no dicritical singularity in $\mathbf{C}^2$ and $f = x^m[x^\ell y + P_\ell(x)]^n$, $m, n \in \mathbf{Z} - \{0\}, P_\ell \in \mathbf{C}[x]$ of degree $\leq \ell - 1$.

Proof. According to Theorem 10 and Lemma 11 we may choose a primitive pair (P, Q) for $\mathcal{F}$. We can assume that the polynomials P and Q are irreducible. Since we have an affine dicritical singularity, we have $(P = 0) \cap (Q = 0) \cap \mathbf{C}^2 \neq \phi$. The reducible affine curve $(P \cdot Q = 0)$ is simply–connected, because it is the union of two $\mathbf{C}^* \cup q_o$, through the (singular) point q_o. Applying Zaidenberg–Lin we obtain either $P.Q = p(x) \prod\limits_{j=1}^{r} (a_j x^k - b_j y^\ell)$, or $P.Q = y^\ell p(x)$, after an affine change of coordinates. Therefore we have the following possibilities:

1. $P \cdot Q = x^p \cdot y^q$, $p, q \in \mathbf{N}$;
2. $P \cdot Q = x^p \cdot (ax^k - by^\ell)^q$, $p, q \in \mathbf{N}, k > 1, \ell > 1, ab \in \mathbf{C}^*$;
3. $P \cdot Q = (ax^k + by^\ell)^p (cx^k + dy^\ell)^q$, $p, q \in \mathbf{N}, k > 1, \ell > 1, (ad - bc)d \in \mathbf{C}^*$.

Assume now that $\mathcal{F}$ has no dicritical singularity in $\mathbf{C}^2$. Then R is of type $\mathbf{C}^*$ and we may choose $R = \frac{P}{Q}$ as above such that $(P = 0) \cap (Q = 0) = \emptyset$. Now, according to Proposition 20 we may have $R = x^m[x^\ell y + P_\ell(x)]^n$, $m, n \in \mathbf{Z} - \{0\}, P_\ell \in \mathbf{C}[x]$ of degree $\leq \ell - 1$, for some affine polynomial coordinates $(x, y) \in \mathbf{C}^2$. ∎

Proposition 44. *Assume that the generic orbit of X is diffeomorphic to $\mathbf{C}^*$, $\mathbf{P}^1_\infty$ is $\mathcal{F}_X$–invariant and the resolution of singularities of $\mathcal{F}_X\big|_{\mathbf{P}^1_\infty}$*

exhibits no saddle–node singularity. If there is a dicritical singularity $p \in \mathrm{sing}(\mathcal{F}_{\mathrm{X}}) \cap \mathbf{P}^1_\infty$ *then* X *admits a meromorphic first integral.*

Proof. Suppose $p \in \mathrm{sing}(\mathcal{F}_{\mathrm{X}}) \cap \mathbf{P}^1_\infty$ is dicritical. Given a leaf $L = L^* \subset \mathbf{C}^2$, if L accumulates $\mathbf{P}^1_\infty$ only at p, then we claim that L is closed in $\mathbf{C}^2 \setminus \mathrm{sing}(\mathcal{F}_{\mathrm{X}})$. Indeed, we consider the resolution of the singularity p. This singularity gives a non–invariant projective line at least, say D_o. Since by hypothesis the singularities at the corners of the resolution divisor are not saddle–nodes, it follows that $\tilde{L}$ = strict transform of L through the resolution morphism, cuts D_o at some regular point $\tilde{q} \in D_o$. Denote by D the exceptional divisor of this resolution. Take a small transverse disk $\tilde{p} \in \tilde{\Sigma} \subset D_o$ and regard the intersections $\tilde{\Sigma} \cap \tilde{L}$. Since $\tilde{L}$ is a Riemann surface and $\tilde{L} \setminus D \simeq L \simeq \mathbf{C}^*$ it follows that $\sharp(\tilde{L} \cap D_o) < \infty$ and $\sharp(\tilde{L} \cap \tilde{\Sigma}) < \infty$. This implies by the Transversal Uniformity Lemma [3] that $\tilde{L}$ is closed outside $D \cup \mathrm{sing}(\tilde{\mathcal{F}}_{\mathrm{X}})$ and since the resolution morphism is a proper mapping it follows that L is closed in $\mathbf{C}^2 \setminus \mathrm{sing}(\mathcal{F}_{\mathrm{X}})$. Now we take any leaf $L \subset \mathbf{C}^2$ which accumulates some non–dicritical singularity $q \in sing(\mathcal{F}_X) \cap \mathbf{P}^1_\infty$, then either L also accumulates some regular point of $\mathbf{P}^1_\infty$ and therefore it accumulates the dicritical singularity p in which case L is closed in $\mathbf{C}^2 \setminus \mathrm{sing}(\mathcal{F}_{\mathrm{X}})$; or $\overline{L} \cap U = L \cup \{q\}$ for some neighborhood $U \ni q$ in $\mathbf{CP}(2)$. In this last case we also conclude that L is closed in $\mathbf{C}^2 \setminus \mathrm{sing}(\mathcal{F}_{\mathrm{X}})$. Since the leaves are parabolic and is a Stein manifold it follows from Suzuki's Theorem (Theorem 14) that X admits a meromorphic first integral on $\mathbf{C}^2$. ∎

LEMMA 45. *If* X *has generic orbit* $\mathbf{C}^*$ *and a non–constant meromorphic first integral, then we have two possibilities:*

(i)X *has no dicritical singularity over* $\mathbf{C}^2$*, and therefore there exists a primitive holomorphic first integral* $F: \mathbf{C}^2 \to \overline{\mathbf{C}}$*. In this case we may choose analytical coordinates on* $\mathbf{C}^2$ *such that* $F = x^m[x^\ell y + P_\ell(x)]^n) = 0$, $m, n \in \mathbf{Z} - \{0\}, P_\ell \in \mathbf{C}[x]$ *of degree* $\leq \ell - 1$.

(ii)X *has a dicritical singularity* $p_o \in \mathbf{C}^2$*. If* p_o *is simple, that is* $DX(p_o)$ *is non–singular, then there exists an entire automorphism* $\psi \in \mathrm{Aut}(\mathbf{C}^2)$*, which linearizes* X*, that is,* $\psi_* X = nx\frac{\partial}{\partial x} + my\frac{\partial}{\partial y}$ *,* $n, m \in \mathbf{N}$*. In particular, in this last case, there exists only one dicritical singularity over* $\mathbf{C}^2$*.*

Proof. Indeed, according to [18] and the remark we made at the beginning of §2.4 we may find a *primitive* meromorphic first integral f for $\mathcal{F}_X$ on $\mathbf{C}^2$. There are two cases to consider:

1. X has no dicritical singularity in $\mathbf{C}^2$. In this case the first integral f has no base points and defines therefore a holomorphic function $f: \mathbf{C}^2 \to \overline{\mathbf{C}}$. Therefore $\sigma(f) = \emptyset$ and $Dom(f) = \mathbf{C}^2$. This implies that f is of

type $\mathbf{C}^*$ on $\mathbf{C}^2$ and we may apply Theorem 19 in order to find analytical coordinates $(x,y) \in \mathbf{C}^2$ such that f writes $f(x,y) = x^m[x^\ell y + P_\ell(x)]^n) = 0$, $m,n \in \mathbf{Z} - \{0\}, P_\ell \in \mathbf{C}[x]$ of degree $\leq \ell - 1$. Since we have $X(f) = 0$ it follows that $X = u.(nx^{\ell+1}\frac{\partial}{\partial x} - [(m+\ell n)x^\ell y + mP_\ell(x) + nxP'_\ell(x)]\frac{\partial}{\partial y})$, for some unity $u \in \mathcal{O}^*(\mathbf{C}^2)$.

2. X has a dicritical singularity $p_o \in \mathbf{C}^2$. In this case it follows that $p_o \in \sigma(f)$, that is p_o is an indeterminacy point for f. This implies that there exists a positive logarithmic capacity set of values $c \in \mathbf{C}$ for which $\overline{\{(x,y) \in Dom(f),\, f(x,y) = c\}} \subset \mathbf{C}^2$ is diffeomorphic to $\mathbf{C}^* \cup \{p_o\} \simeq \mathbf{C}$. We can also assume that the p_o is the origin. Assume now that $DX(0)$ is non–singular. Then, since X admits a meromorphic first integral in a neighborhood of 0 it follows that X is locally linearizable at 0 [16]. We denote by U the *attraction basin* of 0 on $\mathbf{C}^2$. Let $\psi\colon V,0 \to W,0$ be a linearizing diffeomorphism for X. Then as it is plain to see, ψ extends to a biholomorphism $\psi\colon U \to \mathbf{C}^2$, and in particular U is diffeomorphic to $\mathbf{C}^2$. ∎

We want to prove that indeed:

LEMMA 46. $U = \mathbf{C}^2$.

Proof. Assume by contradiction that $U \neq \mathbf{C}^2$. Denote by ∂U the boundary of U. Since U is clearly open it follows that $\partial U = \overline{U} - U$, where $\overline{U}$ means the closure of U on $\mathbf{C}^2$. We study ∂U. Take a point $q_o \in \partial U$. Since the attraction basin of any singularity is open it follows that q_o does not belong to another attraction basin. Since by hypothesis *all orbits of X are diffeomorphic to* $\mathbf{C}^*$, it follows that in particular the orbit L_{q_o} passing through $q_o \in \partial U$ is diffeomorphic to $\mathbf{C}^*$.

We use this fact to prove that the orbit $L^*_{q_o}$ is *not* closed on $\mathbf{C}^2$. In fact, if L_{q_o} is closed then the orbits close to the former are also closed. This is a consequence of the fact that the existence of a meromorphic first integral implies that the holonomy group of $L^*_{q_o}$ is finite, and using the fact that this leaf is closed in $\mathbf{C}^2$ we may argue as in Reeb's Stability Theorem [3] and conclude that the leaves close to $L^*_{q_o}$ are closed on $\mathbf{C}^2$. Therefore these orbits cannot accumulate the singularity 0, but this contradicts the fact that some of them pass by U. Thus $L^*_{q_o}$ is not closed and must accumulate another singularity $q_1 \in \mathbf{C}^2 - \{0\}$. This singularity must be non–dicritical by obvious reasons. Since X has therefore a holomorphic first integral at q_1 it follows that it has a finite number of local separatrices at q_1, one of which is given by $L^*_{q_o}$. Now, since there are only finitely many singularities on $\mathbf{C}^2$ it follows that ∂U is a finite union of analytic curves $\overline{L}_{q_o}$. Therefore $U = \mathbf{C}^2-$(finite union of analytic curves on $\mathbf{C}^2$) - because a finite union of analytic curves is a thin set and hence does not disconnect $\mathbf{C}^2$-. This is not possible by reasons of homotopy (recall that U is diffeomorphic to $\mathbf{C}^2$, and is therefore simply connected). This proves Lemma 46 and finishes the proof of Lemma 45. ▌

Lemma 47. *Suppose X has generic orbit diffeomorphic to $\mathbf{C}^*$, $\mathbf{P}^1_\infty$ is $\mathcal{F}_X$–invariant,* $\operatorname{sing}(\mathcal{F}_{\mathrm{X}}) \cap \mathbf{P}^1_\infty$ *is non–dicritical and consists only of generalized curves. Then all the singular projective holonomy groups of $\mathcal{F}_X\big|_{\mathbf{P}^1_\infty}$ are solvable.*

Proof. According to [2] if a foliation $\mathcal{F}$ on $\mathbf{CP}(2)$ admits an entire invariant curve $\varphi \colon \mathbf{C} \to \mathbf{CP}(2)$ which is not algebraic then $\mathcal{F}$ admits non–trivial holonomy invariant measures ν with support contained in the closure $\overline{\varphi(\mathbf{C})} \subset \mathbf{CP}(2)$. This is certainly the case we have now. Therefore, we have two possibilities for a generic leaf $L \simeq \mathbf{C}^*$ of $\mathcal{F}_X$.

(1) The leaf L is not algebraic and there exists a holonomy invariant measure for $\mathcal{F}_X$ that is not supported only in $\mathbf{P}^1_\infty$.
(2) L accumulates only at the line at the infinity or has algebraic closure in $\mathbf{CP}(2)$.

Obviously, we may assume that case (2) does not occur. Using now [7] we conclude that the singular projective holonomy groups of $\mathcal{F}_X\big|_{\mathbf{P}^1_\infty}$ are solvable. ▌

Lemma 48. *If X admits no meromorphic first integral and every (non–singular) orbit of X is diffeomorphic to $\mathbf{C}^*$ then X has at most two algebraic invariant curves not contained in $\mathbf{P}^1_\infty$. Indeed, there exists an affine change of coordinates such that such that in the new (x, y) we one of the following equations for the algebraic orbits of X:*

(1) $\{y = 0\}$,
(2) $\{x \cdot y = 0\}$ *and*
(3) $\{x = 0\}, \{x - a = 0\}$, *for some* $a \in \mathbf{C}^*$.

Proof. The proof is similar to the proof of Lemma 35 and we just have to remark that for any leaf L^* of $\mathcal{F}^*$ on $\mathbf{C}^2$ we have by hypothesis $\pi_1(L^*) \approx \mathbf{Z}$ (where $\approx$ means group isomorphism) and on the other hand if $D \subset \mathbf{CP}(1)$ is a finite subset with $\sharp D \geq 3$ then $\pi_1(\mathbf{CP}(1) \setminus D)$ contains a subgroup isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ and therefore any holonomy homomorphism $\pi_1(\mathbf{CP}(1) \setminus D) \to \mathbf{Z}$ must have a non–trivial kernel. The proof follows now as in Lemma 35. ▌

Lemma 49. *Let X be given without meromorphic first integral, every orbit diffeomorphic to $\mathbf{C}^*$ and such that $\mathcal{F}$ is given by a closed rational 1–form. There exists an affine change of coordinates such that $\mathcal{F}$ is given by one of the forms:*

(1) $\frac{dy}{y} + d\big(\frac{P(x,y)}{y^m}\big) = 0$;
(2) $\frac{dx}{x} + \lambda\frac{dy}{y} + d\big(\frac{P(x,y)}{x^n y^m}\big) = 0$,
(3) $\frac{dx}{x} + \lambda\frac{dx}{x-a} + d\big(\frac{P(x,y)}{x^n (x-a)^m}\big) = 0$;

where $a, \lambda \in \mathbf{C}^*, n, m \in \mathbf{N}, P \in \mathbf{C}[x,y]$.

Proof. This lemma follows from Lemma 48 above as in the proof of Lemma 37. ▮

Proof (Proof of Theorem **B**). Items B.1, B.2 and B.3 (i) are now straightforward consequences of Proposition 41, Lemma 43, Proposition 44 and Lemma 45. Assume now that we are in case B.3 (but not in situation B.3 (i)) of the statement. According to Lemma 47 the singular projective holonomy groups in the resolution of singularities of $\mathcal{F}\big|_{\mathbf{P}^1_\infty}$ are well–defined and solvable so that (as in the proof of Theorem **A**) it follows from [23] that $\omega = i_X(dx \wedge dy)$ admits a closed (rational) logarithmic derivative say η. As in the proof of Theorem **A**, [23] and Claim 4 imply that $\mathcal{F}_X$ belongs to the list B.3 (ii). This ends the proof of Theorem **B**. ▮

Motivated by the results exposed in §§ 1, 2 one may ask for the classification of the polynomial vector fields on $\mathbf{C}^2$ whose orbits are algebroides and polynomial vector fields with simply connected orbits on $\mathbf{C}^n, n \geq 3$.

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