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# Planar Quadratic Vector Fields with Invariant Lines of total Multiplicity at least Five

Dana Schlomiuk*

*Département de Mathématiques et de Statistiques Université de Montréal.*
E-mail: dash@DMS.UMontral.CA

and

Nicolae Vulpe†

*Institute of Mathematics and Computer Science Academy of Science of Moldova.*
E-mail: nvulpe@math.md

In this article we consider the action of the real affine group and time rescaling on real planar quadratic differential systems. We construct a system of representatives of the orbits of systems with at least five invariant lines, including the line at infinity and including multiplicities. For each orbit we exhibit its configuration. We characterize in terms of algebraic invariants and comitants and also geometrically, using divisors of the complex projective plane, the class of real quadratic differential systems with at least five invariant lines. These conditions are such that no matter how a system may be presented, one can verify by using them whether the system has or does not have at least five invariant lines and to check to which orbit (or family of orbits) it belongs.

*Key Words:* quadratic differential system, Poincaré compactification, algebraic invariant curve, algebraic affine invariant, configuration of invariant lines.

## 1. INTRODUCTION

We consider here real planar differential systems of the form

$$(S) \qquad \frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1}$$

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where $p$, $q \in \mathbb{R}[x, y]$, i.e. $p$, $q$ are polynomials in $x$, $y$ over $\mathbb{R}$, and their associated vector fields

$$\tilde{D} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}. \tag{2}$$

Each such system generates a complex differential vector field when the variables range over $\mathbb{C}$. To the complex systems we can apply the work of Darboux on integrability via invariant algebraic curves (cf. [6]). For a brief introduction to the work of Darboux we refer to the survey article [22]. Some applications of the work of Darboux in connection with the problem of the center are given in [23].

For the system (1) we can use the following definition.

Definition 1. An *affine algebraic invariant curve* of a polynomial system (1) (or an *algebraic particular integral*) is a curve $f(x, y) = 0$ where $f \in \mathbb{C}[x, y]$, $\deg(f) \geq 1$, such that there exists $k(x, y) \in \mathbb{C}[x, y]$ satisfying $\tilde{D}f = fk$ in $\mathbb{C}[x, y]$. We call $k$ the *cofactor* of $f$ with respect to the system.

Poincaré was the first to appreciate the work of Darboux [6], which he called "admirable" (see [15]) and inspired by Darboux's work, Poincaré wrote two articles [16], [17] where he also stated a problem still open today.

With this brilliant work Darboux opened up a whole new area of investigations where one studies how the presence of particular algebraic integrals impacts on global properties of the systems, for example on global integrability. In recent years there has been a surge in activity in this area of research and this article is part of a growing literature in the subject. In particular we mention here [3], [5] and the work of C. Christopher, J.V. Perreira and J. Llibre on the notion of multiplicity of an invariant algebraic curve of a differential system [4].

In this article, which is based on [24], we study systematically the simplest kind of such a structure, i.e. quadratic systems (1) possessing invariant lines. Some references on this topic are: [1, 2, 7, 11–13, 18, 21, 29, 30, 33].

To a line $f(x, y) = ux + vy + w = 0$ we associate its projective completion $F(X, Y, Z) = uX + vY + wZ = 0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}_2(\mathbb{C})$, $(x, y) \mapsto [x : y : 1]$. The line $Z = 0$ is called the line at infinity of the systems (1). It follows from the work of Darboux that each system of differential equations of the form (1) yields a differential equation on the complex projective plane which is the compactification of the complex systems (1) on $\mathbb{P}_2(\mathbb{C})$ (cf. Section 2). The line $Z = 0$ is an invariant manifold of this complex differential equation.

NOTATION 2. *Let us denote by*

$$\mathbf{QS} = \left\{ (S) \,\middle|\, \begin{array}{l} (S) \textit{ is a system } (1) \textit{ such that } \gcd(p(x,y),q(x,y)) = 1 \\ \textit{and} \quad \max\big(\deg(p(x,y)), \deg(q(x,y))\big) = 2 \end{array} \right\};$$

$$\mathbf{QSL} = \left\{ (S) \in \mathbf{QS} \,\middle|\, \begin{array}{l} (S) \textit{ possesses at least one invariant affine line or} \\ \textit{the line at infinity with multiplicity at least two} \end{array} \right\}.$$

For the multiplicity of the line at infinity the reader is refereed to Section 2.

We shall call *degenerate quadratic differential system* a system (1) with $\deg\gcd(p,q) \geq 1$ and $\max\big(\deg(p), \deg(q)\big) = 2$.

To a system (1) in $\mathbf{QS}$ we can associate a point in $\mathbb{R}^{12}$, the ordered tuple of the coefficients of $p(x,y)$, $q(x,y)$ and this correspondence is an injection

$$\begin{array}{rrcl} \mathcal{B}: & \mathbf{QS} & \hookrightarrow & \mathbb{R}^{12} \\ & S & \mapsto & \boldsymbol{a} = \mathcal{B}(S) \end{array} \tag{3}$$

The topology of $\mathbb{R}^{12}$ yields an induced topology on $\mathbf{QS}$.

DEFINITION 3. We say that an invariant straight line $\mathcal{L}(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$, $(u,v,w) \in \mathbb{C}^3$ for a quadratic vector field $\tilde{D}$ has *multiplicity* $m$ if there exists a sequence of real quadratic vector fields $\tilde{D}_k$ converging to $\tilde{D}$, such that each $\tilde{D}_k$ has $m$ distinct (complex) invariant straight lines $\mathcal{L}_k^1 = 0, \ldots, \mathcal{L}_k^m = 0$, converging to $\mathcal{L} = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m+1$.

PROPOSITION 4 (see [2]). *The maximum number of invariant lines (including the line at infinity and including multiplicities) which a quadratic system in* $\mathbf{QS}$ *could have is six.*

DEFINITION 5. We call *configuration of invariant lines* (or simply *configuration*) of a system $(S)$ in $\mathbf{QSL}$ the set of all its invariant lines (real or complex), each endowed with its own multiplicity and together with all the real singular points of $(S)$ located on these lines, each one endowed with its own multiplicity.

We associate to each system in $\mathbf{QSL}$ its configuration of invariant lines. In analogous manner to how we view the phase portraits of the systems on the Poincaré disc (see for example [10]), we can also view the configurations of real lines on the disc. To help imagining the full configurations, we complete the picture by drawing dashed lines whenever these are complex.

On the class of quadratic systems acts the group of real affine transformations and time rescaling. Since quadratic systems depend on 12 parameters and since this group depends on 7 parameters, the class of quadratic systems modulo this group action actually depends on five parameters.

It is clear that the configuration of invariant lines of a system is an affine invariant. The notion of multiplicity defined by Definition 3 is invariant under the group action, i.e. if a quadratic system $(S)$ has an invariant line $l$ of multiplicity $m$, then each system $(\tilde{S})$ in the orbit of $(S)$ under the group action has an invariant line $\tilde{l}$ of the same multiplicity $m$.

In this article we shall consider the case when the system (1) has at least five invariant lines considered with their multiplicities.

The problems which we solve in this article are the following:

*I)* Construct a system of representatives of the orbits of systems with at least five invariant lines, including the line at infinity and including multiplicities. For each orbit exhibit its configuration.

*II)* Characterize in terms of algebraic invariants and comitants and also geometrically, using divisors or zero-cycles of the complex projective plane, the class of quadratic differential systems with at least five invariant lines. These conditions should be such that no matter how a system may be presented to us, we should be able to verify by using them whether the system has or does not have at least five invariant lines and to check to which orbit or perhaps family of orbits it belongs.

Our main results are formulated in Theorems 50 and 57. Theorem 50 gives a total of 11 distinct orbits of systems with a configuration with exactly six invariant lines including the line at infinity and including multiplicities. Theorem 57 gives a system of representatives for 19 distinct orbits of systems with exactly five invariant lines including the line at infinity and including multiplicities. Furthermore theorem 57 gives a complete list of representatives of the remaining orbits which are classified in 11 one-parameter families. We characterize each one of these 11 families in terms of algebraic invariants and comitants and geometrically. As the calculation of invariants and comitants can be implemented on a computer, this verification can be done by a computer.

All quadratic systems with at least five invariant lines including the line at infinity and including multiplicities are integrable via the method of Darboux (see [6]) and hence all of them have elementary first integrals. The phase portraits of these systems can easily be drawn. The issues related to integrability, as well as the drawing of the phase portraits of the systems we consider here are done in [25] and [26].

The invariants and comitants of differential equations used in the classification theorems (Theorems 50 and 57) are obtained following the theory established by K.Sibirsky and his disciples (cf. [27], [28], [31], [19], [20]).

## 2. DIFFERENTIAL EQUATIONS IN $\mathbb{P}_2(\mathbb{C})$ OF FIRST DEGREE AND FIRST ORDER AND THEIR INVARIANT PROJECTIVE CURVES

In [6] Darboux considered differential equations of first degree and first order of the complex projective plane. These are equations of the form

$$\begin{vmatrix} L & M & N \\ X & Y & Z \\ dX & dY & dZ \end{vmatrix} = 0 \tag{$CF$}$$

where $L$, $M$, $N$ are homogeneous polynomials in $X$, $Y$, $Z$ over $\mathbb{C}$, of the same degree $m$. These are called equations in Clebsch form $(CF)$. [1]

We remark that we can have an infinity of such equations yielding the same integral curves. Indeed, for any ordered triple $L, M, N$ of homogeneous polynomials in $X$, $Y$, $Z$ over $\mathbb{C}$, of the same degree $m$ and for any homogeneous polynomial $A$ of degree $m-1$, the $(CF)$-equation corresponding to

$$L' = L + AX, \quad M' = M + AY, \quad N' = N + AZ \tag{4}$$

has the same integral curves as the equation $(CF)$. Two equations $(CF)$ determined by polynomials $L, M, N$ and $L', M', N'$ satisfying (4) are said to be equivalent.

Theorem 6 (see [6]). *Let $L$, $M$, $N$ be homogeneous polynomials of the same degree $m$ over $\mathbb{C}$. Then there exists a unique $A$, more precisely*

$$A = -\frac{1}{m+2}\Big(\frac{\partial L}{\partial X} + \frac{\partial M}{\partial Y} + \frac{\partial N}{\partial Z}\Big)$$

*such that if $L', M', N'$ satisfy (4) for this $A$ then*

$$\frac{\partial L'}{\partial X} + \frac{\partial M'}{\partial Y} + \frac{\partial N'}{\partial Z} \equiv 0.$$

Theorem 7 (see [6]). *Every equation (CF) with* $\deg(L) = \deg(M) = \deg(N) = m$ *is equivalent to an equation*

$$A\,dX + B\,dY + C\,dZ = 0 \tag{5}$$

*where $A$, $B$, $C$ are homogeneous polynomials in $X$, $Y$, $Z$ over $\mathbb{C}$, of degree $m+1$ subject to the identity*

$$A\,X + B\,Y + C\,Z = 0. \tag{6}$$

[1]Darboux used the notion of Clebsch connex to define them.

We shall use the expression "$G \mid H$" where $G, H$ are elements in an integral domain $\mathbf{R}$ whenever $G$ *divides* $H$ in $\mathbf{R}$.

DEFINITION 8. [see [6]] An *algebraic invariant curve* for an equation $(CF)$ is a projective curve $F(X,Y,Z)=0$ where $F$ is a homogeneous polynomial over $\mathbb{C}$ such that $F \mid \hat{D}F$ where $\hat{D}$ is the differential operator

$$\hat{D} = L\frac{\partial}{\partial X} + M\frac{\partial}{\partial Y} + N\frac{\partial}{\partial Z}$$

i.e. $\exists K \in \mathbb{C}[X,Y,Z]$ such that $\hat{D}F = FK$. $K$ is called the *cofactor* of $F$ with respect to the equation $(CF)$.

We now show that this definition is in agreement with Definition 1, i.e. it includes as a particular case Definition 1.

To a system (1) we can associate an equation (5) subject to the identity (6). We first associate to the systems (1) the differential form

$$\omega_1 = q(x,y)dx - p(x,y)dy$$

and its corresponding differential equation $\omega_1 = 0$.

We consider the map $j : \mathbb{C}^3 \setminus \{Z=0\} \to \mathbb{C}^2$, given by $j(X,Y,Z) = (X/Z, Y/Z) = (x,y)$ and suppose that $\max\big(\deg(p), \deg(q)\big) = m > 0$. Since $x = X/Z$ and $y = Y/Z$ we have:

$$dx = (ZdX - XdZ)/Z^2\ , \qquad dy = (ZdY - YdZ)/Z^2\ ,$$

the pull–back form $j^*(\omega_1)$ has poles at $Z=0$ and its associated equation $j^*(\omega_1)=0$ can be written as

$$j^*(\omega_1) = q(X/Z,Y/Z)(ZdX - XdZ)/Z^2 - p(X/Z,Y/Z)(ZdY - YdZ)/Z^2 = 0.$$

Then the 1–form $\omega = Z^{m+2} j^*(\omega_1)$ in $\mathbb{C}^3 \setminus \{Z=0\}$ has homogeneous polynomial coefficients of degree $m+1$, and for $Z \neq 0$ the equations $\omega = 0$ and $j^*(\omega_1) = 0$ have the same solutions. Therefore the differential equation $\omega = 0$ can be written as (5) where

$$\begin{aligned} A(X,Y,Z) &= ZQ(X,Y,Z) = Z^{m+1}q(X/Z,Y/Z)\,, \\ B(X,Y,Z) &= -ZP(X,Y,Z) = -Z^{m+1}p(X/Z,Y/Z)\,, \\ C(X,Y,Z) &= YP(X,Y,Z) - XQ(X,Y,Z) \end{aligned} \tag{7}$$

and $P(X,Y,Z) = Z^m p(X/Z,Y/Z)$, $Q(X,Y,Z) = Z^m q(X/Z,Y/Z)$. Clearly $A$, $B$ and $C$ are homogeneous polynomials of degree $m+1$ satisfying (6).

The equation (5) becomes in this case

$$P(YdZ - ZdY) + Q(ZdX - XdZ) = 0$$

or equivalently

$$\begin{vmatrix} P & Q & 0 \\ X & Y & Z \\ dX & dY & dZ \end{vmatrix} = 0. \tag{8}$$

We observe that $Z = 0$ is an algebraic invariant curve of this equation according to Definition 8, with cofactor $K = 0$. We shall also say that $Z = 0$ is an invariant line for the systems (1).

To an affine algebraic curve $f(x, y) = 0$, $\deg f = n$, we can associate its projective completion $F(X, Y, Z) = 0$ where $F(X, Y, Z) = Z^n f(X/Z, Y/Z)$. The next proposition follows from the correspondence indicated above between systems (1) and their associated equations (8).

Proposition 9. *Let $f = 0$ ($\deg f = n$) be an invariant algebraic curve of (1) according to Definition 1, with cofactor $k(x, y)$. Then its associated projective completion $F(X, Y, Z) = 0$ where $F(X, Y, Z) = Z^n f(X/Z, Y/Z)$ is an invariant algebraic curve according to Definition 8 for the equation (8), with cofactor $K(X, Y, Z) = Z^{m-1} k(X/Z, Y/Z)$.*

Conversely, starting now with an equation in Clebsch form $(CF)$ we can consider its restriction on the affine chart $Z = 1$ and associate a differential system:

$$\begin{vmatrix} L & M & N \\ X & Y & Z \\ dX & dY & dZ \end{vmatrix} = 0 \;\rightarrow\; (\hat{M} - y\hat{N})dx - (\hat{L} - x\hat{N})dy = 0 \;\rightarrow\; \begin{cases} \dot{x} = \hat{L} - x\hat{N} \\ \dot{y} = \hat{M} - y\hat{N}, \end{cases} \tag{9}$$

where $\hat{L} = L(x, y, 1)$, $\hat{M} = M(x, y, 1)$, $\hat{N} = N(x, y, 1)$. The following proposition follows easily by using Euler's formula $XF'_X + YF'_Y + ZF'_Z = nF$ for a homogeneous polynomial $F(X, Y, Z)$ of degree n.

Proposition 10. *Let $F(X, Y, Z) = 0$ ($\deg F = n$) be an invariant algebraic curve (according to Definition 8) for the equation $(CF)$ with cofactor $K(X, Y, Z)$, such that $Z \nmid F$. Then $f(x, y) = F(x, y, 1) = 0$ is an invariant affine algebraic curve (according to Definition 1) of the differential system in (9) corresponding to $(CF)$, with cofactor $k(x, y) = K(x, y, 1) - nN(x, y, 1)$.*

Definition 11. We say that $Z = 0$ is an *invariant line of multiplicity $m$* for a system $(S)$ of the form (1) if and only if there exists a sequence of systems $(S_i)$ of the form (1) such that $(S_i)$ tend to $(S)$ when $i \to \infty$ and the systems $(S_i)$ have $m-1$ distinct invariant affine lines $\mathcal{L}_i^j = u_i^j x + v_i^j y + w_i^j = 0$, $(u_i^j, v_i^j) \neq (0,0)$, $(u_i^j, v_i^j, w_i^j) \in \mathbb{C}^3$ $(j = 1, \ldots, m-1)$ such that for every $j$, $\lim\limits_{i\to\infty} (u_i^j, v_i^j, w_i^j) = (0,0,1)$ and they do not have $m$ invariant such lines $L_i^j$, $j = 1, \ldots, m$ satisfying the above mentioned conditions.

## 3. DIVISORS ASSOCIATED TO INVARIANT LINES CONFIGURATIONS

Consider real differential systems of the form:

$$(S) \qquad \begin{cases} \dfrac{dx}{dt} = p_0 + p_1(x,y) + p_2(x,y) \equiv p(x,y), \\ \dfrac{dx}{dt} = q_0 + q_1(x,y) + q_2(x,y) \equiv q(x,y) \end{cases} \tag{10}$$

with

$$p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2,$$
$$q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2.$$

We stress that in (10) $a_{ij}$ are *variable* parameters and we denote by

$$a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$$

the 12-tuple of the coefficients of systems (10) and denote

$$\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y].$$

In what follows it is important for us to make a distinction between a polynomial $f(a,x,y) \in \mathbb{R}[a,x,y]$ and the polynomial obtained by evaluating $f(a,x,y)$ at the specific value $\boldsymbol{a} = (\boldsymbol{a_{00}, a_{10}, \ldots, b_{02}}) \in \mathbb{R}^{12}$ which is $f(\boldsymbol{a}, x, y) \in \mathbb{R}[x,y]$

Definition 12. We consider formal expressions $\mathbf{D} = \sum n(w)w$ where either all $w$ in $\mathbf{D}$ are points of $\mathbb{P}_2(\mathbb{C})$ or all $w$ in $\mathbf{D}$ are irreducible algebraic curves of $\mathbb{P}_2(\mathbb{C})$. Such an expression is called:

*(i)* a *zero-cycle of* $\mathbb{P}_2(\mathbb{C})$ in the first case,
*(ii)* a *divisor of* $\mathbb{P}_2(\mathbb{C})$ in the second case,
*(iii)* a *divisor of an irreducible algebraic curve* $\mathfrak{C}$ in $\mathbb{P}_2(\mathbb{C})$ if all $w$ in $\mathbf{D}$ belong to the curve $\mathfrak{C}$.

We call *degree* of the expression $\mathbf{D}$ the integer $\deg(\mathbf{D}) = \sum n(w)$. We call *support* of $\mathbf{D}$ the set $\operatorname{Supp}(\mathbf{D})$ of points $w$ such that $n(w) \neq 0$.

In this section we shall assume that systems (10) belong to **QS**.

NOTATION 13. *Let*

$$\begin{aligned}
P(X,Y,Z) =& p_0(\boldsymbol{a})Z^2 + p_1(\boldsymbol{a},X,Y)Z + p_2(\boldsymbol{a},X,Y) = 0;\\
Q(X,Y,Z) =& q_0(\boldsymbol{a})Z^2 + q_1(\boldsymbol{a},X,Y)Z + q_2(\boldsymbol{a},X,Y) = 0;\\
C(X,Y,Z) =& YP(X,Y,Z) - XQ(X,Y,Z);\\
\sigma(P,Q) =& \{w \in \mathbb{P}_2(\mathbb{C}) \mid P(w) = Q(w) = 0\};\\
\mathbf{D}_S(P,Q) =& \sum_{w\in\sigma(P,Q)} I_w(P,Q)w;\\
\mathbf{D}_S(C,Z) =& \sum_{w\in\{Z=0\}} I_w(C,Z)w \quad \text{if} \quad Z \nmid C(X,Y,Z);\\
\mathbf{D}_S(P,Q;Z) =& \sum_{w\in\{Z=0\}} I_w(P,Q)w;\\
\widehat{\mathbf{D}}_S(P,Q,Z) =& \sum_{w\in\{Z=0\}} \Big(I_w(C,Z),\, I_w(P,Q)\Big)w,
\end{aligned}$$

*where $I_w(F,G)$ is the intersection number (see [8]) of the curves defined by homogeneous polynomials $F,\ G \in \mathbb{C}[X,Y,Z]$ and* $\deg(F), \deg(G) \geq 1$.

A complex projective line $uX + vY + wZ = 0$ is invariant for a system $(S)$ if either it coincides with $Z = 0$ or it is the projective completion of an invariant affine line $ux + vy + w = 0$.

NOTATION 14. *Let $(S) \in$ **QSL**. Let us denote*

$$\mathbf{IL}(S) = \left\{ \; l \;\middle|\; \begin{array}{c} l \text{ is a line in } \mathbb{P}_2(\mathbb{C}) \text{ such} \\ \text{that } l \text{ is invariant for } (S) \end{array} \right\};$$
$$M(l) = \text{the multiplicity of the invariant line } l \text{ of } (S).$$

*Remark 15.* We note that the line $l_\infty : Z = 0$ is included in $\mathbf{IL}(S)$ for any $(S) \in$ **QSL**.

Let $l_i \,:\, f_i(x,y) = 0$, $i = 1,\dots,k$, be all the distinct invariant affine lines (real or complex) of a system $(S) \in$ **QSL**. Let $l_i' \,:\, \mathcal{F}_i(X,Y,Z) = 0$ be the complex projective completion of $l_i$.

Notation 16. *We denote*

$$\mathcal{G}\,:\,\prod_i \mathcal{F}_i(X,Y,Z)Z=0;\;\; Sing\mathcal{G}=\{w\in\mathcal{G}|w \text{ is a singular point of}\mathcal{G}\}\,;$$

$$\nu(w) = \text{the multiplicity of the point } w, \text{ as a point of } \mathcal{G}.$$

$$n_{\mathbb{R}}^{\infty} = \#\{w\in Supp\,\mathbf{D}_S(C,Z)\,\big|\, w\in\mathbb{P}_2(\mathbb{R})\}.$$

$$\mathbf{D}_{\mathbf{IL}}(S)=\sum_{l\in\mathbf{IL}(S)} M(l)l,\quad (S)\in\mathbf{QSL};$$

$$Supp\,\mathbf{D}_{\mathbf{IL}}(S) = \{\, l\mid l\in\mathbf{IL}(S)\}.$$

$$\begin{aligned}
M_{\mathbf{IL}} &= \deg \mathbf{D}_{\mathbf{IL}}(S);\\
N_{\mathbb{C}} &= \#Supp\,\mathbf{D}_{\mathbf{IL}};\\
N_{\mathbb{R}} &= \#\{l\in Supp\,\mathbf{D}_{\mathbf{IL}}\,\big|\, l\in\mathbb{P}_2(\mathbb{R})\};\\
n^{\mathbb{R}}_{\mathcal{G},\,\sigma} &= \#\{\omega\in Supp\,\mathbf{D}_S(P,Q)\,|\,\omega\in\mathcal{G}\big|_{\mathbb{R}^2}\};\\
d^{\mathbb{R}}_{\mathcal{G},\,\sigma} &= \sum_{\omega\in\mathcal{G}|_{\mathbb{R}^2}} I_\omega(P,Q);\\
m_{\mathcal{G}} &= \max\{\nu(\omega)\,|\,\omega\in Sing\,\mathcal{G}|_{\mathbb{C}^2}\};\\
m^{\mathbb{R}}_{\mathcal{G}} &= \max\{\nu(\omega)\,|\,\omega\in Sing\,\mathcal{G}|_{\mathbb{R}^2}\}.
\end{aligned}$$

## 4. THE MAIN *T*-COMITANTS ASSOCIATED TO CONFIGURATIONS OF INVARIANT LINES

On the set $\widehat{\mathbf{QS}}$ of all differential systems of the form (10) acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane. Indeed for every $\mathfrak{g}\in Aff(2,\mathbb{R})$, $\mathfrak{g}:\ \mathbb{R}^2\longrightarrow\mathbb{R}^2$ we have:

$$\mathfrak{g}:\ \begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}=M\begin{pmatrix}x\\ y\end{pmatrix}+B;\qquad \mathfrak{g}^{-1}:\ \begin{pmatrix}x\\ y\end{pmatrix}=M^{-1}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}-M^{-1}B,$$

where $M=||M_{ij}||$ is a $2\times 2$ nonsingular matrix, $B$ is a $2\times 1$ matrix over $\mathbb{R}$. For every $(S)\in\widehat{\mathbf{QS}}$ we can form its transformed system $\tilde{S}=\mathfrak{g}S$:

$$\frac{\partial\tilde{x}}{\partial t}=\tilde{p}(\tilde{x},\tilde{y}),\qquad \frac{\partial\tilde{y}}{\partial t}=\tilde{q}(\tilde{x},\tilde{y}),\tag{$\tilde{S}$}$$

where

$$\begin{pmatrix}\tilde{p}(\tilde{x},\tilde{y})\\ \tilde{q}(\tilde{x},\tilde{y})\end{pmatrix}=M\begin{pmatrix}(p\circ\mathfrak{g}^{-1})(\tilde{x},\tilde{y})\\ (q\circ\mathfrak{g}^{-1})(\tilde{x},\tilde{y})\end{pmatrix}.$$

The map

$$\begin{array}{rcl} Aff(2,\mathbb{R})\times\widehat{\mathbf{QS}} & \longrightarrow & \widehat{\mathbf{QS}} \\ (\mathfrak{g},\ S) & \longrightarrow & \tilde{S}=\mathfrak{g}\circ S \end{array}$$

verifies the axioms for a left group action. For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of $G$ on $\widehat{\mathbf{QS}}$. We can identify the set $\widehat{\mathbf{QS}}$ of systems of the form (10) with $\mathbb{R}^{12}$ via the map $\widehat{\mathbf{QS}} \longrightarrow \mathbb{R}^{12}$ which associates to each such system the 12-tuple $\boldsymbol{a}=(\boldsymbol{a}_{00},\boldsymbol{a}_{10}\dots,\boldsymbol{b}_{02})$ of its coefficients.

The action of $Aff(2,\mathbb{R})$ on $\widehat{\mathbf{QS}}$ yields an action of this group on $\mathbb{R}^{12}$. For every $\mathfrak{g}\in Aff(2,\mathbb{R})$ let $r_{\mathfrak{g}}:\ \mathbb{R}^{12}\longrightarrow\mathbb{R}^{12}$, $r_{\mathfrak{g}}(\boldsymbol{a})=\tilde{\boldsymbol{a}}$ where $\tilde{\boldsymbol{a}}$ is the 12-tuple of coefficients of $\tilde{S}$. It is known that $r_{\mathfrak{g}}$ is linear and that the map $r:\ Aff(2,\mathbb{R})\longrightarrow GL(12,\mathbb{R})$ thus obtained is a group homomorphism. For every subgroup $G$ of $Aff(2,\mathbb{R})$, $r$ induces a representation of $G$ onto a subgroup $\mathcal{G}$ of $GL(12,\mathbb{R})$.

Definition 17. A polynomial $U(a\,,x,y)\in\mathbb{R}[a,x,y]$ is called a *comitant* of systems (10) with respect to a subgroup $G$ of $Aff(2,\mathbb{R})$, if there exists $\chi\in\mathbb{Z}$ such that for every $(\mathfrak{g},\,\boldsymbol{a})\in G\times\mathbb{R}^{12}$ and for every $(x,y)\in\mathbb{R}^2$ the following relation holds:

$$U(r_{\mathfrak{g}}(\boldsymbol{a}),\ \mathfrak{g}(x,y)\,)\equiv\ (\det\,\mathfrak{g})^{-\chi}\,U(\boldsymbol{a},x,y),$$

where $\det\mathfrak{g}=\det M$. If the polynomial $U$ does not explicitly depend on $x$ and $y$ then it is called *invariant.* The number $\chi\in\mathbb{Z}$ is called the *weight* of the comitant $U(a,x,y)$. If $G=GL(2,\mathbb{R})$ (or $G=Aff(2,\mathbb{R})$) then the comitant $U(a,x,y)$ of systems (10) is called *$GL$-comitant* (respectively, *affine comitant*).

Definition 18. A subset $X\subset\mathbb{R}^{12}$ will be called *$G$-invariant*, if for every $\mathfrak{g}\in G$ we have $r_{\mathfrak{g}}(X)\subseteq X$.

Let us consider the polynomials

$$\begin{array}{rcl} C_i(a,x,y) & = & yp_i(a,x,y)-xq_i(a,x,y)\in\mathbb{Q}[a,x,y],\ i=0,1,2,\\ D_i(a,x,y) & = & \dfrac{\partial}{\partial x}p_i(a,x,y)+\dfrac{\partial}{\partial y}q_i(a,x,y)\in\mathbb{Q}[a,x,y],\ i=1,2. \end{array}$$

As it was shown in [27] the polynomials

$$\Big\{\ C_0(a,x,y),\quad C_1(a,x,y),\quad C_2(a,x,y),\quad D_1(a),\quad D_2(a,x,y)\ \Big\} \tag{11}$$

of degree one in the coefficients (variable parameters) of systems (10) are $GL$-comitants of these systems.

Notation 19. *Let* $f,\ g \in \mathbb{R}[a,x,y]$ *and*

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}. \tag{12}$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ *is called the transvectant of index* $k$ *of* $(f,g)$ *(cf. [9], [14]).*

Theorem 20 (see [31]). *Any* $GL$*-comitant of systems* (10) *can be constructed from the elements of the set* (11) *by using the operations:* $+$, $-$, $\times$, *and by applying the differential operation* $(f,g)^{(k)}$.

The following statements are direct consequences of the definition of a $GL$-comitant (see Definition 17).

*Remark 21.*

*(i)* Every product of $GL$-comitants is a $GL$-comitant;

*(ii)* A linear combination of $GL$-comitants is a $GL$-comitant if and only if all terms have the same weight.

Let $T(2,\mathbb{R})$ be the subgroup of $Aff(2,\mathbb{R})$ formed by translations. Consider the linear representation of $T(2,\mathbb{R})$ into its corresponding subgroup $\mathcal{T} \subset GL(12,\mathbb{R})$, i.e. for every $\tau \in T(2,\mathbb{R})$, $\tau:\ x = \tilde{x} + \alpha, y = \tilde{y} + \beta$ we consider as above $r_\tau:\ \mathbb{R}^{12} \longrightarrow \mathbb{R}^{12}$.

Definition 22. Consider a polynomial $U(a,x,y) = \sum_{j=0}^{d} U_i(a) x^{d-j} y^j \in \mathbb{R}[a,x,y]$ which is a $GL$-comitant of systems (10). We say that this polynomial is a $T$*-comitant* of systems (10) if for every $(\tau,\ \boldsymbol{a}) \in T(2,\mathbb{R}) \times \mathbb{R}^{12}$ we have $U_j(r_\tau(\boldsymbol{a})) = U_j(\boldsymbol{a})$, $\forall\ j = 0,1,\dots,d$.

Consider $s$ polynomials $U_i(a,x,y) = \sum_{j=0}^{d_i} U_{ij}(a) x^{d_i-j} y^j \in \mathbb{R}[a,x,y]$, $i = 1,\dots,s$ and assume that the polynomials $U_i$ are $GL$-comitants of systems (10) where $d_i$ denotes the degree of the binary form $U_i(a,x,y)$ in $x$ and $y$ with coefficients in $\mathbb{R}[a]$. We denote by

$$\mathcal{U} = \{\, U_{ij}(a) \in \mathbb{R}[a] \mid i = 1,\dots,s,\ j = 0,1,\dots,d_i \,\}$$

the set of the coefficients in $\mathbb{R}[a]$ of the $GL$-comitants $U_i(a,x,y)$, $i = 1,\dots,s$ and by $V(\mathcal{U})$ its zero set:

$$V(\mathcal{U}) = \left\{\, \boldsymbol{a} \in \mathbb{R}^{12} \mid U_{ij}(\boldsymbol{a}) = 0,\ \forall\ U_{ij}(a) \in \mathcal{U} \,\right\}.$$

DEFINITION 23. Let $U_1, U_2, \ldots, U_s$ be $GL$-comitants of systems (10) and homogeneous polynomials in the coefficients of these systems. A $GL$-comitant $U(a,x,y)$ of systems (10) is called a *conditional $T$-comitant* (or *$CT$-comitant*) modulo $\langle U_1, U_2, ..., U_s\rangle$ (i.e. modulo the ideal generated by $U_{ij}(a)$ $(i=1,\ldots,s; j=0,1,\ldots,d_i)$ in the ring $\mathbb{R}[a]$) if the following two conditions are satisfied:

*(i)* The algebraic subset $V(\mathcal{U})\subset\mathbb{R}^{12}$ is $Aff(2,\mathbb{R})$-invariant (see Definition 18);

*(ii)* For every $(\tau,\ \boldsymbol{a})\in T(2,\mathbb{R})\times V(\mathcal{U})$, $U(r_\tau(\boldsymbol{a}),\ \tilde{x},\ \tilde{y}) = U(\boldsymbol{a},\ \tilde{x},\ \tilde{y})$ in $\mathbb{R}[\tilde{x},\ \tilde{y}]$.

DEFINITION 24. A polynomial $U(a,x,y)\in\mathbb{R}[a,x,y]$, homogeneous of even degree in $x$, $y$ has *well determined sign on* $V\subset\mathbb{R}^{12}$ with respect to $x$, $y$ if for every $\boldsymbol{a}\in V$, the binary form $u(x,y)=U(\boldsymbol{a},x,y)$ yields a function of constant sign on $\mathbb{R}^2\setminus\{u=0\}$.

*Remark 25.* We draw the attention to the fact, that if a $CT$-comitant $U(a,x,y)$ of systems (10) of even weight is a binary form of even degree in $x$ and $y$ and of even degree in $a$ and also has well determined sign on some $Aff(2,\mathbb{R})$-invariant algebraic subset $V$, then this sign is conserved after the application of any affine transformation of the plane $x,y$ and time rescaling.

We now construct polynomials $D(a,x,y)$ and $H(a,x,y)$ which will be shown in Lemma 62 to be $T$-comitants.

NOTATION 26. *Consider the polynomial* $\Phi_{\alpha,\beta}=\alpha P+\beta Q\in\mathbb{R}[a,X,Y,Z,\alpha,\beta]$ *where* $P=Z^2p(X/Z,Y/Z)$, $Q=Z^2q(X/Z,Y/Z)$, $p,\ q\in\mathbb{R}[a,x,y]$ *and* $\max(\deg_{(x,y)}p,\deg_{(x,y)}q)=2$. *Then*

$$\begin{aligned}\Phi_{\alpha,\beta} &= c_{11}(a,\alpha,\beta)X^2+2c_{12}(a,\alpha,\beta)XY+c_{22}(a,\alpha,\beta)Y^2+2c_{13}(a,\alpha,\beta)XZ\\ &\quad+2c_{23}(a,\alpha,\beta)YZ+c_{33}(a,\alpha,\beta)Z^2,\\ \Delta(a,\alpha,\beta) &= \det||c_{ij}(a,\alpha,\beta)||_{i,j\in\{1,2,3\}},\quad D(a,x,y)=4\Delta(a,y,-x),\\ H(a,x,y) &= 4\big[\det||c_{ij}(a,y,-x)||_{i,j\in\{1,2\}}\big].\end{aligned}$$

*Remark 27.* The polynomials $D(a,x,y)$ and $H(a,x,y)$ constructed above are $GL$-comitants due to the following relations and to Theorem 20 and Remark 21 above:

$$\begin{aligned}H(a,x,y) &= \big[(C_2,C_2)^{(2)}-8(C_2,D_2)^{(1)}-2D_2^2\big]/18;\\ D(a,x,y) &= \Big[36C_0H-9C_2D_1^2-C_1\big((C_1,C_2)^{(2)}-6(C_1,D_2)^{(1)}\big)\\ &\quad+\big((C_1,C_2)^{(1)},C_1\big)^{(1)}+6D_1\big(C_1D_2-(C_1,C_2)^{(1)}\big)\Big]/36.\end{aligned}$$

Proposition 28. *Consider $m \leq 3$ distinct directions in the affine plane, where by direction we mean a point $[v:-u] \in \mathbb{P}_1(\mathbb{C})$. A necessary condition to have for each one of these directions an invariant line with that direction is that there exist $m$ distinct common factors of the polynomials $C_2(\boldsymbol{a},x,y)$ and $D(\boldsymbol{a},x,y)$ over $\mathbb{C}$.*

*Proof.* Suppose that $\mathcal{L}(x,y) \equiv ux+vy+w=0$, $\mathcal{L}(x,y) \in \mathbb{C}[x,y]$, $(u,v) \neq (0,0)$ is an invariant line for a quadratic system corresponding to $\boldsymbol{a} \in \mathbb{R}^{12}$. Then we must have $r,s,t \in \mathbb{C}$ such that

$$\frac{\partial \mathcal{L}}{\partial x} p(x,y) + \frac{\partial \mathcal{L}}{\partial y} q(x,y) = \mathcal{L}(x,y)(rx+sy+t). \tag{13}$$

Hence

$$up(x,y)+vq(x,y) = (ux+vy+w)(rx+sy+t).$$

So $\Phi_{u,v}(\boldsymbol{a},x,y)=0$ is a reducible conic which occurs if and only if the respective determinant $\Delta(\boldsymbol{a},u,v)=0$. But $D(\boldsymbol{a},v,-u)=4\Delta(\boldsymbol{a},u,v)=0$. The point at infinity of $\mathcal{L}=0$ is $[v:-u:0]$ and so $C_2(\boldsymbol{a},v,-u)=0$. Hence, the two homogeneous polynomials of degree 3 in $x$, $y$ must have the common factor $ux+vy$. ∎

*Remark 29.* Consider two parallel invariant affine lines $\mathcal{L}_i(x,y) \equiv ux+vy+w_i=0$, $(u,v) \neq (0,0)$, $\mathcal{L}_i(x,y) \in \mathbb{C}[x,y]$, $(i=1,2)$ of a quadratic system $(S)$ of coefficients $\boldsymbol{a}$. Then $H(\boldsymbol{a},-v,u)=0$, i.e. the T-comitant $H(a,x,y)$ can be used for determining the directions of parallel invariant lines of systems (10).

Indeed, according to (13) from the hypothesis we must have

$$up(x,y)+vq(x,y) = (ux+vy+w_1)(ux+vy+w_2).$$

Therefore for the quadratic form in $x$ and $y$: $F_2(\boldsymbol{a},x,y) = up_2(\boldsymbol{a},x,y) + vq_2(\boldsymbol{a},x,y)$ we obtain $F_2=(ux+vy)^2$ and hence Discriminant $(F_2)=0$. Then calculations yield: Discriminant $(F_2(\boldsymbol{a},x,y)) = -H(\boldsymbol{a},-v,u)$ and hence $H(\boldsymbol{a},-v,u)=0$.

We construct the following polynomials which will be shown in Lemma 62 to be $T$-comitants:

Notation 30.

$$\begin{aligned} B_3(a,x,y) &= (C_2,D)^{(1)} = Jacob\,(C_2,D)\,, \\ B_2(a,x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,D)^{(3)}, \\ B_1(a) &= \operatorname{Res}_x(C_2,D)/y^9 = -2^{-9}3^{-8}(B_2,B_3)^{(4)}\,. \end{aligned} \tag{14}$$

Proposition 31. *Suppose* $\tilde{d} = \deg\gcd\left(C_2(\boldsymbol{a},x,y), D(\boldsymbol{a},x,y)\right)$. *Then:*

$$\begin{aligned}
\tilde{d} = 0 \quad &\Leftrightarrow \quad B_1(\boldsymbol{a}) \neq 0;\\
\tilde{d} = 1 \quad &\Leftrightarrow \quad B_1(\boldsymbol{a}) = 0,\ B_2(\boldsymbol{a},x,y) \neq 0;\\
\tilde{d} = 2 \quad &\Leftrightarrow \quad B_2(\boldsymbol{a},x,y) = 0,\ B_3(\boldsymbol{a},x,y) \neq 0;\\
\tilde{d} = 3 \quad &\Leftrightarrow \quad B_3(\boldsymbol{a},x,y) = 0.
\end{aligned}$$

*Proof.* Since the polynomial $B_3(a,x,y)$ is the Jacobian of the cubic binary forms $C_2(a,x,y)$ and $D(a,x,y)$ we conclude that $\tilde{d} = 3$ if and only if $B_3(\boldsymbol{a},x,y) = 0$. We assume $B_3(\boldsymbol{a},x,y) \neq 0$ (i.e. $\tilde{d} \le 2$) and consider the two subcases: $B_2(\boldsymbol{a},x,y) = 0$ and $B_2(\boldsymbol{a},x,y) \neq 0$.

1. Assuming that $B_2(\boldsymbol{a},x,y) = 0$ then $\tilde{d} = 2$. Indeed, suppose $\tilde{d} < 2$. From (14) the condition $B_2 = 0$ yields $B_1 = 0$ and since the polynomial $B_1(a)$ is the resultant of the binary forms $C_2(a,x,y)$ and $D(a,x,y)$ we get $\tilde{d} = 1$, i.e. these polynomials have a common linear factor $\alpha x + \beta y$. We may assume $\beta = 0$ (the case $\beta \neq 0$ can be reduced to this one via the transformation $x_1 = \alpha x + \beta y$, $y_1 = x$). Then

$$C_2 = x(a_1x^2 + b_1xy + c_1y^2) \equiv x\tilde{A}(x,y), \ D = x(a_2x^2 + b_2xy + c_2y^2) \equiv x\tilde{B}(x,y).$$

Considering (14), calculations yield $B_2(\boldsymbol{a},x,y) = 3x^4 \cdot \mathrm{Res}_x(\tilde{A},\tilde{B})/y^4$ and we obtain a contradiction: since $B_2 = 0$ according to [32] (see Theorem 10.7 on page 29 of [32]) the polynomials $\tilde{A}$ and $\tilde{B}$ have a common nonconstant factor, i.e. $\tilde{d} > 1$.

Conversely, suppose that $\tilde{d} = 2$. Then clearly we have

$$C_2 = (ax + by)\tilde{C}, \qquad D = (cx + dy)\tilde{C}$$

and taking into account (14) calculations yield $B_2 = 0$.

2. Let us assume now that the condition $B_2(\boldsymbol{a},x,y) \neq 0$ holds. Then $\tilde{d} \le 1$ and since the polynomial $B_1(a)$ is the resultant of the binary forms $C_2(a,x,y)$ and $D(a,x,y)$ we get $\tilde{d} = 1$ if and only if $B_1(\boldsymbol{a}) = 0$.

∎

From the Propositions 28 and 31 the next result follows:

Corollary 32. *Given a direction* $[v : -u]$, *respectively two or three distinct directions* $[v_i : -u_i]$, $i = 1, 2$ *or* $i = 1, 2, 3$ *in the affine plane, for the existence of an invariant straight line having the direction* $[v : -u]$ *respectively two or three invariant straight lines having the given directions* $[v_i : -u_i]$, *it is necessary that* $B_1 = 0$, *respectively* $B_2 = 0$ *or* $B_3 = 0$.

Let us apply a change of variables $x = x' + x_0$, $y = y' + y_0$ to the polynomials $p(a, x, y)$, $q(a, x, y) \in \mathbb{R}[a, x, y]$. We obtain $\tilde{p}(\tilde{a}(a, x_0, y_0), x', y') = p(a, x' + x_0, y' + y_0)$, $\quad \tilde{q}(\tilde{a}(a, x_0, y_0), x', y') = q(a, x' + x_0, y' + y_0)$. Let us construct the following polynomials

$$\Gamma_i(a, x_0, y_0) \equiv \mathrm{Res}_{\,x'}\Big(C_i\big(\tilde{a}(a, x_0, y_0), x', y'\big), C_0\big(\tilde{a}(a, x_0, y_0), x', y'\big)\Big)/(y')^{i+1},$$
$$\Gamma_i(a, x_0, y_0) \in \mathbb{R}[a, x_0, y_0],\ (i = 1, 2).$$

NOTATION 33.

$$\tilde{\mathcal{E}}_i(a, x, y) = \Gamma_i(a, x_0, y_0)|_{\{x_0 = x,\ y_0 = y\}} \in \mathbb{R}[a, x, y] \quad (i = 1, 2). \tag{15}$$

*Remark 34.* It can easily be checked using the Definition 17 that the constructed polynomials $\tilde{\mathcal{E}}_1(a, x, y)$ and $\tilde{\mathcal{E}}_2(a, x, y)$ are affine comitants of systems (10). They are homogeneous polynomials in coefficients $a_{00}, \ldots, b_{02}$ and non-homogeneous polynomials in $x, y$ over $\mathbb{R}$ and

$$\deg_a \tilde{\mathcal{E}}_1 = 3,\ \deg_{(x,y)} \tilde{\mathcal{E}}_1 = 5,\ \deg_a \tilde{\mathcal{E}}_2 = 4,\ \deg_{(x,y)} \tilde{\mathcal{E}}_2 = 6.$$

NOTATION 35. *Let $\mathcal{E}_i(a, X, Y, Z)$ $(i = 1, 2)$ be the homogenization of $\tilde{\mathcal{E}}_i(a, x, y)$, i.e.*

$$\mathcal{E}_1(a, X, Y, Z) = Z^5 \tilde{\mathcal{E}}_1(a, X/Z, Y/Z),\ \mathcal{E}_2(a, X, Y, Z) = Z^6 \tilde{\mathcal{E}}_1(a, X/Z, Y/Z)$$

*and* $\qquad \mathcal{H}(a, X, Y, Z) = \gcd\Big(\mathcal{E}_1(a, X, Y, Z),\ \mathcal{E}_2(a, X, Y, Z)\Big).$

In what follows we shall examine the geometrical meaning of these invariant polynomials. We shall prove the following theorem:

THEOREM 36. *The straight line $\mathcal{L}(x, y) \equiv ux + vy + w = 0$ $u, v, w \in \mathbb{C}$, $(u, v) \neq (0, 0)$ is an invariant line for a system (10) in* **QS** *corresponding to a point $\boldsymbol{a} \in \mathbb{R}^{12}$ if and only if the polynomial $\mathcal{L}$ is a common factor of the polynomials $\tilde{\mathcal{E}}_1(\boldsymbol{a}, x, y)$ and $\tilde{\mathcal{E}}_2(\boldsymbol{a}, x, y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a}, x, y) = (ux + vy + w)\widetilde{W}_i(x, y) \in \mathbb{C}[x, y] \quad (i = 1, 2).$$

To prove this Theorem we first prove the following lemma:

LEMMA 37. *The straight line $\tilde{\mathcal{L}}(x, y) \equiv ux + vy = 0$, $(u, v) \in \mathbb{C}^2 \setminus \{0\}$ is an invariant line of a system (10) of coefficients $\boldsymbol{a}$ with ${\boldsymbol{a_{00}}}^2 + {\boldsymbol{b_{00}}}^2 \neq 0$ if and only if $C_0(\boldsymbol{a}, -v, u) = 0$, $C_1(\boldsymbol{a}, -v, u) = 0$, and $C_2(\boldsymbol{a}, -v, u) = 0$. These conditions are equivalent to the following ones:*

$$\begin{aligned} &\mathrm{Res}_{\,x}(C_0(a, x, y),\ C_1(a, x, y))/y^2\Big|_{(\boldsymbol{a})} = 0,\\ &\mathrm{Res}_{\,x}(C_0(a, x, y), C_2(a, x, y))/y^3\Big|_{(\boldsymbol{a})} = 0. \end{aligned} \tag{16}$$

*Proof.* According to Definition 1 the line $\tilde{\mathcal{L}}(x, y)=0$ is a particular algebraic integral for a system (10) if and only if the identity (13) holds for this system and this line. So in this case

$$u(p_0(\boldsymbol{a}) + p_1(\boldsymbol{a}, x, y) + p_2(\boldsymbol{a}, x, y)) + v(q_0(\boldsymbol{a}) + q_1(\boldsymbol{a}, x, y) + q_2(\boldsymbol{a}, x, y)) = \\ = (ux + vy)(S_0 + S_1(x, y)),$$

for some $S_0 \in \mathbb{C}$ and $S_1 \in \mathbb{C}[x, y]$. Herein we obtain:

*(i)* $up_0(\boldsymbol{a}) + vq_0(\boldsymbol{a}) = 0$;
*(ii)* $up_1(\boldsymbol{a}, x, y) + vq_1(\boldsymbol{a}, x, y) = (ux + vy)S_0(\boldsymbol{a})$;
*(iii)* $up_2(\boldsymbol{a}, x, y) + vq_2(\boldsymbol{a}, x, y) = (ux + vy)S_1(\boldsymbol{a}, x, y)$.

We observe that, if $x = -v$ and $y = u$ then the left-hand sides of $(i)$, $(ii)$ and $(iii)$ become $C_0(\boldsymbol{a}, -v, u)$, $C_1(\boldsymbol{a}, -v, u)$ and $C_2(\boldsymbol{a}, -v, u)$, respectively. At the same time the right-hand sides of these identities vanish. Therefore the following equations are obtained:

$$C_0(\boldsymbol{a}, -v, u) = 0,\ C_1(\boldsymbol{a}, -v, u) = 0,\ C_2(\boldsymbol{a}, -v, u) = 0. \tag{17}$$

As the degree of $C_0(a, x, y)$ is one, the relations (16) hold. ▮

*Proof* (Proof of Theorem 36). Consider the straight line $\mathcal{L}(x, y) = 0$. Let $(x_0, y_0) \in \mathbb{R}^2$ be any fixed non-singular point of the systems (10) (i.e. $p(x_0, y_0)^2 + q(x_0, y_0)^2 \neq 0$) which lies on the line $\mathcal{L}(x, y) = 0$, i.e. $ux_0 + vy_0 + w = 0$. Let $\tau_0$ be the translation $x = x' + x_0$, $y = y' + y_0$, $\tau_0(x', y') = (x, y)$. Then

$$\mathcal{L}(x, y) = \mathcal{L}(x' + x_0, y' + y_0) = ux' + vy' \equiv \tilde{\mathcal{L}}(x', y')$$

and consider the line $ux' + vy' = 0$. By Lemma 37 the straight line $\tilde{\mathcal{L}}(x', y') = 0$ will be an invariant line of systems $(10^{\tau_0})$ if and only if the conditions (16) are satisfied for these systems, i.e. $\Gamma_1(\boldsymbol{a}, x_0, y_0)) = \Gamma_2(\boldsymbol{a}, x_0, y_0) = 0$ for each point $(x_0, y_0)$ situated on the line $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, since the relation $ux_0 + vy_0 + w = 0$ is satisfied.

Thus we have $\Gamma_i(\boldsymbol{a}, x_0, y_0) = (ux_0 + vy_0 + w)\widetilde{W}_i(\boldsymbol{a}, x_0, y_0)$ $(i = 1, 2)$. Taking into account the notations (15) we conclude that the statement of Theorem 36 is true.

▮

We now consider the possibility for a straight line to be a multiple invariant line.

LEMMA 38. *If* $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u, v) \neq (0, 0)$ *is an invariant straight line of multiplicity* $k$ *for a quadratic system* (10) *then*

$[\mathcal{L}(x,y)]^k \mid \gcd(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2)$, *i.e. there exist* $W_i(\boldsymbol{a}, x, y) \in \mathbb{C}[x,y]$ $(i = 1,2)$ *such that*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a}, x, y) = (ux + vy + w)^k W_i(\boldsymbol{a}, x, y), \quad i = 1, 2. \tag{18}$$

*Proof.* Suppose that $\mathcal{L}(x,y) \equiv ux + vy + w = 0$ is an invariant line of multiplicity $k$ for a system (10) which corresponds to point $\boldsymbol{a} \in \mathbb{R}^{12}$. Let us denote by $\boldsymbol{a}_\varepsilon \in \mathbb{R}^{12}$ the point corresponding to the perturbed system $(10_\varepsilon)$, which has $k$ distinct invariant lines: $\mathcal{L}_{i\varepsilon}(x,y)$ $(i = 1, 2, ...k)$.

According to Theorem 36 for systems $(10_\varepsilon)$ the following relations are valid:

$$\tilde{\mathcal{E}}_{j\varepsilon}(\boldsymbol{a}_\varepsilon, x, y) = \mathcal{L}_{1\varepsilon} \cdot \mathcal{L}_{2\varepsilon} ... \cdot \mathcal{L}_{k\varepsilon} \widetilde{W}_j(\boldsymbol{a}_\varepsilon, x, y), \quad j = 1, 2,$$

and according to Definition 3 when $\varepsilon \to 0$ then $\mathcal{L}_{i\varepsilon}(x,y) \to \mathcal{L}(x,y)$, $\forall i = 1, ..k$. At the same time $\tilde{\mathcal{E}}_{j\varepsilon} \to \tilde{\mathcal{E}}_j = [\mathcal{L}(x,y)]^k W_j, \quad j = 1, 2.$ ▮

Corollary 39. *If the line* $l_\infty : Z = 0$ *is of multiplicity* $k > 1$ *then* $Z^{k-1} \mid \gcd(\mathcal{E}_1, \mathcal{E}_2)$.

*Proof.* Indeed, suppose that the line $l_\infty : Z = 0$ is of multiplicity $k > 1$ for a system $(S)$ which corresponds to a point $\boldsymbol{a} \in \mathbb{R}^{12}$. Then by Definition 11 there exists a perturbed system $(S_\varepsilon)$ corresponding to the point $\boldsymbol{a}_\varepsilon \in \mathbb{R}^{12}$ which has $k-1$ distinct invariant affine straight lines: $\mathcal{L}_{i\varepsilon}(x,y) = u_{i\varepsilon}x + v_{i\varepsilon}y + w_{i\varepsilon}$, $(u_{i\varepsilon}, v_{i\varepsilon}) \neq (0,0)$, $(u_{i\varepsilon}, v_{i\varepsilon}, w_{i\varepsilon}) \in \mathbb{C}^3$ $(i = 1, 2, ...k-1)$ such that for every $i$: $\lim\limits_{\varepsilon \to 0}(u_{i\varepsilon}, v_{i\varepsilon}, w_{i\varepsilon}) = (0, 0, 1)$.

By Lemma 38 each of the $k-1$ affine lines $\mathcal{L}_{i\varepsilon}$ must be a factor of the polynomial $\mathcal{H}(\boldsymbol{a}_\varepsilon, X, Y, Z) = \gcd\left(\mathcal{E}_1(\boldsymbol{a}_\varepsilon, X, Y, Z), \mathcal{E}_2(\boldsymbol{a}_\varepsilon, X, Y, Z)\right)$. Therefore we conclude that for the system $(S)$ we have $Z^{k-1} \mid \mathcal{H}(\boldsymbol{a}, X, Y, Z)$. ▮

As a next step we shall determine necessary conditions for the existence of parallel invariant lines. Let us consider the following $GL$-comitants of systems (10):

Notation 40.

$$\begin{aligned}
M(a,x,y) =& 2\,\text{Hess}\left(C_2(x,y)\right) = (C_2, C_2)^{(2)},\\
\eta(a) =& \text{Discriminant}\left(C_2(x,y)\right) = \left(\left((C_2, C_2)^{(2)}, C_2\right)^{(1)}, C_2\right)^{(3)}/576,\\
K(a,x,y) =& \text{Jacob}\left(p_2(x,y), q_2(x,y)\right) = \left[(C_2, C_2)^{(2)} + 4(C_2, D_2)^{(1)} + 4D_2^2\right]/18,\\
\mu(a) =& \text{Discriminant}\left(K(a,x,y)\right) = -(K,K)^{(2)}/2,\\
N(a,x,y) =& K(a,x,y) + H(a,x,y),\\
\theta(a) =& \text{Discriminant}\left(N(a,x,y)\right) = -(N,N)^{(2)}/2,
\end{aligned}$$

the geometrical meaning of which is revealed in the next 3 lemmas below.

*Remark 41.* We note that, by Discriminant $(C_2(x,y))$ of the cubic form $C_2(a,x,y)$, we mean the expression given in Maple program via the function "discrim$(C_2,x)/y^6$".

LEMMA 42. *Let $(S)\in \mathbf{QS}$ and let $\boldsymbol{a}\in\mathbb{R}^{12}$ be its 12-tuple of coefficients. The common points of $P=0$ and $Q=0$ on the line $Z=0$ are given by the common linear factors over $\mathbb{C}$ of $p_2$ and $q_2$. This yields the geometrical meaning of the T-comitants $\mu(a)$ and $K(a,x,y)$:*

$$\deg\gcd(p_2(x,y),q_2(x,y))=\begin{cases}0 \;\; iff \;\; \mu(\boldsymbol{a})\neq 0;\\ 1 \;\; iff \;\; \mu(\boldsymbol{a})=0,\;\; K(\boldsymbol{a},x,y)\neq 0;\\ 2 \;\; iff \;\; K(\boldsymbol{a},x,y)=0.\end{cases}$$

The proof follows from the fact that $K$ is the Jacobian of $p_2(x,y)$ and $q_2(x,y)$ (i.e. $p_2$ and $q_2$ are proportional if and only if $K(\boldsymbol{a},x,y)=0$ in $\mathbb{R}[x,y]$) and $\mu=\operatorname{Res}_x(p_2,q_2)/y^4$.

We shall prove the following assertion:

LEMMA 43. *A necessary condition for the existence of one couple (respectively, two couples) of parallel invariant straight lines of a system* (10) *corresponding to $\boldsymbol{a}\in\mathbb{R}^{12}$ is $\theta(\boldsymbol{a})=0$ (respectively, $N(\boldsymbol{a},x,y)=0$).*

*Proof.* Let $\mathcal{L}_i(x,y)\equiv ux+vy+w_i=0$, $(u,v)\neq(0,0)$, $(u,v,w_i)\in\mathbb{C}^3$ $(i=1,2)$ be two distinct $(w_1\neq w_2)$ parallel invariant lines for a quadratic system (10). Then by (13) we have

$$up(x,y)+vq(x,y)=\xi(ux+vy+w_1)(ux+vy+w_2)$$

and via a time rescaling we may assume $\xi=1$. Therefore for the quadratic homogeneities we obtain

$$(u\,\boldsymbol{a_{20}}+v\,\boldsymbol{b_{20}})x^2+2(u\,\boldsymbol{a_{11}}+v\,\boldsymbol{b_{11}})xy+(u\,\boldsymbol{a_{02}}+v\,\boldsymbol{b_{02}})y^2=(ux+vy)^2, \quad (19)$$

and hence, for the existence of parallel invariant lines the solvability of the following systems of quadratic equations with respect to parameters $u$ and $v$ is necessary:

$$(A_1)\; u\,\boldsymbol{a_{20}}+v\,\boldsymbol{b_{20}}=u^2;\;\; (A_2)\; u\,\boldsymbol{a_{11}}+v\,\boldsymbol{b_{11}}=uv;\;\; (A_3)\; u\,\boldsymbol{a_{02}}+v\,\boldsymbol{b_{02}}=v^2. \quad (20)$$

Without loss of generality we may consider $uv\neq 0$, otherwise a rotation of phase plane can be done. We now consider $vA_1-uA_2$ and $uA_3-vA_2$:

$$\begin{aligned} vA_1-uA_2: &\qquad -\boldsymbol{a_{11}}u^2+(\boldsymbol{a_{20}}-\boldsymbol{b_{11}})uv+\boldsymbol{b_{20}}v^2=0,\\ uA_3-vA_2: &\qquad \boldsymbol{a_{02}}u^2+(\boldsymbol{b_{02}}-\boldsymbol{a_{11}})uv-\boldsymbol{b_{11}}v^2=0.\end{aligned}$$

Let $F_1(u,v)$ and $F_2(u,v)$ be the left hand sides of the above equations. Clearly, for the existence of two directions $(u_1,v_1)$ and $(u_2,v_2)$ such that in each of them there are two parallel invariant straight lines of a system (10) it is necessary that the rank$(U)=1$, where

$$U=\begin{pmatrix} -\boldsymbol{a_{11}} & \boldsymbol{a_{20}}-\boldsymbol{b_{11}} & \boldsymbol{b_{20}} \\ \boldsymbol{a_{02}} & \boldsymbol{b_{02}}-\boldsymbol{a_{11}} & -\boldsymbol{b_{11}} \end{pmatrix}.$$

Hence, it is necessary

$$\tilde{A}=\begin{vmatrix} -a_{11} & a_{20}-b_{11} \\ a_{02} & b_{02}-a_{11} \end{vmatrix}=0,\ \tilde{B}=\begin{vmatrix} -a_{11} & b_{20} \\ a_{02} & -b_{11} \end{vmatrix}=0,\ \tilde{C}=\begin{vmatrix} a_{20}-b_{11} & b_{20} \\ b_{02}-a_{11} & -b_{11} \end{vmatrix}=0.$$

Since the resultant of the binary forms $F_1(u,v)$ and $F_2(u,v)$ is $\tilde{B}^2-\tilde{A}\tilde{C}$, we conclude that for the existence of one couple of parallel invariant lines it is necessary that $\tilde{B}^2-\tilde{A}\tilde{C}=0$. On the other hand calculations yield $N(\boldsymbol{a},x,y)=\tilde{C}x^2+2\tilde{B}xy+\tilde{A}y^2$, $\theta=4(\tilde{B}^2-\tilde{A}\tilde{C})$ and this completes the proof of lemma. ▌

LEMMA 44. *The type of the divisor $D_S(C,Z)$ for systems* (10) *is determined by the corresponding conditions indicated in Table 1, where we write $w_1^c+w_2^c+w_3$ if two of the points, i.e. $w_1^c, w_2^c$, are complex but not real. Moreover, for each type of the divisor $D_S(C,Z)$ given by Table 1 the quadratic systems* (10) *can be brought via a real linear transformation to one of the canonical systems* $(\mathbf{S}_I)-(\mathbf{S}_V)$ *given further below, corresponding to their behavior at infinity.*

*Proof.* The Table 1 follows easily from the construction of $\eta(a)$ and $M(a,x,y)$. It is well known that a cubic binary form in $x,y$ over $\mathbb{R}$ can be brought via a real linear transformation of the plane $(x,y)$: $\mathfrak{g}(x,y)=(\tilde{x},\tilde{y})$ to one of the following five canonical forms

$$I.\ \tilde{x}\tilde{y}(\tilde{x}-\tilde{y});\quad II.\ \tilde{x}(\tilde{x}^2+\tilde{y}^2);\quad III.\ \tilde{x}^2\tilde{y};\quad IV.\ \tilde{x}^3;\quad V.\ 0. \tag{21}$$

Let us consider a system (10) corresponding to a point $\boldsymbol{a}\in\mathbb{R}^{12}$ and let us consider the $GL$-comitant $C_2(\boldsymbol{a},x,y)=yp_2(\boldsymbol{a},x,y)-xq_2(\boldsymbol{a},x,y)$ simply as a cubic binary form in $x$ and $y$. Then the transformed binary form $\mathfrak{g}C_2(\boldsymbol{a},x,y)=C_2(\boldsymbol{a},\mathfrak{g}^{-1}(\tilde{x},\tilde{y}))$ is one of the canonical forms (21) corresponding to cases indicated in Table 1.

On the other hand, according to the Definition 17 of a $GL$-comitant, for $C_2(\boldsymbol{a},x,y)$ whose weight $\chi=-1$, we have for the same linear transformation $\mathfrak{g}\in GL(2,\mathbb{R})$

$$C_2(r_{\mathfrak{g}}(\boldsymbol{a}),\ \mathfrak{g}(x,y))=\det(\mathfrak{g})\,C_2(\boldsymbol{a},\ x,y).$$

$$\begin{cases} \dfrac{dx}{dt} = k + cx + dy + gx^2 + (h-1)xy, \\ \dfrac{dy}{dt} = l + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_I)$$

$$\begin{cases} \dfrac{dx}{dt} = k + cx + dy + gx^2 + (h+1)xy, \\ \dfrac{dy}{dt} = l + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases} \dfrac{dx}{dt} = k + cx + dy + gx^2 + hxy, \\ \dfrac{dy}{dt} = l + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases} \dfrac{dx}{dt} = k + cx + dy + gx^2 + hxy, \\ \dfrac{dy}{dt} = l + ex + fy - x^2 + gxy + hy^2, \end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases} \dfrac{dx}{dt} = k + cx + dy + x^2, \\ \dfrac{dy}{dt} = l + ex + fy + xy. \end{cases} \qquad (\mathbf{S}_V)$$

**TABLE 1.**

| *Case* | *Type of* $D_S(C,Z)$ | *Necessary and sufficient conditions on the comitants* |
|---|---|---|
| 1 | $w_1 + w_2 + w_3$ | $\eta > 0$ |
| 2 | $w_1^c + w_2^c + w_3$ | $\eta < 0$ |
| 3 | $2w_1 + w_2$ | $\eta = 0, \quad M \neq 0$ |
| 4 | $3w$ | $M = 0, \quad C_2 \neq 0$ |
| 5 | $D_S(C,Z)$ *undefined* | $C_2 = 0$ |

Using $\mathfrak{g}(x,y)=(\tilde{x},\tilde{y})$ we obtain $C_2(r_{\mathfrak{g}}(\boldsymbol{a}),\,\tilde{x},\tilde{y})=\det(\mathfrak{g})C_2(\boldsymbol{a},\,\mathfrak{g}^{-1}(\tilde{x},\tilde{y}))$, where we may assume $\det(\mathfrak{g})=1$ via the rescaling: $\tilde{x}\to\tilde{x}/\det(\mathfrak{g})$, $\tilde{y}\to\tilde{y}/\det(\mathfrak{g})$ in the system transformed by $\mathfrak{g}$. Thus, recalling that

$$p_2(\tilde{x},\tilde{y})=\tilde{\boldsymbol{a}}_{20}\tilde{x}^2+2\tilde{\boldsymbol{a}}_{11}\tilde{x}\tilde{y}+\tilde{\boldsymbol{a}}_{02}\tilde{y}^2,\qquad q_2(\tilde{x},\tilde{y})=\tilde{\boldsymbol{b}}_{20}\tilde{x}^2+2\tilde{\boldsymbol{b}}_{11}\tilde{x}\tilde{y}+\tilde{\boldsymbol{b}}_{02}\tilde{y}^2,$$

for the first canonical form in (21) we have

$$C_2(\tilde{\boldsymbol{a}},\tilde{x},\tilde{y})=-\tilde{\boldsymbol{b}}_{20}\tilde{x}^3+(\tilde{\boldsymbol{a}}_{20}-2\tilde{\boldsymbol{b}}_{11})\tilde{x}^2\tilde{y}+(2\tilde{\boldsymbol{a}}_{11}-\tilde{\boldsymbol{b}}_{02})\tilde{x}\tilde{y}^2+\tilde{\boldsymbol{a}}_{02}\tilde{y}^3=\tilde{x}\tilde{y}(\tilde{x}-\tilde{y}).$$

Identifying the coefficients of the above identity we get the canonical form $(\mathbf{S}_I)$.

Analogously for the cases $II$, $III$ and $IV$ we obtain the canonical forms $(\mathbf{S}_{II})$, $(\mathbf{S}_{III})$ and $(\mathbf{S}_{IV})$ associated to the respective polynomials in (21).

Let us consider the case $V$, i.e. $C_2(a,x,y)=0$ in $\mathbb{R}[a,x,y]$. Then we obtain the systems

$$\frac{dx}{dt}=k+cx+dy+gx^2+hxy,\quad \frac{dy}{dt}=l+ex+fy+gxy+hy^2$$

with $g^2+h^2\neq 0$. By interchanging $x$ and $y$ we may assume $g\neq 0$ and then via the linear transformation $\tilde{x}=gx+hy$, $\tilde{y}=y$ we obtain the systems $(S_V)$. ∎

In order to determine the existence of a common factor of the polynomials $\mathcal{E}_1(\boldsymbol{a},X,Y,Z)$ and $\mathcal{E}_2(\boldsymbol{a},X,Y,Z)$ we shall use the notion of the resultant of two polynomials with respect to a given indeterminate (see for instance, [32]).

Let us consider two polynomials $f,g\in R[x_1,x_2,\dots,x_r]$ where $R$ is a unique factorization domain. Then we can regard the polynomials $f$ and $g$ as polynomials in $x_r$ over the ring $\mathcal{R}=R[x_1,x_2,\dots,x_{r-1}]$, i.e.

$$\begin{aligned} f(x_1,x_2,\dots,x_r)&=a_0+a_1x_r+\ldots+a_nx_r^n,\\ g(x_1,x_2,\dots,x_r)&=b_0+a_1x_r+\ldots+b_mx_r^m\quad a_i,b_i\in\mathcal{R}. \end{aligned}$$

Lemma 45 (see [32]). *Assuming $n,m>0$, $a_nb_m\neq 0$ the resultant* $\operatorname{Res}_{x_r}(f,g)$ *of the polynomials $f$ and $g$ with respect to $x_r$ is a polynomial in $R[x_1,x_2,\dots,\dots,x_{r-1}]$ which is zero if and only if $f$ and $g$ have a common factor involving $x_r$.*

We also shall use the following remark:

*Remark 46.* Assume $s,\,\gamma\in\mathbb{R}$, $\gamma>0$. Then the transformation $x=\gamma^s x_1$, $y=\gamma^s y_1$ and $t=\gamma^{-s}t_1$ does not change the coefficients of the quadratic part of a quadratic system, whereas each coefficient of the linear (respectively constant ) part will be multiplied by $\gamma^{-s}$ (respectively by $\gamma^{-2s}$).

## 5. THE CONFIGURATIONS OF INVARIANT LINES OF QUADRATIC DIFFERENTIAL SYSTEMS WITH $M_{IL} = 6$

NOTATION 47. *We denote by* $\mathbf{QSL_6}$ *the class of all quadratic differential systems* (10) *with* $p$, $q$ *relatively prime* ($(p, q) = 1$), $Z \nmid C$ *and possessing a configuration of 6 invariant straight lines including the line at infinity and including possible multiplicities.*

LEMMA 48. *For a quadratic system* $(S)$ *in* $\mathbf{QSL_6}$ *the conditions* $N(\boldsymbol{a}, x, y) = 0$ *and* $B_3(\boldsymbol{a}, x, y) = 0$ *in* $\mathbb{R}[x, y]$*, are satisfied.*

*Proof.* Indeed, if for a system (10) the condition $M_{IL} = 6$ is satisfied, then taking into account the Definitions 3 and 11 we conclude that there exists a perturbation of the coefficients of the system (10) within the class of quadratic systems such that the perturbed systems has 6 distinct invariant lines (real or complex, including the line $Z = 0$). Hence, the perturbed systems must possess two couples of parallel lines with distinct directions and an additional line in a third direction. Then, by continuity and according to Lemma 43 and Corollary 32 we have $B_3(\boldsymbol{a}, x, y) = 0$ and $N(\boldsymbol{a}, x, y) = 0$. ∎

By Theorem 36 and Lemma 38 we obtain the following result:

LEMMA 49. *If* $M_{IL} = 6$ *then*
$$\deg \gcd\big(\mathcal{E}_1(\boldsymbol{a}, X, Y, Z), \mathcal{E}_2(\boldsymbol{a}, X, Y, Z)\big) = 5, \text{ i.e. } \mathcal{E}_1 \mid \mathcal{E}_2.$$

THEOREM 50.

*(i) The class* $\mathbf{QSL_6}$ *splits into 11 distinct subclasses indicated in Diagram 1 with the corresponding Configurations 6.1-6.11 where the complex invariant straight lines are indicated by dashed lines. If an invariant straight line has multiplicity* $k > 1$*, then the number* $k$ *appears near the corresponding straight line and this line is in bold face. We indicate next to the singular points their multiplicities as follows:* $(I_w(p, q))$ *if* $w$ *is a real finite singularity,* $(I_w(C, Z),\ I_w(P, Q))$ *if* $w$ *is a real infinite singularity with* $I_w(P, Q) \neq 0$ *and* $(I_w(C, Z))$ *if* $w$ *is a real infinite singularity with* $I_w(P, Q) = 0$.

*(ii) We consider the orbits of the class* $\mathbf{QSL_6}$ *under the action of the real affine group and time rescaling. The systems (VI.1) up to (VI.11) from the Table 2 form a system of representatives of these orbits under this action. A differential system* $(S)$ *in* $\mathbf{QSL_6}$ *is in the orbit of a system belonging to* $(VI.i)$ *if and only if* $B_3(\boldsymbol{a}, x, y) = 0 = N(\boldsymbol{a}, x, y)$ *and the corresponding conditions in the middle column (where the polynomials* $H_i$ $(i = 1, 2, 3)$ *and* $N_j$ $(j = 1, \ldots, 4)$ *are* $CT$*-comitants to be introduced below) are verified for this system* $(S)$*. The conditions indicated in the middle*

*column are invariant under the action of this group when jointly taken with the conditions* $B_3(\boldsymbol{a},x,y)=0=N(\boldsymbol{a},x,y)$.

*Wherever we have a case with invariant straight lines of multiplicity greater than one, we indicate the corresponding perturbations proving this in the Table 3.*

**TABLE 2.**

| Orbit representative | Necessary and sufficient conditions | Configu-ration |
|---|---|---|
| $(VI.1):\ \dot{x}=x^2-1,\ \dot{y}=y^2-1$ | $\eta>0,\ H_1>0$ | *Config. 6.1* |
| $(VI.2):\ \dot{x}=x^2+1,\ \dot{y}=y^2+1$ | $\eta>0,\ H_1<0$ | *Config. 6.2* |
| $(VI.3):\ \dot{x}=2xy,\ \dot{y}=y^2-x^2-1$ | $\eta<0,\ H_1<0$ | *Config. 6.3* |
| $(VI.4):\ \dot{x}=2xy,\ \dot{y}=y^2-x^2+1$ | $\eta<0,\ H_1>0$ | *Config. 6.4* |
| $(VI.5):\ \dot{x}=x^2,\ \dot{y}=y^2$ | $\eta>0,\ H_1=0$ | *Config. 6.5* |
| $(VI.6):\ \dot{x}=2xy,\ \dot{y}=y^2-x^2$ | $\eta<0,\ H_1=0$ | *Config. 6.6* |
| $(VI.7):\ \dot{x}=x^2-1,\ \dot{y}=2y$ | $MD\neq0, \eta=H=N_1=N_2=0$ | *Config. 6.7* |
| $(VI.8):\ \dot{x}=x^2-1,\ \dot{y}=2xy$ | $MH\neq0, \eta=H_2=0, H_3>0$ | *Config. 6.8* |
| $(VI.9):\ \dot{x}=x^2+1,\ \dot{y}=2xy$ | $MH\neq0, \eta=H_2=0, H_3<0$ | *Config. 6.9* |
| $(VI.10):\ \dot{x}=x^2,\ \dot{y}=1$ | $M\neq0, \eta=H=D=N_1=N_2=0$ | *Config. 6.10* |
| $(VI.11):\ \dot{x}=x,\ \dot{y}=y-x^2$ | $\eta=M=N_3=N_4=0$ | *Config. 6.11* |

*Proof of the Theorem 50.* According to Table 1 we shall consider the subcases corresponding to distinct types of the divisor $D_S(C,Z)$. Since we only discuss the case $C_2\neq0$, in what follows it suffices to consider only the canonical forms $(\mathbf{S}_I)$ to $(\mathbf{S}_{IV})$. The idea of the proof is to perform a case by case discussion for each one of these canonical forms, for which according to Lemma 48 the conditions $B_3=0=N$ must be fulfilled. These conditions yield specific conditions on the parameters. The discussion proceeds further by breaking these cases in more subcases determined by more restrictions

**Diagram 1** ($M_{\text{IL}} = 6$)

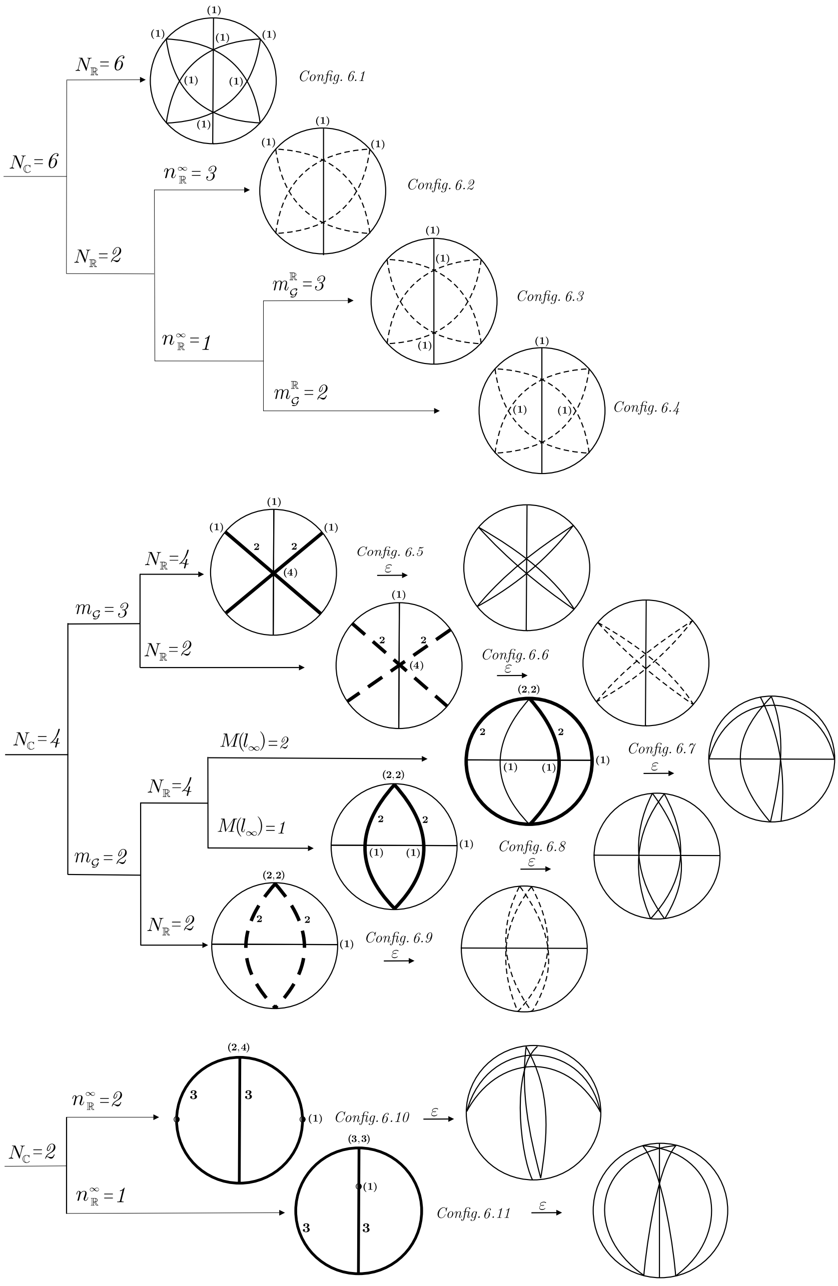

**TABLE 3.**

| Perturbations | Invariant straight lines |
|---|---|
| $(VI.5_\varepsilon): \ \dot{x}=x^2-\varepsilon^2,\ \dot{y}=y^2-\varepsilon^2$ | $x\pm\varepsilon,\ y\pm\varepsilon,\ x-y$ |
| $(VI.6_\varepsilon): \ \dot{x}=2xy,\ \dot{y}=\varepsilon^2-x^2+y^2$ | $x,\ x\pm iy-\varepsilon,\ x\pm iy+\varepsilon$ |
| $(VI.7_\varepsilon): \ \dot{x}=-1+x^2,\ \dot{y}=2y(\varepsilon y+1)$ | $x\pm 1,\ \varepsilon y+1,\ y,\ x-2\varepsilon y-1$ |
| $(VI.8_\varepsilon): \ \dot{x}=x^2-1,\ \dot{y}=2xy+\varepsilon y^2$ | $y,\ x=\pm 1,\ x+\varepsilon y\pm 1$ |
| $(VI.9_\varepsilon): \ \dot{x}=x^2+1,\ \dot{y}=2xy+\varepsilon y^2$ | $y,\ x\pm i,\ x+\varepsilon y\pm i$ |
| $(VI.10_\varepsilon): \begin{cases} \dot{x}=(1-\varepsilon)^2x^2-\varepsilon^2, \\ \dot{y}=\big(2\varepsilon^2y+1\big)\,(2\varepsilon y+1) \end{cases}$ | $(1-\varepsilon)x\pm\varepsilon,\ 2\varepsilon y+1,\ 2\varepsilon^2y+1,$ $(\varepsilon-1)^2x-4\varepsilon^3y-\varepsilon(\varepsilon+1)$ |
| $(VI.11_\varepsilon): \begin{cases} \dot{x}=x+\varepsilon x^2, \\ \dot{y}=y-x^2-2\varepsilon xy-2\varepsilon^2y^2 \end{cases}$ | $x,\ \varepsilon x+1,\ x+\varepsilon y,$ $x+2\varepsilon y,\ \varepsilon x+2\varepsilon^2y-1$ |

on the parameters. Finally we construct new invariants or T-comitants which put these conditions in invariant form.

For constructing the invariant polynomials included in the statement of Theorem 50 we shall use the $T$-comitants $D(a,x,y)$ and $H(a,x,y)$ indicated before as well as the $GL$-comitants (11).

### 5.1. Systems with the divisor $\boldsymbol{D_S(C,Z)} = 1\cdot \boldsymbol{w}_1 + 1\cdot \boldsymbol{w}_2 + 1\cdot \boldsymbol{w}_3$

For this case we shall later need the following polynomial which is shown to be an affine invariant in Lemma 62.

NOTATION 51. *Let us denote* $H_1(a) = -\Big(\big((C_2,C_2)^{(2)},C_2\big)^{(1)},D\Big)^{(3)}.$

According to Lemma 44 a system with this type of divisor can be brought by linear transformations to the canonical form $(\mathbf{S}_I)$ for which we have:

$$N(\boldsymbol{a},x,y) \;=\; (g^2-1)x^2+2(g-1)(h-1)xy+(h^2-1)y^2.$$

Hence the condition $N=0$ yields $(g-1)(h-1)=g^2-1=h^2-1=0$ and we obtain 3 possibilities: $(a)$ $g=1=h$; $(b)$ $g=1=-h$; $(c)$ $g=-1=-h$. The cases $(b)$ and $(c)$ can be brought by linear transformations to the case $(a)$. Hence the resulting polynomials are: $p_2(x,y)=x^2$ and $q_2(x,y)=y^2$. Then the term in $x$ of the first equation and the term in $y$ in the second equation can be eliminated via a translation. Thus we obtain the systems

$$\dot{x} = k + dy + x^2, \quad \dot{y} = l + ex + y^2 \tag{22}$$

for which we have $B_3 = 3[-e^2x^4 + 2e^2x^3y + 4(l-k)x^2y^2 - 2d^2xy^3 + d^2y^4]$. Hence, the condition $B_3 = 0$ yields $d = e = k - l = 0$ and we get the systems of the form:

$$\dot{x} = l + x^2, \quad \dot{y} = l + y^2. \tag{23}$$

By Remark 46 ( $\gamma = |l|,\ s = 1/2$) for systems (23) we can consider $l \in \{-1, 0, 1\}$. Clearly these systems possess the invariant straight lines $x = \pm\sqrt{-l}$, $y = \pm\sqrt{-l}$, $y = x$. Therefore, we obtain Config. 6.1 (respectively, Config. 6.2) for $l < 0$ (respectively, for $l > 0$) and Config. 6.5 for $l = 0$. For systems (23) the affine invariant $H_1(\boldsymbol{a}) = -2^9 3^3 l$ and hence $\operatorname{sign}(l) = -\operatorname{sign}(H_1(\boldsymbol{a}))$.

### 5.2. Systems with the divisor $\boldsymbol{D_S(C, Z)} = 1 \cdot \boldsymbol{w}_1^c + 1 \cdot \boldsymbol{w}_2^c + 1 \cdot \boldsymbol{w}_3$

In this case by Lemma 44 the systems (10) can be brought by linear transformations to the canonical form $(\mathbf{S}_{II})$ for which we have:

$$N(\boldsymbol{a}, x, y) \;=\; (g^2 - 2h + 2)x^2 + 2g(h+1)xy + (h^2 - 1)y^2.$$

Hence the condition $N = 0$ yields $g = h - 1 = 0$ and we may consider $c = d = 0$ due to the translation $x = x_1 - d/2$, $y = y_1 - c/2$. We thus obtain the systems

$$\dot{x} = k + 2xy, \qquad \dot{y} = l + ex + fy - x^2 + y^2 \tag{24}$$

for which $B_3 = 6\left[(ef - 2k)x^4 + (f^2 - e^2)x^3y - (4k + ef)x^2y^2 - 2ky^4\right]$. Hence, the condition $B_3 = 0$ yields $k = e = f = 0$ and we obtain the following form

$$\dot{x} = 2xy, \qquad \dot{y} = l - x^2 + y^2 \tag{25}$$

where $l \in \{-1, 0, 1\}$ by the Remark 46 ( $\gamma = |l|,\ s = 1/2$). It is not difficult to convince ourselves that these systems possess as invariant straight lines the components over $\mathbb{C}$ of:

$$x = 0, \quad x^2 + 2\,i\,xy - y^2 - l = 0, \quad x^2 - 2\,i\,xy - y^2 - l = 0,$$

with the intersection points: $p_{1,2} = (0, \pm\sqrt{-l})$, $p_{3,4} = (\pm\sqrt{l}, 0)$. On the other hand for systems (25) we have $H_1 = 2^{10}\,3^2\,l$. Therefore, if $H_1 \neq 0$ we get Config. 6.3 for $H_1 < 0$ and Config. 6.4 for $H_1 > 0$, whereas for $H_1 = 0$ we obtain Config. 6.6.

### 5.3. Systems with the divisor $\boldsymbol{D_S(C, Z)} = 2 \cdot \boldsymbol{w}_1 + 1 \cdot \boldsymbol{w}_2$

For this case we shall later need the following polynomials which are shown to be $CT$-comitants in Lemma 62.

NOTATION 52. *Let us denote*

$$H_2(a, x, y) = (C_1,\ 2H - N)^{(1)} - 2D_1N, \quad H_3(a, x, y) = (C_2, D)^{(2)},$$
$$N_1(a, x, y) = C_1(C_2, C_2)^{(2)} - 2C_2(C_1, C_2)^{(2)},$$
$$N_2(a, x, y) = D_1(C_1, C_2)^{(2)} - \big((C_2, C_2)^{(2)}, C_0\big)^{(1)}.$$

We are in the case of the canonical form $(\mathbf{S}_{III})$ for which we have:

$$\begin{aligned} N(a, x, y) &= (g^2 - 1)x^2 + 2h(g - 1)xy + h^2y^2, \\ H(a, x, y) &= -(g - 1)^2x^2 - 2h(g + 1)xy - h^2y^2. \end{aligned} \tag{26}$$

The condition $N = 0$ yields $h = g^2 - 1 = 0$ and we shall examine two subcases: $H(a, x, y) \neq 0$ and $H(a, x, y) = 0$.

#### 5.3.1. *The case* $\boldsymbol{H(a, x, y)} \neq 0$

In this case for $h = 0$ we have $H(a, x, y) = -(g - 1)^2x^2 \neq 0$ and hence the condition $N = 0$ yields $g = -1$. Moreover, for systems $(\mathbf{S}_{III})$ we can consider $e = f = 0$ due to the translation of the origin of coordinates to the point $(f/2, e/2)$. Thus, the systems $(\mathbf{S}_{III})$ can be brought to the form

$$\dot{x} = k + cx + dy - x^2, \qquad \dot{y} = l - 2xy, \tag{27}$$

for which $B_3 = 6x(-2lx^3 + cdxy^2 + d^2y^3)$. So, the condition $B_3 = 0$ yields $l = d = 0$ and we obtain the systems

$$\dot{x} = k + cx - x^2, \qquad \dot{y} = -2xy \tag{28}$$

with $k \neq 0$ (otherwise systems (28) become degenerate).

So far we have only used the necessary conditions $N = 0$ and $B_3 = 0$ for this particular case. These are not sufficient for having 6 invariant lines. According to Lemma 49 we must have $\mathcal{E}_1 \mid \mathcal{E}_2$. Calculations yield :

$$\mathcal{E}_1 = (kZ^2 - X^2)\mathcal{H}, \quad \mathcal{E}_2 = X(X^2 - cXZ - kZ^2)\mathcal{H},$$
$$\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = 2Y\left(kZ^2 + cXZ - X^2\right).$$

Since $k \neq 0$ according to Lemma 45 the condition $\operatorname{Res}_Z(\mathcal{E}_1/\mathcal{H}, \mathcal{E}_2/\mathcal{H}) = -c^2kX^6 \equiv 0$ must hold. This yields $c = 0$ and the systems (28) become

$$\dot{x} = k - x^2, \qquad \dot{y} = -2xy. \tag{29}$$

By Remark 46 ($\gamma = |k|,\ s = 1/2$) we may assume $k \in \{-1, 1\}$.

For the systems (29) we have $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = 2Y\left(kZ^2 - X^2\right)^2$ and according to Lemma 38 each one of the two invariant lines $x = \pm\sqrt{k}$ of the systems (29) could be of the multiplicity two. And they are indeed of multiplicity two as it is shown by the perturbations ($VI.8_\varepsilon$) (for $k = 1$) and ($VI.9_\varepsilon$) (for $k = -1$) from Table 3. Thus, we obtain Config. 6.8 for $k = 1$ and Config. 6.9 for $k = -1$.

On the other hand for the systems (29) we have $H_2 = 16cx^2$, $H_3 = 32kx^2$. Hence the $T$-comitants $H_2$ and $H_3$ capture exactly the conditions $c = 0$ and $k > 0$ or $c = 0$ and $k < 0$ and this leads to the corresponding conditions in Table 2, where in the respective canonical systems we change $t \to -t$.

*5.3.2. The case* $\boldsymbol{H}(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}) = 0$

According to (26) the conditions $N = H = 0$ yield $h = 0$, $g = 1$ and translating the origin of coordinates to the point $(-c/2, 0)$ the systems ($\mathbf{S}_{III}$) can be brought to the form

$$\dot{x} = k + dy + x^2, \qquad \dot{y} = l + ex + fy. \tag{30}$$

For these systems we have $B_3 = 6dxy^2(fx - dy)$ and the condition $B_3 = 0$ yields $d = 0$. So, we obtain the systems

$$\dot{x} = k + x^2, \qquad \dot{y} = l + ex + fy \tag{31}$$

for which we have $D(\boldsymbol{a}, x, y) = -f^2x^2y$.

1. If $D \neq 0$ then $f \neq 0$ and by Remark 46 ($\gamma = f/2,\ s = 1$) we can consider $f = 2$. Then via the translation we may assume $l = 0$ and we obtain the systems

$$\dot{x} = k + x^2, \qquad \dot{y} = ex + 2y, \tag{32}$$

for which calculations yield

$$\begin{aligned} &\mathcal{E}_1 = \left[4Y(X - Z) + e(X^2 - 2XZ - kZ^2)\right]\mathcal{H}, \quad \mathcal{H} = Z\left(X^2 + kZ^2\right), \\ &\mathcal{E}_2 = (eX + 2Y)(X^2 + kZ^2)\mathcal{H}, \\ &\operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = -2e(X^2 + kZ^2)^2. \end{aligned} \tag{33}$$

Hence for $\mathcal{E}_1 \mid \mathcal{E}_2$ the condition $\operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H})=0$ must be fulfilled in $\mathbb{R}[X,Z]$. This yields $e=0$, then we obtain $\operatorname{Res}_X\big((\mathcal{E}_1/\mathcal{H}),(\mathcal{E}_2/\mathcal{H})\big)|_{e=0}=32(k+1)Y^3Z^2=0$. Hence $k+1=0$ and for $e=k+1=0$ we obtain the system

$$\dot{x}=x^2-1,\qquad \dot{y}=2y, \tag{34}$$

for which $\mathcal{H}=\gcd(\mathcal{E}_1,\mathcal{E}_2)=YZ(X-Z)^2(X+Z)$. This system possesses the invariant affine lines $x=\pm1,\quad y=0$. Moreover, taking into account the polynomial $\mathcal{H}$, by Lemma 38 and Corollary 39 the line $x=1$ as well as the line $l_\infty: Z=0$ could be of multiplicity two. This is confirmed by the perturbations ($VI.7_\varepsilon$) from Table 3. Since this system possesses only two finite singularities $(\pm1,0)$ which are simple, we conclude that the configuration of the invariant lines of the system (34) is Config. 6.7.

It remains to observe that the conditions $e=0=k+1$ are equivalent to $N_1=N_2=0$, as for systems (32) we have $N_1=8e\,x^4,\quad N_2=16(k+1)x$.

2. The condition $D=0$ implies $f=0$ and we obtain the systems

$$\dot{x}=k+x^2,\qquad \dot{y}=l+ex. \tag{35}$$

Calculations yield:

$$\mathcal{E}_1=\big[2\,lXZ+e(X^2-kZ^2)\big]\,\mathcal{H},\quad \mathcal{E}_2=(eX+lZ)(X^2+kZ^2)\,\mathcal{H}, \tag{36}$$

where $\mathcal{H}=Z\left(X^2+kZ^2\right)$. Hence for $\mathcal{E}_1 \mid \mathcal{E}_2$ according to Lemma 45 at least one of the following conditions must hold:

$$\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H})=-4\,ek(e^2k+l^2)^2Z^6=0,$$
$$\operatorname{Res}_Z(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H})=-4\,ek(e^2k+l^2)^2X^6=0,$$

and we obtain that either $ek=0$ or $e^2k+l^2=0$. Since the second case yields a degenerate system we obtain the necessary condition $ek=0$. It is easy to observe that for $e^2+k^2\neq0$ we obtain $\mathcal{E}_1\nmid\mathcal{E}_2$. Therefore $k=e=0$ (then $l\neq0$) and via the additional rescaling $y\to l\,y$ we obtain the system:

$$\dot{x}=x^2,\qquad \dot{y}=1 \tag{37}$$

for which $\mathcal{H}=\gcd(\mathcal{E}_1,\mathcal{E}_2)=X^3Z^2$. By Lemma 38 and Corollary 39 the line $x=0$ as well as the line $Z=0$ could be of multiplicity three. This is confirmed by the perturbations ($VI.10_\varepsilon$) from Table 3. It remains to note that for systems (35) we obtain $N_1=8e\,x^4$, $N_2=16kx$ and, hence in this case we obtain Config. 6.10 if and only if $N_1=0=N_2$.

### 5.4. Systems with the divisor $D_S(C, Z) = 3 \cdot w$

For this case we shall later need the following polynomials which are shown to be $CT$-comitants in Lemma 62.

NOTATION 53. *Let us denote* $N_3(a, x, y) = (C_2, C_1)^{(1)}$, $N_4(a, x, y) = 4\,(C_2, C_0)^{(1)} - 3C_1D_1$.

We are in the case of the canonical form $(\mathbf{S}_{IV})$ for which we have:

$$N = (g^2 - 2h)x^2 + 2ghxy + h^2y^2.$$

So, the condition $N = 0$ yields $h = g = 0$ and due to the translation $x = x_1 + e/2$, $y = y_1$ we may assume $e = 0$. Hence the systems $(\mathbf{S}_{IV})$ become

$$\dot{x} = k + cx + dy, \qquad \dot{y} = l + fy - x^2, \tag{38}$$

for which $B_3 = 6dx^3(fx - dy)$. The condition $B_3 = 0$ yields $d = 0$ and we shall examine the systems of the form

$$\dot{x} = k + cx, \qquad \dot{y} = l + fy - x^2. \tag{39}$$

Calculations yield

$$\begin{aligned} \mathcal{E}_1 &= \big[(c+f)X^2 + 2kXZ + (c-f)(fY + lZ)Z\big]\,\mathcal{H}, \\ \mathcal{E}_2 &= Z(cX + kZ)^2\,\mathcal{H}, \quad \mathcal{H} = Z^2(cX + kZ). \end{aligned} \tag{40}$$

Since the polynomial $\mathcal{E}_2/\mathcal{H}$ depends only on $X$ and $Z$ for $\mathcal{E}_1 \mid \mathcal{E}_2$ the following condition must hold: $f(c - f) = 0$. We claim that for $f = 0$ we cannot have $\mathcal{E}_1 \mid \mathcal{E}_2$. Indeed, assuming $f = 0$ we obtain the quadratic form $\mathcal{E}_1/\mathcal{H} = cX^2 + 2kXZ + clZ^2$ in $X$ and $Z$, which must divide $Z(cX + kZ)^2$. This clearly implies that the discriminant of this form must be zero, i.e. $4(k^2 - c^2l) = 0$. However this leads to degenerate systems.

Therefore we must have $c - f = 0$ and for the systems (39) with $f = c$ calculations yield: $\mathcal{E}_1 = X\,\tilde{\mathcal{H}}$, $\mathcal{E}_2 = Z(cX + kZ)\,\tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}} = Z^2(cX + kZ)^2$. Therefore $\mathcal{E}_1 \mid \mathcal{E}_2$ if and only if $k = 0$ and we obtain the systems $\dot{x} = cx$, $\dot{y} = l + cy - x^2$ with $c \neq 0$. We may assume $c = 1$ by Remark 46 ( $\gamma = c$, $s = 1$) and via the translation of the origin of coordinates to the point $(0, -l)$ we obtain $l = 0$. This leads to the following system

$$\dot{x} = x, \qquad \dot{y} = y - x^2, \tag{41}$$

with $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = X^3Z^2$ and by Lemma 38 and Corollary 39 each one of the invariant lines $x = 0$ and $Z = 0$ is of multiplicity 3. This is confirmed by the perturbed systems ($VI.11_\varepsilon$) from Table 3. On the other hand for systems (39) $N_3 = 3(c-f)x^3$, $N_4 = 3x[4kx + (f^2 - c^2)y]$ and hence, the conditions $c - f = k = 0$ are equivalent to $N_3 = N_4 = 0$. Taking into account the existence of the simple singular point $(0, 0)$ placed on the line $x = 0$ we obtain Config. 6.11.

All the cases in Theorem 50 are thus examined. To finish the proof of the Theorem 50 it remains to show that the conditions occurring in the middle column of Table 2, jointly taken with the conditions $B_3(\boldsymbol{a}, x, y) = 0 = N(\boldsymbol{a}, x, y)$, are affinely invariant. This follows from the proof of Lemma 62.

## 6. THE CONFIGURATIONS OF INVARIANT LINES OF QUADRATIC DIFFERENTIAL SYSTEMS WITH $M_{\text{IL}} = 5$

NOTATION 54. *We denote by* $\mathbf{QSL_5}$ *the class of all quadratic differential systems* (10) *with* $p$, $q$ *relatively prime* ($(p, q) = 1$), $Z \nmid C(X, Y, Z)$ *and possessing a configuration of five invariant straight lines including the line at infinity and including possible multiplicities.*

LEMMA 55. *If for a quadratic system* $(S)$ $M_{\text{IL}} = 5$, *then for this system one of the two following conditions are satisfied:*

*(i)* $N(\boldsymbol{a}, x, y) = 0 = B_2(\boldsymbol{a}, x, y)$ *in* $\mathbb{R}[x, y]$;
*(ii)* $\theta(\boldsymbol{a}) = 0 = B_3(\boldsymbol{a}, x, y)$ *in* $\mathbb{R}[x, y]$.

*Proof.* Indeed, if for a system (10) the condition $M_{\text{IL}} = 5$ is satisfied then taking into account the Definition 3 we conclude that there exists a perturbation of the coefficients of the system (10) within the class of quadratic systems such that the perturbed systems have five distinct invariant lines (real or imaginary, including the line $Z = 0$). Hence, the perturbed systems must possess either two couples of parallel lines with distinct directions or one couple of parallel lines and two additional lines with distinct directions. Then, by continuity and according to Lemma 43 and Corollary 32 we respectively have either the conditions $(i)$ or $(ii)$. ∎

By Theorem 36 and Lemmas 38 and 48 we obtain the following result:

LEMMA 56.

*(i) If for a system* $(S)$ *of coefficients* $\boldsymbol{a} \in \mathbb{R}^{12}$, $M_{\text{IL}} = 5$, *then*

$$\deg \gcd\big(\mathcal{E}_1(\boldsymbol{a}, X, Y, Z), \mathcal{E}_2(\boldsymbol{a}, X, Y, Z)\big) = 4.$$

*(ii) If $N(\boldsymbol{a}, x, y) \not\equiv 0$ then $M_{\mathbf{IL}} \leq 5$.*

THEOREM 57.

*(i) The class $\mathbf{QSL_5}$ splits into 30 distinct subclasses indicated in Diagram 2 with the corresponding Configurations 5.1-5.30 where the complex invariant straight lines are indicated by dashed lines. Nineteen of these subclasses are orbits under the group action and each one of the remaining eleven subclasses is a family of orbits depending on a parameter $g \in \mathbb{R} \setminus A$ where $A$ is a finite set of points. If an invariant straight line has multiplicity $k > 1$, then the number $k$ appears near the corresponding straight line and this line is in bold face. We indicate next to the singular points their multiplicities as follows: $(I_w(p,q))$ if $w$ is a real finite singularity, $(I_w(C,Z),\ I_w(P,Q))$ if $w$ is a real infinite singularity with $I_w(P,Q) \neq 0$ and $(I_w(C,Z))$ if $w$ is a real infinite singularity with $I_w(P,Q) = 0$.*

*(ii) We consider the orbits of the class $\mathbf{QSL_5}$ under the action of the real affine group and time rescaling. The systems (V.1) up to (V.30) from the Table 4 form a system of representatives of these orbits under this action. A differential system $(S)$ in $\mathbf{QSL_5}$ is in the orbit of a system belonging to $(V.i)$ if and only if the corresponding conditions in the middle column (where the polynomials $H_i$ $(i = 7, \ldots, 11)$ and $N_j$ $(j = 5, 6)$ are $CT$-comitants to be introduced below) are verified for this system $(S)$. The conditions indicated in the middle column are invariant under the action of this group when jointly taken.*

*Wherever we have a case with invariant straight lines of multiplicity greater than one, we indicate the corresponding perturbations proving this in the Table 5.*

*Remark 58.* We observe that in the middle column of the Table 4 (and of the Table 2) there occur conditions of the form $\mathcal{M}(a,x,y) = 0$ in $\mathbb{R}[x,y]$ or of the form $\mathcal{M}(a,x,y) > 0$ (or $< 0$), where $\mathcal{M}(a,x,y)$ is a homogeneous polynomial in $a$ and separately in $x$ an $y$, which is a $CT$-comitant. All polynomials occurring in conditions of the second type are of even weight, of even degree in $a_{00}, \ldots, b_{02}$ and have a well determined sign on the corresponding variety indicated in the Lemma 62.

*Proof of Theorem 57.* Since we only discuss the case $C_2 \neq 0$, in what follows it suffices to consider only the canonical forms $(\mathbf{S}_I)$ to $(\mathbf{S}_{IV})$. The idea of the proof is the same as in the proof of the Theorem 50. We shall perform a case by case discussion for each one of these canonical forms, for which according to Lemma 55 we must examine two subcases: $(i)$ $N =$

**Diagram 2** ($M_{\mathbf{IL}} = 5$)

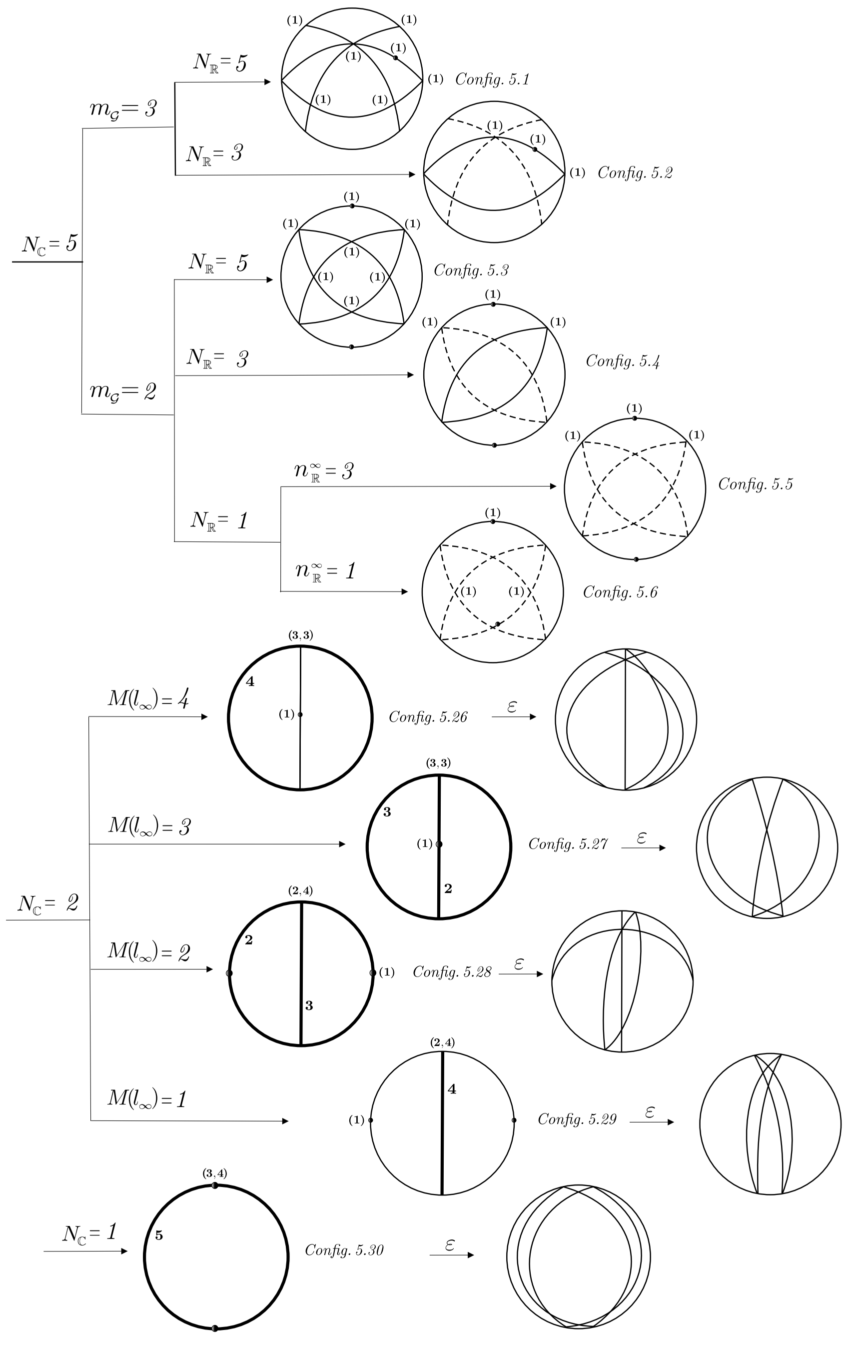

**Diagram 2** ($M_{\mathbf{IL}} = 5$) *(continued)*

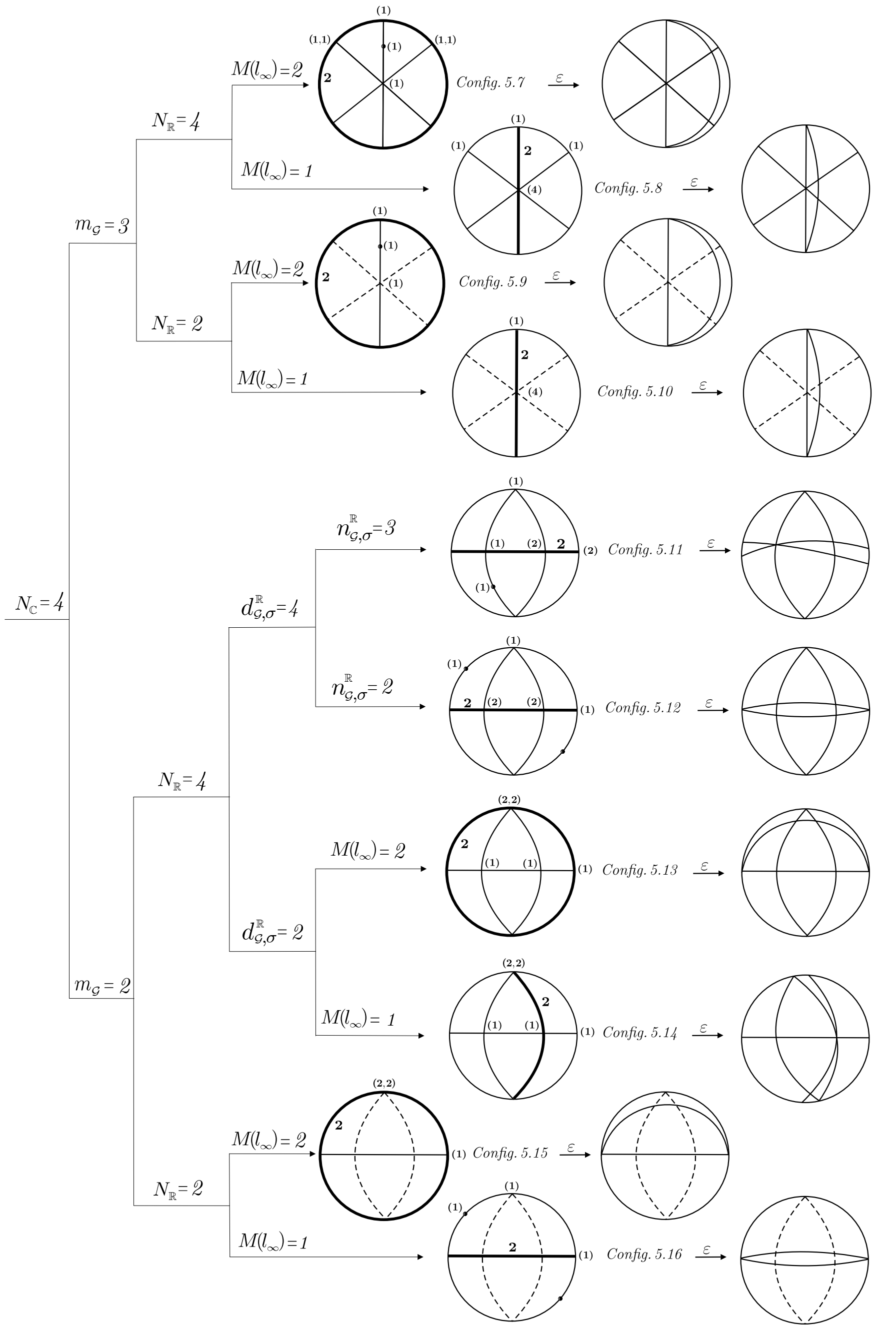

**Diagram 2** ($M_{\mathbf{IL}} = 5$) *(continued)*

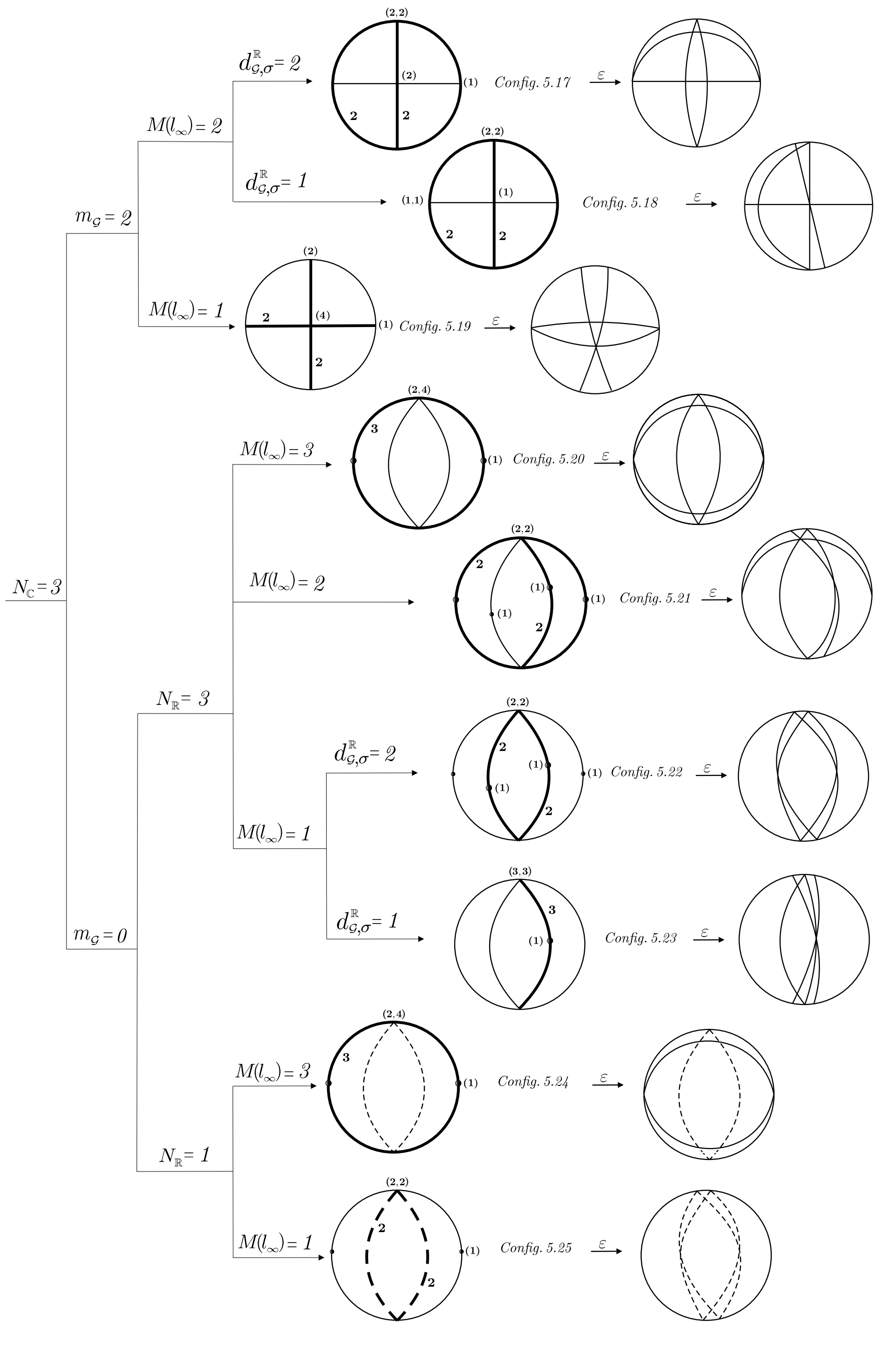

**TABLE 4.**

| Orbit representative | Necessary and sufficient conditions | Configu-ration |
|---|---|---|
| (V.1) $\begin{cases}\dot{x}=(x+1)(gx+1),\\ \dot{y}=(g-1)xy+y^2,\\ \quad g(g^2-1)\neq 0\end{cases}$ | $\eta>0,\ B_3=\theta=0,$ $N\neq 0,\ \mu\neq 0,\ H_1\neq 0$ | Config. 5.1 |
| (V.2) $\begin{cases}\dot{x}=g(x^2-4),\ g\neq 0\\ \dot{y}=(g^2-4)+(g^2+4)x\\ \quad -x^2+gxy-y^2\end{cases}$ | $\eta<0,\ B_3=\theta=0,$ $N\neq 0,\ \mu\neq 0,\ H_1\neq 0$ | Config. 5.2 |
| (V.3) $\begin{cases}\dot{x}=-1+x^2,\ g(g^2-1)\neq 0\\ \dot{y}=g(y^2-1),\end{cases}$ | $\eta>0,\ B_2=N=0,\ B_3\neq 0,$ $H_1>0,\ H_4=0,\ H_5>0$ | Config. 5.3 |
| (V.4) $\begin{cases}\dot{x}=-1+x^2,\\ \dot{y}=g(1+y^2),\quad g\neq 0\end{cases}$ | $\eta>0,\ B_2=N=0,\ B_3\neq 0,$ $H_4=0,\ H_5<0$ | Config. 5.4 |
| (V.5) $\begin{cases}\dot{x}=1+x^2,\ g(g^2-1)\neq 0\\ \dot{y}=g(1+y^2)\end{cases}$ | $\eta>0,\ B_2=N=0,\ B_3\neq 0,$ $H_1<0,\ H_4=0,\ H_5>0$ | Config. 5.5 |
| (V.6) $\begin{cases}\dot{x}=1+2xy,\\ \dot{y}=g-x^2+y^2,\quad g\in\mathbb{R}\end{cases}$ | $\eta<0,\ B_3\neq 0,\ B_2=N=0$ | Config. 5.6 |
| (V.7) $\begin{cases}\dot{x}=1+x,\\ \dot{y}=-xy+y^2\end{cases}$ | $\eta>0,\ B_3=\theta=0,$ $N\neq 0,\ \mu=H_6=0$ | Config. 5.7 |
| (V.8) $\begin{cases}\dot{x}=gx^2,\quad g(g^2-1)\neq 0\\ \dot{y}=(g-1)xy+y^2\end{cases}$ | $\eta>0,\ B_3=\theta=0,$ $N\neq 0,\ \mu\neq 0,\ H_1=0$ | Config. 5.8 |
| (V.9) $\begin{cases}\dot{x}=2x,\\ \dot{y}=1-x^2-y^2\end{cases}$ | $\eta<0,\ B_3=\theta=0,$ $N\neq 0,\ \mu=H_6=0$ | Config. 5.9 |
| (V.10) $\begin{cases}\dot{x}=gx^2,\quad g\neq 0\\ \dot{y}=-x^2+gxy-y^2\end{cases}$ | $\eta<0,\ B_3=\theta=0,$ $N\neq 0,\ \mu\neq 0,\ H_1=0$ | Config. 5.10 |
| (V.11) $\begin{cases}\dot{x}=x^2+xy,\\ \dot{y}=y+y^2\end{cases}$ | $\eta=0,\ M\neq 0,\ B_3=\theta=0,$ $\mu\neq 0,\ N\neq 0,\ D\neq 0$ | Config. 5.11 |
| (V.12) $\begin{cases}\dot{x}=-1+x^2,\\ \dot{y}=y^2\end{cases}$ | $\eta>0,\ B_2=N=0,\ B_3\neq 0,$ $H_1>0,\ H_4=H_5=0$ | Config. 5.12 |
| (V.13) $\begin{cases}\dot{x}=g(x^2-1),\\ \dot{y}=2y,\quad g(g^2-1)\neq 0\end{cases}$ | $\eta=0,\ M\neq 0, B_3=N=0,$ $H=N_1=0,\ N_2D\neq 0,\ N_5>0$ | Config. 5.13 |
| (V.14) $\begin{cases}\dot{x}=(x+1)(gx+1),\\ \dot{y}=(g-1)xy,\ g(g^2-1)\neq 0\end{cases}$ | $\eta=0,\ M\neq 0, B_3=\theta=0,$ $NK\neq 0,\ \mu=H_6=0$ | Config. 5.14 |
| (V.15) $\begin{cases}\dot{x}=g(x^2+1),\\ \dot{y}=2y,\quad g\neq 0\end{cases}$ | $\eta=0,\ M\neq 0, B_3=N=0,$ $H=N_1=0,\ N_2D\neq 0,\ N_5<0$ | Config. 5.15 |

| Orbit representative | Necessary and sufficient conditions | Configuration |
|---|---|---|
| $(V.16)$ $\begin{cases}\dot{x}=1+x^2,\\ \dot{y}=y^2\end{cases}$ | $\eta>0,\ B_2=N=0,\ B_3\neq 0,$ $H_1<0,\ H_4=H_5=0$ | Config. 5.16 |
| $(V.17)$ $\begin{cases}\dot{x}=x^2,\\ \dot{y}=2y\end{cases}$ | $\eta=0,\ M\neq 0, B_3=N=0,$ $H=N_1=N_5=0,\ N_2D\neq 0$ | Config. 5.17 |
| $(V.18)$ $\begin{cases}\dot{x}=1+x,\\ \dot{y}=-xy\end{cases}$ | $\eta=0,\ M\neq 0, B_3=\theta=0,$ $N\neq 0,\ \mu=K=H_6=0$ | Config. 5.18 |
| $(V.19)$ $\begin{cases}\dot{x}=x^2+xy,\\ \dot{y}=y^2\end{cases}$ | $\eta=0,\ M\neq 0, B_3=\theta=0,$ $\mu\neq 0,\ N\neq 0,\ D=0$ | Config. 5.19 |
| $(V.20)$ $\begin{cases}\dot{x}=-1+x^2,\\ \dot{y}=1\end{cases}$ | $\eta=0,\ M\neq 0, B_3=N=H=0,$ $D=N_1=0, N_2\neq 0, N_5>0$ | Config. 5.20 |
| $(V.21)$ $\begin{cases}\dot{x}=-1+x^2,\\ \dot{y}=x+2y\end{cases}$ | $\eta=0,\ M\neq 0, B_3=N=0,$ $H=N_2=0, D\neq 0,\ N_1\neq 0$ | Config. 5.21 |
| $(V.22)$ $\begin{cases}\dot{x}=1-x^2,\\ \dot{y}=1-2xy\end{cases}$ | $\eta=0,\ M\neq 0, B_2=N=0,$ $B_3\neq 0,\ H_2=0,\ H_3>0$ | Config. 5.22 |
| $(V.23)$ $\begin{cases}\dot{x}=-1+x^2,\\ \dot{y}=-3+y-x^2+xy\end{cases}$ | $\eta=M=0,\ N\neq 0,$ $B_3=\theta=N_6=0$ | Config. 5.23 |
| $(V.24)$ $\begin{cases}\dot{x}=1+x^2,\\ \dot{y}=1\end{cases}$ | $\eta=0,\ M\neq 0, B_3=N=H=0,$ $D=N_1=0, N_2\neq 0, N_5<0$ | Config. 5.24 |
| $(V.25)$ $\begin{cases}\dot{x}=-1-x^2,\\ \dot{y}=1-2xy\end{cases}$ | $\eta=0,\ M\neq 0, B_2=N=0,$ $B_3\neq 0,\ H_2=0,\ H_3<0$ | Config. 5.25 |
| $(V.26)$ $\begin{cases}\dot{x}=-x,\\ \dot{y}=y-x^2\end{cases}$ | $\eta=M=0,\ N_3\neq 0,$ $B_3=N=D_1=0$ | Config. 5.26 |
| $(V.27)$ $\begin{cases}\dot{x}=1+x,\\ \dot{y}=y-x^2\end{cases}$ | $\eta=M=0,\ N_4\neq 0,$ $B_3=N=N_3=0,\ D_1\neq 0$ | Config. 5.27 |
| $(V.28)$ $\begin{cases}\dot{x}=x^2,\\ \dot{y}=1+x\end{cases}$ | $\eta=0,\ M\neq 0, B_3=N=0,$ $H=D=N_2=0, N_1\neq 0$ | Config. 5.28 |
| $(V.29)$ $\begin{cases}\dot{x}=-x^2,\\ \dot{y}=1-2xy\end{cases}$ | $\eta=0,\ M\neq 0,\ B_2=N=0,$ $B_3\neq 0,\ H_2=H_3=0$ | Config. 5.29 |
| $(V.30)$ $\begin{cases}\dot{x}=1,\\ \dot{y}=-x^2\end{cases}$ | $\eta=M=0,\ N_4\neq 0,$ $B_3=N=N_3=D_1=0$ | Config. 5.30 |

**TABLE 5.**

| Perturbations | Invariant straight lines |
|---|---|
| $(V.7_\varepsilon): \dot{x}=(x+1)(\varepsilon x+1),\ \dot{y}=(\varepsilon-1)xy+y^2$ | $y,\ x+1,\ y-x-1,\ \varepsilon x+1$ |
| $(V.8_\varepsilon): \dot{x}=(x+\varepsilon)(gx+\varepsilon),\ \dot{y}=(g-1)xy+y^2$ | $y,\ x+\varepsilon,\ y-x-\varepsilon,\ gx+\varepsilon$ |
| $(V.9_\varepsilon): \begin{cases}\dot{x}=2x(\varepsilon x+1),\\ \dot{y}=1+2\varepsilon x-x^2+2\varepsilon xy-y^2\end{cases}$ | $x,\ \varepsilon x+1,\ y\pm ix+1$ |
| $(V.10_\varepsilon): \begin{cases}\dot{x}=4g\varepsilon^2+\varepsilon(g^2+4)x+gx^2,\\ \dot{y}=\varepsilon^2(4-g^2)-x^2+gxy-y^2\end{cases}$ | $x+g\varepsilon,\quad gx+4\varepsilon,$ <br> $x+g\varepsilon\pm i(y+2\varepsilon)$ |
| $(V.11_\varepsilon): \dot{x}=\varepsilon x+x^2+(1+\varepsilon)xy,\ \dot{y}=y+y^2$ | $x,\ y+1,\ y,\ x+\varepsilon y+\varepsilon$ |
| $(V.12_\varepsilon): \ \dot{x}=x^2-1,\ \dot{y}=y^2-\varepsilon^2$ | $x\pm1,\quad y\pm\varepsilon$ |
| $(V.13_\varepsilon): \ \dot{x}=g(x^2-1),\ \dot{y}=2y(\varepsilon y+1)$ | $y,\quad x\pm1,\quad \varepsilon y+1$ |
| $(V.14_\varepsilon): \dot{x}=(x+1)(gx+1),\ \dot{y}=(g-1)xy-\varepsilon y^2$ | $x+1,\ gx+1,\ y,\ x+\varepsilon y+1$ |
| $(V.15_\varepsilon): \ \dot{x}=g(x^2+1),\ \dot{y}=2y(\varepsilon y+1)$ | $y,\quad x\pm i,\quad \varepsilon y+1$ |
| $(V.16_\varepsilon): \ \dot{x}=x^2+1,\ \dot{y}=y^2-\varepsilon^2$ | $x\pm i,\quad y\pm\varepsilon$ |
| $(V.17_\varepsilon): \dot{x}=x^2-\varepsilon^2,\ \dot{y}=2y(\varepsilon y+1)$ | $y,\quad x\pm\varepsilon,\quad \varepsilon y+1$ |
| $(V.18_\varepsilon): \dot{x}=(x+1)(\varepsilon x+1),\ \dot{y}=(\varepsilon-1)xy-\varepsilon y^2$ | $x+1,\ \varepsilon x+1,\ y,\ x+\varepsilon y+1$ |
| $(V.19_\varepsilon): \ \dot{x}=\varepsilon^2x+x^2+(1+\varepsilon)xy,\ \dot{y}=\varepsilon y+y^2$ | $x,\ y,\ y+\varepsilon,\ x+\varepsilon y+\varepsilon^2$ |
| $(V.20_\varepsilon): \ \dot{x}=x^2-1,\ \dot{y}=1-\varepsilon^2y^2$ | $x\pm1,\quad \varepsilon y\pm1$ |
| $(V.21_\varepsilon): \begin{cases}\dot{x}=(x+1)(x+4\varepsilon x-1),\\ \dot{y}=(x+2y)(1+4\varepsilon y)\end{cases}$ | $x+1,\ x(1+4\varepsilon)-1,$ <br> $4\,\varepsilon y+1,\ x-8\,\varepsilon y-1$ |
| $(V.22_\varepsilon): \ \dot{x}=1-x^2,\ \dot{y}=1-2xy-\varepsilon y^2$ | $x\pm\ 1,\quad x+\varepsilon y\pm\sqrt{1+\varepsilon}$ |
| $(V.23_\varepsilon): \begin{cases}\dot{x}=(1+\varepsilon)(x-1+2\varepsilon)(x+1-2\varepsilon),\\ \dot{y}=(4\varepsilon^2-3)+(1+2\varepsilon)y-x^2\\ \qquad +(1-2\varepsilon)xy-2\varepsilon^2y^2\end{cases}$ | $x\pm(1-2\varepsilon),\ x+\varepsilon y-1,$ <br> $x+2\varepsilon y-1-2\varepsilon$ |
| $(V.24_\varepsilon): \ \dot{x}=x^2+1,\ \dot{y}=1-\varepsilon^2y^2$ | $x\pm i,\quad \varepsilon y\pm1$ |
| $(V.25_\varepsilon): \ \dot{x}=-1-x^2,\ \dot{y}=1-2xy-\varepsilon y^2$ | $x\pm\ i,\quad x+\varepsilon y\pm\ i\sqrt{1-\varepsilon}$ |
| $(V.26_\varepsilon): \begin{cases}\dot{x}=-x-2\varepsilon x^2,\\ \dot{y}=y-x^2+3\varepsilon^2y^2\end{cases}$ | $x,\ 3\varepsilon(x+\varepsilon y)+1,$ <br> $2\varepsilon x+1,\ \varepsilon(x-3\varepsilon y)-1$ |

| Perturbations | Invariant straight lines |
|---|---|
| $(V.27_\varepsilon): \begin{cases} \dot x = 1+x+\varepsilon x^2, \\ \dot y = y - x^2 - 2\varepsilon xy - 2\varepsilon^2 y^2 \end{cases}$ | $1+x+\varepsilon x^2$, $\varepsilon(x+2\varepsilon y)^2-(x+2\varepsilon y)-1$ |
| $(V.28_\varepsilon): \begin{cases} \dot x = (\varepsilon-1)\varepsilon^2+2\varepsilon^3 x + \\ \qquad +(1-\varepsilon)(1-2\varepsilon+3\varepsilon^2)x^2, \\ \dot y = (1-\varepsilon)(2\varepsilon^2 y+1)(x+2\varepsilon y+1) \end{cases}$ | $(\varepsilon-1)x-\varepsilon,\ \ 2\varepsilon^2 y+1$, $(1-2\varepsilon+3\varepsilon^2)x-\varepsilon(1-\varepsilon)$, $(\varepsilon-1)^2x-4\varepsilon^3 y-\varepsilon(\varepsilon+1)$ |
| $(V.29_\varepsilon):\ \ \dot x = \varepsilon^2 - x^2,\ \dot y = 1-2xy-\varepsilon y^2$ | $x\pm\ \varepsilon,\ x+\varepsilon y\pm\sqrt{\varepsilon^2+\varepsilon}$ |
| $(V.30_\varepsilon): \begin{cases} \dot x = 1+\varepsilon x+\varepsilon^3 x^2, \\ \dot y = \varepsilon y - x^2 - 2\varepsilon^3 xy - 2\varepsilon^6 y^2 \end{cases}$ | $1+\varepsilon x+\varepsilon^3 x^2$, $\varepsilon^3(x+2\varepsilon^3 y)^2-\varepsilon(x+2\varepsilon^3 y)-1$ |

$B_2 = 0$ and $(ii)$ $N \neq 0$, $\theta = B_3 = 0$. Each one of these conditions yields specific conditions on the parameters. The discussion proceeds further by breaking these cases in more subcases determined by more restrictions on the parameters. Finally we construct invariants or T-comitants which put these conditions in invariant form.

### 6.1. Systems with the divisor $\boldsymbol{D_S(C,Z)} = 1\cdot \boldsymbol{w}_1 + 1\cdot \boldsymbol{w}_2 + 1\cdot \boldsymbol{w}_3$

For this case we shall later need the following polynomials which are shown to be $T$-comitants in Lemma 62.

NOTATION 59. *Let us denote*

$$H_4(a) = \big((C_2,D)^{(2)},(C_2,D_2)^{(1)}\big)^{(2)},$$
$$H_5(a) = \big((C_2,C_2)^{(2)},(D,D)^{(2)}\big)^{(2)} + 8\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)},$$
$$H_6(a,x,y) = 16N^2(C_2,D)^{(2)} + H_2^2(C_2,C_2)^{(2)},$$

#### 6.1.1. *The case* $\boldsymbol{N} = 0 = \boldsymbol{B}_2$

It was shown above (see page 161) that the systems $(S_I)$ with $N(\boldsymbol{a},x,y) = 0$ can be brought by an affine transformation to the systems (22) for which we have

$$B_2 = 648\big[e^2(4k-4l-e^2)x^4 + 2d^2e^2(2x^2-3xy+2y^2) - d^2(4k-4l+d^2)y^4\big].$$

Hence the condition $B_2 = 0$ yields $de = e(4k-4l-e^2) = d(4k-4l+d^2) = 0$. According to Lemma 56, in order to have $M_{\text{IL}} = 5$ we must satisfy the condition $\deg\gcd(\mathcal{E}_1,\mathcal{E}_2) = 4$.

*We claim*, that for this it is necessary that $d = e = 0$. Indeed, let us suppose, that $de = 0$ but $d^2 + e^2 \neq 0$. Then by interchanging $x$ and $y$ we may assume $d = 0,\ e = 2$ via Remark 46 ($\gamma = e/2,\ s = 1$). Then we obtain the systems

$$\dot{x} = k + x^2, \quad \dot{y} = l + 2x + y^2, \tag{42}$$

for which the condition $B_2 = 2^7 3^4 (k - l - 1)x^4 = 0$ yields $k = l + 1$. Then for the systems (42) with $k = l + 1$ we obtain

$$\begin{aligned} \mathcal{E}_1 &= -2\big[Y^2 - YZ + Z(X + Z + lZ)\big]\mathcal{H}, \\ \mathcal{E}_2 &= -(X + Y - Z)\big[Y^2 + Z(2X + lZ)\big]\mathcal{H}, \end{aligned}$$

where $\mathcal{H} = (Y - X + Z)(X^2 + Z^2 + lZ^2)$. Thus, $\deg \mathcal{H} = 3$ and we shall show that for all values given to the parameter $l$ the degree of $\gcd(\mathcal{E}_1, \mathcal{E}_2)$ remains three. Indeed, since the common factor of the polynomials $\mathcal{E}_1/\mathcal{H}$ and $\mathcal{E}_2/\mathcal{H}$ must depend on $Y$, according to Lemma 45 it is sufficient to observe that $\mathrm{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = -8Z^2[X^2 + (1 + l)Z^2]^2 \neq 0$. This proves our claim and hence, the condition $d = e = 0$ must hold. Since for systems (22) we have $H_4 = 96(d^2 + e^2)$ this condition is equivalent to $H_4 = 0$.

Assuming $H_4 = 0$ (i.e. $d = e = 0$) the systems (22) become

$$\dot{x} = k + x^2, \quad \dot{y} = l + y^2, \tag{43}$$

and calculations yield $\mathcal{E}_1 = 2(X - Y)\,\mathcal{H}$, $\mathcal{E}_2 = \big[X^2 - Y^2 + (k - l)Z^2\big]\mathcal{H}$, where $\mathcal{H} = (X^2 + kZ)(Y^2 + lZ^2)$. Hence by Theorem 36 each system in the family (43) possesses four invariant affine lines which means that for these systems $M_{\mathrm{IL}} \geq 5$. We observe that to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ it is necessary and sufficient that $k - l = 0$. So, to have $M_{\mathrm{IL}} = 5$ the condition $k - l \neq 0$ must be satisfied. This condition is equivalent to $B_3 \neq 0$, since for the systems (43) we have $B_3 = 4(l - k)x^2y^2$.

Systems (43) possess the invariant lines, components of the conics: $x^2 + k = 0$, $y^2 + l = 0$, and then we obtain the following configurations of invariant straight lines (Diagram 2):
($i$) Config. 5.3 for $k < 0$ and $l < 0$; ($ii$) Config. 5.4 for $kl < 0$; ($iii$) Config. 5.5 for $k > 0$ and $l > 0$; ($iv$) Config. 5.12 for $kl = 0$ and $k + l < 0$; ($v$) Config. 5.16 for $kl = 0$ and $k + l > 0$.

On the other hand for the systems (43) we have: $H_1 = -2^7 3^2\,(k + l)$, $H_5 = 2^{11} 3\,kl$. Herein we conclude that these two $T$-comitants capture in invariant form exactly the conditions for distinguishing the Configurations 5.3–5.5, 5.12 and 5.16 as it is indicated in Table 4.

We observe that if for the systems (43) the condition $H_5 \leq 0$ holds (i.e. $kl \leq 0$) then interchanging $x$ and $y$ we may assume $l \geq 0$. Moreover, by Remark 46 ( $\gamma = |k|,\ s = 1/2$) we may assume $k \in \{-1, 1\}$. We also note that for $l < 0$ (respectively $l > 0$) we may set $l = -g^2$ (respectively, $l = g^2$) and due to the substitution $y \to gy$ we obtain the canonical system ($V.3$) (respectively ($V.4$) and ($V.5$) ) from Table 4. We note that for the systems ($V.3$) (respectively, ($V.5$) ) we have $B_3 = -12g(g^2-1)x^2y^2$ (respectively $B_3 = 12g(g^2-1)x^2y^2$) and hence in both cases the condition $B_3 \neq 0$ yields $g(g^2-1) \neq 0$ .

*6.1.2. The case* $\boldsymbol{N} \neq 0$, $\boldsymbol{\theta} = 0 = \boldsymbol{B}_3$

For the canonical systems ($\mathbf{S}_I$) we calculate $\theta = -8(h-1)(g-1)(g+h)$. Hence the condition $\theta = 0$ yields $(h-1)(g-1)(g+h) = 0$ and without loss of generality we can consider $h = 1$. Indeed, if $g = 1$ (respectively, $g + h = 0$) we can apply the linear transformation which will replace the straight line $x = 0$ with $y = 0$ (respectively, $x = 0$ with $y = x$) reducing this case to $h = 1$. Assume $h = 1$. Then $N = (g^2-1)x^2 \neq 0$ and we may assume $e = f = 0$ via a translation. Thus the systems ($\mathbf{S}_I$) become

$$\dot{x} = k + cx + dy + gx^2, \quad \dot{y} = l + (g-1)xy + y^2 \tag{44}$$

and calculations yield $\mu = 32g^2$ and $B_3 = -3l(g-1)^2x^4 + 6l(g-1)^2x^3y - 6d^2gxy^3 + 3d^2gy^4 + 3\left[(4gl - k(g+1)^2 + c^2 + cd - cdg\right]x^2y^2$. The condition $B_3$=0 implies $dg = 0$. We shall examine two subcases: $\mu \neq 0$ and $\mu = 0$.

*The subcase* $\mu \neq 0$. In this case we obtain $g \neq 0$, and from $g - 1 \neq 0$ the condition $B_3 = 0$ for the systems (44) yields $d = l = c^2 - k(g+1)^2 = 0$. Since $N \neq 0$ then $g + 1 \neq 0$ and we may set $c = u(g+1)$ where $u$ is a new parameter. Then $k = u^2$ and we obtain the systems

$$\dot{x} = u^2 + u(g+1)x + gx^2, \quad \dot{y} = (g-1)xy + y^2, \tag{45}$$

for which $H_1 = 576u^2(g-1)^2$.

1. If $H_1 \neq 0$ then $u \neq 0$ and we may assume $u = 1$ via Remark 46 ($\gamma = u$, $s = 1$). This leads to the systems

$$\dot{x} = (x+1)(gx+1), \quad \dot{y} = (g-1)xy + y^2, \tag{46}$$

for which $g(g^2-1) \neq 0$ and calculations yield:

$$\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = Y(Y - X - Z)(X + Z)(gX + Z). \tag{47}$$

Hence $\deg \mathcal{H} = 4$. By hypothesis $N \neq 0$ and hence, according to Lemma 56 for every $g$ such that $g(g^2-1) \neq 0$, $M_{\text{IL}} \leq 5$. By Theorem 36, from (47) the systems (46) possess the following four distinct invariant affine lines: $y = 0$, $x+1 = 0$, $x-y+1 = 0$, $gx+1 = 0$. Thus we obtain the Config. 5.1.

2. For $H_1 = 0$ we have $u = 0$ and the systems (45) become

$$\dot{x} = gx^2, \quad \dot{y} = (g-1)xy + y^2, \tag{48}$$

with $g(g^2-1) \neq 0$ and we calculate: $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = gX^2Y(X-Y)$. Hence $\deg \mathcal{H} = 4$ and we obtain $M_{\text{IL}} \geq 5$. Since $N \neq 0$ by Lemma 56, $M_{\text{IL}}$ cannot be equal to 6. The systems (48) possess the invariant lines $x = 0$, $y = 0$ and $x = y$. Moreover, according to Lemma 38 the line $x = 0$ could be of multiplicity two and the perturbations $(V.8_\varepsilon)$ from Table 5 show this. Hence, for $H_1 = 0$ we obtain Config. 5.8.

*The subcase* $\mu = 0$. The condition $\mu = 32g^2 = 0$ yields $g = 0$, and for the systems (44) the condition $B_3 = 0$ yields $g = l = c(c+d) - k = 0$. Thus, $g = l = 0$, $k = c(c+d) \neq 0$, otherwise we get degenerate systems (44). Hence, we may assume $c = 1$ via Remark 46 ($\gamma = c$, $s = 1$) and we obtain the systems

$$\dot{x} = d+1+x+dy, \quad \dot{y} = -xy + y^2. \tag{49}$$

Calculations yield:

$$\begin{aligned}
&\mathcal{E}_1 = \left[-X^2 + 2XY + d(Y+Z)^2 + Z(2Y+Z)\right]\mathcal{H},\\
&\mathcal{E}_2 = (Y-X)(Y+Z)\left[X+Z+d(Y+Z)\right]\mathcal{H}, \quad \mathcal{H} = YZ(X-Y+Z+dZ),\\
&\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = -9d(d+1)^2(Y+Z)^6.
\end{aligned}$$

Hence $\deg \mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = 3$ and the condition on the parameter $d$ so as to have an additional common factor of $\mathcal{E}_1$, $\mathcal{E}_2$, according to Lemma 45 is $\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) \equiv 0$. Since $d+1 \neq 0$ (otherwise we get the degenerate system (49)) this condition yields $d = 0$. Then we obtain the following system

$$\dot{x} = 1+x, \qquad \dot{y} = -xy + y^2 \tag{50}$$

for which $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = YZ(X+1)(X-Y+Z)$. We observe that this system possesses the invariant affine straight lines: $y = 0$, $x+1 = 0$, $x-y+1 = 0$. Taking into account that $Z \mid \mathcal{H}$, we have by Corollary 39 that the line $Z = 0$ could be of multiplicity two. This is confirmed by the perturbations $(V.7_\varepsilon)$ from Table 5. On the other hand for the systems (49) calculations yields $H_6 = 128dx^2(x^2 - xy + y^2)(x^2 - 2xy - dy^2)$. Hence, the conditions $g = 0 = d$ are equivalent to $\mu = 0$ and $H_6 = 0$. In this case we obtain Config. 5.7.

### 6.2. Systems with the divisor $D_S(C, Z) = 1 \cdot w_1^c + 1 \cdot w_2^c + 1 \cdot w_3$

We are in the case of the canonical form $(\mathbf{S}_{II})$.

#### 6.2.1. *The case* $N = 0 = B_2$

It was shown above (see page 161) that the systems $(\mathbf{S}_{II})$ with $N(\boldsymbol{a}, x, y) = 0$ can be brought by an affine transformation to the systems (24) for which we have

$$\begin{aligned} B_2 =& 648\left[(8efk - (f^2 + e^2)^2\right]x^4 - 16k(e^2 - f^2)xy(x^2 - y^2) - \\ &- 48efkx^2y^2 + 8efky^4, \\ B_3 =& 6\left[(ef - 2k)x^4 + (f^2 - e^2)x^3y - (4k + ef)x^2y^2 - 2ky^4\right]. \end{aligned}$$

If $B_3 = 0$ then $k = e = f = 0$ and we obtain the systems (25) for which $M_{\mathbf{IL}} = 6$ (see page 161). Hence $B_3 \neq 0$ and this implies $k \neq 0$, otherwise from $B_2 = 0$ we obtain again $e = f = 0$. Therefore $k \neq 0$ and we may consider $k > 0$ via the change $x \to -x$ and by Remark 46 $(\gamma = k,\ s = 1/2)$ we may assume $k = 1$. Then the condition $B_2 = 0$ yields $e = f = 0$ and we obtain the systems

$$\dot{x} = 1 + 2xy, \qquad \dot{y} = l - x^2 + y^2 \tag{51}$$

which possess the invariant lines $y + ix = \pm\sqrt{-l - i}$, $y - ix = \pm\sqrt{i - l}$. This leads to the Config. 5.6.

#### 6.2.2. *The case* $N \neq 0$, $\theta = B_3 = 0$

For the systems $(\mathbf{S}_{II})$ we calculate

$$\theta = 8(h+1)[(h-1)^2 + g^2], \quad N = (g^2 - 2h + 2)x^2 + 2g(h+1)xy + (h^2 - 1)y^2$$

and hence by $N \neq 0$, the condition $\theta = 0$ yields $h = -1$. Then we may assume $f = 0$ due to a translation and the systems $(\mathbf{S}_{II})$ become

$$\dot{x} = k + cx + dy + gx^2, \quad \dot{y} = l + ex - x^2 + gxy - y^2. \tag{52}$$

For these systems calculations yield Coefficient$[B_3,\ y^4] = -3d^2g$ and $\mu = 32\,g^2$. We shall examine two subcases: $\mu \neq 0$ and $\mu = 0$.

*The subcase* $\mu \neq 0$. This yields $g \neq 0$ and then the condition $B_3 = 0$ implies $d = 0$. Moreover, we may assume $c = 0$ via the translation of the origin of coordinates to the point $(-c/(2g), -c/4)$. Thus, the systems (52) become

$$\dot{x} = k + 2gx^2, \quad \dot{y} = l + ex - x^2 + 2gxy - y^2,$$

for which we calculate

$$B_3 = 3\left[k(4-g^2) - 4gl\right]x^2(x^2-y^2) + 6\left[l(4-g^2) + 4gk + e^2\right]x^3y.$$

Hence, the condition $B_3 = 0$ yields the following linear system of equations with respect to parameters $k$ and $l$:

$$k(4-g^2) - 4gl = 0, \quad 4gk + l(4-g^2) + e^2 = 0.$$

Setting $e = u(g^2+4)$ ( $u$ is a new parameter) we have the following solution of this system: $k = -4gu^2$, $l = (g^2-4)u^2$. Thus we obtain the systems

$$\dot{x} = -4gu^2 + gx^2, \;\; \dot{y} = (g^2-4)u^2 + u(g^2+4)x - x^2 + gxy - y^2, \quad (53)$$

for which $H_1 = -2^{12}3^2u^2g^2$.

1. If $H_1 \neq 0$ we have $u \neq 0$ and we can assume $u = 1$ via the Remark 46 ($\gamma = u$, $s = 1$). Hence the systems (53) become

$$\dot{x} = g(x^2-4), \;\; \dot{y} = (g^2-4) + (g^2+4)x - x^2 + gxy - y^2 \quad (54)$$

and calculations yield: $\mathcal{H} = g(X-2Z)(X+2Z)(X^2+Y^2-4XZ+2\,gYZ+4Z^2+g^2Z^2)$. Hence, $M_{\mathbf{IL}} \geq 5$ and since $N \neq 0$ by Lemma 56 $M_{\mathbf{IL}}$ cannot be equal to 6. By Theorem 36 the systems (54) possess the following four distinct invariant straight lines: $y - ix + g + 2i = 0$, $y + ix - 2i + g = 0$, $x = \pm 2$. Thus we obtain the Config. 5.2.

2. For $H_1 = 0$ we have $u = 0$ and the systems (53) become

$$\dot{x} = gx^2, \quad \dot{y} = -x^2 + gxy - y^2, \quad (55)$$

with $g \neq 0$. We calculate $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = gX^2(X^2+Y^2)$ and hence $\deg \mathcal{H} = 4$. Since $N \neq 0$ by Lemma 56 we obtain that $M_{\mathbf{IL}}$ equals exactly 5. The systems (55) possess the following invariant straight lines: $x = 0$, $y = \pm ix$ and the line $x = 0$ could be of multiplicity two. This is confirmed by the perturbations ($V.10_\varepsilon$) from Table 5. Thus we get Config. 5.10.

*The subcase* $\mu = 0$. Then we obtain $g = 0$ and we may assume $e = 0$ via a translation. Therefore the systems (52) become $\dot{x} = k + cx + dy$, $\dot{y} = l - x^2 - y^2$ and we calculate $B_3 = 12kx^4 + 6(4l - c^2 - d^2)x^3y - 12kx^2y^2$. Hence the condition $B_3 = 0$ yields $k = 4l - c^2 - d^2 = 0$. We replace $c$ by $2c$ and $d$ by $2d$ and then we obtain $l = c^2 + d^2$. This leads to the systems:

$$\dot{x} = 2cx + 2dy, \quad \dot{y} = c^2 + d^2 - x^2 - y^2 \tag{56}$$

for which calculations yield:

$$\begin{aligned}
\mathcal{E}_1 &= \big[dX^2 - 2\,cXY - dY^2 + 2\,(c^2 + d^2)XZ + d(c^2 + d^2)Z^2\big]\,\mathcal{H},\\
\mathcal{E}_2 &= (cX + dY)[X^2 + Y^2 + 2\,dXZ - 2\,cYZ + (c^2 + d^2)Z^2]\,\mathcal{H},\\
\mathcal{H} &= 2Z[X^2 + Y^2 - 2\,dXZ + 2\,cYZ + (c^2 + d^2)Z^2].
\end{aligned}$$

Thus, $\deg \mathcal{H} = 3$ and we need an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$. Since $c^2 + d^2 \neq 0$ we observe that such a common factor of the polynomials $\mathcal{E}_1/\mathcal{H}$ and $\mathcal{E}_2/\mathcal{H}$ must depend on $X$. Hence, by Lemma 45 the following condition must hold:

$$\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 4\,d(c^2 + d^2)^2(Y - cZ)^6 = 0.$$

Therefore the condition $d = 0$ must be satisfied and then $c \neq 0$ (otherwise we get degenerate system from (56)). On the other hand for the systems (56) we have $H_6 = -2^{13}dx^3(3x^2 - y^2)(dx^2 - 2cxy - dy^2)$ and hence the condition $d = 0$ is are equivalent to $H_6 = 0$. We may assume $c = 1$ via the Remark 46 ($\gamma = c$, $s = 1$) and then we obtain the system

$$\dot{x} = 2x, \quad \dot{y} = 1 - x^2 - y^2. \tag{57}$$

For this system we calculate $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = 4\,XZ(X^2 + Y^2 + 2YZ + Z^2)$ and according to Theorem 36 the system (57) possesses the invariant affine lines: $x = 0$ and $y \pm ix + 1 = 0$. Moreover, by Corollary 39 the line $l_\infty : Z = 0$ could be of multiplicity two. This is confirmed by the perturbations ($V.9_\varepsilon$) from Table 5. Therefore we obtain the Config. 5.9.

### 6.3. Systems with the divisor $\boldsymbol{D_S(C, Z)} = 2 \cdot \boldsymbol{w}_1 + 1 \cdot \boldsymbol{w}_2$

We are in the case of the canonical form ($\mathbf{S}_{III}$). For this case we shall later need the following polynomial which is shown to be a $CT$-comitant in Lemma 62.

NOTATION 60. *Let us denote* $N_5(\boldsymbol{a}, x, y) = \big((D_2, C_1)^{(1)} + D_1D_2\big)^2 - 4\big(C_2, C_2\big)^{(2)}\big(C_0, D_2\big)^{(1)}$.

*6.3.1. The case* $\boldsymbol{N} = 0 = \boldsymbol{B}_2$

It was previously shown (see page 162) that to examine the systems $(\mathbf{S}_{III})$ with $N(\boldsymbol{a}, x, y) = 0$ we have to consider two subcases: $H(\boldsymbol{a}, x, y) \neq 0$ and $H(\boldsymbol{a}, x, y) = 0$.

*The subcase* $H(\boldsymbol{a}, x, y) \neq 0$. In this case the systems $(\mathbf{S}_{III})$ with $N = 0$ can be brought by an affine transformation to the systems (27) (see page 162) for which we have: $B_2 = -648d(8clx^4 + 16dlx^3y + d^3y^4)$. Therefore, the condition $B_2 = 0$ yields $d = 0$ and we obtain the systems

$$\dot{x} = k + cx - x^2, \qquad \dot{y} = l - 2xy, \tag{58}$$

for which calculations yield: $\mathcal{E}_1 = \left[-2X^2Y + Z^2(2kY + clZ)\right]\mathcal{H}$, $\mathcal{H} = (kZ^2 + cXZ - X^2)$, $\mathcal{E}_2 = (X^2 - cXZ - kZ^2)(2XY - lZ^2)\mathcal{H}$. Thus, $\deg \mathcal{H} = 2$ and to have $M_{\text{IL}} = 5$ the polynomials $\mathcal{E}_1/\mathcal{H}$ and $\mathcal{E}_2/\mathcal{H}$ must have a common factor of degree two. We observe, that this common factor necessarily depends on $X$ and hence by Lemma 45 the following condition must hold:

$$\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = -2c^2YZ^6(4kY^2 + 2clYZ - l^2Z^2)^2 \equiv 0.$$

Herein we obtain either $c = 0$ or $k = l = 0$, but the second case yields degenerate systems. So, we assume $c = 0$ and then for the systems (58) we obtain $\mathcal{E}_1 = 2Y\mathcal{H}$, $\mathcal{H} = (kZ^2 - X^2)^2$, $\mathcal{E}_2 = (-6XY + 3lZ^2)\mathcal{H}$. We observe, that $\deg \mathcal{H} = 4$ and that the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ do not have an additional common factor if and only if $l \neq 0$. This condition is equivalent to $B_3 \neq 0$, since for the systems (58) we have $B_3 = -12lx^4$. As $l \neq 0$ we may consider $l = 1$ via the rescaling $y \to ly$ and we obtain the systems

$$\dot{x} = k - x^2, \qquad \dot{y} = 1 - 2xy. \tag{59}$$

Moreover, due to the rescaling $x \to |k|^{1/2}x$, $y \to |k|^{-1/2}y$ and $t \to |k|^{-1/2}t$ (for $k \neq 0$) we may assume $k \in \{-1, 0, 1\}$. These systems possess two invariant lines $x = \pm\sqrt{k}$. By Lemma 38 for $k \neq 0$ each one of these lines could be of multiplicity two and for $k = 0$ the invariant line $x = 0$ of the system (59) is of the multiplicity four. This is confirmed by the perturbations $(V.22_\varepsilon)$ (respectively $(V.25_\varepsilon)$; $(V.29_\varepsilon)$) from Table 5 for $k = 1$ (respectively $k = -1$; $k = 0$). Thus, we obtain Config. 5.22 (respectively, Config. 5.25; Config. 5.29).

On the other hand for the systems (58) calculations yield: $H_2 = 16cx^2$ and $H_3 = 32kx^2$. Hence, these $T$-comitants capture exactly the conditions

$c = 0$ and $k < 0$ (respectively $c = 0$, $k = 0$ or $c = 0$, $k > 0$) . It remains to observe that the condition $B_3 \neq 0$ implies $H \neq 0$, since for $H = 0$ the condition $B_2 = 0$ implies $B_3 = 0$ (see the subcase $H(a, x, y) = 0$ below).

*The subcase* $H(a, x, y) = 0$. It is previously shown (see page 163) that if $N = H = 0$ then the systems $(\mathbf{S}_{III})$ can be brought by an affine transformation to the systems (30). For these systems we have $B_2 = -648d^4y^4$, $B_3 = 6dxy^2(fx - dy)$ and hence the condition $B_2 = 0$ yields $d = 0$. Therefore the conditions $B_2 = 0$ and $B_3 = 0$ are equivalent and since for any quadratic system (10) the condition $B_3 = 0$ implies $B_2 = 0$ (see the formulas (14) on page 148), we shall use in this case the condition $B_3 = 0$.

Assuming $d = 0$ we obtain the systems (31) for which $D(x, y) = -f^2x^2y$ and we shall consider two subcases: $D \neq 0$ and $D = 0$.

*(1)* For $D \neq 0$ the systems (31) can be brought by an affine transformation to the systems (32) and calculations yield the values (33) of the affine comitants $\mathcal{E}_1$ and $\mathcal{E}_2$. We observe that deg $\mathcal{H} = 3$ and taking into account that the polynomials $\mathcal{E}_1/\mathcal{H}$ and $\mathcal{E}_2/\mathcal{H}$ cannot have the common factor $Z$, to have an additional factor of these polynomials according to Lemma 45 at least one of the following two conditions must hold: $\text{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = -36e(k+1)(4Y^2Z+e^2kZ^3)^2 = 0$, $\text{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_1/\mathcal{H}) = -6e(X^2+kZ^2)^2 = 0$. Thus we obtain either the condition $e = 0$ or $k = -1$. On the other hand for systems (32) we obtain $N_1 = 8ex^4$ and $N_2 = 16(k + 1)x$ and we shall consider two subcases: $N_1 = 0$ and $N_1 \neq 0$, $N_2 = 0$.

*(1a)* Assume $N_1 = 0$. Then $e = 0$ and the systems (32) become

$$\dot{x} = k + x^2, \qquad \dot{y} = 2y. \tag{60}$$

Calculations yield: $\mathcal{E}_1 = 2(X - Z)\mathcal{H}$, $\mathcal{E}_2 = 3(X^2 + kZ^2)\mathcal{H}$, $\mathcal{H} = 4YZ(kZ^2 + X^2)$, $\text{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 12(k + 1)Z^2$. Hence $\deg \mathcal{H} = 4$ and we observe that in order not to have an additional common factor of the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ we must have $k + 1 \neq 0$ (i.e. $N_2 \neq 0$). The systems (60) possess the invariant affine lines $y = 0$, $x = \pm\sqrt{-k}$. According to Corollary 39 the line $l_\infty : Z = 0$ could be of multiplicity 2 and the line $x = 0$ also could be of multiplicity 2 in the case when $k = 0$. Since for systems (32) we have $N_5 = -64kx^2$, we obtain Config. 5.13 for $N_5 > 0$, Config. 5.15 for $N_5 < 0$ and Config. 5.17 for $N_5 = 0$.

Note that for $k < 0$ (respectively, $k > 0$) one can set $k = -g^2$ (respectively, $k = g^2$) and due to the substitution $x \to gx$ we obtain the canonical system $(V.13)$ (respectively, $(V.15)$) from Table 4. It remains to observe that the perturbations ($V.13_\varepsilon$), ($V.15_\varepsilon$) and ($V.17_\varepsilon$) from Table 5 confirm the validity of the Config. 5.13, 5.15 and 5.17, respectively.

*(1b)* For $N_1 \neq 0$, $N_2 = 0$ we have $e \neq 0$, $k = -1$ and then for the systems (32) calculations yield: $\mathcal{E}_1 = [4Y + e(X - Z)]\,\mathcal{H}$, $\mathcal{E}_2 = (ceX + 2Y)(X + Z)\,\mathcal{H}$, $\mathcal{H} = Z(X + Z)(X - Z)^2$, $\operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = -2e(X + Z)^2$. Hence $\deg \mathcal{H} = \deg\gcd(\mathcal{E}_1, \mathcal{E}_2) = 4$ and since $N_1 \neq 0$ (i.e. $e \neq 0$) the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ could not have an additional common factor. Assuming $e = 1$ via the rescaling $y \to ey$ the systems (32) become

$$\dot{x} = -1 + x^2, \qquad \dot{y} = x + 2y. \tag{61}$$

This system possesses the invariant lines $x = \pm 1$. According to Lemma 38 and Corollary 39 the line $x = 1$ as well as the line $Z = 0$ could be of multiplicity two. This is confirmed by the perturbations ($V.21_\varepsilon$) from Table 5. Thus for $N_1 \neq 0$ and $N_2 = 0$ we obtain Config. 5.21.

*(2)* If $D = 0$ then we have $f = 0$ and the systems (31) become the systems (35) (see page 164) for which calculations yield the corresponding expressions (36) for the affine comitants $\mathcal{E}_1$ and $\mathcal{E}_2$. As $\deg \mathcal{H} = 3$ we need an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$. Taking into account that these polynomials depend only on $X$ and $Z$, according to Lemma 45 at least one of the following two conditions must hold:

$$\begin{aligned}\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) &= -4\,ek(e^2k + l^2)^2 Z^6 = 0,\\ \operatorname{Res}_Z(\mathcal{E}_1/\mathcal{H}, \mathcal{E}_2/\mathcal{H}) &= -4\,ek(e^2k + l^2)^2 X^6 = 0.\end{aligned}$$

Hence we obtain either $ek = 0$ or $e^2k + l^2 = 0$. Since the second condition leads to degenerate systems, we must examine the conditions $e = 0$ and $k = 0$. For systems (35) we have $N_1 = 8ex^4$ and $N_2 = 16kx$ and we shall consider two subcases: $N_1 = 0$ and $N_1 \neq 0$, $N_2 = 0$.

*(2a)* Assume $N_1 = 0$. Then $e = 0$ and the systems (35) become $\dot{x} = k + x^2$ $\dot{y} = l$. Calculations yield $\mathcal{E}_1 = X\,\mathcal{H}$, $\mathcal{E}_2 = (X^2 + kZ^2)\,\mathcal{H}$, $\mathcal{H} = lZ^2(X^2 + kZ^2)$, $\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 4\,kZ^2$. Therefore $\deg \mathcal{H} = 4$ and the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ could not have an additional common factor if and only if $k \neq 0$ (i.e. $N_2 \neq 0$). Since $l \neq 0$ (otherwise we get degenerate systems) after the rescaling $x \to |k|^{1/2}x$, $y \to l|k|^{-1/2}y$ and $t \to |k|^{-1/2}t$ we get the systems

$$\dot{x} = k + x^2, \qquad \dot{y} = 1 \tag{62}$$

with $k \in \{-1, 1\}$. The systems (62) possess two invariant affine lines: $x = \pm\sqrt{-k}$ which are distinct due to the condition $k \neq 0$. Moreover, by Corollary 39 the line $Z = 0$ could be of multiplicity three. This is confirmed by the perturbations $(V.20_\varepsilon)$ (for $k < 0$) and $(V.24_\varepsilon)$ (for $k > 0$) from Table 5. On the other hand for systems (62) we have $N_5 = -64kx^2$. Therefore, for $N_1 = 0$ and $N_2 \neq 0$ we obtain the Config. 5.20 if $N_5 > 0$ and the Config. 5.24 if $N_5 < 0$.

*(2b)* Let $N_1 \neq 0$, $N_2 = 0$. In this case we have $e \neq 0$, $k = 0$ and systems (35) become

$$\dot{x} = x^2, \qquad \dot{y} = l + ex. \tag{63}$$

For the systems (63) calculations yield: $\mathcal{E}_1 = (eX + 2lZ)\mathcal{H}$, $\mathcal{E}_2 = X(eX + lZ)\mathcal{H}$, $\mathcal{H} = X^3Z$. Therefore deg $\mathcal{H} = 4$ and since $l \neq 0$ (otherwise we get the degenerate systems (63)) and $e \neq 0$ ( $N_1 \neq 0$) we conclude that the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ could not have an additional common factor, i.e. each non-degenerate system of the family (63) belongs to $\mathbf{QSL_5}$. Via the rescaling $x \to le^{-1}x$, $y \to e\,y$ and $t \to el^{-1}t$ systems (63) become

$$\dot{x} = x^2, \qquad \dot{y} = 1 + x. \tag{64}$$

This system possesses the invariant line $x = 0$. Taking into account the polynomial $\mathcal{H}$, by Lemma 38 and Corollary 39 we obtain that the line $x = 0$ could be of multiplicity three whereas the line $Z = 0$ could be of multiplicity two. This is confirmed by the perturbations $(V.28_\varepsilon)$ from Table 5. Thus we obtain the Config. 5.28.

#### 6.3.2. *The case* $\boldsymbol{N} \neq 0$, $\boldsymbol{\theta} = 0 = \boldsymbol{B}_3$

Since for the systems $(\mathbf{S}_{III})$ we have

$$\theta = -8h^2(g-1), \quad \mu = 32gh^2, \quad N = (g^2-1)x^2 + 2h(g-1)xy + h^2y^2 \tag{65}$$

we shall consider two cases: $\mu \neq 0$ and $\mu = 0$.

*The subcase* $\mu \neq 0$. Then $gh \neq 0$ and the condition $\theta = 0$ yields $g = 1$. Then the systems $(\mathbf{S}_{III})$ with $g = 1$ by the transformation $x \to x - d/h$, $y \to (hy + 2d - ch)/h^2$ will be brought to the systems: $\dot{x} = k + x^2 + xy$, $\dot{y} = l + ex + fy + y^2$, for which $B_3 = -3e^2x^4 + (3l - 12k)x^2y^2 - 6kxy^3$. Hence the condition $B_3 = 0$ yields $e = k = l = 0$ and we obtain the systems

$$\dot{x} = x^2 + xy, \quad \dot{y} = fy + y^2, \tag{66}$$

for which we can assume $f \in \{0, 1\}$ via Remark 46 ($\gamma = f,\ s = 1$). For the systems (66) calculations yield: $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = X^2Y(Y + fZ)$. Hence, deg $\mathcal{H} = 4$, i.e $M_{\text{IL}} \geq 5$ and since $N \neq 0$ by Lemma 56 we have $M_{\text{IL}} < 6$. By Theorem 36 the systems (66) possess the invariant lines $x = 0$, $y = 0$ and $y + f = 0$. Moreover, according to Lemma 38 the line $x = 0$ of could be of the multiplicity two, and the lines $y = 0$ and $y = -f$ are distinct if and only if $f \neq 0$. Since for the systems (66) we have $D = -f^2x^2y$, the condition $f \neq 0$ can be expressed by using this $T$-comitant.

If $D \neq 0$ (then $f = 1$) the perturbed systems $(V.11_\varepsilon)$ from Table 5 show that the invariant line $x = 0$ is double one. Thus, for $D \neq 0$ we obtain Config. 5.11.

Assume $D = 0$. Then $f = 0$ and the invariant line $x = 0$ as well as the line $y = 0$ is of multiplicity two. This is confirmed by the perturbed systems $(V.19_\varepsilon)$ from Table 5. Therefore the case $D = 0$ leads to the Config. 5.19.

*The subcase* $\mu = 0$. In this case from (65) we obtain $h = 0$ and the condition $N \neq 0$ yields $g^2 - 1 \neq 0$. Then the systems $(\mathbf{S}_{III})$ with $h = 0$ will be brought via the translation $x \to x + f/(1-g)$, $y \to y + e/(1-g)$ to the systems:

$$\dot{x} = k + cx + dy + gx^2, \quad \dot{y} = l + (g-1)xy, \tag{67}$$

for which $B_3 = -3\,l\,(g-1)^2x^4 - 3\,c\,d(g-1)x^2y^2 - 6\,d^2g\,xy^3$. Hence, as $N \neq 0$ the condition $B_3 = 0$ yields $l = c\,d = d\,g = 0$. We claim that if $d \neq 0$ then for the systems (67) we have $M_{\text{IL}} < 5$. Indeed, suppose $d \neq 0$. Hence the condition $B_3 = 0$ yields $l = c = g = 0$. Thus we obtain the systems $\dot{x} = k + dy$, $\dot{y} = -xy$, for which calculations yield:

$$\mathcal{E}_1 = (-kX^2 + d^2Y^2 + 2dkYZ + k^2Z^2)\,\mathcal{H},\ \ \mathcal{E}_2 = -X(dY + kZ)^2\mathcal{H},\ \ \mathcal{H} = YZ^2.$$

Thus deg $\mathcal{H} = 3$ and since $d \neq 0$ to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$ by Lemma 45 the following condition must hold: $\operatorname{Res}_Y(\mathcal{E}_1/\mathcal{H}, \mathcal{E}_2/\mathcal{H}) = d^4k^2X^6 = 0$. Therefore $dk = 0$ and since $d \neq 0$ we obtain $k = 0$. However this condition leads to the degenerate systems. Our claim is proved.

Let us assume $d = 0$. Then the condition $B_3 = 0$ yields $l = 0$ and the systems (67) become

$$\dot{x} = k + cx + gx^2, \quad \dot{y} = (g-1)xy. \tag{68}$$

Calculations yield: $\mathcal{E}_1 = (X^2 - kZ^2)\,\mathcal{H}$, $\mathcal{E}_2 = X(gX^2 + cXZ + kZ^2)\,\mathcal{H}$, $\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = k^2[c^2 - k(1+g)^2]Z^6$, where $\mathcal{H} = (g-1)Y(gX^2 + cXZ + kZ^2)$. Hence deg $\mathcal{H} = 3$ and we need an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$. For this, according to Lemma 45, the condition $\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = 0$ is necessary, i.e. $k\left[(c^2 - k(g+1)^2\right] = 0$. As

$k \neq 0$ (otherwise we get degenerate systems) we obtain the condition $c^2 - k(g+1)^2 = 0$.

Assume $c^2 = k(g+1)^2$. Since $N \neq 0$ (i.e. $g+1 \neq 0$) we may set $c = u(g+1)$, where $u$ is a new parameter. Then $k = u^2 \neq 0$ and via the Remark 46 ($\gamma = u$, $s = 1$) the systems (68) will be brought to the form:

$$\dot{x} = 1 + (g+1)x + gx^2, \quad \dot{y} = (g-1)xy. \tag{69}$$

For systems (69) we obtain $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = 2(g-1)Y(X+Z)^2(gX+Z)$. Hence $\deg \mathcal{H} = 4$, i.e. $M_{\mathbf{IL}} \geq 5$ and since $N \neq 0$ by Lemma 56, $M_{\mathbf{IL}} \neq 6$ for any system (69).

On the other hand the conditions $d = 0$ and $c^2 - k(g+1)^2 = 0$ are equivalent to $B_3 = H_6 = 0$. Indeed, the condition $B_3 = 0$ implies $dg = 0$ for system (67) (see above) and then $H_6 = 64(g-1)^2x^4\big[2(g-1)^2[k(g+1)^2 - c^2]x^2 - 5cdxy - 2d^2y^2\big]$. Hence as $N \neq 0$ (i.e. $g-1 \neq 0$) the condition $H_6 = 0$ yields $d = c^2 - k(g-1)^2 = 0$.

We observe that $Z \mid \mathcal{H}$ if and only if $g = 0$. So, since by $N \neq 0$ the condition $g = 0$ is equivalent to $K = 2g(g-1)x^2 = 0$, we shall examine two subcases: $K \neq 0$ and $K = 0$.

1. If $K \neq 0$ then $g \neq 0$ and according to Theorem 36 the systems (69) possess the following invariant affine straight lines: $y = 0$, $x+1 = 0$, $gx+1 = 0$. By $g-1 \neq 0$ the invariant line $gx+1 = 0$ cannot coincide with $x = -1$. Moreover, the line $x = -1$ could be of multiplicity two and this is confirmed by the perturbations ($V.14_\varepsilon$) from Table 5. Thus for $K \neq 0$ we obtain the Config. 5.14.

2. For $K = 0$ we obtain $g = 0$ and then the line $l_\infty : Z = 0$ appears as a component of a conic in the pencil of conics corresponding to systems (69). Hence, the invariant line $x+1 = 0$ as well as the line $Z = 0$ is of multiplicity two, as is shown by the perturbations ($V.18_\varepsilon$) from Table 5. Thus for $K = 0$ we obtain the Config. 5.18.

### 6.4. Systems with the divisor $D_S(C, Z) = 3 \cdot w$

We are in the case of the canonical form $(\mathbf{S}_{IV})$ and we shall later need the following polynomial which is shown to be a $CT$-comitant in Lemma 62.

Notation 61. *Let us denote*

$$N_6(\boldsymbol{a}, x, y) = 8D + C_2\left[8(C_0, D_2)^{(1)} - 3(C_1, C_1)^{(2)} + 2D_1^2\right].$$

*6.4.1. The case* $\boldsymbol{N} = 0 = \boldsymbol{B}_2$

It was previously shown (see page 165) that for $N(\boldsymbol{a}, x, y) = 0$ we have to examine the systems (38) for which we have $B_2 = -648\, d^4x^4$, $B_3 = 6dx^3(fx - dy)$. Thus the condition $B_2 = 0$ is equivalent to $B_3 = 0$ and this yields $d = 0$. Then we obtain the systems (39) for which the expressions of the affine comitants $\mathcal{E}_1$ and $\mathcal{E}_2$ are given in (40). We observe that deg $\mathcal{H} = 3$ and we need to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$. Since the polynomial $\mathcal{E}_2/\mathcal{H}$ does not depend of $Y$, to have such a common factor by Lemma 45 at least one of two following conditions must hold:

$$\begin{aligned} &\text{Res}\,_X(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = (c-f)^2(c^2fY - k^2Z + c^2lZ)^2Z^4 = 0;\\ &\text{Res}\,_Z(\mathcal{E}_1/\mathcal{H},\ \mathcal{E}_2/\mathcal{H}) = (c-f)^2(c+f)X^4(k^2X - c^2lX + cfkY)^2 = 0. \end{aligned}$$

Hence we obtain either $(c-f)(c+f) = 0$ or $k = cl = cf = 0$; however the second case leads to the degenerate systems (39). On the other hand for systems (39) we obtain $N_3 = 3(c-f)x^3$, $D_1 = c+f$. So, the condition $(c-f)(c+f) = 0$ is equivalent to $N_3D_1 = 0$ and we shall consider two subcases: $N_3 = 0$ and $N_3 \neq 0$, $D_1 = 0$.

*The subcase* $N_3 = 0$. Then $f = c$ and we get the systems:

$$\dot{x} = k + cx, \qquad \dot{y} = l + cy - x^2 \tag{70}$$

for which calculations yield: $\mathcal{E}_1 = 2\,X\,\mathcal{H}$, $\mathcal{E}_2 = Z(cX + kZ)\,\mathcal{H}$, $\mathcal{H} = Z^2(cX + kZ)^2$. So deg $\mathcal{H} = 4$. It is easy to observe that the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ do not have an additional common factor if and only if $k \neq 0$ and the polynomial $\mathcal{H}$ has a factor of multiplicity four if $c = 0$. On the other hand for systems (70) $D_1 = 2c$ and $N_4(\boldsymbol{a}, x, y) = 12kx^2$. Hence the conditions $c = 0$ and $k \neq 0$ can be expressed by using the $CT$-comitants $D_1$ and $N_4$. So, we shall consider two cases: $D_1 \neq 0$ and $D_1 = 0$.

1. Assume $D_1 \neq 0$. Then $c \neq 0$ and the systems (70) can be brought by the affine transformation $x = c^{-1}kx_1$, $y = c^{-3}k^2y_1 - c^{-1}l$, $t = c^{-1}t_1$ to the system

$$\dot{x} = 1 + x, \qquad \dot{y} = y - x^2 \tag{71}$$

for which $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = Z^2(X+Z)^2$. By Lemma 38 and Corollary 39 the line $x = -1$ could be of multiplicity 2, whereas the line $Z = 0$ could be of multiplicity 3. This is confirmed by the perturbations ($V.27_\varepsilon$) from Table 5. Thus, for $N_3 = 0$, $N_4 \neq 0$ and $D_1 \neq 0$ we obtain the Config. 5.27.

2. If $D_1 = 0$ for systems (70) we have $c = 0$ and since $k \neq 0$ via the linear transformation $x_1 = k^{-1}x$ and $y_1 = k^{-3}(-lx + ky)$ we obtain the systems

$$\dot{x} = 1, \qquad \dot{y} = -x^2. \tag{72}$$

For the systems (72) we have $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = Z^4$ and hence by Corollary 39 the line $Z = 0$ could be of multiplicity 5. This is confirmed by the perturbation ($V.30_\varepsilon$) from Table 5. Thus, we obtain the Config. 5.30.

*The subcase* $N_3 \neq 0$, $D_1 = 0$. These conditions yield $f = -c \neq 0$ and we may assume $c = -1$ via the Remark 46 ($\gamma = -c$, $s = 1$). Then via the affine transformation $x_1 = x - k$, $y_1 = -kx + y + l$ systems (39) will be brought to the systems:

$$\dot{x} = -x, \qquad \dot{y} = y - x^2. \tag{73}$$

For system (6.32) we calculate: $\mathcal{H} = -XZ^3$, i.e. $\deg \mathcal{H} = 4$. By Corollary 39 the line $Z = 0$ could be of multiplicity four. This is confirmed by the perturbations ($V.26_\varepsilon$)from Table 5. Thus, for $D_1 = 0$ and $N_3 \neq 0$ we obtain the Config. 5.26.

*6.4.2. The case* $\boldsymbol{N} \neq 0$, $\boldsymbol{\theta} = 0 = \boldsymbol{B}_3$

For the systems ($\mathbf{S}_{IV}$) we calculate: $\theta = 8h^3$, $N = (g^2 - 2h)x^2 + 2ghxy + h^2y^2$. From $\theta = 0$, $N \neq 0$ we obtain $h = 0$, $g \neq 0$ and we may assume $g = 1$ via the rescaling $x \to g^{-1}x$, $y \to g^{-2}y$. Then the systems ($\mathbf{S}_{IV}$) with $h = 0$ and $g = 1$ will be brought by the translation $x \to x - c/2$, $y \to y - (c + e)$ to the systems: $\dot{x} = k + dy + x^2$, $\dot{y} = l + fy - x^2 + xy$. For these systems we have $B_3 = 3(2df - k - f^2)x^4 - 6d(d - 4f)x^3y - 9d^2x^2y^2$. Hence the condition $B_3 = 0$ yields $d = 0$, $k = -f^2$ and we obtain the systems

$$\dot{x} = -f^2 + x^2, \quad \dot{y} = l + fy - x^2 + xy, \tag{74}$$

for which calculations yield: $\mathcal{E}_1 = (X^2 + 2fXZ + lZ^2)\mathcal{H}$, $\mathcal{E}_2 = 3(X - fZ)(X + fZ)^2\mathcal{H}$, where $\mathcal{H} = 2(X - fZ)^2(X + fZ)$. Hence $\deg \mathcal{H} = 3$ and to have an additional common factor of $\mathcal{E}_1$ and $\mathcal{E}_2$, according to Lemma 45, the condition $\operatorname{Res}_X(\mathcal{E}_1/\mathcal{H}, \ \mathcal{E}_2/\mathcal{H}) = 9(f^2 - l)^2(3f^2 + l)Z^6 = 0$ must hold. We observe that for $l = f^2$ the systems (74) become degenerate. Therefore $l = -3f^2$ and since $f \neq 0$ (otherwise we get the degenerate system) we may assume $f = 1$ via Remark 46 ($\gamma = f$, $s = 1$). Thus we obtain the canonical system:

$$\dot{x} = -1 + x^2, \quad \dot{y} = -3 + y - x^2 + xy \tag{75}$$

and calculations yield $\mathcal{H} = \gcd(\mathcal{E}_1, \mathcal{E}_2) = 2(X-Z)^3(X+Z)$. Therefore, according to Theorem 36, the system (75) possesses the invariant straight lines $x = 1$ and $x = -1$ and the line $x = 1$ could be of multiplicity three. This is confirmed by the perturbations ($V.23_\varepsilon$) from Table 5. Thus we obtain Config. 5.23. Since for systems (74) we have $N_6 = 8(l+3f^2)x^3$, the condition $l + 3f^2 = 0$ is equivalent to $N_6 = 0$.

All the cases in Theorem 57 are thus examined. To finish the proof of the Theorem 57 it remains to show that the conditions occurring in the middle column of Table 4 are affinely invariant. This follows from the proof of Lemma 62.

LEMMA 62. *The polynomials which are used in Theorems 50 or 57 have the properties indicated in the Table 6. In the last column are indicated the algebraic sets on which the GL-comitants on the left are CT-comitants.*

*Proof.* Firstly we draw attention to the fact that all polynomials in the second column of Table 6 are $GL$-comitants in view of their definition (see Notations 26, 53, 30, 40, 51, 52, 59, 60, 61), of Theorem 20 and of Remark 21.

*I. Cases 1,…,14.* Let us consider the action of the translation group $T(2,\mathbb{R})$ on systems in $\widehat{\mathbf{QS}}$. If $\tau \in T(2,\mathbb{R})$, i.e. $\tau:\ x = \tilde{x} + x_0,\ y = \tilde{y} + y_0$ and $S_{\boldsymbol{a}}$ is a system in $\widehat{\mathbf{QS}}$ of coefficients $\boldsymbol{a} \in \mathbb{R}^{12}$, then applying this action to $S_{\boldsymbol{a}}$ we obtain the system $S_{\tilde{\boldsymbol{a}}}$ of coefficients $\tilde{\boldsymbol{a}} \in \mathbb{R}^{12}$, i.e.

$$S_{\tilde{\boldsymbol{a}}}:\quad \begin{cases} \dot{\tilde{x}} = P(\boldsymbol{a}, x_0, y_0) + P_x(\boldsymbol{a}, x_0, y_0)\tilde{x} + P_y(\boldsymbol{a}, x_0, y_0)\tilde{y} + p_2(\boldsymbol{a}, \tilde{x}, \tilde{y}), \\ \dot{\tilde{y}} = Q(\boldsymbol{a}, x_0, y_0) + Q_x(\boldsymbol{a}, x_0, y_0)\tilde{x} + Q_y(\boldsymbol{a}, x_0, y_0)\tilde{y} + q_2(\boldsymbol{a}, \tilde{x}, \tilde{y}). \end{cases}$$

Then calculations yield:

$$\begin{aligned} &U(\tilde{\boldsymbol{a}}) = U(\boldsymbol{a}) \quad \text{for each} \quad U \in \{\eta, \mu, \theta, B_1, H_1, H_4, H_5\}, \\ &W(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = W(\boldsymbol{a}, \tilde{x}, \tilde{y}) \quad \text{for each} \\ &\qquad\qquad W \in \{C_2, K, H, M, N, D, B_2, B_3, H_2, H_3, H_6\}. \end{aligned}$$

Since this holds for every $\boldsymbol{a} \in \mathbb{R}^{12}$, according to Definition 22 we conclude that the GL- comitants indicated in the lines 1–14 of Table 6 are $T$-comitants for systems (10).

*II. Cases 15,…,21.* 1. We consider firstly $N_1(a,x,y)$, $N_2(a,x,y)$, $N_5(a,x,y)$, the $GL$-comitants which according to Tables 2 and 4 were used only when the conditions $\eta = 0 = H$ are satisfied. According to Lemma 44 for $\eta = 0$ there correspond three canonical forms: $(\mathbf{S}_{II})$, $(\mathbf{S}_{III})$ and $(\mathbf{S}_V)$.

**TABLE 6.**

| Case | $GL$-comitants | Degree in $a$ | Degree in $x$ and $y$ | Weight | Algebraic subset $V(*)$ |
|---|---|---|---|---|---|
| 1 | $\eta(a),\ \mu(a),\ \theta(a)$ | 4 | 0 | 2 | $V(0)$ |
| 2 | $C_2(a,x,y)$ | 1 | 3 | −1 | $V(0)$ |
| 3 | $H(a,x,y),\ K(a,x,y)$ | 2 | 2 | 0 | $V(0)$ |
| 4 | $M(a,x,y),\ N(a,x,y)$ | 2 | 2 | 0 | $V(0)$ |
| 5 | $D(a,x,y)$ | 3 | 3 | −1 | $V(0)$ |
| 6 | $B_1(a)$ | 12 | 0 | 3 | $V(0)$ |
| 7 | $B_2(a,x,y)$ | 8 | 4 | 0 | $V(0)$ |
| 8 | $B_3(a,x,y)$ | 4 | 4 | −1 | $V(0)$ |
| 9 | $H_1(a)$ | 6 | 0 | 2 | $V(0)$ |
| 10 | $H_2(a,x,y))$ | 3 | 2 | 0 | $V(0)$ |
| 11 | $H_3(a,x,y)$ | 4 | 2 | 0 | $V(0)$ |
| 12 | $H_4(a)$ | 6 | 0 | 2 | $V(0)$ |
| 13 | $H_5(a)$ | 8 | 0 | 2 | $V(0)$ |
| 14 | $H_6(a,x,y))$ | 8 | 6 | 0 | $V(0)$ |
| 15 | $N_1(a,x,y)$ | 3 | 4 | −1 | $V(\eta,H)$ |
| 16 | $N_2(a,x,y)$ | 3 | 1 | 0 | $V(\eta,H,B_3)$ |
| 17 | $N_3(a,x,y)$ | 2 | 3 | −1 | $V(M,N)$ |
| 18 | $N_4(a,x,y)$ | 2 | 2 | −1 | $V(M,N,N_3)$ |
| 19 | $N_5(a,x,y)$ | 4 | 2 | 0 | $V(\eta,H,B_3)$ |
| 20 | $N_6(a,x,y)$ | 3 | 3 | −1 | $V(M,\theta,B_3)$ |
| 21 | $D_1(a)$ | 1 | 0 | 0 | $V(M,N)$ |

Since for the systems $(\mathbf{S}_V)$ we have $H = -x^2 \neq 0$, we need to consider the following cases: $(i)$ $\eta = 0$ and $M \neq 0$; $(ii)$ $\eta = 0$ and $M = 0$ and $C_2 \neq 0$.

*(i)* For $\eta = 0$ and $M \neq 0$ we are in the class of systems $(\mathbf{S}_{III})$, for which the condition $H = -(g-1)^2x^2 - 2h(g+1)xy - hy^2 = 0$ yields $h = g - 1 = 0$ and this leads to systems (30) (see page 163). On the other hand for any system corresponding to a point $\tilde{\boldsymbol{a}} \in \mathbb{R}^{12}$ in the orbit under the translation group action of a system (30) calculations yield:

$$N_1(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 8\tilde{x}^2(e\tilde{x}^2 - 2d\tilde{y}^2), \quad B_3(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 6d\tilde{x}\tilde{y}^2(f\tilde{x} - d\tilde{y})$$
$$N_2(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 4(f^2 + 4k)\tilde{x} + 4df\tilde{y} + 8d(x_0\tilde{y} + 2y_0\tilde{x}),$$
$$N_5(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = -16(4k\tilde{x}^2 - d^2\tilde{y}^2) + 64d\tilde{x}(x_0\tilde{y} - y_0\tilde{x}).$$

Hence the value of $N_1$ does not depend of the vector defining the translation and for $B_3 = 0$ the same occurs for $N_2$ and $N_5$. Therefore we conclude that for $M \neq 0$ the polynomial $N_1$ is a $CT$-comitant modulo $\langle \eta, H \rangle$, whereas the polynomials $N_2$ and $N_5$ are $CT$-comitants modulo $\langle \eta, H, B_3 \rangle$.

*(ii)* Assume now that $M = 0$ and $C_2 \neq 0$. Then we are in the class of systems $(\mathbf{S}_{IV})$, for which the condition $H = -(g^2+4h)x^2 - 2ghxy - hy^2 = 0$ yields $g = h = 0$. In this case using an additional translation (see page 165) we obtain the systems (38). Then for any system corresponding to a point $\tilde{\boldsymbol{a}} \in \mathbb{R}^{12}$ in the orbit under the translation group action of a system (38) calculations yield: $N_1(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = -24d\tilde{x}^4$, $N_2(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 12d(c+f)\tilde{x}$, $N_5(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 0$. Since the condition $M = 0$ implies $\eta = 0$, considering the case $(i)$ above we conclude that independently of either $M \neq 0$ or $M = 0$, the $GL$-comitant $N_1$ is a $CT$-comitant modulo $\langle \eta, H \rangle$ and $N_2$ and $N_5$ are $CT$-comitants modulo $\langle \eta, H, B_3 \rangle$.

2. Let us now consider the $GL$-comitants $D_1(a)$, $N_3(a, x, y)$, $N_4(a, x, y)$ and $N_6(a, x, y)$. According to Tables 2 and 4 the polynomials $N_3$, $N_4$ and $D_1$ (respectively $N_6$) were used only when the conditions $M = N = 0$ (respectively $M = \theta = 0, N \neq 0$) are satisfied. In both cases we are in the class of systems $(\mathbf{S}_{IV})$ and we shall consider the two subcases: $N = 0$ and $N \neq 0$, $\theta = 0$.

*(i)* If for the system $(\mathbf{S}_{IV})$ the condition $N = 0$ is fulfilled then as it was shown on the page 165 we obtain systems (38). Then for any system corresponding to a point $\tilde{\boldsymbol{a}} \in \mathbb{R}^{12}$ in the orbit under the translation group action of a system (38) calculations yield:

$$N_3(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 3(c-f)\tilde{x}^3 + 2d\tilde{x}^2\tilde{y}, \quad B_3(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 6d\tilde{x}^3(f\tilde{x} - d\tilde{y}),$$
$$N_4(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 12k\tilde{x}^2 + 3(f^2 - c^2)\tilde{x}\tilde{y} - 3d(c+f)\tilde{y}^2 + 6\tilde{x}^2[(c-f)x_0 + 2dy_0],$$
$$N_6(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 8c(c-f)\tilde{x}^3 + 16df\tilde{x}^2\tilde{y} - 8d^2\tilde{x}\tilde{y}^2 - 48dx_0\tilde{x}^3, \; D_1(\tilde{\boldsymbol{a}}) = c + f.$$

These relations show us that: ($\alpha$) the $GL$-comitants $N_3$ and $D_1$ are $CT$-comitants modulo $\langle M, N\rangle$; ($\beta$) the $GL$-comitant $N_4$ is a $CT$-comitant modulo $\langle M, N, N_3\rangle$; ($\gamma$) the $GL$-comitant $N_6$ is a $CT$-comitant modulo $\langle M, N, B_3\rangle$.

*(ii)* Assume that for the system ($\mathbf{S}_{IV}$) the conditions $\theta = 0$ and $N \neq 0$ are fulfilled. As it was shown on the page 188 for $B_3 = 0$ we obtain systems (74). For any system corresponding to a point $\tilde{\boldsymbol{a}} \in \mathbb{R}^{12}$ in the orbit under the translation group action of a system (74) we have $N_6(\tilde{\boldsymbol{a}}, \tilde{x}, \tilde{y}) = 8(l + 3f^2)\tilde{x}^3$. Therefore, since the condition $N = 0$ implies $\theta = 0$, considering the case $(i)$ above we conclude that independently of either $N \neq 0$ or $N = 0$, the $GL$-comitant $N_6$ is a $CT$-comitant modulo $\langle M, \theta, B_3\rangle$.

The Table 6 shows us that all the conditions indicated in the middle column of Tables 2 and 4 are affinely invariant. Indeed, the $CT$-comitants $N_i$, $i = 1, \ldots, ..., 6$ and $D_1$ are used in Tables 2 and 4 only for the varieties indicated in the last column of Table 6.

This completes the proof of the Theorems 50 and 57. ▌

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