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Completely Symmetric Centers

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Dedicated to Jorge Sotomayor on his $60th$ birthday

A phase portrait of a vector field on a plane is called completely symmetric if it is invariant with respect to the group consisting of four involutions $i_1, i_2, i_1 i_2, id$. The simplest example is a local center defined by the germ of an analytic vector field with a non-degenerate linear approximation. By the Poincare-Lyapunov theorem such a center is diffeomorphic to the center defined by the vector field $\dot{x}_1 = x_2, \dot{x}_2 = -x_1$ and consequently it is is completely symmetric. The paper is devoted to the classification of completely symmetric centers defined by germs of vector fields with a nilpotent linear approximation and by germs of vector fields with zero 2-jet and generic 3-jet.

Key Words: involution, symmetric center, reversible vector field

1. INTRODUCTION AND MAIN RESULTS

In the sequel all objects are germs at $0 \in \mathbf{R}^2$ and belong to a fixed category which is either C^∞ or C^ω (real analytic).

By a phase portrait of a vector field on the plane will be understood the *non-oriented* phase portraits, i.e., the foliation of the plane by the phase curves of the vector field.

An involution of the plane is a local diffeomorphism $i : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ such that $i^2 = id$. Simplest examples of involutions:

$$i_1 : \ (x_1, x_2) \to (-x_1, x_2), \quad i_2 : (x_1, x_2) \to (x_1, -x_2). \tag{1}$$

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Note that the composition of two involutions is as involution if and only if these involutions commute. The involutions i_1 and i_2 commute, and their composition is the involution $(x_1, x_2) \to (-x_1, -x_2)$ preserving orientation of the plane. The involutions i_1 and i_2 are orientation-reversing.

Note also that if i is an involution and ϕ is a local diffeomorphism then $\phi^{-1} \circ i \circ \phi$ is also an involution (conjugate to i), therefore an involution does not need to be linear.

A phase portrait is symmetric with respect to an involution i if the i-image of any phase curve is also a phase curve.

Definition 1. A phase portrait will be called completely symmetric if it is symmetric with respect to a group consisting of four involutions (including id).

The explanation of this definition is as follows: *any* group of involutions of the plane consists either of two or of four involutions (including id). This is a corollary of the following classical Bochner theorem.

Theorem 2. (Bochner, see [5].) *Let G be a group consisting of a finite number of local diffeomorphisms $\phi_1, \dots \phi_p$ of $\mathbf{R}^n$. There exists a local coordinate system $x = (x_1, \dots, x_n)$ in which each of the diffeomorphisms $\phi_i \in G$ is linear: $\phi_i(x) = A_i x,\ i = 1, \dots, p$, where A_i are matrices.*

In the case $n = 2$ this theorem implies the following corollary which will be used throughout the paper.

Theorem 3. (corollary of Theorem 2). *Any group $G \neq \{id\}$ of involutions of the plane consists either of two involutions i, id or of four commuting involutions. In the latter case there exists a local coordinate system (x_1, x_2) such that $G = \{i_1, i_2, i_1 i_2, id\}$, where i_1 and i_2 have form (1).*

Within phase portraits defined by linear vector fields $\dot{x} = Ax$ with a non-degenerate matrix A the saddle, the node with different eigenvalues, and the center are completely symmetric. In fact, in these cases a linear change of coordinates brings the vector field to the form $X : \dot{x}_1 = \lambda_1 x_1,\ \dot{x}_2 = \lambda_2 x_2$ or $X : \dot{x}_1 = -\lambda x_2, \dot{x}_2 = \lambda x_1$. In the first case $(i_1)_* X = (i_2)_* X = X$ and in the second case $(i_1)_* X = (i_2)_* X = -X$. In both cases the involutions (1) preserve the phase portrait of X. The phase portrait of the vector field $\dot{x}_1 = \lambda x_1, \dot{x}_2 = \lambda x_2$ is of course also completely symmetric. It is easy to see that in all other cases (focus and non-diagonalizable node) the phase portrait of a linear vector field with a non-degenerate matrix A is not completely symmetric.

It follows that in the C^∞ category the phase portrait of a vector field X on the plane with the linear approximation $\dot{x} = Ax$ at the singular point $0 \in \mathbf{R}^2$ is completely symmetric provided that the eigenvalues of A are real and different and there are no resonant relations (in this case X

reduces to its linear part by a change of coordinates, see [1]). In the C^ω category the same is true under additional assumption that there are no "small denominators" (Siegel or Brjuno condition, see [1]).

Another case where one has a complete symmetry is the case of a C^ω non-degenerate center (center defined by a vector field X with a non-degenerate linear approximation, i.e., with pure imaginary eigenvalues at the singular point.) In this case X is in general not linearizable by a change of coordinates, but by the classical Poincare-Lyapunov theorem (see [7] and references there) it is *orbitally equivalent to its linear approximation.* By definition two vector fields are orbitally equivalent if one of them can be brought to the other by a local diffeomorphism (change of coordinates) and multiplication by a non-vanishing function. The multiplication of a vector field by a non-vanishing function Q does not change its phase portrait (remind that we consider non-oriented phase portraits, the oriented phase portrait will be the same only if $Q > 0$). Therefore the Poincare-Lyapunov theorem implies the following corollary.

THEOREM 4. (corollary of Poincare-Lyapunov theorem). *In the C^ω category any non-degenerate center is completely symmetric and diffeomorphic to the center given by the vector field $\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1$.*

Completely symmetric centers are not exhausted by non-degenerate centers. For example, the phase portrait of any vector field which is orbitally equivalent to the Hamiltonian vector field:

$$\dot{x}_1 = \frac{\partial H}{\partial x_2}, \quad \dot{x}_2 = -\frac{\partial H}{\partial x_1}, \qquad H(x_1, x_2) = x_2^2 + x_1^{2m},\ m \geq 2 \tag{2}$$

is a completely symmetric center: involutions (1) preserve the phase portrait $\{H(x_1, x_2) = const\}$. Another example is any vector field which is orbitally equivalent to

$$X: \quad \dot{x}_1 = (\lambda + \mu) x_1^2 x_2 + x_2^3, \quad \dot{x}_2 = -x_1^3 + \lambda x_1 x_2^2, \quad \mu > -2. \tag{3}$$

The phase portrait is completely symmetric since (3) is reversible with respect to involutions (1) (a vector field X is *reversible* with respect an involution i if $i_* X = -X$). One of the ways to check that the phase portrait of (3) is a center is to pass to polar coordinates (r, θ) and to check that the assumption $\mu > -2$ guarantees that $\frac{d\theta}{dt} \neq 0$ for any trajectory $(r(t), \theta(t))$. This and the reversibility implies that any phase curve is closed.

In the present paper the "unfolding" techniques for studying degenerate completely symmetric centers (and other completely symmetric phase portraits) is developed.

The first step is the reduction of the classification of completely symmetric centers to the orbital classification of completely reversible vector fields (section 2).

The second step is the extension of the approach of M.A. Teixeira [10, 11] from topological to C^∞ and C^ω category. The orbital classification of completely reversible vector fields is reduced to the classification of triples consisting of unfolded vector field and two transversal regular curves (section 3). This reduction is a sort of "inverting" the method of studying the constrained system of ODE's developed by J. Sotomayor and the author in [9, 12]. The advantage of the reduction to the triples is that the unfolded vector field has much simpler singularity than the original vector field.

The beginning of the classification of the triples is given in section 4 (Theorems 15-17), where the techniques developed in [9] are used. A part of results of section 4 can be deduced from a table of normal forms obtained by R. Bogdanov in [3], but the approach in section 4 is different. The starting point is the classification of generic couples (vector field,involution) obtained by A. Davydov in [4], and the main tool is the definition of the order of tangency between a curve in the plane and a foliation of the plane by curves (including singular foliations), comparable with the multiplicity of a singular point of a function. Theorems 15 and 16 are easy, Theorem 17 requires using the homotopy method, it is proved in section 6.

The unfolding approach gives a number of smooth normal forms. As an illustration, I present the simplest ones, classifying the first two occurring singularities of completely symmetric centers - completely symmetric centers defined by vector fields with a nilpotent linear approximation (i.e., vector fields $\dot{x} = Ax + \cdots$, where A is a non-zero matrix with zero eigenvalues) and completely symmetric centers defined by vector fields with zero 2-jet and generic terms of order 3.

THEOREM 5. *Any completely symmetric center defined by a vector field with a nilpotent linear approximation at an algebraically isolated singular point* $0 \in \mathbf{R}^2$ *is diffeomorphic to the center defined by vector field (2) for some* $m \geq 2$.

I show in section 5 that this theorem and Theorem 6 below follow immediately form the results in sections 2-4.

It was proved in [6] that the phase portrait of a vector field with zero 1-jet and non-zero 2-jet cannot be the center. Therefore Theorem 4 and Theorem 5 give a complete classification of all completely symmetric centers defined by vector fields with non-zero 2-jet. Completely symmetric centers defined by a vector field with zero 2-jet and generic terms of order 3 are described by the following theorem.

In what follows the assumption "no small denominators" means the Siegel or (weaker) Brjuno condition on the tuple of eigenvalues of a matrix A under which any vector field of the form $\dot{x} = Ax + \cdots$ is linearizable in the C^ω category, see [1].

THEOREM 6. *Assume that F is a completely symmetric center defined by a vector field X with zero 2-jet at at an algebraically isolated singular point $0 \in \mathbf{R}^2$. Then in suitable coordinates the 3-jet of X has the form*

$$\dot{x}_1 = a_{1,1}x_1^2x_2 + a_{1,2}x_2^3, \quad \dot{x}_2 = a_{2,1}x_1^3 + a_{2,2}x_1x_2^2. \tag{4}$$

If $a_{1,2} \neq 0$, $a_{2,1} \neq 0$ and the tuple of eigenvalues of the matrix $A = (a_{ij})$ is non-resonant then in the C^∞ category the phase portrait F is diffeomorphic to the phase portrait defined by a vector field of the form (3). The same is true in the C^ω category under additional assumption that there are no small denominators.

In sections 3-4 all the assumption of this theorem are explained. Our techniques also allow to obtain normal forms in the case where some of these assumptions are violated, see section 7.

In view of Theorems 5 and 6 it is natural to ask what is the place of completely symmetric centers in the set of all degenerate centers of a certain class (defined by vector fields with a nilpotent linear approximation, homogeneous degree 3 vector fields, a vector field with zero 2-jet).

Given a set W of vector fields with an algebraic isolated singularity at the singular point $0 \in \mathbf{R}^2$ denote by $C(W)$, $SC(W)$, $CSC(W)$ the subset of W consisting of vector fields whose phase portrait is respectively a center, symmetric center, completely symmetric center. Here symmetric center means a center which is symmetric with respect to *one* non-trivial involution. Then the following statements hold.

1. Let W be the set of vector fields with non-degenerate linear approximation. By Theorem 4 in the analytic category $C(W) = SC(W) = CSC(W)$.

2. Let W is the set of vector fields with nilpotent linear approximation at the algebraically isolated singular point $0 \in \mathbf{R}^2$. Berthier and Moussu proved in [2] that in this case in the analytic category $C(W) = SC(W)$. Nevertheless, $CSC(W)$ is a subset of $C(W)$ of infinite codimension. The latter will be explained below.

3. If W is the set of homogeneous degree 3 vector fields then $SC(W) = CSC(W)$ because the phase portrait of any vector field in W is symmetric with respect to the involution $(x_1, x_2) \to (-x_1, -x_2)$. But $SC(W) \neq C(W)$. It is well known that $C(W)$ is a codimension 1 subset of W, and one can prove that $SC(W) = CSC(W)$ is a codimension 2 subset of W. Consequently $SC(W) = CSC(W)$ is a codimension 1 subset of $C(W)$.

4. If W is the set of all vector fields with zero 2-jet then one can prove that any inclusion in the chain $CSC(W) \subset SC(W) \subset C(W) \subset W$ has infinite codimension.

A complete proof of these and related results will be published elsewhere, in this paper I will explain why in the case 2 (nilpotent centers) the set $CSC(W)$ has infinite codimension in $SC(W)$. It was proved in [2] that any vector field in $C(W)$ is formally orbitally equivalent to a vector field of the form

$$\dot{x}_1 = x_2 + F(x_1^2), \quad \dot{x}_2 = -(2m)x_1^{2m-1}, \quad n \geq 2, \tag{5}$$

where F is a function of one variable, $F(0) = 0$. The symmetry is given by the involution $i_1 : x_1 \to -x_1,\ x_2 \to x_2$. One can prove that (5) defines a completely symmetric center if and only if $F(z)$ has the form $a_1 z^m + a_2 z^{2m} + a_3 z^{3m} + \cdots$. For example, any vector field of the family

$$\dot{x}_1 = x_2 + ax_1^2 + bx_1^4, \quad \dot{x}_2 = -4x_1^3,\ a, b \in \mathbf{R}$$

defines a symmetric center, but this center is completely symmetric if and only if $a = 0$. The family of vector fields

$$\dot{x}_1 = x_2 + bx_1^4,\ \dot{x}_2 = -4x_1^3$$

is an example of a family of vector fields defining a completely symmetric center for any value of the parameter b. By Theorem 5 all these centers are diffeomorphic to a single center given by the same vector field with $b = 0$.

2. REDUCTION TO REVERSIBLE VECTOR FIELDS.

A vector field which is reversible with respect to involutions i_1 and i_2 has the form

$$\dot{x}_1 = x_2 U_1(x_1^2, x_2^2), \quad \dot{x}_1 = x_1 U_2(x_1^2, x_2^2). \tag{6}$$

PROPOSITION 7. *If a vector field X has an algebraically isolated singular point $0 \in \mathbf{R}^2$ and the phase portrait of X is a completely symmetric center then X is orbitally equivalent to a vector field of the form (6).*

Of course, not any vector field (6) defines a center (see section 3), but if (6) defines a center then this center is completely-symmetric. The family (6), parameterized by functions U_1 and U_2 can be considered as the universal family satisfying this condition.

Proof. By Theorem 3 we may assume that the phase portrait of X is symmetric with respect to the involutions i_1 and i_2. It follows that

$$(i_1)_* X \wedge X \equiv (i_2)_* X \wedge X \equiv 0, \tag{7}$$

where the relation $X \wedge Y \equiv 0$ between vector fields X and Y means that the vectors $X(p)$ and $Y(p)$ are linearly dependent at any point p. One of the simplest division properties (see, for example [8]) states that if X has an algebraically isolated singular point and $X \wedge Y \equiv 0$ then $Y = QX$ for some function Q. Since the vector field X and consequently the vector fields $(i_1)_*X$ and $(i_2)_*X$ have algebraically isolated singularity at 0 then by this division property

$$(i_1)_*X = Q_1X, \quad (i_2)_*X = Q_2X, \tag{8}$$

where Q_1 and Q_2 are non-vanishing functions.

Let p be a point on the line of fixed points of i_1 (the x_2-axis), and let $X(p) = a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2}$. Then $((i_1)_*X)(p) = -a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2}$. If $Q_1 > 0$ then we obtain that $a = 0$, and it follows that the x_2-axes is an invariant line of the vector field X. This contradicts to the assumption that the phase portrait of X is a center. Therefore $Q_1 < 0$. Similarly $Q_2 < 0$.

Let $\hat{X} = (1-Q_1)X$. Since $Q_1 < 0$ then $1-Q_1$ is a non-vanishing function and therefore $\hat{X}$ is orbitally equivalent to X. Since $i_1^2 = id$ then the first relation in (8) implies that $(i_1)_*(Q_1X) = X$. Consequently

$$(i_1)_*\hat{X} = -\hat{X}, \;\; (i_2)_*\hat{X} = \hat{Q}_2\hat{X}, \tag{9}$$

where $\hat{Q}_2 < 0$. Let $\tilde{X} = (1-\hat{Q}_2)\hat{X}$. Then $\tilde{X}$ is orbitally equivalent to X, and the same arguments as above allow to deduce from the second relation in (9) that $(i_2)_*\tilde{X} = -\tilde{X}$.

Note now that (9) implies the relations

$$(i_2)_*((i_1)_*\hat{X}) = -(i_2)_*\hat{X} = -\hat{Q}_2\hat{X}$$

$$(i_1)_*((i_2)_*\hat{X}) = ((i_1)^*\hat{Q}_2)\cdot(i_1)_*\hat{X} = -((i_1)^*\hat{Q}_2)\hat{X}.$$

Since $i_1i_2 = i_2i_1$ it follows that $(i_1)^*\hat{Q}_2 = Q_2$. This and the first relation in (9) imply that $(i_1)_*\tilde{X} = -\tilde{X}$. We have proved that $\tilde{X}$ is reversible with respect to involutions i_1 and i_2. Therefore $\tilde{X}$ has form (6). ▮

Remark 8. Proposition 7 remains true if one replaces the assumption that the phase portrait of X is a completely symmetric center by a weaker assumption that it is completely symmetric with respect to a group of involutions i_1, i_2, i_1i_2, id such that the fixed curves of the orientation-reversing involutions i_1, i_2 are not X-invariant. The proof remains the same.

3. REDUCTION TO UNFOLDED VECTOR FIELD

Proposition 7 reduces the classification of completely symmetric centers to the orbital classification of vector fields X of the form (6). Of course, not any vector field (6) defines a center. To study topological properties of the phase portraits of vector fields (6) (in particular to distinguish centers) it is natural to make "unfolding"

$$u_1 = x_1^2, \quad u_2 = x_2^2 \tag{10}$$

which is a diffeomorphism from the domain $\{x_1 > 0, x_2 > 0\}$ of the plane $\mathbf{R}^2(x_1, x_2)$ to the domain $\{u_1 > 0, u_2 > 0\}$ of the plane $\mathbf{R}^2(u_1, u_2)$. Let U be the vector field obtained by the unfolding (10). Due to the symmetries, in most cases the topological properties of the phase portrait of (6) in the whole plane $\mathbf{R}^2(x_1, x_2)$ are determined by the phase portrait of U in the domain $\{u_1 > 0, u_2 > 0\}$, see [11].

Considering u_1 and u_2 as functions of time t and differentiating them along a vector field X of the form (6) we obtain

$$\frac{du_1}{dt} = 2x_1(t)x_2(t)U_1(u_1(t), u_2(t)), \tag{11}$$

$$\frac{du_2}{dt} = 2x_1(t)x_2(t)U_2(u_1(t), u_2(t)).$$

Let $t \to (x_1(t), x_2(t)),\ t \in I$ be any trajectory of X defined on on open interval I such that $x_1(t) > 0, x_2(t) > 0$ for any $t \in I$. Then the image of the corresponding trajectory $t \to (u_1(t), u_2(t)),\ t \in I$ of the system (11) is a phase curve of the vector field

$$\dot{u}_1 = U_1(u_1, u_2), \quad \dot{u}_2 = U_2(u_1, u_2) \tag{12}$$

in the domain $\{u_1 > 0, u_2 > 0\}$. To study smooth or analytic properties of X one has to consider the vector field (12) in the whole neighborhood of the origin of the plane $\mathbf{R}^2(u_1, u_2)$, not only in the domain $\{u_1 > 0, u_2 > 0\}$.

Definition 9. Given a vector field of form (6) we will say that the vector field (12) defined on the whole neighborhood of the origin of the plane $\mathbf{R}^2(u_1, u_2)$ is the unfolded vector field corresponding to (6).

Proposition 10. *Let X and $\tilde{X}$ be vector fields of the form (6), and let U and $\tilde{U}$ be corresponding unfolded vector fields. Assume that U and $\tilde{U}$ are orbitally equivalent via a local diffeomorphism of the form*

$$\psi: \ (u_1, u_2) \to (u_1\psi_1(u_1, u_2),\ u_2\psi_2(u_1, u_2))\,,\ \psi_1(0) = \psi_2(0) = 1, \tag{13}$$

i.e., via a local diffeomorphism preserving the u_1-axes, the u_2-axes and having identity linear approximation. Then the vector fields X and $\tilde{X}$ are orbitally equivalent.

Proof. We will show that the local diffeomorphism

$$\phi: \quad (x_1, x_2) \to \left(x_1\sqrt{\psi_1(x_1^2, x_2^2)}, \quad x_2\sqrt{\psi_2(x_1^2, x_2^2)}\right) \tag{14}$$

brings X to $\tilde{X}$ multiplied by a non-vanishing function.

Let us pass from vector fields to 1-forms. Given a vector field X : $A(x_1, x_2)\frac{\partial}{\partial x_1} + B(x_1, x_2)\frac{\partial}{\partial x_2}$ one can associate to it a differential 1-form ω such that $\Omega(X, Y) = \omega(Y)$ for any vector field Y, where Ω is a volume form. If, for example, $\Omega = dx_1 \wedge dx_2$ then $\omega = A(x_1, x_2)dx_2 - B(x_1, x_2)dx_1$. Let $X, \tilde{X}$ be vector fields on the plane, and let $\omega, \tilde{\omega}$ be corresponding 1-forms defined via the same volume form. We will use a simple fact that X and $\tilde{X}$ are orbitally equivalent via a local diffeomorphism ϕ if and only if $\phi^*\omega = Q\tilde{\omega}$, where Q is a non-vanishing function.

Consider the 1-forms

$$\theta = U_1(u_1, u_2)du_2 - U_2(u_1, u_2)du_1, \quad \tilde{\theta} = \tilde{U}_1(u_1, u_2)du_2 - \tilde{U}_2(u_1, u_2)du_1$$

corresponding to the unfolded vector fields U and $\tilde{U}$ via the volume form $du_1 \wedge du_2$. Consider also the 1-forms

$$\omega = x_2U_1(x_1^2, x_2^2)dx_2 - x_1U_2(x_1^2, x_2^2)dx_1,$$

$$\tilde{\omega} = x_2\tilde{U}_1(x_1^2, x_2^2)dx_2 - x_1\tilde{U}_2(x_1^2, x_2^2)dx_1$$

corresponding to X and $\tilde{X}$ via the volume form $dx_1 \wedge dx_2$. Denote by i the map from $\mathbf{R}^2(x_1, x_2)$ to $\mathbf{R}^2(u_1, u_2)$ defined by (10). Then

$$i^*\theta_1 = 2\omega_1, \quad i^*\theta_2 = 2\omega_2, \tag{15}$$

where i^* is the pullback. The orbital equivalence of U_1 and U_2 implies that

$$\psi^*\tilde{\theta} = Q\theta, \tag{16}$$

where Q is a non-vanishing function and ψ is the local diffeomorphism (13) of the plane $\mathbf{R}^2(u_1, u_2)$. Note now that

$$i(\phi(p)) = \psi(i(p)), \quad p \in \mathbf{R}^2(x_1, x_2), \tag{17}$$

where ϕ is the local diffeomorphism (14) of the plane $\mathbf{R}^2(x_1, x_2)$. Relations (15) - (17) imply

$$\phi^*\tilde{\omega} = \frac{1}{2}\phi^*(i^*\tilde{\theta}) = \frac{1}{2}i^*(\psi^*\tilde{\theta}) = \frac{1}{2}i^*(Q\theta) = \tilde{Q}\omega,$$

where $\tilde{Q} = i^*Q$. Therefore the diffeomorphism ϕ brings $\tilde{\omega}$ to ω multiplied by a non-vanishing function and consequently the vector fields X and $\tilde{X}$ are orbitally equivalent. ▮

4. CLASSIFICATION OF TRIPLES CONSISTING OF A VECTOR FIELD AND TWO TRANSVERSAL CURVES

Proposition 10 reduces the classification of completely symmetric centers to the orbital classification of vector fields with respect to local diffeomorphisms preserving two fixed transversal lines. The latter classification problem coincides with the classification of triples consisting of a vector field, defined up to multiplication by a non-vanishing function, and two transversal curves with respect to the group of all local diffeomorphisms. In this section we present the beginning of this classification. In what follows by a regular curve we mean a curve given by equation $f(x_1, x_2) = 0$, where $f(0) = 0,\ df(0) \neq 0$.

Definition 11. The order of tangency of a vector field U on $\mathbf{R}^2(u_1, u_2)$ and a regular curve $\gamma = \{f(u_1, u_2) = 0\}$ is the dimension of the factor-space $\mathbf{R}[[u_1, u_2]]/(f, U(f))$, where $\mathbf{R}[[u_1, u_2]]$ is the ring of all formal series, $U(f)$ is the Lie derivative of f along U, and $(f, U(f))$ is the ideal generated by the formal series of the functions f and $U(f)$.

Example 12. If $U(0) \neq 0$ then in suitable coordinates $U = \frac{\partial}{\partial u_1}$. Let, in the same coordinates, $\gamma = \{f(u_1, u_2) = u_2 + g(u_1) = 0\}$. Assume that $g(u_1) = cu_1^m + \cdots,\ c \neq 0$. Then $U(f) = g'(u_1) = mcu_1^{m-1}$. Therefore the ideal $(f, U(f))$ coincides with the ideal (u_2, u_1^{m-1}), and the order of tangency of U and γ is equal to $m - 1$.

Note that if the order of tangency of U and $\gamma = \{f = 0\}$ is equal to zero then $(U(f))(0) \neq 0$. Therefore the zero order of tangency means that $U(0) \neq 0$ and U is transversal to γ.

Note also that if $U(0) \neq 0$ then an equivalent definition of the order of tangency is as follows: the order of tangency is equal to $m-1$ if $(U(f))(0) = (U^2(f))(0) = \cdots = (U^{m-1}(f))(0) = 0, (U^m(f))(0) \neq 0$. Here $U^i(f)$ is defined by induction: $U^i(f) = U(U^{i-1}(f))$.

Example 13. If $U(0) = 0$ then the minimal possible order of tangency of U and any regular curve $\gamma = \{f = 0\}$ is equal to 1 since $f(0) = (U(f))(0) = 0$. The order of tangency is equal to 1 if and only if no eigenvector of the linear approximation of U is tangent to γ. To check this take local coordinates such that $f = u_1$ and $j^1U = (a_{11}u_1 + a_{12}u_2)\frac{\partial}{\partial u_1} + (a_{21}u_1 + a_{22}u_2)\frac{\partial}{\partial u_2}$. Then $(f, U(f)) = (u_1, a_{11}u_1 + a_{12}u_2 + \cdots)$. The order of tangency is equal to 1 if and only if this ideal is maximal, i.e., contains all formal series with zero free term. This is so if and only if $a_{12} \neq 0$. The latter is equivalent to the condition that j^1U has no eigenvectors tangent to $\gamma = \{u_1 = 0\}$.

Example 14. Let $U = \lambda_1 u_1 \frac{\partial}{\partial u_1} + \lambda_2 u_2 \frac{\partial}{\partial u_2}$. Then U has tangency of order 1 with the curve $au_1 + bu_2 = 0$ provided that $\lambda_1 \neq \lambda_2$ and $ab \neq 0$. If one of these conditions is violated then the order of tangency is ∞. Let now $\gamma = \{f(u_1, u_2) = u_1 + u_2^m + o(u_2^m) = 0\}$. Then $(f, U(f)) = (u_1 + u_2^m + o(u_2^m),\ \lambda_1 u_1 + m\lambda_2 u_2^m + o(u_2^m))$. Changing the coordinate u_1 to $\tilde{u}_1 = u_1 + u_2^m + o(u_2^m)$ we reduce this ideal to $(\tilde{u}_1,\ (m\lambda_2 - \lambda_1)u_2^m + o(u_2^m))$. Therefore the order of tangency of U with γ is equal to m provided that $\lambda_1 \neq m\lambda_2$.

Theorem 15. *Let γ_1, γ_2 be regular curves in the plane and let U be a vector field which is transversal to γ_1 and has tangency of order $m \geq 0$ with γ_2. Then in suitable coordinate system*

$$U = Q\frac{\partial}{u_1}, \quad \gamma_1 = \{u_1 = 0\}, \quad \gamma_2 = \{u_2 - u_1^{m+1} = 0\},$$

where Q is a non-vanishing function.

Proof. Since U is transversal to γ_1 then $U(0) \neq 0$ and in suitable coordinates $U = \frac{\partial}{\partial u_1}$ and $\gamma_1 = \{f(u_1, u_2) = 0\}$, where $\frac{\partial f}{\partial u_1}(0) \neq 0$. Changing the coordinate u_1 by $\tilde{u}_1 = f(u_1, u_2)$ we bring γ_1 to the form $u_1 = 0$ preserving U up to multiplication by a function. The curve γ_2 is transversal to γ_1, therefore it can be given by the equation $u_2 = g(u_1)$. The Taylor series of the function $g(u_1)$ starts with terms of order $m + 1$ since the order of tangency of U and γ_2 is equal to m (see Example 12). A local diffeomorphism $(u_1, u_2) \to (\phi(u_1), \delta u_2)$ with a suitable $\phi(u_1)$ and $\delta \in \{1, -1\}$ brings γ_2 to $\{u_2 - u_1^{m+1} = 0\}$. Such a local difeomorphism preserves the curve γ_1 and, up to multiplication by a function, the vector field U. ▮

Theorem 16. *Let U be a vector field transversal to the u_1-axes and having tangency of order m with the u_2-axes. There exists a local diffeomorphism with identity linear approximation which preserves the u_1-axes and the u_2-axes and brings U to the form*

$$a\frac{\partial}{\partial u_1} + bu_1^m \frac{\partial}{\partial u_2}$$

up to multiplication by a non-vanishing function.

Proof. Apply Theorem 15 with γ_1 and γ_2 being the u_1-axes and the u_2-axes, and make a change of coordinates $(u_1, u_2) \to (u_1, u_2 - u_1^{m+1})$. It brings U to the form $U_0 = \frac{\partial}{\partial u_1} - (m+1)u_1^m \frac{\partial}{\partial u_2}$. It follows that there exists a local diffeomorphism preserving the u_1-axes and the u_2-axes and bringing

U to U_0 up to multiplication by a non-vanishing function. The composition of this diffeomorphism and a scale transformation $(u_1, u_2) \to (k_1 u_1, k_2 u_2)$ with suitable non-zero k_1, k_2 is the required local diffeomorphism with identity linear approximation. ∎

THEOREM 17. *Let γ_1, γ_2 be regular transversal curves in the plane. Let U be a vector field vanishing at* 0 *and having the minimal possible order* 1 *of tangency with γ_1 and with γ_2. Let the straight lines l_1, l_2 be the linear approximations of γ_1, γ_2. There exists a local diffeomorphism with the identity linear approximation preserving U up to multiplication by a non-vanishing function and bringing γ_1, γ_2 to l_1, l_2 respectively.*

This theorem is proved by the homotopy method in section 6.

5. PROOFS OF THEOREMS 5 AND 6

Theorems 5 and 6 are almost immediate corollaries of the results in sections 2-4.

Proof of Theorem 5. By Proposition 7 the vector field X is orbitally equivalent to a vector field U of the form (6). Since the linear approximation of X is nilpotent then $U(0) \neq 0$, U is transversal to one of the axes $\{u_1 = 0\}$, $\{u_2 = 0\}$ and is tangent to the other axes. The order of tangency is finite, this follows form the assumption that 0 is an algebraically isolated singular point of X. By Proposition 10 and Theorem 16 X is orbitally equivalent to a vector field of the form $\dot{x}_1 = ax_2$, $\dot{x}_2 = bx_1^{2m+1}$, where $a, b \neq 0$. A scale transformation reduces a to 2 and b to $\pm(2m+2)$. Therefore X is orbitally equivalent to a vector field $\dot{x}_1 = \frac{\partial H}{\partial x_2}$, $\dot{x}_2 = -\frac{\partial H}{\partial x_1}$, $H(x_1, x_2) = x_2^2 \pm x_1^{2m+2}$. Since the phase portrait of X is a center then one has the sign $+$.

Proof of Theorem 6. By Proposition 7 X is orbitally equivalent to a vector field of the form (6). The unfolded vector field U has linear approximation with the matrix A, therefore it can be brought to its linear part by a local diffeomorphism with the identity linear approximation. The assumptions $a_{12} \neq 0$ and $a_{21} \neq 0$ imply that U has tangency of minimal possible order 1 with the lines $\{u_1 = 0\}$ and $\{u_2 = 0\}$ (see Example 13). By Proposition 10 and Theorem 17 (with with γ_1 and γ_2 being the u_1-axes and the u_2-axes) X is orbitally equivalent to a vector field of form (4). A scale transformation reduces a_{12} to 1 and a_{21} to ± 1. Passing to polar coordinates it it easy to check that the phase portrait of this vector field is a center if and only if $a_{21} = -1$ and $a_{11} - a_{22} > 2$.

6. PROOF OF THEOREM 17

Let

$$\gamma_1 = \{A_1 + a_1 = 0\}, \quad \gamma_2 = \{A_2 + a_2 = 0\},$$

where A_1, A_2 are linear functions and a_1, a_2 are functions with zero 1-jet. We use the homotopy method, like in [9]. Let $A_{i,t} = A_i + t(a_i - A_i), i = 1, 2$. In what follows t is a parameter varying on $[0, 1]$. To prove Theorem 17 it suffices to solve the system

$$h_t U(A_{i,t}) + q_{i,t} A_{i,t} = -a_i, \quad i = 1, 2 \tag{18}$$

with respect to families of functions $h_t, q_{1,t}, q_{2,t}$ such that $h_t(0) = 0$. If $(h_t, q_{1,t}, q_{2,t})$ is a solution then the family of local diffeomorphisms ϕ_t defined by the system of ODE's $\frac{d\phi_t}{dt} = (h_t U)(\phi_t)$ (where a point of $\mathbf{R}^2$ is a parameter) and the initial condition $\phi_0 = id$ brings the family of curves $\{A_{i,t} = 0\}$ to the curve $\{A_i = 0\}$, $i = 1, 2$. The construction of ϕ_t implies that this family preserves U up to multiplication by a non-vanishing function. Since $h_t(0) = 0$ then the vector fields $h_t U$ have zero 1-jet and it follows that the diffeomorphisms ϕ_t have the identity linear approximation.

The first equation of the system (18) (with $i = 1$) can be solved due to the assumption that U has tangency of order 1 with γ_1. This assumption implies that the functions $A_{1,t}$ and $U(A_{1,t})$ are differentially independent for any $t \in [0, 1]$ and it follows that the first equation has a solution $h_t, q_{1,t}$ such that $h_t(0) = 0$.

Since the system (18) is linear and the first equation is solvable then to prove the solvability of (18) there is no loss of generality to assume that $a_1 \equiv 0$. Then for any family of functions r_t the families $h_t = r_t A_{1,t}$, $q_{1,t} = -r_t U(A_t)$ give a solution of the first equation of the system such that $h_t(0) = 0$. The second equation reduces to the equation

$$r_t A_{1,t} U(A_{2,t}) + q_{2,t} A_{2,t} = -a_2 \tag{19}$$

with respect to the families r_t and $q_{1,t}$. The transversality of the curves γ_1 and γ_2 implies that there exists a coordinate system, depending on t, such that $A_{1,t} = u_1, A_{2,t} = u_2$. Let $U = (a_{11}u_1 + a_{12}u_2 + \cdots)\frac{\partial}{\partial u_1} + (a_{21}u_1 + a_{22}u_2 + \cdots)\frac{\partial}{\partial u_2}$, where the dots denote non-linear terms. The equation (19) takes the form

$$r_t u_1(a_{21}u_1 + a_{22}u_2 + \cdots) + q_{2,t}u_2 = f_t(u_1, u_2), \tag{20}$$

where f_t is a family of functions with zero 1-jet. Since any function with zero 1-jet belongs to the ideal generated by u_2 and u_1^2 then (20) is solvable provided that $a_{21} \neq 0$. The latter follows from the assumption that U has tangency of order 1 with the curve γ_2.

7. FURTHER CLASSIFICATION RESULTS

The developed techniques leads to many other classification results. Consider, for example, the case where a completely symmetric center is defined by a vector field X with the 3-jet (4) at the algebraically isolated singular point $0 \in \mathbf{R}^2$, such that $a_{12} \neq 0$, $a_{21} \neq 0$, and there is a resonant relation $\lambda_2 = p\lambda_1,\ p \geq 2$ between the eigenvalues of the matrix $A = (a_{ij})$. To obtain an orbital normal form for X one has to construct an orbital normal form for the class of vector fields $\dot{u} = Au + \cdots,\ u \in \mathbf{R}^2$ with respect to changes of coordinates with identity linear approximation. Using the orbital classification of vector fields on $\mathbf{R}^2$ with respect to complete group of changes of coordinates (see, for example, [1]), one can obtain a normal form $\dot{u} = Au + (cu_1^p,\ 0)^t,\ c \in \mathbf{R}$. Together with the results of sections 2-4 (Propositions 7, 10 and Theorem 17) it leads to the following orbital normal form for X:

$$\dot{x}_1 = (\lambda + \mu)x_1^2x_2 + x_2^3 + cx_1^{2p}x_2, \quad \dot{x}_2 = -x_1^3 + \lambda x_1 x_2^2, \quad \mu > -2.$$

Of course, normal forms for other types of resonant relations also can be constructed using Theorem 17 provided $a_{12} \neq 0, a_{21} \neq 0$. If one of these numbers is equal to zero then we need further classification results for the triples consisting of a vector field and two curves. One of results in this direction is as follows.

THEOREM 18. *Let U be a vector field on the plane $\mathbf{R}^2(u_1, u_2)$ such that $U(0) = 0$. Assume that the eigenvalues of the linear approximation of U are different and that there are no resonant relations. In the C^ω category we additionally assume that there are no small denominators. Let γ_1 and γ_2 be regular transversal curves. If U has tangency of order 1 with γ_1 and tangency of order $m \geq 2$ with γ_2 then in suitable coordinates*

$$U = Q\left(\lambda_1 u_1 \frac{\partial}{\partial u_1} + \lambda_2 u_2 \frac{\partial}{\partial u_2}\right), \quad \gamma_1 = \{u_1 + u_2 = 0\}, \ \ \gamma_2 = \{u_2 + u_1^m = 0\},$$

where Q is a non-vanishing function.

Proof. Since the order of tangency of U with γ_2 is bigger than 1 then the eigenvalues are real (see Example 13.) By the assumption of the theorem, U is linearizable, therefore we can assume that $U = U_0 = \lambda_1 u_1 \frac{\partial}{\partial u_1} + \lambda_2 u_2 \frac{\partial}{\partial u_2},\ \ \lambda_1 \neq \lambda_2$. Let $\gamma_1 = \{au_1 + bu_2 + f(u_1, u_2) = 0\},\ \ j^1 f = 0$. Since U has minimal order 1 of tangency with γ_1 then $ab \neq 0$. We can reduce a and b to 1 by scaling the coordinates (preserving U_0). Applying Theorem 17 with γ_2 replaced by any regular curve having tangency of order 1 with U we reduce f to 0. Now $U = QU_0$ and $\gamma_1 : \{u_1 + u_2 = 0\}$, Q is a non-vanishing function. In these coordinates γ_2 can be described by the equation $\gamma_2 = \{u_2 - g(u_1) = 0\}$. Since the order of tangency of U and γ_2 is equal to m and there are no resonant relation $\lambda_2 = p\lambda_1$ with any $p \geq 1$ then, by Example 14, $g(u_1) = cu_1^m + h(u_1), \quad c \neq 0, h(u_1) = o(u_1^m)$. The coefficient c can be reduced to 1. Using the homotopy method like in section 6 one can reduce $h(u_1)$ to 0. ▮

Using Theorem 18 and Propositions 7 and 10 it is easy to obtain the following corollary.

THEOREM 19. *Assume that a completely symmetric phase portrait is defined by a vector field with an algebraically isolated singular point at the origin and the* 3*-jet (4), where* $a_{2,1} = 0, a_{1,2} \neq 0$. *If the eigenvalues* $a_{1,1}, a_{2,2}$ *of the matrix* $A = (a_{ij})$ *are different, there are no resonant relations and in the* C^ω *category there are no small denominators then the phase portrait is diffeomorphic to the phase portrait of the vector field*

$$\dot{x}_1 = x_2(x_1^2 + \delta_1 x_2^2), \quad \dot{x}_2 = x_1(\lambda x_2^2 + \delta_2 x_1^{2m}),$$

where $\delta_1, \delta_2 \in \{1, -1\}, \ \lambda \in \mathbf{R}, \quad m \geq 2.$

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