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A Note on a Standard Family of Twist Mappings

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We investigate the break up of the last invariant curve for analytic families of standard mappings

$$S_\lambda : \begin{cases} y' = \lambda g(x) + y, \\ x' = x + y' \; mod \; 1, \end{cases}$$

where $g : S^1 \to \mathbb{R}$ is an analytic function such that $\int_{S^1} g(x)dx = 0$. Our main result is another evidence of how hard this problem is. We give an example of a particular function g as above such that the mapping S_λ associated to it has a "pathological" behavior, namely the set of parameters λ for which the mapping has at least one rotational invariant curve does not "seem" to be an interval.

Key Words: twist mappings, rotational invariant curves, topological methods, vertical rotation number, piecewise linear standard mappings.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this paper, we investigate the following problem:

Let $\widetilde{g} : \mathbb{R} \to \mathbb{R}$ be an analytic, non-zero, periodic function, $\widetilde{g}(x+1) = \widetilde{g}(x)$, such that $\int_0^1 \widetilde{g}(x)dx = 0$. We define the following one parameter family (λ) of analytic diffeomorphisms of the annulus:

$$S_\lambda : \begin{cases} y' = \lambda g(x) + y, \\ x' = x + y' \; mod \; 1, \end{cases} \tag{1}$$

where $g : S^1 \to \mathbb{R}$ is the map induced by $\widetilde{g}$.

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For all $\lambda \in$IR, S_λ is an area-preserving twist mapping, because $\partial_y x' = 1$, for any $(x, y) \in S^1\times$IR$=$(IR$/\mathbb{Z})\times$IR and $\det[DS_\lambda] = 1$. Also, the fact that $\int_0^1 \widetilde{g}(x)dx = 0$ implies that S_λ is an exact mapping, which means that given any homotopically non-trivial simple closed curve $C \subset S^1 \times$ IR, the area above C and below $S_\lambda(C)$ is equal the area below C and above $S_\lambda(C)$. Another obvious fact about this family is that S_0 is an integrable mapping, that is, the cylinder is foliated by invariant curves $y = y_0$.

So, KAM theory applies to S_λ and we can prove that there is a parameter $\lambda_0 > 0$, such that for any $\lambda \in [0, \lambda_0]$ S_λ has at least one rotational invariant curve. On the other hand, if we choose $x_0 \in S^1$ such that $g(x) \leq g(x_0)$ for all $x \in S^1$, we get that S_λ does not have rotational invariant curves for all $\lambda \geq \lambda^* = \frac{1}{g(x_0)} > 0$. The proof of this classical fact is very simple, so we present it here:

Given $\lambda \geq \lambda^*$, choose $x_\lambda \in S^1$ such that $\lambda = \frac{1}{g(x_\lambda)}$. A computation shows that $S_\lambda^n(x_\lambda, 0) = (x_\lambda, n)$, for all $n \in \mathbb{Z}$. So there can be no rotational invariant curves.

A result due to Birkhoff implies that the set

$$A_g = \{\lambda \geq 0 : \ S_\lambda \text{ has at least one rotational invariant curve}\} \tag{2}$$

is closed. So a very "natural" conjecture would be the following (see [5]):

Conjecture 1. $A_g = [0, \lambda_{cr}]$, for some $\lambda_{cr} > 0$.

Another interesting one parameter family is the following:

$$T_\lambda : \begin{cases} y' = g(x) + y + \lambda \\ x' = x + y' \ mod\ 1 \end{cases} \tag{3}$$

Of course T_λ is also an area-preserving twist mapping, the difference is that it is exact if and only if $\lambda = 0$, so when $\lambda \neq 0$ there is no rotational invariant curve.

It can be proved (see section 2) that there is a closed interval, called vertical rotation interval, $\rho_V = [\rho_V^{\min}, \rho_V^{\max}]$ associated to S_λ (and to T_λ) with the following property: Given $\omega \in \rho_V$, there is a point $X \in S_1 \times$ IR such that as $n \to \infty$

$$\lim \frac{p_2 \circ S_\lambda^n(X) - p_2(X)}{n} = \omega,$$

where $p_1(x, y) = x$ and $p_2(x, y) = y$. From the exactness of S_λ we get that $0 \in \rho_V(S_\lambda)$ for all $\lambda \in$IR, something that may not hold for T_λ.

In section 3 we prove a result which implies that $\rho_V^{\max}$ and $\rho_V^{\min}$ are continuous functions of the parameter λ. A first difference between S_λ and T_λ is that $\rho_V^{\max}(S_\lambda) = 0$ for any $\lambda \in [0, \lambda_0]$ while $\rho_V^{\max}(T_\lambda) \neq 0$ for all $\lambda \neq 0$. In fact, in a certain sense, the behavior of the function $\lambda \to \rho_V^{\max}(T_\lambda)$ is similar to the one of the rotation number of certain families of homeomorphisms of the circle.

Given a circle homeomorphism $f : S^1 \to S^1$, a well studied family (see for instance [6]) is the one given by translations of f :

$$x' = f_\lambda(x) = f(x) + \lambda$$

In this case it is easy to prove that the rotation number of f_λ is a non-decreasing function of the parameter. We have a similar result for T_λ:

Lemma 2. *$\rho_V^{\max}(T_\lambda)$ is a non-decreasing function of λ.*

As the proof will show, this fact is an easy consequence of proposition 3, page 466 of [9].

If we had a similar result for S_λ, then Conjecture 1 would trivially be true, because $A_g = (\rho_V^{\max})^{-1}(0)$ (see Theorem 4) and this set is an interval if $\rho_V^{\max}(S_\lambda)$ is a non-decreasing function.

The main result of this note goes in the opposite direction; we present an example in the analytic topology such that we do not know whether or not A_g is a closed interval (although we believe it is not), but for this example $\rho_V^{\max}(S_\lambda)$ is not a non-decreasing function of λ. More precisely, we have:

Theorem 3. *There exists an analytic function g^* as above such that $\rho_V^{\max}(S_\lambda)$ is not a non-decreasing function of λ.*

The proof of the theorem implies that we can choose

$$g^*(x) = \sum a_n . \cos(2\pi n x), \text{for } n = 1 \text{ to some } N.$$

Although this choice of g^* is a finite sum of cosines obtained as the truncation of a certain Fourier series of a continuous function, it is still possible that for $g_S(x) = \cos(2\pi x)$, $\rho_V^{\max}(S_\lambda)$ is in fact a non-decreasing function, as numerical experiments suggest. Nevertheless, this shows how subtle the problem is. Moreover, the proof of the main theorem shows that a lot of pathological families can be constructed. We just have to take any analytic function g, which is periodic, has zero mean and is sufficiently C^0 close to the example that appears in [4].

The proof of this theorem is based on a result previously obtained by the author, on a paper due to S.Bullett [4] on piecewise linear standard mappings and on some consequences of results from [9].

2. BASIC TOOLS

First we present a theorem which is a consequence of some results from [1]. Before we need to introduce some definitions:

1) $D_0(\mathrm{T}^2)$ is the set of torus homeomorphisms $T : \mathrm{T}^2 \to \mathrm{T}^2$ of the following form:

$$T : \begin{cases} y' = g(x) + y \ mod \ 1 \\ x' = x + y' \ mod \ 1 \end{cases}, \tag{4}$$

where $g : S^1 \to \mathrm{I\!R}$ is a Lipschitz function such that $\int_{S^1} g(x)dx = 0$.

2) $D_0(S^1 \times \mathrm{I\!R})$ is the set of lifts to the cylinder of elements from $D_0(\mathrm{T}^2)$, the same for $D_0(\mathrm{I\!R}^2)$. Given $T \in D_0(\mathrm{T}^2)$ as in (4), its lifts $\widehat{T} \in D_0(S^1 \times \mathrm{I\!R})$ and $\widetilde{T} \in D_0(\mathrm{I\!R}^2)$ write as ($\widetilde{g}$ is a lift of g)

$$\widehat{T} : \begin{cases} y' = g(x) + y \\ x' = x + y' \ mod \ 1 \end{cases} \text{ and } \widetilde{T} : \begin{cases} y' = \widetilde{g}(x) + y \\ x' = x + y' \end{cases}$$

3) We say that $T \in D_0(\mathrm{T}^2)$ has a $\frac{p}{q}$-vertical periodic orbit (set) if there is a point $A \in S^1 \times \mathrm{I\!R}$ such that $\widehat{T}^q(A) = A + (0, p)$. It is clear that $T^q(\pi_2(A)) = \pi_2(A)$, where $\pi_2 : S^1 \times \mathrm{I\!R} \to \mathrm{T}^2$ is given by $\pi_2(x, y) = (x, y \ mod \ 1)$. The periodic orbit that contains $\pi_2(A)$ is said to have vertical rotation number $\rho_V = \frac{p}{q}$.

4) Given an irrational number ω, we say that $T \in D_0(\mathrm{T}^2)$ has a ω-vertical quasi-periodic set if there is a compact T-invariant set $X_\omega \subset \mathrm{T}^2$, such that for any $X \in X_\omega$ and any $Z \in \pi_2^{-1}(X)$,

$$\rho_V(X_\omega) = \lim \frac{p_2 \circ \widehat{T}^n(Z) - p_2(Z)}{n} = \omega, \text{ as } n \to \infty$$

5) We say that $T \in D_0(\mathrm{T}^2)$ has a rotational invariant curve if there is a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times \mathrm{I\!R}$, such that $\widehat{T}(\gamma) = \gamma$.

Now we have the following:

THEOREM 4. *Given $T \in D_0(\mathrm{T}^2)$, there exists a closed interval $0 \in [\rho_V^{\min}, \rho_V^{\max}]$ such that for any $\omega \in]\rho_V^{\min}, \rho_V^{\max}[$, there is a periodic orbit or quasi-periodic set X_ω with $\rho_V(X_\omega) = \omega$, depending on whether ω is rational or not. Moreover, $\rho_V^{\min} < 0 < \rho_V^{\max}$ if and only if, T does not have any rotational invariant curve.*

When $\omega \in \{\rho_V^{\min}, \rho_V^{\max}\}$ a standard argument in ergodic theory (see the discussion below) proves that there is an orbit with that rotation number. In fact, much more can be said, see [3].

Following Misiurewicz and Ziemann [10], we can define another set that is equal to the limit of all the convergent sequences

$$\left\{ \frac{p_2 \circ \widehat{T}^{n_i}(Z_i) - p_2(Z_i)}{n_i},\ Z_i \in S^1 \times \mathrm{R},\, n_i \to \infty \right\},$$

which we call $\rho_V(T)^*$. In the following we present a sketch of the proof that $\rho_V(T) = \rho_V(T)^*$.

First note that the definition of $\rho_V(T)^*$ implies $\rho_V(T) \subseteq \rho_V(T)^*$. Now if we define $\omega^- = \inf\ \rho_V(T)^*$ and $\omega^+ = \sup\ \rho_V(T)^*$, Theorem 2.4 of [10] gives two ergodic T-invariant measures μ_- and μ_+ with vertical rotation numbers ω^- and ω^+, respectively. This means that

$$\int_{\mathrm{T}^2} [p_2 \circ T(X) - p_2(X)]\, d\mu_{-(+)} = \omega^{-(+)}.$$

Therefore from the Birkhoff ergodic theorem, there are points Z^+ and Z^- with $\rho_V(Z^+) = \omega^+$ and $\rho_V(Z^-) = \omega^-$. Finally, applying Theorem 6 of the appendix of [2], we get that $[\omega^-, \omega^+] \subseteq \rho_V(T)$, so $\rho_V(T) = \rho_V(T)^*$.

In the following we recall some topological results for twist mappings essentially due to Le Calvez (see [7] and [8] for proofs), that are used in some proofs contained in this paper. Let $\widehat{T} : S^1 \times \mathbb{R} \hookleftarrow$ be a twist diffeomorphism and $\widetilde{T} : \mathbb{R}^2 \hookleftarrow$ be one of its lifts. We are not assuming area-preservation or any other hypothesis, besides the twist condition, which can be expressed as $\partial_y p_1 \circ \widehat{T} \geq K > 0$, for some $K > 0$.

For every pair (s, q), $s \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ we define the following sets:

$$\widetilde{K}(s,q) = \left\{ (x,y) \in \mathbb{R}^2\colon\ p_1 \circ \widetilde{T}^q(x,y) = x + s \right\} \quad \text{and} \quad K(s,q) = \pi_1 \circ \widetilde{K}(s,q), \tag{5}$$

where $\pi_1 : \mathbb{R}^2 \to \mathrm{S}^1 \times \mathbb{R}$ is given by $\pi_1(x,y) = (x\ mod\ 1,\ y)$.

Then we have the following:

LEMMA 5. *For every $s \in \mathbb{Z}$ and $q \in \mathbb{N}^*$, $K(s,q) \supset C(s,q)$, which is a connected compact set that separates the cylinder.*

Now let us define the following functions on S^1:

$$\mu^-(x) = \min\{p_2(Q)\colon\ Q \in K(s,q) \text{ and } p_1(Q) = x\}$$
$$\mu^+(x) = \max\{p_2(Q)\colon\ Q \in K(s,q) \text{ and } p_1(Q) = x\}$$

We also have have similar functions for $\widehat{T}^q(K(s,q))$:

$$\nu^-(x) = \min\{p_2(Q)\colon\ Q \in \widehat{T}^q \circ K(s,q) \text{ and } p_1(Q) = x\},$$
$$\nu^+(x) = \max\{p_2(Q)\colon\ Q \in \widehat{T}^q \circ K(s,q) \text{ and } p_1(Q) = x\}.$$

The following are important results:

LEMMA 6. *Defining Graph*$\{\mu^\pm\}$=$\{(x,\mu^\pm(x)) : x \in S^1\}$ *we have:*

$$Graph\{\mu^-\} \cup Graph\{\mu^+\} \subset C(s,q).$$

So for all $x \in S^1$ *we have* $(x,\mu^\pm(x)) \in C(s,q)$.

LEMMA 7. $\widehat{T}^q(x,\mu^-(x)) = (x,\nu^+(x))$ *and* $\widehat{T}^q(x,\mu^+(x)) = (x,\nu^-(x))$.

Now we remember some ideas and results from [9]. In the following, $\widehat{T}$ and $\widetilde{T}$ are lifts of a torus twist map which is homotopic to the Dehn twist $(\phi, I) \to (\phi + I \bmod 1, I \bmod 1)$.

Given a triplet $(s,p,q) \in \mathbb{Z}^2 \times \mathbb{N}^*$, if there is no point $(x,y) \in \mathbb{R}^2$ such that $\widetilde{T}^q(x,y) = (x+s, y+p)$, it can be proved that the sets $\widehat{T}^q \circ K(s,q)$ and $K(s,q)+(0,p)$ can be separated by the graph of a continuous function from S^1 to $\mathbb{R}$, essentially because from all the previous results, either one of the following inequalities must hold:

$$\nu^-(x) - \mu^+(x) > p \tag{6}$$
$$\nu^+(x) - \mu^-(x) < p \tag{7}$$

for all $x \in S^1$, where $\nu^+, \nu^-, \mu^+, \mu^-$ are associated to $K(s,q)$.

Following Le Calvez [9], we say that the triplet (s,p,q) is positive (resp. negative) for $\widetilde{T}$ if $\widehat{T}^q \circ K(s,q)$ is above (6) (resp. below (7)) the graph. Given $\widetilde{T} \in D_0(\mathbb{R}^2)$, we have:

$$\widetilde{T}(x,y) = (x',y') \Leftrightarrow y = m(x,x') \text{ and } y' = m'(x,x'),$$

where m and m' are continuous maps from $\mathbb{R}^2$ to $\mathbb{R}$ with some especial properties. In particular, if $\widehat{T}$ is area-preserving then there exists a function $h(x,x')$ (called generating function) which satisfies:

$$m(x,x') = -\partial_x h(x,x') \text{ and } m'(x,x') = \partial_{x'} h(x,x').$$

For S_λ we get the following:

$$m(x,x') = x' - x - \lambda g(x) \text{ and } m'(x,x') = x' - x.$$

If $\widetilde{T}, \widetilde{T^*}$ are lifts to $\mathbb{R}^2$ of two twist mappings of the torus, both homotopic to Dehn twists, we say that $\widetilde{T} \leq \widetilde{T^*}$ if $m^* \leq m$ and $m' \leq m^{*\prime}$, where (m, m') is associated to $\widetilde{T}$ and $(m^*, m^{*\prime})$ to $\widetilde{T^*}$.

PROPOSITION 8. *If (s,p,q) is a positive (resp. negative) triplet of $\widetilde{T}$ and if $\widetilde{T} \leq \widetilde{T^*}$ (resp. $\widetilde{T} \geq \widetilde{T^*}$), then (s,p,q) is a positive (resp. negative) triplet of $\widetilde{T^*}$.*

Now we present an amazing example of a twist homeomorphism from $D_0(\mathrm{T}^2)$. First, let $g' : S^1 \to \mathbb{R}$ be given by $g'(x) = \left|x - \frac{1}{2}\right| - \frac{1}{4}$ and so the lift $\widetilde{g}' : \mathbb{R} \to \mathbb{R}$ is continuous, $\widetilde{g}'(x+1) = \widetilde{g}'(x)$, $\int_0^1 \widetilde{g}'(x)dx = 0$, $Lip(\widetilde{g}') = 1$ and $\widetilde{g}'(x) = \widetilde{g}'(-x)$. Also, $\widetilde{g}'$ is differentiable everywhere, except at points of the form $\frac{n}{2}$, $n \in \mathbb{Z}$. The one parameter family $S'_\lambda \in D_0(\mathrm{T}^2)$ is given by:

$$S'_\lambda : \begin{cases} y' = \lambda g'(x) + y \ mod\ 1, \\ x' = x + y' \ mod\ 1. \end{cases} \tag{8}$$

In [4] this family is studied in detail and among other things, the following theorem is proved:

THEOREM 9. *There are no rotational invariant curves for S'_λ when $\lambda \in]0.918, 1[\bigcup]4/3, \infty[$ and for $\lambda = 4/3$ there are "lots" of rotational invariant curves.*

3. PROOFS

3.1. Preliminary results

Proof (Proof of Lemma 2). This result is a trivial consequence of Proposition 8. Given $\lambda_1 < \lambda_2$, we get from expression (3) that $\widetilde{T}_{\lambda_1} \leq \widetilde{T}_{\lambda_2}$. So if $\rho_V^{\max}(T_{\lambda_2}) < p/q < \rho_V^{\max}(T_{\lambda_1})$ for a certain rational number p/q, then for any $s \in \mathbb{Z}$ the triplet (s,p,q) is negative for $\widetilde{T}_{\lambda_2}$, which implies by Proposition 8 that it is also negative for $\widetilde{T}_{\lambda_1}$, which contradicts the fact that $\rho_V^{\max}(T_{\lambda_1}) > p/q$. ∎

Now we prove the following theorem that has its own interest. It is easy to see from the proof that it is valid in a more general context.

THEOREM 10. *The functions $\rho_V^{\max}, \rho_V^{\min} : D_0(\mathrm{T}^2) \to \mathbb{R}$ are continuous.*

Remark 11. The proofs are analogous, so we do it only for $\rho_V^{\max}$.

Proof. Suppose that there is a $T_0 \in D_0(\mathrm{T}^2)$ such that $\rho_V^{\max}$ is not continuous at T_0. This means that there is an $\epsilon > 0$ and a sequence $D_0(\mathrm{T}^2) \ni T_n \overset{n\to\infty}{\to} T_0$ in the C^0 topology, such that either:

1) $\rho_V^{\max}(T_n) > \rho_V^{\max}(T_0) + \epsilon$, for all n, or
2) $\rho_V^{\max}(T_n) < \rho_V^{\max}(T_0) - \epsilon$, for all n.

The first possibility means that there exists a rational number p/q such that $\rho_V^{\max}(T_n) > p/q > \rho_V^{\max}(T_0)$. This implies that for any $s \in \mathbb{Z}$, the triplet (s, p, q) is non-negative for $\widetilde{T}_n$ (as the value of s is irrelevant in this setting, we fix $s = 0$). But as $\rho_V^{\max}(T_0) < p/q$, $(0, p, q)$ is negative for $\widetilde{T}_0$. As $T_n \overset{n\to\infty}{\to} T_0$, we get from the upper semi-continuity in the Hausdorff topology of the maps

$$T \to K(0, q) \text{ and } T \to \widehat{T}^q(K(0, q)) \tag{9}$$

that $(0, p, q)$ is a negative triplet for all mappings sufficiently close to $\widetilde{T}_0$, which is a contradiction.

In the same way, the second possibility means that there exists a rational number p/q such that $\rho_V^{\max}(T_n) < p/q < \rho_V^{\max}(T_0)$. This implies that there exists $Q \in C(0, q)$ such that

$$p_2 \circ \widehat{T_0}^q(Q) - p_2(Q) > p. \tag{10}$$

Now we prove the following claim, which implies the theorem:

Claim 12. Any mapping $T \in D_0(\mathrm{T}^2)$ sufficiently close to T_0 will satisfy an inequality similar to (10).

Proof. First of all, let us define $P_0 = (x_Q, \mu^-(x_Q))$, where $x_Q = p_1(Q)$. From lemma 7 and the definition of μ^- and ν^+, we get that $\nu^+(x_Q) = p_2 \circ \widehat{T_0}^q(P_0) > p_2(P_0) + p = \mu^-(x_Q) + p$. So there exists $\delta > 0$ such that for any $Z \in \overline{B_\delta(P_0)}$ we have

$$p_2 \circ \widehat{T_0}^q(Z) > p_2(Z) + p.$$

Therefore, there exists a neighborhood $T_0 \in \mathcal{U} \subset D_0(\mathrm{T}^2)$ in the C^0 topology such that for any $T \in \mathcal{U}$, we get $p_2 \circ \widehat{T}^q(Z) > p_2(Z) + p$, for all $Z \in B_\delta(P_0)$. Now defining $\overline{AB} = \{x_Q \times \mathbb{R}\} \cap B_\delta(P_0)$, lemma 6 implies that if we choose a sufficiently small neighborhood V of $C(0, q)$, then for all homotopically non-trivial simple closed curves $\gamma \subset V$, we get that $\gamma \cap \overline{AB} \neq \emptyset$. By the upper semi-continuity in the Hausdorff topology of the maps in (9), if we choose a sufficiently small sub-neighborhood $\mathcal{U}' \subset \mathcal{U}$ we get for any $T \in \mathcal{U}'$ that the set $C(0, q)$ associated to T is also contained in V. Therefore it must cross $\overline{AB}$.

So given any mapping $T \in \mathcal{U}' \subset \mathcal{U}$, there is a point $Q' \in C(0, q) \cap \overline{AB}$ which therefore satisfies $p_2 \circ \widehat{T}^q(Q') > p_2(Q') + p$. ▮

Finally, the above claim implies that $\rho_V^{\max}(T_n) \geq p/q$ for sufficiently large n, which is a contradiction. ▌

3.2. Main theorem

In this section we prove Theorem 3.

First of all we note that from Theorem 9, the mapping $S'_\lambda \in D_0(\mathrm{T}^2)$ (see (8)) has no rotational invariant curve for $\lambda = 0.95$ and has "lots" of rotational invariant curves for $\lambda = 4/3$. Using Theorem 4 one gets that $\rho_V^{\max}(S'_{0.95}) = \epsilon > 0$ and $\rho_V^{\max}(S'_{4/3}) = 0$. A classical result in Fourier analysis implies that the Fourier series $\widetilde{g}'_N(x) = \sum a_n \cos(2\pi n x)$, n going from 1 to some N, converges uniformly to $\widetilde{g}'$, as $N \to \infty$. So if we choose $N > 0$ sufficiently large, we get from Theorem 10 that $\rho_V^{\max}(S'_{N,0.95}) > \epsilon/2$ and $\rho_V^{\max}(S'_{N,4/3}) < \epsilon/10$, where $S'_{N,\lambda}$ is the twist mapping associated to g'_N.

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