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Generalized Liénard Equations, Cyclicity and Hopf–Takens Bifurcations

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We investigate the bifurcation of small–amplitude limit cycles in generalized Liénard equations. We use the simplicity of the Liénard family, to illustrate the advantages of the approach based on Bautin ideals. Essentially, this Bautin ideal is generated by the so–called Lyapunov quantities, that are computed for generalized Liénard equations and used to detect the presence of a Hopf–Takens bifurcation. Furthermore, the cyclicity is computed exactly.

Key Words: limit cycles, cyclicity, centers, Hopf bifurcations, Liénard equations, Lyapunov quantities.

1. INTRODUCTION

Computation of Lyapunov quantities is the key step for solving the stability problem of a plane system which is a perturbation of a linear focus at the origin. Recently many contributions have been devoted to analysing Bautin's approach to the local Hilbert's 16th problem. This involves algebraic techniques such as the Bautin ideal and bifurcation techniques such as normal forms.

There are many interests to consider Liénard equations (see for instance [6, 8, 7]). Furthermore, the (generalized) Liénard equation provides the simplest settings for calculations and applications of the Bautin ideals. This article first recalls the results obtained for the classical Liénard equations concerning the Bautin ideal, the cyclicity and the presence of Hopf–Takens bifurcations. To the best of our knowledge, these three aspects have not been jointly discussed previously. But perhaps the most original contribution of the article is in the second part where we discuss the same topics for generalized Liénard equations.

In article [7], the cyclicity was computed following the Françoise–Yomdin approach based on a recurrency relation for the coefficients of the return mapping and a complex analysis method (Bernstein's inequality) for the classical Liénard equations. In this article, we used instead techniques of R. Roussarie [13] based on a special system of generators for the Bautin ideal which provides a lower bound for the cyclicity. Finding this lower bound is closely related to the existence of Hopf–Takens bifurcations. This last approach is developed latter for generalized Liénard equations. It would be certainly interesting to discuss also the Françoise–Yomdin approach for generalized Liénard equations in the future.

The paper is organised as follows. In Section 2.1, we first recall the definition of a generic Hopf–Takens bifurcation (degenerate or not). In Section 2.2, we recall quickly some techniques involving the Bautin ideal [13], which are used to compute the cyclicity near centers. In Section 2.3, we recall the definition of Lyapunov quantities [4]. Furthermore, we give here a characterisation of the generic Hopf–Takens bifurcation in terms of Lyapunov quantities [4]. In Section 3.1, Lyapunov quantities are computed in classical Liénard equations to investigate the presence of a generic Hopf–Takens bifurcation and to compute the cyclicity in this case (Section 3.2). Finally, we consider the generalized Liénard equation. Such a system can be transformed into a classical Liénard equation, up to a positive factor (Section 4.1). From this reduction it is possible to derive structure formulas for the Lyapunov quantities in generalized Liénard equations (Section 4.2). Then the presence of a generic Hopf–Takens bifurcation and the cyclicity is studied as in case of the classical Liénard equations (Section 4.3).

2. PRELIMINARY DEFINITIONS AND PROPERTIES

2.1. Standard generic Hopf–Takens bifurcation

The Hopf bifurcation is a very well–known generic and structurally stable 1–parameter bifurcation. It unfolds a non–degenerate singularity of codimension 1 and it gives birth to a limit cycle. A generalization giving rise to more than 1 limit cycle and related multiple limit cycle bifurcations has been studied in [15]. The generic l–parameter structurally stable bifurcation is nowadays called Hopf–Takens bifurcation of codimension l; F. Takens defined two standard models:

Definition 1. The *standard generic generalized Hopf bifurcation* or *standard generic Hopf–Takens bifurcations of codimension l* is given by the normal form:

$$X_{\pm}^{(l)} = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) \pm \left(\left(x^2+y^2\right)^l + a_{l-1}\left(x^2+y^2\right)^{l-1} + \ldots + a_0\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right).$$

We denote by $\mathcal{D}_{\pm}^{(l)}$ the bifurcation diagram of the limit cycles of $X_{\pm}^{(l)}$ in function of the parameter $(a_0, \ldots, a_{l-1})$ in $\mathbb{R}^l$.

This family is a weak topological versal deformation of a certain type of elliptic singularity, sometimes called a weak focus of order l. In close analogy with singularity theory of functions, we consider more generally an unfolding $(X_\lambda), \lambda \in \Lambda$ defined as a C^∞ (or C^ω) family of plane vector fields so that there is a submanifold $\Lambda' \in \Lambda$ of the parameter space where $(X_\lambda), \lambda \in \Lambda'$ is topologically conjugated to one of the two models. In this article, we will consider as particular examples of such unfoldings some families of plane vector fields associated with second-order differential equations. In such cases, we will write shortly that the family contains a generic Hopf–Takens bifurcation.

Definition 2. Let $(X_\lambda)_\lambda$ be a C^∞ (or C^ω) family of planar vector fields such that the origin $e = (0,0) \in \mathbb{R}^2$ is a non-degenerate elliptic singularity. Then we say that

1. $(X_\lambda)_{\lambda\sim\lambda_0}$ *is a generic Hopf–Takens bifurcation of codimension l* if and only if the bifurcation diagram of the limit cycles of X_λ in a (fixed) neighbourhood of e in function of $\lambda \in W$ is topologically equivalent to $\mathcal{D}_{\pm}^{(l)}$ (where W is a neighbourhood of λ_0 in $\mathbb{R}^l$).

2. $(X_\lambda)_{\lambda\sim\lambda_0}$ *contains a generic Hopf–Takens bifurcation of codimension l* if and only if there exists a local submersion at λ_0,

$$h : (\mathbb{R}^p, \lambda_0) \to \left(\mathbb{R}^l, \mu_0\right), \mu_0 = h(\lambda_0)$$

such that $\left(X_{h(\lambda)}\right)_{\lambda\sim\lambda_0}$ is a generic Hopf–Takens bifurcation of codimension l.

3. a C^∞ (or C^ω) family of autonomous second order differential equations $(E_\lambda)_\lambda$ *contains a generic Hopf–Takens bifurcation of codimension l* if and only if the associated family of planar vector fields does.

2.2. Bautin Ideal

In this section we quickly recall a technique involving the Bautin ideal to bound the maximum number of limit cycles that can arise after small perturbations of the parameter [13].

Let $(X_\lambda)_\lambda$ be an analytic family of plane vector fields of the form:

$$X_\lambda = (d(\lambda)x - y + f(x,y,\lambda))\frac{\partial}{\partial x} + (x + d(\lambda)y + g(x,y,\lambda))\frac{\partial}{\partial y}, \quad (1)$$

where $f, g = O\left(\|(x,y)\|^2\right)$. Let σ be an analytic section transverse to X_{λ_0}, endowed with an analytic regular parameter $s \in \mathbb{R}$, such that the elliptic point $e = (0,0)$ corresponds to $s = 0$.

Denote the Poincaré-map of first return in σ by P_λ and the associated displacement function by $\delta_\lambda = P_\lambda - Id$. In this way, limit cycles of X_λ correspond to isolated zeroes of δ_λ.

If the vector field X_{λ_0} is of *center type,* i.e. the displacement function is identically zero:

$$\delta_{\lambda_0} \equiv 0,$$

then the elliptic point is contained in a disc full of periodic orbits. In this case an important tool to study the bifurcation set is the Bautin Ideal [6, 13]. Not only does it directly serve to define the set of parameter values near λ_0 at which we have centers, it can also be used in calculating an upperbound for the cyclicity $Cycl(X_\lambda, e)$. Recall that this number represents the maximum number of limit cycles γ that can perturb from e; more precisely, it is defined by

$$\mathrm{Cycl}(X_\lambda, e) = \limsup_{\lambda_1 \to \lambda_0, \gamma_1 \to e} \{\text{number of limit cycles } \gamma_1 \text{ of } X_{\lambda_1}\},$$

where the convergence $\gamma_1 \to e$ is in the sense of the Hausdorff metric on the set of non-empty compact subsets of the phase plane.

Let us recall the definition of the Bautin Ideal. Write the displacement function δ_λ as an expansion in terms of s :

$$\delta_\lambda(s) = \sum_{i=1}^{\infty} \alpha_i(\lambda) s^i, \qquad s \to 0, \lambda \sim \lambda_0. \quad (2)$$

Consider the local ring $\mathcal{O}_{\lambda_0}$ of analytic function germs at λ_0, with unique maximal ideal, which we denote by $\mathcal{M}$. We will use twiddles$^\sim$ to denote the germ of a certain analytic function in λ_0. The Bautin ideal is defined as the ideal generated by the germs of the analytic functions $\widetilde{\alpha}_j, j \in \mathbb{N}_1$. Remark that the local ring $\mathcal{O}_{\lambda_0}$ is Noetherian (cfr. [12]); therefore this ideal is finitely generated: there exists a number $M \in \mathbb{N}$ such that

$$\mathcal{I} = (\widetilde{\alpha}_j : j \in \mathbb{N}_1) = (\widetilde{\alpha}_j : j = 1, \ldots, M).$$

Let us recall a number of properties whose proofs can be found in [13]. The Bautin ideal does not depend on the chosen transverse section σ, neither on the chosen (regular) parametrization of this section σ. The Bautin ideal is generated by the odd coefficients in expansion (2) and more precisely:

$$\forall p \geq 1 : \widetilde{\alpha}_{2p} \in (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_{2p-1}) . \tag{3}$$

There always exists *a minimal system of generators* $\{\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_l\}$ for $\mathcal{I}$, i.e. a system of generators for $\mathcal{I}$ such that the set $\{\widetilde{\varphi}_1 \mathrm{mod} \mathcal{MI}, \ldots, \widetilde{\varphi}_l \mathrm{mod} \mathcal{MI}\}$ forms a basis of the real vector space $\mathcal{I}/\mathcal{MI}$. The existence of such a minimal system of generators is ensured by *Nakayama's lemma* (which holds in a local ring, see [12]):

$$\mathcal{I} = \mathcal{I}' + \mathcal{MI} \Longrightarrow \mathcal{I} = \mathcal{I}'.$$

If $\{\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_l\}$ is a set of generators for $\mathcal{I}$ such that the mapping

$$\varphi : (\mathbb{R}^p, \lambda_0) \to \mathbb{R}^l : \lambda \mapsto (\varphi_1(\lambda), \ldots, \varphi_l(\lambda))$$

is a submersion at λ_0, then it can be easily checked that $\{\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_l\}$ is a *minimal* set of generators for $\mathcal{I}$. The displacement function can always be expanded in this system of generators: there exist analytic functions $h_j, j = 1, \ldots l$ defined on a neighborhood of λ_0 such that on this neighborhood we have:

$$\delta(s, \lambda) = \sum_{j=1}^{m} \varphi_j(\lambda) h_j(s, \lambda) .$$

The analytic functions H_j, defined by $H_j := h_j(\cdot, \lambda_0)$, are called *the factor functions* associated to the minimal system of generators $\{\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_l\}$. These functions $H_j, j = 1, \ldots, l$ are independent over $\mathbb{R}$ (in the sense of germs). Furthermore, the minimal system can be chosen such that the associated factor functions have a strictly increasing order at $s = 0$:

$$\mathrm{order} H_1(0) < \mathrm{order} H_2(0) < \ldots < \mathrm{order} H_l(0) .$$

We will refer to such a system of generators as *a minimal system of generators adapted to* $s = 0$.

In [13] it is proven that

$$\mathrm{Cycl}(X_{\lambda_0}, e) \leq \frac{\mathrm{order} H_l(0) - 1}{2}.$$

The Bautin ideal is called *regular* if the mapping $\lambda \mapsto (\varphi_1(\lambda), \ldots, \varphi_l(\lambda))$ is a submersion at λ_0. In that case, it is proven in [13] that $l - 1$ is a lower bound for the cyclicity:

$$\mathrm{Cycl}(X_{\lambda_0}, e) \geq l - 1. \tag{4}$$

In the rest of the paper, we don't use the twiddles any more to make a distinction between the germ of an analytic function at λ_0 and its representative. When we deal with ideals, we will always work in the local ring $\mathcal{O}_{\lambda_0}$ of analytic function germs at λ_0, without mentioning it.

2.3. Lyapunov quantities

We first recall the definition of Lyapunov quantities in Lemma 3. Then we recall from [4] a relation between Lyapunov coefficients and the coefficients of the displacement mapping; therefore these coefficients are also called Lyapunov–Poincaré quantities. From [4], we can define a generic Hopf–Takens bifurcation in terms of the Lyapunov quantities (it is classically defined in terms of normal forms as it was recalled in the introduction, cfr. [15]).

2.3.1. Definition and properties

LEMMA 3 (see [14]). *Suppose a family of vector fields as given in* (1) *. Then there exists a formal power series F_λ,*

$$F_\lambda(x,y) = \frac{1}{2}\left(x^2+y^2\right) + \sum_{j=3}^{\infty} F_j(x,y,\lambda),$$

where F_j is a homogenous polynomial of degree j in x and y,

$$F_j(x,y,\lambda) = \sum_{i=0}^{j} f_{ij}(\lambda)\, x^i y^{j-i}$$

and there exist coefficients $V_i(\lambda)$ such that:

$$X_\lambda . F_\lambda(x,y) = \sum_{i=0}^{\infty} V_i(\lambda)\left(x^2+y^2\right)^{i+1}. \tag{5}$$

Moreover, if F_λ and V_i $(i \in \mathbb{N})$ are solutions satisfying (5)*, then the functions f_{ij} and V_i are C^∞ in λ.*

Such functions $\{V_i(\lambda) : i \in \mathbb{N}\}$ are called *Lyapunov quantities* (or *focal values*) of the vector field X_λ. Let us remark that the *moreover* part in Lemma 3 can be generalized in the following sense: if the family given in (1) is of class C^γ $(\gamma \in \mathbb{N} \cup \{\infty, \omega\})$ in λ, then also the functions f_{ij} and V_i are C^γ.

We call the ideal generated by the germs of the Lyapunov quantities V_i in λ_0 the *Lyapunov Ideal* and denote it by $\mathcal{L} = (\tilde{V}_i : i \in \mathbb{N})$. The following proposition explains why the coefficients of the displacement mapping can be considered as Lyapunov quantities. The following proposition was proven in [4]:

PROPOSITION 4 (see [4]). *Consider a C^ω family of vector fields $(X_\lambda)_\lambda$ as given in* (1) .

1. Then the Bautin ideal and the Lyapunov ideal coincide, i.e. $\mathcal{I} = \mathcal{L}$.

2. Moreover, the coefficients $\alpha_{2j+1}, j \in \mathbb{N}$ in (2) *and the Lyapunov quantities are related by*

$$\begin{cases} \alpha_1(\lambda) & = V_0(\lambda)\left(\frac{e^{2\pi V_0(\lambda)}-1}{V_0(\lambda)}\right) \text{ and } \forall j \geq 1, \\ \alpha_{2j+1}(\lambda) & = V_j(\lambda)\, e^{2\pi V_0(\lambda)}\left(\frac{e^{4j\pi V_0(\lambda)}-1}{2jV_0(\lambda)}\right) mod\,(V_0, \ldots, V_{j-1}). \end{cases} \tag{6}$$

3. In terms of ideals in the ring of C^ω function germs, we have:

$$(\tilde{V}_0, \tilde{V}_1, \ldots, \tilde{V}_N) = (\tilde{\alpha}_1, \tilde{\alpha}_3, \ldots, \tilde{\alpha}_{2N+1}). \tag{7}$$

As a consequence, the Lyapunov quantities are uniquely defined in the following sense: if $W_i, i \in \mathbb{N}$ also is a set of Lyapunov quantities, then for each $k \in \mathbb{N}$, there exists a $c_k > 0$ such that

$$W_k = c_k V_k \text{mod}\,(V_0, \ldots, V_{k-1}). \tag{8}$$

Hence, the order and the sign of the first non identical vanishing quantity is uniquely determined. Another consequence is that the Lyapunov quantities are 'invariant' under coordinate transformations, in the same sense as above: let $\{V_i : i \in \mathbb{N}\}$ be a set of Lyapunov quantities for the family $(X_\lambda)_\lambda$ and let $\{W_i : i \in \mathbb{N}\}$ be a set of Lyapunov quantities for the family $(Y_\lambda)_\lambda$, where (Y_λ) is the family of vector fields obtained after a coordinate transformation, then these Lyapunov quantities are related by (8) . In the rest of the paper, we will refer to any set of functions $\{W_k : k \in \mathbb{N}\}$ satisfying property (8) as a set of Lyapunov quantities.

2.3.2. Generic Hopf–Takens bifurcations

Here we recall a slightly generalized result from [4] that expresses the presence of a generic Hopf–Takens bifurcation in terms of the Lyapunov quantities:

THEOREM 5 (Generic Hopf–Takens bifurcation, [4]). *Let* $(X_\lambda)_\lambda$ *be a* C^∞ *family of planar vector fields as given in* (1) *with a given set of Lyapunov quantities* $V_i, 0 \leq i \leq k+l$. *Suppose that*

$$\begin{cases} V_j(\lambda) \equiv 0 & \forall 0 \leq j \leq k, \\ V_{k+j}(\lambda_0) = 0, & \forall 0 \leq j \leq l-1, \\ V_{k+l}(\lambda_0) \neq 0. \end{cases}$$

1.Then, the displacement mapping can be written as

$$\delta(s,\lambda) = s^{2k+1}\left(\sum_{j=0}^{l} c_j V_{k+j}(\lambda)\, s^{2j} + O\left(s^{2l+1}\right)\right), s \to 0 \tag{9}$$

where $c_j > 0, \forall, = 0, \ldots, l$. *In particular,*

$$Cycl\,(X_\lambda, (e, (v_0, 0))) \leq l$$

2.If, furthermore, the mapping $V := (V_k, V_{k+1}, \ldots, V_{k+l-1})$ *is a submersion at* λ_0, *then the family* $(X_\lambda)_\lambda$ *contains a generic Hopf–Takens bifurcation of codimension* l *in the origin. Moreover, the sign of its type* $X_\pm^{(l)}$ *is given by the sign of* $V_{k+l}(\lambda_0)$.

Proof. 1. Using Proposition 4, we find expansion (9) for the displacement mapping δ. By a division-derivation algorithm based on Rolle's theorem, we then find that l is an upperbound for the cyclicity [13]. 2. This result is proven in [4]. ∎

2.3.3. Degenerate generic Hopf–Takens bifurcations and general Bautin ideals

Suppose that $\lambda = (\nu, \varepsilon)$, where ε is small, and that the Lyapunov quantities all are divisible by some power of ε; more precisely we now assume that $\{V_i : i \in \mathbb{N}\}$ is a set of Lyapunov quantities and that k is the biggest integer such that $\forall i \in \mathbb{N}$:

$$V_i(\nu,\varepsilon) = \varepsilon^k \bar{V}_i(\nu) + O\left(\varepsilon^{k+1}\right), \varepsilon \to 0.$$

Then the Bautin ideal is generated by ε^k; moreover, the displacement mapping can be divided by ε^k to define the so-called reduced displacement mapping $\bar{\delta}$:

$$\delta(s,\nu,\varepsilon) = \varepsilon^k \bar{\delta}(s,\nu,\varepsilon).$$

We will refer to the functions $\{\bar{V}_i : i \in \mathbb{N}\}$ as a set of reduced Lyapunov quantities. From [4] we recall the following generalized theorem:

THEOREM 6 (Degenerate generic Hopf–Takens bifurcation, [4]). *Assume that* $\{V_i : i \in \mathbb{N}\}$ *is a set of Lyapunov quantities for the analytic family* $(X_\lambda)_\lambda$ *such that* $\forall i \in \mathbb{N}$

$$V_i(\nu, \varepsilon) = \varepsilon^k \bar{V}_i(\nu) + O\left(\varepsilon^{k+1}\right), \varepsilon \to 0.$$

Assume further that $\exists r \in \mathbb{N}$ *such that*

$$\begin{cases} \bar{V}_i(\nu) & \equiv & 0, \ \forall 0 \le i \le r-1, \\ \bar{V}_{r+i}(\nu_0) & = & 0, \ \forall 0 \le i \le l-1, \\ \bar{V}_{r+l}(\nu_0) & \neq & 0. \end{cases}$$

1. Then, the reduced displacement mapping can be written as

$$\bar{\delta}(s, \nu, \varepsilon) = s^{2r+1} \left(\sum_{j=0}^{l} \bar{V}_{r+j}(\nu) c_j s^{2j} + o\left(s^{2j}\right) \right) + O(\varepsilon), s \to 0, \varepsilon \to 0$$

for certain $c_j > 0, \forall j = 0, \ldots, l$. *In particular,*

$$Cycl(X_\lambda, (e, (v_0, 0))) \le l.$$

2. If, furthermore, the mapping $\bar{V} := \left(\bar{V}_r, \bar{V}_{r+1}, \ldots, \bar{V}_{r+l-1}\right)$ *is a submersion at* ν_0, *then the family* $\left(X_{(\nu,\varepsilon)}\right)_{(\nu,\varepsilon)}$ *contains a generic Hopf–Takens bifurcation of codimension* l, *uniformly in* $\varepsilon \neq 0$; *i.e. there exists a neighborhood* W *of* ν_0 *and* $\varepsilon_0 > 0$ *such that* $\forall 0 < |\varepsilon| < \varepsilon_0 : \left(X_{(\nu,\varepsilon)}\right)_{\nu \in W}$ *contains a Hopf–Takens bifurcation of codimension* l. *Moreover the sign of its type* $X_{\pm}^{(l)}$ *is given by the sign of* $\varepsilon^k \bar{V}_{r+l}(\nu_0)$.

Proof. 1. Using Proposition 4, we find expansion (10) for the reduced displacement mapping $\bar{\delta}$. By a division-derivation algorithm based on Rolle's theorem, we then find that N is an upperbound for the cyclicity [13]. 2. This result is proven in [4]. ▮

In previous theorem, the degeneracy is caused only by the variable ε; once divided by a certain factor of ε, the degeneracy is removed and we are left with a regular situation. However, it is possible that, after division by this factor of ε, we still remain with a degeneracy in the variable ν. Then the Bautin ideal is not any more that simple; in that case, there are more variables involved to generate the Bautin ideal. In the case of such a general Bautin ideal, we have the following (weaker) result:

THEOREM 7 (General Bautin Ideal). *Assume that* $\{V_i : i \in \mathbb{N}\}$ *is a set of Lyapunov quantities for the family* $(X_\lambda)_\lambda$ *such that* k *is the biggest integer such that* $\forall i \in \mathbb{N}$

$$V_i(\nu, \varepsilon) = \varepsilon^k \bar{V}_i(\nu) + O\left(\varepsilon^{k+1}\right), \varepsilon \to 0.$$

Assume that $\forall i \in \mathbb{N} : \bar{V}_i(\nu_0) = 0$. *Then we define a* reduced Bautin ideal $\mathcal{I}^r$ *to be the ideal generated by the reduced Lyapunov quantities* $\bar{V}_i, i \in \mathbb{N}$. *Suppose that* $\bar{V}_j \equiv 0, \forall 0 \le j < t$.

1. Then, if $\left\{\bar{V}_{t+i} : 0 \le i \le N\right\}$ *is a set of generators for* $\mathcal{I}^r$, *then the displacement mapping for* $(X_\lambda)_\lambda$ *can be written as*

$$\bar{\delta}(s, \nu, \varepsilon) = s^{2t+1} \sum_{j=0}^{N} \bar{V}_{t+j}(\nu)\, h_j(s, \nu) + O(\varepsilon), \varepsilon \to 0, \tag{10}$$

where h_j *are analytic functions with*

$$h_j(s, \nu) = \eta_j s^{2j} + o\left(s^{2j}\right), s \to 0,$$

with $\eta_j > 0, \forall 0 \le j \le N$. *In particular,*

$$Cycl\left(X_\lambda, (e, (v_0, 0))\right) \le N.$$

2. If $\left\{\bar{V}_{t+i} : 0 \le i \le N\right\}$ *is a set of generators for* $\mathcal{I}^r$ *such that the mapping*

$$\nu \mapsto \left(\bar{V}_t(\nu), \bar{V}_{t+1}(\nu), \ldots, \bar{V}_{t+N}(\nu)\right) \tag{11}$$

is a submersion at ν_0, *then*

$$Cycl\left(X_\lambda, (e, (v_0, 0))\right) = N.$$

Proof. 1. Using Proposition 4, we find expansion (10) for the reduced displacement mapping $\bar{\delta}$. By a division-derivation algorithm based on Rolle's theorem, we then find that N is an upperbound for the cyclicity [13].

2. In case the mapping (11) is a local submersion at ν_0 (i.e. the reduced Bautin ideal is regular), then there exists a sequence of parameter values $((\nu^n, \varepsilon_n))_{n\in\mathbb{N}}$ tending to $(\nu_0, 0)$ such that the corresponding displacement

mapping $\bar{\delta}(\cdot, \nu^n, \varepsilon_n)$ has at least N small zeroes (the proof of this fact is analogous to the one of (4) that one can find in [13]). As a consequence, $Cycl\,(X_\lambda, (e, (v_0, 0))) \geq N$ and by the first assertion, we obtain the equality. ▌

3. CLASSICAL LIÉNARD EQUATIONS

Throughout this section, we will study (locally) the family of Liénard equations:

$$\ddot{x} + f(x, \lambda)\,\dot{x} + x = 0, \tag{12}$$

where $f(x, \lambda)$ is a function of class C^γ ($\gamma \in \mathbb{N} \cup \{\infty, \omega\}$) with $f(0, \lambda) \equiv 0$. If f is sufficiently differentiable, then we can write

$$f(x, \lambda) = \sum_{j=1}^{2N} f_j(\lambda)\, x^j + O\left(x^{2N+1}\right), x \to 0,$$

for $N \in \mathbb{N}$ and certain functions $f_j, j \in \mathbb{N}$ of class C^γ.

For this family, we will compute the cyclicity and the Bautin ideal (or equivalently, the Lyapunov quantities, by Proposition 4). In particular, we investigate the presence of Hopf–Takens bifurcations (respectively degenerate Hopf–Takens bifurcations) using Theorem 5 (respectively theorems 6 and 7). To use these theorems, we need to calculate the Lyapunov quantities and to be able to express them in terms of the family (12). In Section 3.1, we derive such expressions for the Lyapunov quantities. Then, in Section 3.2, the results about the cyclicity and the presence of a (degenerate) Hopf–Takens bifurcation are summarized. First, in Section 3.2.1, we deal with the most general case: the case that the Liénard equations are not necessarily polynomial. The poynomial case is dealt with in Section 3.2.2.

3.1. Calculation of Lyapunov quantities

The Lyapunov quantities of the Liénard equation (12) are defined to be the Lyapunov quantities of the corresponding system of first order differential equations:

$$X_\lambda \leftrightarrow \begin{cases} \dot{x} &= y \\ \dot{y} &= -x - f(x, \lambda)\, y. \end{cases} \tag{13}$$

To separate the variables x and y, we use the transformation

$$(x,y)\mapsto (x, y-F_1(x,\lambda)), \text{ where } F_1(x,\lambda) = -\int_0^x f(u,\lambda)\,\mathrm{d}u. \quad (14)$$

In this way, system (12) is transformed into:

$$\left\{\begin{array}{lcl}\dot{X} & = & Y + F_1(X,\lambda)\\ \dot{Y} & = & -X.\end{array}\right. \quad (15)$$

Next proposition states that the Lyapunov quantities of (15) are essentially given by the coefficients of odd order in x of $F_1(x,\lambda)$. This result is due to C. Christopher and N. Lloyd (see [5]) but we include a proof here to make this article self-contained.

PROPOSITION 8. *For the system*

$$\left\{\begin{array}{lcl}\dot{x} & = & y + \sum_{i=1}^{N} a_{2i}x^{2i} + \sum_{i=k}^{N} a_{2i+1}x^{2i+1} + O\left(\|(x,y)\|^{2N+2}\right),\\ \dot{y} & = & -x + O\left(\|(x,y)\|^{2N+2}\right), \|(x,y)\| \to 0,\end{array}\right.$$

with $k, N \in \mathbb{N}, N \gg k$, *the Lyapunov quantities* $V_l, l \in \mathbb{N}$ *are given by*

$$\left\{\begin{array}{lcl}V_l & = & 0, \forall 0 \le l < k\\ V_k & = & c_k a_{2k+1}\end{array}\right.$$

for a certain $c_k \in \mathbb{Q}^+ \setminus \{0\}$.

Proof. The system

$$\left\{\begin{array}{lcl}\dot{x} & = & y + \sum_{i=1}^{N} a_{2i}x^{2i}\\ \dot{y} & = & -x\end{array}\right.$$

is time–reversible under the transformation $(t,x,y) \mapsto (-t,-x,y)$. Then, there exists an analytic function $F(x,y) = O\left(\|(x,y)\|^3\right), \|(x,y)\| \to 0$ such that

$$X_a^R\left(\frac{x^2+y^2}{2} + F(x,y)\right) = 0,$$

where $X_a^R = \left(y + \sum_{i=1}^{N} a_{2i}x^{2i}\right)\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. To compute V_k, we need to find a homogeneous polynomial P of degree $2k+2$ of the form

$$P(x,y) = \sum_{j=0}^{k} \beta_j x^{2(k-j)+1} y^{2j+1}$$

such that, for $\|(x,y)\| \to 0$,

$$X_a\left(\frac{x^2+y^2}{2}+F(x,y)+P(x,y)\right)=V_k\left(x^2+y^2\right)^{k+1}+O\left(\left(x^2+y^2\right)^{k+2}\right),$$

where

$$\begin{aligned} X_a &= \left(y+\sum_{i=1}^{N}a_{2i}x^{2i}+a_{2k+1}x^{2k+1}+O\left(\|(x,y)\|^{2k+2}\right)\right)\frac{\partial}{\partial x}- \\ &\quad \left(x+O\left(\|(x,y)\|^{2k+2}\right)\right)\frac{\partial}{\partial y}. \end{aligned}$$

Therefore,

$$\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\sum_{j=0}^{k}\beta_j x^{2(k-j)+1}y^{2j+1}+a_{2k+1}x^{2k+2}=V_k\left(x^2+y^2\right)^{k+1}.$$

This yields in the following linear system in $(\beta_0,\beta_1,\ldots,\beta_k)$:

$$\begin{aligned} a_{2k+1}-\beta_0 &= V_k \\ (2(k-j)+1))\beta_j-(2j+3)\beta_{j+1} &= V_kC_{j+1}^{k+1}, \forall j=0,\ldots,k-1 \\ \beta_k &= V_k, \end{aligned}$$

where C_{j+1}^{k+1} are the binomial coefficients:

$$C_{j+1}^{k+1}=\binom{k+1}{j+1}=\frac{(k+1)!}{(j+1)!\,(k-j)!}.$$

By backward substitution, one finds the solution:

$$\beta_j=d_jV_k, \forall j=0,1,\ldots,k,$$

where $d_k=1, d_{k-j}=\frac{1}{2(k-j)+1}\left(C_{j+1}^{k+1}+(2j+3)\,d_{k-j+1}\right), \forall j=1,\ldots,k$. As a consequence,

$$a_{2k+1}=\beta_0+V_k=(d_0+1)\,V_k,$$

and the required result follows with $c=\frac{1}{d_0+1}\in\mathbb{Q}^+\setminus\{0\}$. ▮

This proposition shows that the Lyapunov quantities of (12) are essentially given by the coefficients of even order in x of the function $f(x)$. More precisely:

Corollary 9. *Let* $\{V_i : i \in \mathbb{N}\}$ *be a set of Lyapunov quantities of* (12) . *Then* $V_0 \equiv 0$, *and* $\forall 1 \le i \le N$:

$$V_i = c_i f_{2i} mod\,(f_2, f_4, \ldots, f_{2i-2}),$$

where $c_i \in \mathbb{Q}^- \setminus \{0\}$.

3.2. Conclusions

Combining Corollary 9 and Theorems 5, 6 and 7, we are now able to estimate the cyclicity and to detect the presence of a generic Hopf–Takens bifurcation (degenerate or not) in the family of classical Liénard equations. First, in Section 3.2.1, we state these results in the most general case, in the sense that we consider families that do not necessarily depend in a polynomial way on (x, y) , but are of a certain differentiability class in (x, y). Afterwards, in Section 3.2.2, these results are formulated in a polynomial setting.

3.2.1. General case

Theorem 10 (Generic Hopf–Takens bifurcation). *Consider the Liénard equation:*

$$\ddot{x} + f(x, \lambda)\,\dot{x} + x = 0,$$

where $f(x, \lambda)$ *is a function of class* C^γ $(\gamma \in \mathbb{N} \cup \{\infty, \omega\})$ *with*

$$f(x, \lambda) = \sum_{j=1}^{2N} f_j(\lambda)\, x^j + O\left(x^{2N+1}\right), x \to 0,$$

for $N \in \mathbb{N}$ *and certain functions* $f_j, 1 \le j \le 2N$ *of class* C^γ. *Suppose that* $\lambda_0 \in \mathbb{R}^p$ *such that*

$$f_{2j}(\lambda_0) = 0, \forall 1 \le j \le N-1, \text{ and } f_{2N}(\lambda_0) \ne 0.$$

1.Then there are at most $N-1$ *limit cycles in system* (13) *that bifurcate from the focus; i.e.*

$$Cycl\,(X_\lambda, (e, \lambda_0)) \le N - 1.$$

2.Furthermore, if the mapping

$$\lambda \mapsto (f_2(\lambda), f_4(\lambda), \ldots, f_{2N-2}(\lambda))$$

is a submersion at λ_0*, then the family* $(X_\lambda)_\lambda$ *contains a* generic Hopf–Takens bifurcation of codimension $N-1$ *at the origin* e*. Moreover, the sign of its type* $X_{\pm}^{(N-1)}$ *is given by the sign of* $-f_{2N}(\lambda_0)$*.*

Proof. This result follows from Corollary 9 and Theorem 5. ∎

Theorem 11 (General Bautin Ideal). *Suppose that* $f(x,\lambda)$ *is analytic with*

$$f(x,\lambda)=\sum_{j=1}^{\infty} f_j(\lambda)\,x^j, x\to 0$$

for certain analytic functions $f_j, j\in\mathbb{N}_1$*. Suppose that* $\lambda_0\in\mathbb{R}^p$ *is such that* $f_{2j}(\lambda_0)=0, \forall j\in\mathbb{N}$*. In other words, the vector field* X_{λ_0} *defined by* (13) *is of center type. Let* $N\in\mathbb{N}$ *be such that* $\forall j>N: f_{2j}\in(f_2,f_4,\ldots,f_{2N})$*.*

1.Then the Bautin ideal is generated by the germs of the analytic functions $f_2,f_4,\ldots,f_{2N}$ *at* λ_0*, and the displacement mapping can be written as:*

$$\delta(s,\lambda)=s\sum_{j=1}^{N} f_{2j}(\lambda)\,h_j(s,\lambda),$$

for analytic functions h_j *with*

$$h_j(s,\lambda)=\eta_j s^{2j}+O\left(s^{2j+1}\right), s\to 0,$$

for certain $\eta_j<0, \forall 1\le j\le N$*.*

2.If $\{f_{2j}:1\le j\le N\}$ *is a set of generators, then*

$$Cycl\,(X_\lambda,(e,\lambda_0))\le N-1.$$

3.If, furthermore, $\{f_{2j}:1\le j\le N\}$ *is a set of generators, such that the mapping*

$$\lambda\mapsto(f_2,f_4,\ldots,f_{2N})$$

is a submersion at λ_0*, then*

$$Cycl\,(X_\lambda,(e,\lambda_0))=N-1.$$

Proof. This result follows from Corollary 8 and Theorem 7 (with $k=0$). ∎

3.2.2. Polynomial case

Denote the integer part of $N/2$ by $[N/2]$.

THEOREM 12 (Generic Hopf–Takens bifurcation). *Consider a family of polynomial Liénard equations*

$$\ddot{x} + f(x,\lambda)\dot{x} + x = 0,$$

with

$$f(x,\lambda) = \sum_{j=1}^{N} \lambda_j x^j, \lambda = (\lambda_1, \ldots, \lambda_N).$$

Fix a parameter $\lambda_0 = \left(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_N\right)$.

1.If $1 \leq l \leq [N/2]$ *such that* $\bar{\lambda}_{2i} = 0, \forall 1 \leq i \leq l-1$ *and* $\bar{\lambda}_{2l} \neq 0$, *then the family of Liénard equations contains a generic Hopf–Takens bifurcation of codimension* $l-1$ *at e. Moreover, the sign of its type* $X_{\pm}^{(l-1)}$ *is given by the sign of* $-\bar{\lambda}_{2l}$. *Particularly,*

$$Cycl(X_\lambda, (e, \lambda_0)) = l - 1.$$

2.If $\forall 1 \leq l \leq [N/2] : \bar{\lambda}_{2l} = 0$, *then the Bautin ideal is generated by the germs of the analytic functions* $\lambda_2, \lambda_4, \ldots, \lambda_{2[N/2]}$ *at* λ_0, *and the displacement mapping can be written as:*

$$\delta(s,\lambda) = s \sum_{j=1}^{[N/2]} \lambda_{2j} h_j(s,\lambda),$$

for analytic functions h_j *with*

$$h_j(s,\lambda) = \eta_j s^{2j} + O\left(s^{2j+1}\right), s \to 0,$$

for certain $\eta_j < 0, \forall 1 \leq j \leq [N/2]$. *Particularly,*

$$Cycl(X_\lambda, (e, \lambda_0)) = [N/2] - 1.$$

Proof. This result follows from Corollary 8 and Theorem 5. ∎

THEOREM 13 (Deg. Hopf–Takens bif. and General Bautin Ideal). *Consider a family of polynomial Liénard equations*

$$\ddot{x} + f(x,\lambda)\dot{x} + x = 0,$$

with

$$f(x,\nu,\varepsilon)=\varepsilon^k\sum_{j=1}^{N}\nu_j x^j+O\left(\varepsilon^{k+1}\right),\nu=(\nu_1,\ldots,\nu_N),\lambda=(\nu,\varepsilon).$$

Fix a parameter $\nu_0=(\bar{\nu}_1,\bar{\nu}_2,\ldots,\bar{\nu}_N),\lambda_0=(\nu_0,0)$.

1.If $1\leq l\leq[N/2]$ *such that* $\bar{\nu}_{2i}=0,\forall 1\leq i\leq l-1$ *and* $\bar{\nu}_{2l}\neq 0$, *then the family of Liénard equations contains a generic Hopf–Takens bifurcation of codimension* $l-1$ *at* e, *uniformly in* $\varepsilon\neq 0$. *Moreover, the sign of its type* $X_{\pm}^{(l-1)}$ *is given by the sign of* $-\varepsilon^k\bar{\nu}_{2l}$. *Particularly,*

$$Cycl\,(X_\lambda,(e,\lambda_0))=l-1.$$

2.If $\forall 1\leq l\leq[N/2]:\bar{\nu}_{2l}=0$, *then the Bautin ideal is generated by the germs of the analytic functions* $\nu_2,\nu_4,\ldots,\nu_{2[N/2]}$ *at* λ_0, *and the displacement mapping can be written as:*

$$\delta(s,\nu,\varepsilon)=s\varepsilon^k\sum_{j=1}^{[N/2]}\nu_{2j}h_j(s,\nu)+O\left(\varepsilon^{k+1}\right),\varepsilon\to 0$$

for analytic functions h_j *with*

$$h_j(s,\nu)=\eta_j s^{2j}+O\left(s^{2j+1}\right),s\to 0,$$

where $\eta_j<0,\forall 1\leq j\leq[N/2]$. *Particularly,*

$$Cycl\,(X_\lambda,(e,\lambda_0))=[N/2]-1.$$

Proof. The first (respectively second) result follows from Corollary 8 and Theorem 6 (respectively Theorem 7). ∎

4. GENERALISED LIÉNARD EQUATIONS

In this section we consider a family of generalized Liénard equations:

$$\ddot{x}+f(x,\lambda)\dot{x}+g(x,\lambda)=0, \tag{16}$$

where f,g are C^γ with $\gamma\in\{\infty,\omega\}$, and

$$\begin{cases} f(0,\lambda) &= 0,\forall\lambda \\ g(x,\lambda) &= x+o(x),x\to 0.\end{cases}$$

For this family of generalized Liénard equations, we will give parallel results as in case of the classical family of Liénard equations: investigating the presence of a generic Hopf–Takens bifurcation and bounding the cyclicity near the focus $e = (0,0)$, again with help of theorems 5,6 and 7. However, this time, more elaborate calculations are involved. We would like to compute the Lyapunov quantities using Proposition 8. Therefore, in Section 4.1, the generalized Liénard equation is reduced to the classical one by a transformation similar to the so-called Cherkas transform. In Section 4.2, we prove by computations that this generalized Cherkas transformation induces a 'triangular' transformation on the generators of the ideal; we conclude this section by giving explicit expressions for the Lyapunov quantities (modulo the preceeding ones), in case the function f is odd in the variable x and f starts with a non-vanishing linear term in x. Finally, in Section 4.3, we state results analogous to Section 3.2.

Let us write the (possibly formal) series

$$\begin{cases} f(x,\lambda) &= \sum_{i=1}^{\infty} f_i(\lambda)\, x^i \\ g(x,\lambda) &= x + \sum_{i=2}^{\infty} g_i(\lambda)\, x^i. \end{cases}$$

4.1. Reduction to form (15)

The generalized Liénard equation can be written as the following system of first order differential equations:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -g(x,\lambda) - f(x,\lambda)\, y. \end{cases}$$

Again, we perform transformation (14) to obtain the system:

$$\begin{cases} \dot{x} &= Y + F_1(x,\lambda) \\ \dot{Y} &= -g(x,\lambda). \end{cases} \tag{17}$$

Now, inspired by Cherkas' article [3], we will introduce a new (locally defined) coordinate $X = xA(x,\lambda)$ such that (17) is equivalent to a system as given in (15). In the coordinates (X,Y) system (17) is transformed into

$$\begin{cases} \dot{X} &= (Y + F_1(XB(X),\lambda))\left(A(x,\lambda) + x\frac{\partial}{\partial x}A(x,\lambda)\right) \\ \dot{Y} &= -X\left(A(x,\lambda) + x\frac{\partial}{\partial x}A(x,\lambda)\right)\frac{g(x,\lambda)}{X\left(A(x,\lambda)+x\frac{\partial}{\partial x}A(x,\lambda)\right)}, \end{cases}$$

where $x = XB(X, \lambda)$ denotes the inverse transformation of $X = xA(x, \lambda)$. Hence A has to be defined such that

$$\frac{g(x, \lambda)}{xA(x, \lambda)\left(A(x, \lambda) + x\frac{\partial}{\partial x}A(x, \lambda)\right)} = 1.$$

This is equivalent to

$$2g(x, \lambda) = \frac{\partial}{\partial x}\left((xA(x, \lambda))^2\right).$$

Therefore, A can formally be written as

$$A(x, \lambda) = \sqrt{1 + \sum_{i=1}^{\infty} q_i(\lambda) x^i},$$

where $\forall i \geq 1$:

$$q_i = \frac{2g_{i+1}}{i+2}. \tag{18}$$

After division by $\left(A(x, \lambda) + x\frac{\partial}{\partial x}A(x, \lambda)\right)$, we obtain the desired system with

$$F(X, \lambda) = F_1(XB(X), \lambda).$$

4.2. Calculation of the Lyapunov quantities

¿From the previous section it is clear that the Lyapunov quantities of the generalized Liénard equation are given by the coefficients of odd order in X of the mapping F. We can write the (possibly formal) series

$$\begin{cases} A(x, \lambda) &= \sum_{i=0}^{\infty} a_i(\lambda) x^i \text{ with } a_0 = 1 \\ B(X, \lambda) &= \sum_{i=0}^{\infty} b_i(\lambda) X^i \text{ with } b_0 = 1 \\ F(X, \lambda) &= \sum_{i=2}^{\infty} d_i(\lambda) X^i \\ F_1(x, \lambda) &= \sum_{i=2}^{\infty} p_i(\lambda) x^i \end{cases}$$

where $\forall i \geq 2$:

$$p_i = -\frac{f_{i-1}}{i}.$$

In this section we will derive successively general structure formulas for the coefficients a_i, b_i and d_i. Furthermore, we investigate for each of these coefficients what happens when we start from the case where $f(x,\lambda)$ is an odd function of x with $f(x,\lambda) = 2x + o(x), x \to 0$ (in this case, $f_1 = 2$, or $p_2 = -1$). Notice that this fact corresponds to the fact that $F_1(x,\lambda)$ is even in x and starts in quadratic terms of x :

$$\begin{cases} F_1(x,\lambda) &= F_1(-x,\lambda) \\ F_1(x,\lambda) &= -x^2 + o(x^2), x \to 0. \end{cases} \tag{19}$$

During the rest of this section, we do not mention any more the dependence of the coefficients on λ explicitly.

Proposition 14. *The coefficients of $A(x)$ are given by the following recurrence relations:*

$$\begin{cases} a_0 &= 1 \text{ and } \forall l \geq 1 : \\ a_l &= \sum_{k=1}^{l} \bar{a}_k \sum_{\substack{1 \leq i_s \leq l \\ i_1+\ldots+i_k=l}} q_{i_1} \cdot \ldots \cdot q_{i_k} \end{cases} \tag{20}$$

where $\bar{a}_k \in \mathbb{Q} \setminus \{0\}$ arise from the Taylor expansion of $\sqrt{1+u}$ at $u = 0$; hence they are given by

$$\bar{a}_k = (-1)^{k-1} \left(\frac{1}{2}\right)^k \frac{1 \cdot 3 \cdot \ldots \cdot (2k-3)}{k!}, \forall k \geq 1.$$

As a consequence, the coefficients a_l of $A(x)$ are polynomials of degree at most l in $q_1, q_2, \ldots, q_l$. We also have the following corollary:

Corollary 15. *There exists the following relation between the coefficients $a_l, l \geq 1$ and the coefficients $q_l, l \geq 1$:*

$$\begin{cases} a_1 &= \frac{1}{2} q_1, \text{ and } \forall l \geq 2 : \\ a_l &= \frac{1}{2} q_l \, mod (q_1, \ldots, q_{l-1}). \end{cases}$$

Moreover,

$$\begin{cases} a_1 &= \frac{1}{2} q_1, \text{ and } \forall k \geq 1 : \\ a_{2k+1} &= \frac{1}{2} q_{2k+1} \, mod (q_1, q_3, \ldots, q_{2k-1}). \end{cases}$$

Proof. The relation between the coefficients of odd order follows from the observation that if the sum $i_1 + \ldots + i_k$ is odd, then there is at least one odd index i_s. ∎

By identifying coefficients of equal powers in X of the equation:

$$XB(X)\,A(XB(X)) = X,$$

we can deduce recurrence formulas for the coefficients b_i of B :

PROPOSITION 16. *The coefficients of $B(X)$ are given by the following recurrence formulas:*

$$\begin{cases} b_0 = 1, \;\; \textit{and } \forall l \geq 1 : \\ b_l + b_0 C_l + b_1 C_{l-1} + \ldots + b_{l-1} C_1 = 0 \end{cases} \tag{21}$$

where $C_l, l \geq 1$ is defined by

$$\begin{aligned} C_l \quad &= a_1 b_{l-1} + a_2 \sum_{\substack{0 \leq i_s \leq l-2 \\ i_1+i_2=l-2}} b_{i_1} b_{i_2} + \cdots \\ &+ a_r \sum_{\substack{0 \leq i_s \leq l-r \\ i_1+\ldots+i_r=l-r}} b_{i_1} \cdot \ldots \cdot b_{i_r} + \ldots + a_l. \end{aligned} \tag{22}$$

As a consequence, the coefficients b_l of $B(X)$ are polynomials of degree at most l in $q_1, \ldots, q_l$. We also have the following corollary:

COROLLARY 17. *There exists the following relation between the coefficients $a_l, l \geq 1$ and $b_l, l \geq 1$:*

$$\begin{cases} b_1 & = \; -a_1, \;\; \textit{and } \forall k \geq 1 \\ b_{2k+1} & = \; -a_{2k+1} \textit{mod}\,(b_1, b_3, \ldots, b_{2k-1}) \end{cases}$$

Proof. It can be checked easily that $b_1 = -a_1$ and $C_1 = -b_1$. Using (21) and (22) we prove by induction on k that $\forall k \geq 1$:

$$\begin{cases} C_{2k+1} & = \; a_{2k+1}\mathrm{mod}\,(b_1, b_3, \ldots, b_{2k-1}) \\ b_{2k+1} & = \; -a_{2k+1}\mathrm{mod}\,(b_1, b_3, \ldots, b_{2k-1}). \end{cases} \tag{23}$$

Suppose that the property (23) holds for all $1 \leq k \leq j$. From (22), we have

$$\begin{aligned} C_{2j+3} \quad &= a_1 b_{2j+2} + a_2 \sum_{\substack{0 \leq i_s \leq 2j+1 \\ i_1+i_2=2j+1}} b_{i_1} b_{i_2} + \cdots \\ &+ a_r \sum_{\substack{0 \leq i_s \leq 2j+3-r \\ i_1+\ldots+i_r=2j+3-r}} b_{i_1} \cdot \ldots \cdot b_{i_r} + \ldots + a_{2j+3}. \end{aligned}$$

We now show that the terms

$$a_r \sum_{\substack{0 \le i_s \le 2j+3-r \\ i_1+\ldots+i_r=2j+3-r}} b_{i_1} \cdot \ldots \cdot b_{i_r}$$

belong to the ideal $(b_1, b_3, \ldots, b_{2k+1})$. For $1 \le r \le 2j+1$ odd, by Corollary 15 and the induction hypothesis, these terms belong to the ideal $(a_r) \subset (b_1, b_3, \ldots, b_{2k+1})$. For $2 \le r \le 2j+2$ even, the sum $i_1 + \ldots + i_r = 2j+3-r$ is odd, and hence there has to be at least one odd index i_s, and $0 \le i_s \le 2j+3-r \le 2j+1$. Hence, it follows that the property (23), concerning the coefficients C_{2k+1}, holds for all $k \ge 1$.

For the result on the coefficient b_{2j+3}, using (21), we can write:

$$b_{2j+3} = -C_{2j+3} - \sum_{\substack{r \text{ odd} \\ r=1}}^{2j+1} b_j C_{2j+3-r} - \sum_{\substack{r \text{ even} \\ r=2}}^{2j+2} b_j C_{2j+3-r}$$

It is clear that the second term in the right–hand side of this equation belongs to the ideal

$$(b_1, b_3, \ldots, b_{2j+1});$$

by the induction hypothesis, also the third term belongs to this ideal, since $2j+3-r$ is odd and $2j+3-r \le 2k+1$. ▌

¿From Corollaries 15 and 17, we have now a relation between the coefficients b_{2k+1} and $q_{2k+1}, \forall k \ge 1$:

Corollary 18. *There exists the following relation between the coefficients b_{2k+1} and the coefficients $q_{2k+1}, k \in \mathbb{N}$:*

$$\begin{cases} b_1 &= -\frac{1}{2} q_1 \\ b_{2k+1} &= -\frac{1}{2} q_{2k+1} mod\,(q_1, q_3, \ldots, q_{2k+1}). \end{cases}$$

The next proposition states that the coefficients d_l of the function F are polynomials of degree $\le l$ in $p_2, \ldots, p_l, q_1, \ldots, q_{l-2}$; the structure is given more precisely in the following proposition:

Proposition 19.

1. The coefficients d_l of $F(X)$ are given by the formulas:

$$d_l = \sum_{\substack{j \ge 2, t \in \mathbb{N} \\ t+j=l}} p_j \sum_{i_1+\ldots+i_j=t} b_{i_1} \cdot \ldots \cdot b_{i_j}, \forall l \ge 2. \tag{24}$$

2.Moreover, if $p_{2j+1} = 0, \forall j \in \mathbb{N}$, *then* $\forall k \geq 1$:

$$d_{2k+1} = p_2 b_{2k-1} \, mod \, (b_1, b_3, \ldots, b_{2k-3}) \, .$$

Proof. The first part follows by the fact that d_l corresponds to the coefficient of X^l in

$$F(X, \lambda) = \sum_{j=2}^{\infty} p_j X^j B(X, \lambda)^j \, .$$

The second part follows from the following observation. If $l = 2k+1$ is odd in (24), then t has to be odd (since j even); as a consequence, there must be at least one odd index i_s. ▌

¿From Proposition 19 and Corollary 18, we can deduce the following corollaries.

COROLLARY 20. *Suppose that* $p_{2j+1} = 0, \forall j \in \mathbb{N}$.

1.Then

$$\begin{cases} d_3 & = -\frac{1}{2} p_2 q_1, \ \textit{and} \ \forall k \geq 2 : \\ d_{2k+1} & = -\frac{1}{2} p_2 q_{2k-1} \, mod \, (q_1, q_3, \ldots, q_{2k-3}) \, . \end{cases}$$

2.If, for instance, $p_2 = -1$, *then*

$$\begin{cases} d_3 & = \frac{1}{2} q_1, \ \textit{and} \ \forall k \geq 2 : \\ d_{2k+1} & = \frac{1}{2} q_{2k-1} \, mod \, (q_1, q_3, \ldots, q_{2k-3}) \, . \end{cases}$$

COROLLARY 21. *The Lyapunov quantities* $V_i, i \geq 1$, *in the generalized Liénard system*

$$\ddot{x} + f(x, \lambda) \dot{x} + g(x, \lambda) = 0$$

with

$$\begin{cases} f(x, \lambda) & = 2x + o(x), x \to 0 \\ f(x, \lambda) & = -f(-x, \lambda) \\ g(x, \lambda) & = x + \sum_{i=2}^{\infty} g_i(\lambda) x^i \end{cases}$$

are given by:

$$\begin{cases} V_1(\lambda) & = \frac{1}{3} g_2(\lambda), \ \textit{and} \ \forall k \geq 2 \\ V_k(\lambda) & = \frac{1}{2k+1} g_{2k}(\lambda) \, mod \, (g_2(\lambda), g_4(\lambda), \ldots, g_{2k-2}(\lambda)) \, . \end{cases}$$

4.3. Conclusions

In all our conclusions, we consider the following generalized Liénard system:

$$\ddot{x} + f(x,\lambda)\,\dot{x} + g(x,\lambda) = 0 \tag{25}$$

with

$$\begin{cases} f(x,\lambda) &= 2x + o(x),\, x \to 0 \\ f(x,\lambda) &= -f(-x,\lambda) \\ g(x,\lambda) &= x + \sum_{i=2}^{\infty} g_i(\lambda)\, x^i. \end{cases}$$

¿From Corollary 21 and Theorem 5, we can derive conclusions about the presence of Hopf–Takens bifurcations in these generalized Liénard equations, as we did in Section 3.2 for the classical Liénard equations.

4.3.1. General case

THEOREM 22 (Generic Hopf–Takens bifurcation). *Suppose we are given a generalized Liénard system as in* (25). *Suppose* $\lambda_0 \in \mathbb{R}^p$ *such that*

$$g_{2j}(\lambda_0) = 0, \forall 1 \le j \le N-1, \text{ and } g_{2N}(\lambda_0) \ne 0.$$

1. Then there are at most $N-1$ *limit cycles in system* (25) *that bifurcate from the focus; i.e.*

$$Cycl\,(X_\lambda, (e, \lambda_0)) \le N - 1.$$

2. Furthermore, if the mapping

$$\lambda \mapsto (g_2(\lambda), g_4(\lambda), \ldots, g_{2N-2}(\lambda))$$

is a submersion at λ_0, *then the family* $(X_\lambda)_\lambda$ *contains a* generic Hopf–Takens bifurcation of codimension $N-1$ *at the origin* e. *Moreover, the sign of its type* $X_\pm^{(N-1)}$ *is given by the sign of* $g_{2N}(\lambda_0)$.

Proof. This result follows from Theorem 5 and Corollary 21. ∎

THEOREM 23 (General Bautin Ideal). *Suppose we are given a generalized Liénard system as in* (25); *we suppose that the functions* f *and* g *are analytic. Assume that for a given* $\lambda_0 \in \mathbb{R}^p$ *all* $g_{2j}(\lambda_0) = 0, j \in \mathbb{N}$; *in other words, the vector field* X_{λ_0} *defined by* (25) *is of center type. Let* $N \in \mathbb{N}$ *such that* $\forall j > N : g_{2j} \in (g_2, g_4, \ldots, g_{2N})$.

1. Then the Bautin ideal is generated by $g_2, g_4, \ldots, g_{2N}$, and the displacement mapping can be written as:

$$\delta(s,\lambda) = s\sum_{j=1}^{N} g_{2j}(\lambda)\, h_j(s,\lambda),$$

for analytic functions h_j with

$$h_j(s,\lambda) = \eta_j s^{2j} + O\left(s^{2j+1}\right), s \to 0,$$

with $\eta_j > 0, \forall 1 \leq j \leq N$.

2. If $\{g_{2j} : 1 \leq j \leq N\}$ is a set of generators, then

$$Cycl\left(X_\lambda, (e, \lambda_0)\right) \leq N - 1.$$

3. If, furthermore, $\{g_{2j} : 1 \leq j \leq N\}$ is a set of generators, such that the mapping

$$\lambda \mapsto (g_2, g_4, \ldots, g_{2N})$$

is a submersion at λ_0, then

$$Cycl\left(X_\lambda, (e, \lambda_0)\right) = N - 1.$$

Proof. This result follows from Theorem 7 (with $k = 0$) and Corollary 21. ∎

4.3.2. Polynomial case

THEOREM 24 (Generic Hopf–Takens bifurcation). *Consider the polynomial generalized Liénard system*

$$\ddot{x} + f(x,\lambda)\dot{x} + g(x,\lambda) = 0$$

with

$$\begin{cases} f(x,\lambda) &= 2x + o(x), x \to 0 \\ f(x,\lambda) &= -f(-x,\lambda) \\ g(x,\lambda) &= x + \sum_{i=2}^{N} \lambda_i x^i, \lambda = (\lambda_2, \lambda_3, \ldots, \lambda_N) \in \mathbb{R}^{N-1} \end{cases}$$

Fix $\lambda_0 = (\bar{\lambda}_2, \ldots, \bar{\lambda}_N) \in \mathbb{R}^{N-1}$.

1.If $1 \leq l \leq [N/2]$ *is such that*

$$\bar{\lambda}_{2j} = 0, \forall 1 \leq j \leq l-1, \text{ and } \bar{\lambda}_{2l} \neq 0,$$

then the family $(X_\lambda)_\lambda$ *contains a* generic Hopf–Takens bifurcation of codimension $l-1$ *at the origin* e. *Moreover, the sign of its type* $X_{\pm}^{(l-1)}$ *is given by the sign of* $\bar{\lambda}_{2l}$. *Particularly,*

$$Cycl\,(X_\lambda, (e, \lambda_0)) = l - 1.$$

2.If $\forall 1 \leq l \leq [N/2] : \bar{\lambda}_{2l} = 0$. *Then the Bautin ideal is generated by* $\{\lambda_{2i} : 1 \leq i \leq [N/2]\}$, *and the displacement mapping can be written as:*

$$\delta(s,\lambda) = s \sum_{j=1}^{[N/2]} \lambda_{2j} h_j(s,\lambda),$$

for analytic functions h_j *with*

$$h_j(s,\lambda) = \eta_j s^{2j} + O\left(s^{2j+1}\right), s \to 0,$$

with $\eta_j > 0, \forall 1 \leq j \leq [N/2]$. *Moreover,*

$$Cycl\,(X_\lambda, (e, \lambda_0)) = [N/2] - 1.$$

Proof. This result follows from Theorem 5 and Corollary 21. ∎

THEOREM 25 (Deg. Hopf–Takens bif. and General Bautin Ideal). *Consider the polynomial generalized Liénard system*

$$\ddot{x} + f(x,\lambda)\,\dot{x} + g(x,\lambda) = 0$$

where $\lambda = (\varepsilon, \nu)$, $\nu = (\nu_2, \ldots, \nu_N) \in \mathbb{R}^{N-1}$ *and*

$$\begin{cases} f(x,\lambda) &=& (2x + o(x)) + O(\varepsilon), x \to 0, \varepsilon \to 0 \\ f(x,\lambda) &=& -f(-x,\lambda) \\ g(x,\lambda) &=& x + \varepsilon^k \sum_{i=2}^{N} \nu_i x^i + O\left(\varepsilon^{k+1}\right), \varepsilon \to 0. \end{cases}$$

Fix $\lambda_0 = (0, \bar{\nu}_2, \ldots, \bar{\nu}_N) \in \mathbb{R}^N$.

1.If $1 \leq l \leq [N/2]$ *is such that*

$$\bar{\nu}_{2j} = 0, \forall 1 \leq j \leq l-1, \text{ and } \bar{\nu}_{2l} \neq 0,$$

then the family $(X_\lambda)_\lambda$ *contains a* generic Hopf–Takens bifurcation of codimension $l-1$ *at the origin* e, *uniformly in* $\varepsilon \neq 0$. *Moreover, the sign of its type* $X_\pm^{(l)}$ *is given by the sign of* $\varepsilon^k \bar{\nu}_{2l}$. *Particularly,*

$$Cycl\,(X_\lambda, (e, \lambda_0)) = l - 1.$$

2.If $\forall 1 \leq l \leq [N/2] : \bar{\nu}_{2l} = 0$. *Then the Bautin ideal is generated by* $\{\nu_{2i} : 1 \leq i \leq [N/2]\}$, *and the displacement mapping can be written as:*

$$\delta(s,\lambda) = s\varepsilon^k \sum_{j=1}^{[N/2]} \nu_{2j} h_j(s,\nu) + O\left(\varepsilon^{k+1}\right),$$

for analytic functions h_j *with*

$$h_j(s,\nu) = \eta_j s^{2j} + O\left(s^{2j+1}\right), s \to 0,$$

with $\eta_j > 0, \forall 1 \leq j \leq [N/2]$. *Moreover,*

$$Cycl\,(X_\lambda, (e, \lambda_0)) = [N/2] - 1.$$

Proof. The first (respectively second) result follows from Theorem 6 (respectively Theorem 7) and Corollary 21. ∎

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