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# Some Dynamical Properties of $F(z) = z^2 - 2\overline{z}$

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We study in this note some dynamical properties of $F(z) = z^2 - 2\overline{z}$, $F : \mathbb{C} \to \mathbb{C}$ . Let $K = K(F)$ denote the set of all points whose orbit is bounded. We prove that $F$ restricted to $\mathbb{C} \setminus K$ behaves as $\varphi(z) = z^2$ does in the complement of the unit disk; $K$ has positive area; $F$ restricted to $K$ is transitive; the set of periodic points of $F$ is dense in $K$ and the topological entropy of $F|_K$ is positive.

*Key Words:* Periodic points, transitive mapping, topological entropy.

## 1. INTRODUCTION

The dynamical study of quadratic transformations of the plane has been indeed a very popular one; it includes such examples as the Henon maps or the most famous holomorphic family $z \to z^2 + c$ (see also [1], [8], [7]).

The problem of whether or not it is possible to provide a certain classification of these maps according to its dynamical behaviour was raised

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in [8]. A partial answer was given there, establishing equivalence classes of these maps not in the dynamical sense (conjugation classes) but, we could say, in a geometrical sense (Whitney's equivalence). This work can be considered as a useful starting point for the above mentioned classification problem. One of those (Whitney's) equivalence classes contains the non-holomorphic (nor antiholomorphic) family $z \to z^2 + 2a\overline{z}$, where $a$ is a complex parameter.

It was pointed out then *(ibid.)* that this family is an interesting example to study for several reasons, all of them related to the fact that as $a$ varies, a fairly wide range of very different dynamical phenomena occur. In [11] it was shown that when the parameter $a$ is a real number, the set $K_a$ of bounded orbits is connected if, and only if, the critical point of the map restricted to $\mathbb{R}$ has a bounded orbit, which, in turn, happens when $a \in [-1, 2]$.

This resembles the well known *Julia-Fatou* result for the holomorphic case, but when $a \in \mathbb{C} - \mathbb{R}$ the resemblence ends, due to the fact that the singular set of the map is not a finite set of points, but a whole circunference!

Some time later, I. Petersen, J. C. Alexander, J. Yorke and others, established the surprising existence of a very simple attractor (in fact there were three attractors made of straight lines that formed sort of a triangle) appearing in maps with "very simple equations" but with incredibly involved basins of attraction ("thoroughly intermingled basins" they said; see [2], [1], [10]). It turns out that the maps were members of the family in question when the parameter $a$ is of the form $1 + i\lambda$ with $\lambda$ in the vecinity of certain specific value. This sort of things encouraged the authors of the present paper to study this family of maps having by now arrived to several partial conclusions (see [9]) while work is still in progress.

The aim of this paper is to describe the dynamics that take place for the specific parameter $a = -1$. This case is special because it is the only one for which the set of bounded orbits has a very simple geometrical shape and its border is the set of critical values of the map. It becomes possible, then, to make a (simpler) geometrical model of the map which, in turn, renders as absolutely obvious its complicated dynamical properties. It is also an important bifurcation parameter because as $a \in \mathbb{R}$ crosses $-1$ and becomes less than $-1$, the set $K_a$ sort of "explodes" and from being connected becomes disconnected with infinite components (most surely a Cantor set). Finally, we believe that, aside from the research problem itsef, specially the geometrical model is an example that can be used in a classroom to illustrate, in a simple way, difficult dynamical concepts.

## 2. PRELIMINARY DEFINITIONS

The family of mappings $F_a : \mathbb{C} \to \mathbb{C}$, $F_a(z) = z^2 + 2a\overline{z}$, $a \in \mathbb{C}$, has been recently studied (see [1], [8] and [11]). We present here some properties of the discrete dynamical system generated by one member of this family, the one corresponding to $a = -1$, $F_{-1}(z) = z^2 - 2\overline{z}$.

The study of dynamics of rational mappings on the Riemann Sphere is an important area in dynamical systems. The concepts of Julia Set and Fatou Set are basic in this theory. Here we are interested in the study of such equivalent sets for non holomorphic maps such as $F_{-1}(z)$.

A main difference between real dynamics in two variables and holomorphic dynamics in one variable occurs in the singular set. In the first case it can be a collection of curves while in the second case it consists of a discrete set of points. For the family $F_a(z)$, $a \neq 0$, it is easily seen that the singular set $\left\{z \in \mathbb{C} : \det\left(DF_{a(z)}\right) = 0\right\}$ is the circle $|z| = |a|$, (see [8] or [11]).

One of the results of this paper is that the mapping $F_{-1}(z)$ exhibits a filled "Julia Set" which has nonempty interior without being the whole space $\mathbb{C}$. This is not the case for rational maps.

From here on we denote with $F$ the mapping $F(z) = z^2 - 2\overline{z}$, $F : \mathbb{C} \to \mathbb{C}$. We define $F^1 = F$ and $F^n = F \circ F^{n-1}$ for each $n \geq 2$. Let $z \in \mathbb{C}$, the *orbit of $z$ under $F$* is the set $\left\{z, F(z), F^2(z), \ldots\right\}$ and it is denoted by $o(z, F)$. If there is a positive integer, $n \in \mathbb{N}$, such that $F^n(z) = z$ it is said that $z$ is a *periodic point* of $F$. In such a case, the *period* of $z$ is the smallest $n$ for which $F^n(z) = z$. Let $Per(F)$ denote the set of all periodic points of $F$. If $z \in Per(F)$ is of period $n$ we say that $z$ is *expansive* provided that $|\lambda_1| > 1$ and $|\lambda_2| > 1$, where $\lambda_1$ and $\lambda_2$ are the eigenvalues of the Jacobian matrix of $F^n$ in $z = (x, y)$, $DF^n_{(x,y)}$. Given $z \in \mathbb{C}$ and $\varepsilon > 0$, let $B_\varepsilon(z)$ denote the set $\{\omega \in \mathbb{C} : |\omega - z| < \varepsilon\}$. Given a subset of $\mathbb{C}$, $E \subset \mathbb{C}$, $int(E)$, $\partial E$ and $cl(E)$ denote the interior, the boundary and the closure of $E$ respectively. An orbit $o(z, F)$ is bounded if there exists $M > 0$ such that $o(z, F) \subset B_M(0)$. Let $K = K(F)$ be the set of points whose orbit is bounded. As in holomorphic dynamics we call $K$ the *filled Julia set* of $F$. It follows that $F(K) = K$ and $F(\mathbb{C} \setminus K) = \mathbb{C} \setminus K$. In fact, $\mathbb{C} \setminus K$ is the *basin of atraction of infinity*, i.e. $\mathbb{C} \setminus K = \{z : |F^n(z)| \to \infty\}$. We denote this basin by $B(\infty)$. Our goal is to describe the set $K$, the dynamics of $F$ restricted to $K$, $F|_K : K \to K$, and the dynamics of $F$ restricted to $\mathbb{C} \setminus K = B(\infty)$.

Let $A$ and $B$ be two metric spaces, and let $f : A \to A$ and $g : B \to B$ be continuous mappings. It is said that $f$ and $g$ are *conjugate* provided that there exists a one to one and onto continuous mapping $h : A \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ h\downarrow & & \downarrow h \\ B & \xrightarrow{g} & B \end{array}$$

That is, $h \circ f = g \circ h$.

In section 4 we prove the following: If $\varphi : \mathbb{C} \to \mathbb{C}$ denotes the map $\varphi(z) = z^2$ and $D$ the unit disk, then $F\left|_{\mathbb{C}\setminus int(K)}\right.$ and $\varphi\left|_{\mathbb{C}\setminus int(D)}\right.$ are conjugate.

Let $A$ be a metric space, and let $f : A \to A$ be a continuous mapping. It is said that $f$ is *transitive* in $A$ provided that for every pair of nonempty open subsets of $A$, say $U$ and $W$, there exists a point $x$ in $U$, and a positive integer, $n$, such that $f^n(x) \in W$.

In section 5 we produce a piecewise linear mapping wich we will see is conjugate to $F|_K$ . It allows us to prove, in section 6, that $F|_K$ is transitive in $K$ and $Per(F|_K)$ is a dense set in $K$. Note that these two conditions imply that $F|_K$ is chaotic in $K$ in the sense proposed by R. Devaney (see [3] and [6]). We also show that if $z$ is a periodic point of $F$ and $z \in int(K)$, then it is expansive.

We recall the definition of topological entropy in section 5. In section 6 we prove that the topological entropy of $F|_K$ is positive, and furthermore, that it equals $\log(4)$.

## 3. AN INVARIANT FAMILY OF CURVES

The mapping $F(z) = z^2 - 2\overline{z}$ can be represented as a function of $\mathbb{R}^2$:

$$F(x,y) = \left(x^2 - y^2 - 2x, 2xy + 2y\right),$$

where $z$ is represented by the point $(x,y)$.

It is said that $(u,v)$ is a *critical point* of $F$ provided that the determinant of $DF_{(u,v)}$ is zero. It is immediate that the set of all critical points of $F$ is the unit circle, $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

The point $(s,t)$ is a *critical value* of $F$ if $(s,t) = F(u,v)$ and $(u,v)$ is a critical point. Given $\theta \in [0, 2\pi]$, $z = e^{i\theta}$ is a critical point of $F$. Since $F\left(e^{i\theta}\right) = e^{i2\theta} - 2e^{-i\theta}$ it follows that the set of all critical values is a hypocycloid (see figure 1).

Let us denote the set of critical values with $\Lambda$,

$$\Lambda = \left\{w = e^{i2\theta} - 2e^{-i\theta} : \theta \in [0, 2\pi]\right\}.$$

Proposition 1. *The set $\Lambda$ is invariant under $F$, $F(\Lambda) = \Lambda$.*

*Proof.* Let $e^{i2\theta} - 2e^{-i\theta} \in \Lambda$.

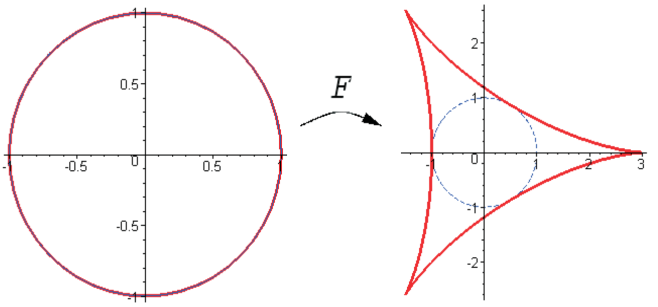


**FIG. 1.**

$$F\left(e^{i2\theta} - 2e^{-i\theta}\right) = e^{i4\theta} - 4e^{i\theta} + 4e^{-i2\theta} - 2e^{-i2\theta} + 4e^{i\theta}$$
$$= e^{i4\theta} + 2e^{-i2\theta}.$$

Taking $\gamma = 2\theta + \pi$, we have

$$F\left(e^{i2\theta} - 2e^{-i\theta}\right) = e^{i2(\gamma-\pi)} + 2e^{-i(\gamma-\pi)}$$
$$= e^{i2\gamma} - 2e^{-i\gamma}.$$

Thus, $F(\Lambda) = \Lambda$. ∎

Note that while $\theta$ goes from zero to $2\pi$, $z = e^{i2\theta} - 2e^{-i\theta}$ goes around $\Lambda$ once and $F(z)$ goes around $\Lambda$ twice.

For each $r$ in the real line, $r \in \mathbb{R}$, consider the curve

$$\Lambda_r = \left\{z = re^{i\theta} + e^{-i2\theta} : \theta \in [0, 2\pi]\right\}.$$

Clearly for each $r \in \mathbb{R}$, $\Lambda_{-r} = \Lambda_r$. Also $\Lambda = \Lambda_2$ and $S^1 = \Lambda_0$. Let $\Gamma = \{\Lambda_r \,|r \in \mathbb{R}\}$. We say that the family of curves $\Gamma$ is invariant under $F$ provided that for each $s \in \mathbb{R}$, there exists $t \in \mathbb{R}$ such that $F(\Lambda_s) = \Lambda_t$.

PROPOSITION 2. *The family* $\Gamma$ *is invariant under* $F$*.*

*Proof.* Let $s \in \mathbb{R}$, and consider $\Lambda_s \in \Gamma$. Take $\theta \in [0, 2\pi]$. Then

$$F\left(se^{i\theta} + e^{-i2\theta}\right) = s^2e^{i2\theta} + 2se^{-i\theta} + e^{-i4\theta} - 2se^{-i\theta} - 2e^{i2\theta}$$
$$= \left(s^2 - 2\right)e^{i2\theta} + e^{-i4\theta}.$$

Taking $\gamma = 2\theta$ and $t = s^2 - 2$, we have

$$F\left(se^{i\theta} + e^{-i2\theta}\right) = te^{i\gamma} + e^{-i2\gamma}.$$

Thus, $F(\Lambda_s) = \Lambda_t$. ∎

We present below some elements $\Lambda_r$ of the family $\Gamma$, corresponding to $r = 3, 2, 1.5, 1, 0.5$ and $0$ (see figures 2, 3 and 4).

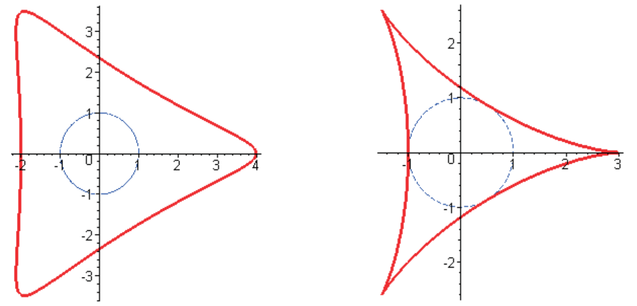


**FIG. 2.**

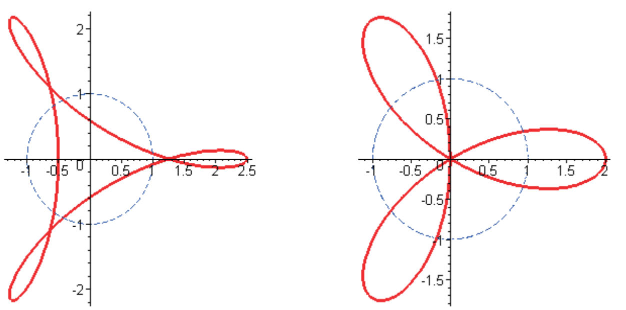


**FIG. 3.**

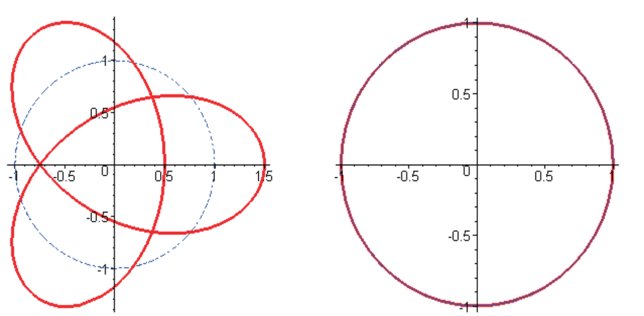


**FIG. 4.**

Note that all points of the plane belong to $\Lambda_s$ for some $s$.

Since $F(\Lambda_s) = \Lambda_{s^2-2}$, if $z \in \Lambda_s$ with $-2 \leq s \leq 2$, then $F(z)$ lies in $\Lambda_t$ with $-2 \leq t \leq 2$. Also, by proposition 2, if $z \in \Lambda_s$ with $s \notin [-2,2]$, $F(z) \in \Lambda_t$ with $t = s^2 - 2 > |s|$. Writing $\phi(s) = s^2 - 2$, we have that $F^n(z) \in \Lambda_{\phi^n(s)}$ and $|F^n(z)| \geq \phi^n(s) - 1$. Therefore $\lim_{n\to\infty} |F^n(z)| = \infty$. It follows that given a point $z \in \mathbb{C}$, $o(z, F)$ is bounded if and only if there exists $s \in [-2,2]$ with $z \in \Lambda_s$.

It is not difficult to see that $\Lambda_s$ is a simple closed curve whenever $s \geq 2$. Also, if $2 \leq s < t$, then $\Lambda_s \cap \Lambda_t = \phi$.

Let $C$ be the bounded component of $\mathbb{C} \setminus \Lambda$. Note that $C \cup \Lambda = cl(C)$, and $z \in cl(C)$ if and only if $z \in \Lambda_s$ for some $s \in [-2,2]$. It follows that $cl(C) = K$, the set of all points $z \in \mathbb{C}$ with bounded orbit under $F$.

## 4. THE QUADRATIC MAPPING

Let $\varphi : \mathbb{C} \to \mathbb{C}$, given by $\varphi(z) = z^2$, and let $D$ be the unit disk. In this section we prove that $\varphi\left|_{\mathbb{C}\setminus int(D)}\right.$ and $F\left|_{\mathbb{C}\setminus int(K)}\right.$ are conjugate.

First note that the interval $[3,\infty) \subset \mathbb{R}$ is invariant under $F$ an that the interval $[1,\infty) \subset \mathbb{R}$ is invariant under $\varphi$. Also, $F\left|_{[3,\infty)}\right. : [3,\infty) \to [3,\infty)$ is a homeomorphism, $F\left|_{[3,\infty)}\right.(x) = x^2 - 2x$, $F\left|_{[3,\infty)}\right.(3) = 3$, and given $x > 3$,

$$\lim_{n\to\infty} \left(F\left|_{[3,\infty)}\right.\right)^n(x) = \infty \text{ and } \lim_{n\to\infty} \left(F\left|_{[3,\infty)}\right.\right)^{-n}(x) = 3.$$

The following lemma shows that the behaviour of $F\left|_{[3,\infty)}\right.$ in $[3,\infty)$ is, essentially, the same as the behaviour of $\varphi$ restricted to $[1,\infty)$. The proof follows a standard procedure (see page 55 of [6]), so we leave it to the reader.

Lemma 3. *There exists a homeomorphism $h : [1,\infty) \to [3,\infty)$ such that the following diagram commutes:*

$$\begin{array}{ccc} [1,\infty) & \xrightarrow{\varphi|_{[1,\infty)}} & [1,\infty) \\ h\downarrow & & \downarrow h \\ [3,\infty) & \xrightarrow{F|_{[3,\infty)}} & [3,\infty) \end{array}.$$

*That is, for each $x \in [1,\infty)$, $h(x^2) = (h(x))^2 - 2h(x)$.*

With $h$ at hand we define a mapping

$$H : \mathbb{C} \setminus int(D) \to \mathbb{C} \setminus int(K)$$

in this way: Take $z = \rho e^{i\theta}$, $\rho \geq 1$, $\theta \in [0, 2\pi]$, let

$$H\left(\rho e^{i\theta}\right) = \left(h\left(\rho\right) - 1\right) e^{i\theta} + e^{-i2\theta}.$$

Another way to represent $H$ is the following:

$$H\left(z\right) = \left(h\left(\left|z\right|\right) - 1\right) \frac{z}{\left|z\right|} + \frac{\overline{z}}{z}.$$

Notice that $H$ maps circles onto members of family $\Gamma$. In particular, $H\left(S^1\right) = \Lambda$. Since $h\left(\rho\right) - 1 \geq 2$, it follows that the image of the circle $\rho e^{i\theta}$ under $H$ (i.e., the curve $\Lambda_{h(\rho)-1}$) is a simple closed curve, hence $H$ is one to one. It is not difficult to see that $H$ is continuous and onto as well.

Proposition 4. *The following diagram commutes:*

$$\begin{array}{ccc} \mathbb{C} \setminus int\left(D\right) & \xrightarrow{\varphi} & \mathbb{C} \setminus int\left(D\right) \\ H \downarrow & & \downarrow H \\ \mathbb{C} \setminus int\left(K\right) & \xrightarrow{F} & \mathbb{C} \setminus int\left(K\right) \end{array}.$$

*Proof.* Let $z$ be a point in $\mathbb{C}$ such that $|z| \geq 1$.

$$\begin{aligned} H\left(\varphi\left(z\right)\right) = H\left(z^2\right) &= \left(h\left(\left|z^2\right|\right) - 1\right) \frac{z^2}{\left|z^2\right|} + \frac{\overline{z^2}}{z^2} \\ &= \left(h\left(\left|z\right|^2\right) - 1\right) \frac{z^2}{\left|z\right|^2} + \left(\frac{\overline{z}}{z}\right)^2 \\ &= \left(\left(h\left(\left|z\right|\right)\right)^2 - 2h\left(\left|z\right|\right) - 1\right) \frac{z^2}{\left|z\right|^2} + \left(\frac{\overline{z}}{z}\right)^2. \end{aligned}$$

On the other hand we have:

$$F\left(H\left(z\right)\right) = F\left(\left(h\left(\left|z\right|\right) - 1\right) \frac{z}{\left|z\right|} + \frac{\overline{z}}{z}\right) =$$

$$\left(h\left(\left|z\right|\right) - 1\right)^2 \frac{z^2}{\left|z\right|^2} + 2\left(h\left(\left|z\right|\right) - 1\right) \frac{z}{\left|z\right|} \frac{\overline{z}}{z} + \left(\frac{\overline{z}}{z}\right)^2 - 2\left(\left(h\left(\left|z\right|\right) - 1\right) \frac{\overline{z}}{\left|z\right|} + \frac{z}{\overline{z}}\right) =$$

$$\left(h\left(\left|z\right|\right) - 1\right)^2 \frac{z^2}{\left|z\right|^2} + \left(\frac{\overline{z}}{z}\right)^2 - 2\frac{z}{\overline{z}} =$$

$$\left(\left(h\left(\left|z\right|\right)\right)^2 - 2h\left(\left|z\right|\right) + 1\right) \frac{z^2}{\left|z\right|^2} + \left(\frac{\overline{z}}{z}\right)^2 - 2\frac{z}{\overline{z}} =$$

$$\left(\left(h\left(\left|z\right|\right)\right)^2 - 2h\left(\left|z\right|\right) - 1\right) \frac{z^2}{\left|z\right|^2} + \left(\frac{\overline{z}}{z}\right)^2.$$

Therefore $\varphi$ in $\mathbb{C} \setminus int\,(D)$ is conjugate with $F$ in $\mathbb{C} \setminus int\,(K)$. Also, note that $\varphi$ in $S^1$ is conjugate with $F$ in $\Lambda$.

∎

It remains only to study the dynamics of $F$ in $K$.

## 5. A PIECEWISE LINEAR MAPPING

In this section we produce an auxiliary piecewise linear mapping defined in an equilateral triangle $\Delta \subset \mathbb{R}^2$, $G : \Delta \to \Delta$. We present some dynamical properties of $G$. Then, in the next section, we show that $G$ in $\Delta$ and $F$ in $K$ are conjugate.

Let

$$\Delta = \left\{ (x,y) : x \in [-1,2]\,, \frac{\sqrt{3}}{3}\,(x-2) \leq y \leq \frac{\sqrt{3}}{3}\,(2-x) \right\}.$$

We subdivide $\Delta$ in the following four equilateral triangles (see figure 5).

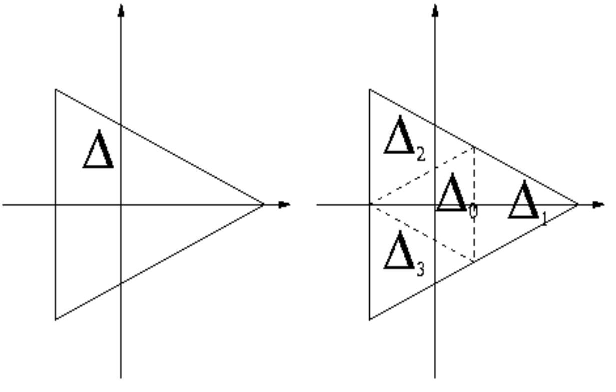


**FIG. 5.**

$$\Delta_0 = \left\{ (x,y) : x \in \left[-1, \frac{1}{2}\right], -\frac{\sqrt{3}}{3}\,(x+1) \leq y \leq \frac{\sqrt{3}}{3}\,(x+1) \right\},$$

$$\Delta_1 = \left\{ (x,y) : x \in \left[\frac{1}{2}, 2\right], \frac{\sqrt{3}}{3}\,(x-2) \leq y \leq \frac{\sqrt{3}}{3}\,(2-x) \right\},$$

$$\Delta_2 = \left\{ (x,y) : x \in \left[-1, \frac{1}{2}\right], \frac{\sqrt{3}}{3}\,(x+1) \leq y \leq \frac{\sqrt{3}}{3}\,(2-x) \right\},$$

$$\Delta_3 = \left\{ (x,y) : x \in \left[-1, \frac{1}{2}\right], \frac{\sqrt{3}}{3}\,(x-2) \leq y \leq -\frac{\sqrt{3}}{3}\,(x+1) \right\}.$$

Before we give the definition of $G$ in detail let us describe its action on $\Delta$. First, for $i = 1,2,3$, fold the triangle $\Delta_i$ onto $\Delta_0$ leaving the segment $\Delta_i \cap \Delta_0$ fixed. Denote by $G_1$ this piecewise folding of the three triangles $\Delta_i$. Second, reflect $\Delta_0 = G_1(\Delta_i)$ about the $y-axis$. Denote this reflection by $G_2$. Finally, linearly expand the reflected triangle by a factor of 2. The mapping $G : \Delta \to \Delta$ is then given by $G = G_3 \circ G_2 \circ G_1$, where $G_3$ is the linear expansion just mentioned (see figure 6).

Take $(x,y) \in \Delta$. Let $G : \Delta \to \Delta$ be the piecewise linear mapping defined by

$$G(x,y) = \begin{cases} (-2x, 2y) & \text{if } (x,y) \in \Delta_0 \\ (2x-2, 2y) & \text{if } (x,y) \in \Delta_1 \\ \left(1 - x - \sqrt{3}y, \sqrt{3} + \sqrt{3}x - y\right) & \text{if } (x,y) \in \Delta_2 \\ \left(1 - x + \sqrt{3}y, \sqrt{3} + \sqrt{3}x + y\right) & \text{if } (x,y) \in \Delta_3 \end{cases}.$$

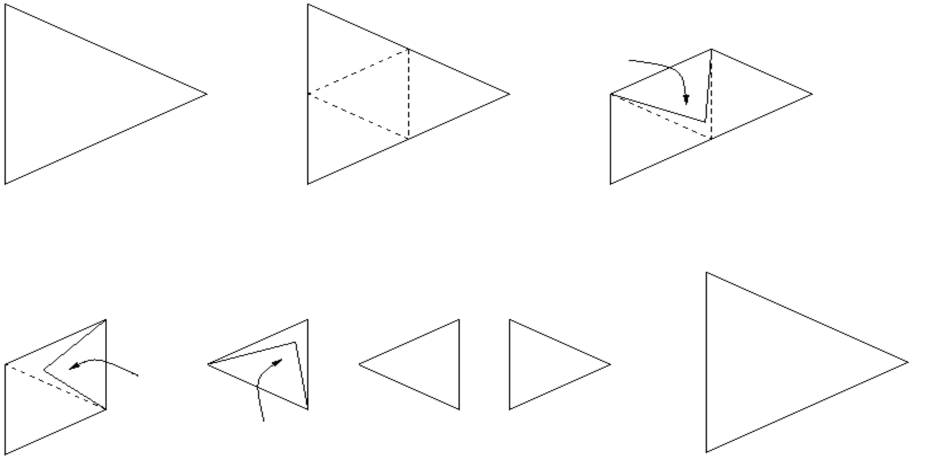

**FIG. 6.**

*Remark 5.* It is a fact that the above "folding", "reflection" and "expansion" produce the exact same result when performed in any order whatsoever, so there are other ways to describe the action of the mapping $G$.

PROPOSITION 6. *The set $Per(G)$ is dense in $\Delta$.*

*Proof.* In $\Delta$ we have four equilateral triangles, $\Delta_0$, $\Delta_1$, $\Delta_2$ and $\Delta_3$, such that $\Delta = \cup_{i=0}^{3}\Delta_i$, and $G(\Delta_i) = \Delta$, $i = 0,1,2,3$. For each $i$ fixed there exist four equilateral triangles, $\Delta_{0,i}$, $\Delta_{1,i}$, $\Delta_{2,i}$ and $\Delta_{3,i}$, such that $\Delta_{j,i} \subset \Delta_j$ and $G(\Delta_{j,i}) = \Delta_i$. Now, for each pair $j,i$ fixed there exist four equilateral triangles, $\Delta_{0,j,i}$, $\Delta_{1,j,i}$, $\Delta_{2,j,i}$ and $\Delta_{3,j,i}$, such that $\Delta_{k,j,i} \subset \Delta_k$ and $G(\Delta_{k,j,i}) = \Delta_{j,i}$ and $G^2(\Delta_{k,j,i}) = \Delta_i$. This procedure leads us to the following: Given $n \in \mathbb{N}$ consider the set

$$\Psi_n = \{w = (w_1, w_2, \ldots, w_n) : w_i \in \{0, 1, 2, 3\}\}.$$

For each $w = (w_1, w_2, \ldots, w_n)$ in $\Psi_n$, there exists an equilateral triangle in $\Delta$, namely $\Delta_w = \Delta_{w_1, w_2, \ldots, w_n} \subset \Delta_{w_1, w_2, \ldots, w_{n-1}}$, such that

$$G^j(\Delta_{w_1, w_2, \ldots, w_n}) = \Delta_{w_{j+1}, \ldots, w_n}, \ 1 \le j \le n-1,$$

and $G^n(\Delta_w) = \Delta$.

Notice that each $G^j\big|_{\Delta_w} : \Delta_w \to \Delta_{w_{j+1}, \ldots, w_n}$, $1 \le j \le n$, is a homeomorphism that enlarges the distance between any two points by a factor of $2^j$, the length of the sides of the triangle $\Delta_w$ is $\frac{1}{2^n}\left(2\sqrt{3}\right)$, the area of $\Delta_w$ is $\frac{a}{4^n}$ where $a$ is the area of $\Delta$, and $\Delta = \cup_{w \in \Psi_n} \Delta_w$.

Given $(x, y) \in \Delta$ and $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $w \in \Psi_n$ such that $\Delta_w \subset B_\varepsilon(x, y) \cap \Delta$. Since $G^n : \Delta_w \to \Delta$ is a homeomorphism, there is a point $(u, v) \in \Delta_w$ with $G^n(u, v) = (u, v)$. ▮

*Remark 7.* Note that the boundary of $\Delta$ is invariant under $G$, $G(\partial\Delta) = \partial\Delta$. Take $(x, y) \in Per(G) \cap int(\Delta)$ of period $n$. Then,

$$o((x, y), G) \subset int(\Delta) \setminus G^{-1}(\partial\Delta).$$

Since in $int(\Delta) \setminus G^{-1}(\partial\Delta)$ the mapping $G$ is a linear expansion by a factor of 2, then $G$ is differentiable in each point of $o((x, y), G)$ and the eigenvalues of $DG^n_{(x,y)}$, $\lambda_1$ and $\lambda_2$, satisfy $|\lambda_1| = 2^n = |\lambda_2|$. Therefore each periodic point of $G$ that lies in $int(\Delta)$ is expansive, also commonly known as a *repellor*.

In fact, if $(x, y) \in Per(G) \cap int(\Delta)$, then, for every $n \in \mathbb{N}$, $(x, y) \notin G^{-n}(\partial\Delta)$; that is,

$$(x, y) \in \Delta \setminus G^{-n}(\partial\Delta) = U_n.$$

Note that $U_n$ is open and dense in $\Delta$ for every $n \in \mathbb{N}$ and that the whole orbit of the periodic point $(x, y)$ lies in the set

$$U_\infty = \cap_{n=1}^{\infty} U_n,$$

which, according to Baire's theorem, is dense in $\Delta$. Also, $U_\infty$ is invariant under $G$, $G(U_\infty) = U_\infty$.

PROPOSITION 8. *$G$ is transitive in $\Delta$.*

*Proof.* Let $U$ and $W$ be two nonempty open subsets of $\Delta$. Then there exist $n \in \mathbb{N}$ and $w \in \Psi_n$, as in proposition 6, such that $\Delta_w \subset U$. It follows that $G^n(U) = \Delta$ and, therefore, there exists $(u, v) \in U$ with $G^n(u, v) \in W$. ▌

Let $\Sigma$ be the following set:

$$\Sigma = \left\{\widehat{t} = (t_1, t_2, \ldots) : t_i \in \{0, 1, 2, 3\}, i \in \mathbb{N}\right\}.$$

Given $\widehat{t} = (t_1, t_2, \ldots)$ and $\widehat{s} = (s_1, s_2, \ldots)$ in $\Sigma$ define

$$d\left(\widehat{t}, \widehat{s}\right) = \Sigma_{i=1}^{\infty} \frac{|t_i - s_i|}{2^i}.$$

It is known that $(\Sigma, d)$ is a compact metric space. Let $\sigma : \Sigma \to \Sigma$ be the shift mapping, $\sigma(t_1, t_2, \ldots) = (t_2, t_3, \ldots)$. The dynamics of mappings like $\sigma$ are well known (see [4] and [12]). In particular we are interested in the topological entropy of $\sigma$.

DEFINITION 9. Let $X$ be a compact topological space and $f : X \to X$ a continuous map. If $\alpha$ is an open cover of $X$, let $N(\alpha)$ denote the number of sets in a finite subcover of $\alpha$ with smallest cardinality. If $\alpha$ and $\beta$ are two open covers of $X$, let $\alpha \vee \beta = \{A \cap B \,|A \in \alpha, B \in \beta\}$ and $f^{-1}(\alpha) = \left\{f^{-1}(A)\,|A \in \alpha\right\}$. For an open cover $\alpha$ and $n \in \mathbb{N}$ let

$$\vee_{i=0}^{n-1} f^{-i}(\alpha) = \alpha \vee f^{-1}(\alpha) \vee f^{-2}(\alpha) \vee \cdots \vee f^{-(n-1)}(\alpha)$$

and

$$ent(f, \alpha) = \lim_{n \to \infty} \frac{1}{n} \log N\left(\vee_{i=0}^{n-1} f^{-i}(\alpha)\right).$$

The topological entropy of $f$ is defined by

$$ent(f) = \sup\{ent(f, \alpha)\,|\alpha \text{ is an open cover of } X\}.$$

The following two propositions are results already known. We refer the reader to [4] and [12] for detailed proofs. For the second part of proposition 11 see also theorem 17 in [5].

PROPOSITION 10. $ent(\sigma) = \log(4)$.

Proposition 11. *Let $f : X \to X$ and $g : Y \to Y$ be two mappings defined on compact topological spaces. Let $h : X \to Y$ be a surjective mapping. If the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array},$$

*then $ent(f) \geq ent(g)$. Furtheremore, if there exists $M$ such that for each $y \in Y$, $\#\left(h^{-1}(y)\right) \leq M$, then $ent(f) = ent(g)$.*

Let $k : \Sigma \to \Delta$ be the following mapping: For each $\widehat{t} = (t_1, t_2, \ldots) \in \Sigma$ the set $\cap_{n=1}^{\infty} \Delta_{t_1,t_2,\ldots,t_n}$ is just one point in $\Delta$. For the sides of the triangle $\Delta_{t_1,t_2,\ldots,t_n}$ have length $\frac{1}{2^n}\left(2\sqrt{3}\right)$. Define

$$k\left(\widehat{t}\right) = \cap_{n=1}^{\infty} \Delta_{t_1,t_2,\ldots,t_n}.$$

Let $(x, y)$ be a point in $\Delta$. For each $k \geq 1$ choose $w_k \in \{0, 1, 2, 3\}$ according to the following rule: Let $w_k = \min\left\{s : G^{k-1}(x, y) \in \Delta_s\right\}$. Hence $(x, y) \in \Delta_{w_1,w_2,\ldots,w_n}$ for each $n$. Let

$$\widehat{t} = (w_1, w_2, \ldots, w_n, \ldots) \in \Sigma.$$

Let us call $\widehat{t} = (w_1, w_2, \ldots, w_n, \ldots)$ an *itinerary* of $(x, y)$. It follows that $k\left(\widehat{t}\right) = (x, y)$. Thus $k : \Sigma \to \Delta$ is a surjective mapping.

*Remark 12.* Notice that given $(x, y)$ in $\Delta$ the following three conditions hold:

1. For each $k \in \mathbb{N}$, $\#\left\{s : G^{k-1}(x, y) \in \Delta_s\right\} \leq 3$.

2. If for some $k$, $\#\left\{s : G^{k-1}(x, y) \in \Delta_s\right\} = 3$, then $G^{k-1}(x, y) \in \partial\Delta$ and for each $n > k$, $\#\left\{s : G^{n-1}(x, y) \in \Delta_s\right\} = 1$.

3. If for some $k$, $\#\left\{s : G^{k-1}(x, y) \in \Delta_s\right\} = 2$, then $G^{k-1}(x, y) \in G^{-1}(\partial\Delta)$ and for each $n > k$, $\#\left\{s : G^{n-1}(x, y) \in \Delta_s\right\} = 1$ with, at most, one exception $m > k$ where the cardinality of $\left\{s : G^{m-1}(x, y) \in \Delta_s\right\}$ could be 3.

Thus the cardinality of $\left\{\widehat{t} \in \Sigma : k\left(\widehat{t}\right) = (x, y)\right\}$ is at most 6.

The next proposition contains two other properties of $k$. The proofs are not difficult, so we leave them to the reader.

Proposition 13. *The mapping $k : \Sigma \to \Delta$ is continuous. Furtheremore, the following diagram commutes:*

$$\begin{array}{ccc} \Sigma & \underrightarrow{\sigma} & \Sigma \\ k\downarrow & & \downarrow k \\ \Delta & \underrightarrow{G} & \Delta \end{array}.$$

Corollary 14. *The entropy of $\sigma$ is equal to the entropy of $G$.*

*Proof.* It follows immediately from propositions 11 and 13, and the remark following proposition 11. ▮

## 6. $G$ IN $\Delta$ AND $F$ IN $K$ ARE CONJUGATE

The interval $[-1,2] \subset \mathbb{R}$ is invariant under $G$. Let us denote $G\left|_{[-1,2]}\right.$ with $g$. Then for each $x$ in $[-1,2]$,

$$g(x) = \begin{cases} -2x & \text{if } x \in \left[-1, \frac{1}{2}\right] \\ 2x-2 & \text{if } x \in \left[\frac{1}{2}, 2\right] \end{cases}.$$

The graph of $g$ resembles the graph of a very well known function defined in the interval $[0,1]$: the *tent map.*

$$T(x) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right] \\ 2-2x & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Lemma 15. *The functions $g$ and $T$ are conjugate.*

*Proof.* Consider $l : [-1,2] \to [0,1]$ given by $l(x) = \frac{2}{3} - \frac{1}{3}x$. It is easy to see that $l \circ g = T \circ l$. ▮

The interval $[-1,3] \subset \mathbb{R}$ is invariant under $F$. Let us denote $F\left|_{[-1,3]}\right.$ with $f$. That is, $f(x) = x^2 - 2x$. Let $L : [0,1] \to [0,1]$ be the logistic function, $L(x) = 4x(1-x)$.

Lemma 16. *The functions $L$ and $f$ are conjugate.*

*Proof.* Consider $k : [0,1] \to [-1,3]$ given by $k(x) = 3 - 4x$. It readily follows that $k \circ L = f \circ k$. ▮

*Remark 17.* It is known that $T$ and $L$ are conjugated via the homeomorphism

$$j(x) = \sin^2\left(\frac{\pi x}{2}\right).$$

That is, the diagram

$$\begin{array}{ccc} [0,1] & \xrightarrow{T} & [0,1] \\ j \downarrow & & \downarrow j \\ [0,1] & \xrightarrow{L} & [0,1] \end{array}$$

commutes.

The homeomorphisms $l$, $k$ and $j$ allows us to see that $g$ and $f$ are conjugate. That is, restricted to the intervals $[-1,2]$ and $[-1,3]$ our piecewise linear model $G$ and $F$ have essentialy equal behaviour.

Proposition 18. *Let $h : [-1,2] \to [-1,3]$ given by $h = k \circ j \circ l$. That is, for each $x \in [-1,2]$,*

$$h(x) = -4\sin^2\left(\frac{\pi}{2}\left(\frac{2}{3} - \frac{1}{3}x\right)\right) + 3.$$

*Then for each $x \in [-1,2]$, $h \circ g(x) = f \circ h(x)$.*

*Proof.* It follows from lemmas 15 and 16, and the remark following lemma 16. ▮

Now we extend $h$ in order to produce a homeomorphism from $\Delta$ onto $K$ which allows us to show that $G$ and $F$ are conjugate. Consider the mapping $H : \Delta \to H(\Delta)$ given by

$$H(x,y) = (h(x) - 1)\, e^{i\frac{\pi y}{3\sqrt{3}}} + e^{-i\frac{2\pi y}{3\sqrt{3}}}.$$

Note the following:

1. $H$ is continuous, and $H(x,0) = h(x)$. Since $x \in [-1,2]$, $H(x,0) \in [-1,3]$.
2. For each $x \in [-1,2]$ fixed, $(x,y) \in \Delta$ lies in a vertical segment and $H(x,y)$ lies in a subarc of a member of the family $\Gamma$, $H(x,y) \in \Lambda_{h(x)-1}$. That is, $H$ maps the interval $\{x\} \times \left[\frac{\sqrt{3}}{3}(x-2), \frac{\sqrt{3}}{3}(2-x)\right]$ onto the subarc of $\Lambda_{h(x)-1}$ that goes from the point $2e^{i\frac{2\pi}{9}(x-2)} + e^{-i2\frac{2\pi}{9}(x-2)}$ to the point $2e^{i\frac{2\pi}{9}(x-2)} + e^{-i2\frac{2\pi}{9}(x-2)}$. These two points lie in the boundary of $K$. It follows that $H$ is one to one and $H(\Delta) = K$ (see figure 7).
3. Replacing $h(x)$ in the formula of $H(x,y)$ we have that

$$H(x,y) = 2\cos\left(\frac{\pi}{3}(2-x)\right) e^{i\frac{\pi y}{3\sqrt{3}}} + e^{-i\frac{2\pi y}{3\sqrt{3}}}.$$

It follows that in each point of $int(\Delta)$ the mapping $H$ is a local diffeomorphism.

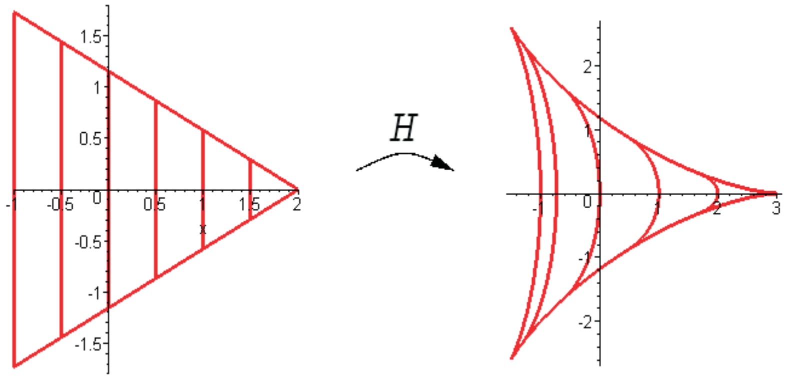


**FIG. 7.**

PROPOSITION 19. *The diagram*

$$\begin{array}{ccc} \Delta & \underrightarrow{G} & \Delta \\ H\downarrow & & \downarrow H \\ K & \underrightarrow{F} & K \end{array}$$

*commutes.*

Due to the way $G$ is defined we divide the proof of this proposition in several steps, according to whether the point $(x, y)$ is in one triangle $\Delta_i$ or another. We warn the reader that, although somehow elementary, the details of this proof can become highly cumbersome. For the sake of clarity we postpone this proof till the end of this section.

COROLLARY 20. *The entropy of* $F\,|_K$ *is* $\log(4)$.

*Proof.* It follows immediately from corollary 14 and proposition 19 ∎

COROLLARY 21. *Each periodic point of* $F$ *that lies in* $int\,(K)$ *is expansive.*

*Proof.* Let $(x, y) \in Per\,(F) \cap int\,(K)$ of period $n$. Since $H \circ G = F \circ H$, $H \circ G^n \circ H^{-1} = F^n$.

Note that $H^{-1}(x, y)$ is a periodic point of $G$ of period $n$. Also

$$o\left(H^{-1}(x, y), G\right) \subset int\,(\Delta).$$

We already know that every periodic point of $G$ that lies in $int\,(\Delta)$ is expansive. Since $H$ is a local diffeomorphism in each point of the orbit

$$o\left(H^{-1}(x,y), G\right),$$

the eigenvalues of $DF^n$ in $(x,y)$ are the same as the eigenvalues of $DG^n$ in $H^{-1}(x,y)$. ∎

*Proof* (Proof of Proposition 19).

*Step 1.* Let $(x,y) \in \Delta_0$. Then

$$\begin{aligned} H \circ G(x,y) &= H(-2x, 2y) \\ &= (h(-2x) - 1)\, e^{i\frac{2y\pi}{3\sqrt{3}}} + e^{-i\frac{4y\pi}{3\sqrt{3}}}. \end{aligned}$$

On the other hand.

$$\begin{aligned} F \circ H(x,y) &= F\left((h(x) - 1)\, e^{i\frac{y\pi}{3\sqrt{3}}} + e^{-i2\frac{y\pi}{3\sqrt{3}}}\right) \\ &= \left(h(x)^2 - 2h(x) + 1\right) e^{i\frac{2y\pi}{3\sqrt{3}}} + 2(h(x) - 1)\, e^{-i\frac{y\pi}{3\sqrt{3}}} \\ &\quad + e^{-i4\frac{y\pi}{3\sqrt{3}}} - 2(h(x) - 1)\, e^{-i\frac{y\pi}{3\sqrt{3}}} - 2e^{i2\frac{y\pi}{3\sqrt{3}}}. \end{aligned}$$

Since $x \in \left[-1, \frac{1}{2}\right]$, $h(x)^2 - 2h(x) = h(-2x)$. Hence

$$\begin{aligned} F \circ H(x,y) &= (h(-2x) + 1)\, e^{i\frac{2y\pi}{3\sqrt{3}}} - 2e^{i2\frac{y\pi}{3\sqrt{3}}} + e^{-i4\frac{y\pi}{3\sqrt{3}}} \\ &= (h(-2x) - 1)\, e^{i\frac{2y\pi}{3\sqrt{3}}} + e^{-i\frac{4y\pi}{3\sqrt{3}}}. \end{aligned}$$

*Step 2.* Let $(x,y) \in \Delta_1$. Then

$$\begin{aligned} F \circ H(x,y) &= F\left((h(x) - 1)\, e^{i\frac{y\pi}{3\sqrt{3}}} + e^{-i2\frac{y\pi}{3\sqrt{3}}}\right) \\ &= \left(h(x)^2 - 2h(x) + 1\right) e^{i\frac{2y\pi}{3\sqrt{3}}} + e^{-i4\frac{y\pi}{3\sqrt{3}}} - 2e^{i2\frac{y\pi}{3\sqrt{3}}}. \end{aligned}$$

Since $x \in \left[\frac{1}{2}, 2\right]$, $h(x)^2 - 2h(x) = h(2x - 2)$. Hence

$$\begin{aligned} F \circ H(x,y) &= (h(2x - 2) - 1)\, e^{i\frac{2y\pi}{3\sqrt{3}}} + e^{-i4\frac{y\pi}{3\sqrt{3}}} \\ &= H \circ G(x,y). \end{aligned}$$

*Step 3.* Let $(x,y)\in\Delta_2$. Also, let

$$\alpha=\frac{\pi}{3}\left(1+x-\frac{y}{\sqrt{3}}\right)$$

and

$$\beta=\frac{\pi}{3}\left(1+x+\sqrt{3}y\right).$$

By definition of $H$ and $G$,

$$\begin{aligned}H\circ G\,(x,y)&=2\cos\left(\frac{\pi}{3}\left(2-\left(1-x-\sqrt{3}y\right)\right)\right)e^{i\alpha}+e^{-i2\alpha}\\&=2\cos(\beta)\left(\cos(\alpha)+i\sin(\alpha)\right)+\cos(2\alpha)+i\sin(-2\alpha)\\&=2\cos(\beta)\cos(\alpha)+\cos(2\alpha)+i\left(2\cos(\beta)\sin(\alpha)-\sin(2\alpha)\right)\\&=\cos(\beta+\alpha)+\cos(\beta-\alpha)+\cos(2\alpha)\\&+i\left(\sin(\beta+\alpha)+\sin(-(\beta+\alpha))-\sin(2\alpha)\right).\end{aligned}$$

Now, noticing that $\beta+\alpha=\frac{2\pi}{3}\left(1+x+\frac{y}{\sqrt{3}}\right)$ and writing $a=\frac{2\pi}{3}(1+x)$, $b=\frac{\pi}{3}\left(\frac{y}{\sqrt{3}}\right)$, we have

$$\begin{aligned}H\circ G\,(x,y)&=\cos(a+b)+\cos(2b)+\cos(a-b)\\&+i\left(\sin(a+b)+\sin(-2b)-\sin(a-b)\right)\\&=2\cos(a)\cos(b)+\cos(2b)\\&+i\left(2\sin(b)\cos(a)+\sin(-2b)\right)\\&=2\cos(a)\,e^{ib}+e^{-i2b}.\end{aligned}$$

On the other hand,

$$G\circ H\,(x,y)=\left((h(x))^2-2h(x)-1\right)e^{ib}+e^{-i2b}.$$

Since $x\in\left[-1,\frac{1}{2}\right]$, replaicing $h(x)$ by its value and using again an obvious trigonometrical identity,

$$(h(x))^2-2h(x)-1=h(-2x)-1=2\cos\left(\frac{\pi}{3}(2+2x)\right).$$

Thus,

$$G \circ H(x, y) = 2\cos(a)\, e^{ib} + e^{-i2b}.$$

*Step 4.* Let $(x, y) \in \Delta_3$. Let $\gamma$ be the map given by $\gamma(x, y) = (x, -y)$. It is easy to see that the following diagram commutes:

$$\begin{array}{ccc} \Delta_3 & \xrightarrow{G} & \Delta_3 \\ \gamma \downarrow & & \downarrow \gamma \\ \Delta_2 & \xrightarrow{G} & \Delta_2 \end{array}.$$

Also, in $K$, $\gamma \circ F = F \circ \gamma$, and in $\Delta$, $\gamma \circ H \circ \gamma = H$. Then given $(x, y) \in \Delta_3$ we have the following:

$$H \circ G(x, y) = \gamma \circ H \circ \gamma \circ G(x, y) = \gamma \circ H \circ G \circ \gamma(x, y).$$

Since $\gamma(x, y) \in \Delta_2$,

$$\begin{aligned} H \circ G(x, y) &= \gamma \circ H \circ G \circ \gamma(x, y) \\ &= \gamma \circ F \circ H \circ \gamma(x, y) \\ &= F \circ \gamma \circ H \circ \gamma(x, y) \\ &= F \circ H(x, y). \end{aligned}$$

∎

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