QUALITATIVE THEORY OF DYNAMICAL SYSTEMS **4**, 439–454 (2004)
ARTICLE NO. 71

Convex Energy Levels of Hamiltonian Systems

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Submitted: July 27, 2003 *Accepted*: March 7, 2004
To Professor Jorge Sotomayor for his 60^{th} birthday

We give a simple necessary and sufficient condition for a non-regular energy level of a Hamiltonian system to be strictly convex. We suppose that the Hamiltonian function is given by kinetic plus potential energy. We also show that this condition holds for several Hamiltonian functions, including the Hénon-Heiles one.

Key Words: Hamiltonian systems, convexity, positive curvature, saddle-center, Hénon-Heiles Hamiltonian.

1. INTRODUCTION

An important theorem due to Hofer, Zehnder and Wysocki [11] for Hamiltonian systems with two degrees of freedom states that a strictly convex energy level diffeomorphic to S^3 always has a periodic orbit which is the boundary of a global surface of section of disk-type. Moreover, this implies the existence of 2 or infinitely many periodic orbits in that energy level. This geometric property turns out to be of great importance to understand the topological structure of its orbits.

When the Hamiltonian function is of the form $\frac{p_x^2+p_y^2}{2} + V(x,y)$ then the conditions of convexity can be expressed in terms of the potential function V. In this paper, we give these geometric conditions in the regular case

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and in the case where the energy level has a singularity. The main result of this paper is the following theorem

THEOREM 1. *Let $H : \mathbb{R}^4 \to \mathbb{R}$ be a Hamiltonian function given by $H(x, y, p_x, p_y) = \frac{p_x^2+p_y^2}{2} + V(x, y)$ where $V : \mathbb{R}^2 \to \mathbb{R}$ is a $C^{k\geq 2}$ function. Suppose that $S \subset H^{-1}(E)$ is homeomorphic to S^3, invariant by the flow and has at most one singularity p_c. Let $B \stackrel{def}{=} \pi(S)$ be the disk given by the canonical projection of the surface S on the plane (x, y). Then S is a strictly convex hypersurface in $\mathbb{R}^4$ if and only if*

$$2(E - V)(V_{xx}V_{yy} - V_{xy}^2) + V_{xx}V_y^2 + V_{yy}V_x^2 - 2V_xV_yV_{xy} > 0 \qquad (1)$$

for all points in $B\backslash\pi(p_c)$.

The inequality (1) represents the curvature of the hypersurface. Notice that an analogous result is false in the plane. There are closed curves with positive curvature except at one singular point such that the domain it bounds is non-convex (See Figure 1).

This paper is organized as follows. In section 2 we give some definitions and facts related to convex hypersurfaces in $\mathbb{R}^4$. In section 3 we prove the global convexity property for hypersurfaces homeomorphic to S^3 with positive curvature and one singularity. In section 4 we express the condition for an energy level to have positive curvature in terms of the potential function V and prove Theorem 1. Finally, in section 5 we give some examples where the convexity hypothesis is satisfied, including the Hénon-Heiles Hamiltonian.

2. REGULAR STRICTLY CONVEX HYPERSURFACES IN $\mathbb{R}^4$

DEFINITION 2. We say that a non-empty set $K \subset \mathbb{R}^4$ is convex if given $x, y \in K$, then $(1 - t)x + ty \in K$ for $t \in [0, 1]$.

DEFINITION 3. A hypersurface $S \subset \mathbb{R}^4$, $C^{k\geq 2}$, has contact of order 1 with the hyperplane F at $p \in S$ if given a C^2 curve $\alpha : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$ and $\alpha'(0) \neq 0$, then $\langle\alpha'(0), N_F\rangle = 0$ and $\langle\alpha''(0), N_F\rangle \neq 0$ where N_F is a normal vector to F.

DEFINITION 4. A hyperplane F is a non-singular support hyperplane of S if $F \cap S = \{p\}$, where $p \in S$, and S has contact of order 1 with F at p.

DEFINITION 5. A hypersurface $S \subset \mathbb{R}^4$, $C^{k\geq 2}$, is strictly convex if all the hyperplanes tangent to S are non-singular support hyperplanes of S.

Given a regular strictly convex hypersurface $S \subset \mathbb{R}^4$, a point $p \in S$ and a curve $\alpha : (-\varepsilon, \varepsilon) \to S$, with $\alpha(0) = p$ and $\alpha'(0) \neq 0$, we have

$\langle \alpha''(0), N(p) \rangle = -\langle dN(p)\alpha'(0), \alpha'(0) \rangle \neq 0$ where $dN(p)$ is the differential at p of the Gauss map $N : S \to S^3$. It follows that $dN(p)$ is definite (i.e., the quadratic form associated to $dN(p)$ is definite) and its sign depends on the orientation N.

If a hypersurface S is given by $S = H^{-1}(c)$ where $H : \mathbb{R}^4 \to \mathbb{R}$ is a $C^{k\geq 2}$ function and c a regular value, then $N(p) \stackrel{def}{=} \frac{H_x(p)}{\|H_x(p)\|}$ is a unitary normal vector to S at p. For $v \in T_pS$ ($v \perp H_x(p)$), we have

$$\langle dN(p)v, v \rangle = \frac{1}{\|H_x(p)\|} \langle H_{xx}(p)v, v \rangle$$

It follows that a necessary condition for a regular hypersurface $S = H^{-1}(c)$ to be strictly convex is that the Hessian of H for all $p \in S$ must be definite when restricted to T_pS. When the hypersurface S is diffeomorphic to S^3, then this necessary condition is also sufficient and S is called an ovaloid (see [15]). In the next section, we show an analogous result for non-regular hypersurfaces in $\mathbb{R}^4$.

More about convex sets and strictly convex hypersurfaces can be seen in [2], [6] and [7].

3. NON-REGULAR STRICTLY CONVEX HYPERSURFACES IN $\mathbb{R}^4$

Let $H : \mathbb{R}^4 \to \mathbb{R}$ be a $C^{k\geq 2}$ function. In this section we consider hypersurfaces $S \subset H^{-1}(a)$, $a \in Im(H)$, satisfying the following hypotheses:

(H1) S is homeomorphic to S^3, i.e, there exists a homeomorphism h of $\mathbb{R}^4$ such that $h(S) = S^3$;

(H2) there exists only one singular point $p_c \in S$ ($H_x(p_c) = 0$), i.e., $q \in S$ is a regular point of S if $q \neq p_c$;

(H3) for all regular points $q \in S$, we have $\langle H_{xx}(q)v, v \rangle > 0$ for all $v \in T_qS$;

By assumption, $S_0 \stackrel{def}{=} S \backslash \{p_c\}$ is a regular hypersurface in $\mathbb{R}^4$.

Definition 6. Let S be a hypersurface satisfying the hypotheses (H1) and (H2). We say that the non-regular hypersurface S is a strictly convex hypersurface in $\mathbb{R}^4$ if: (i) S is the boundary of a convex set in $\mathbb{R}^4$; (ii) All the hyperplanes tangent to S_0 are non-singular support hyperplanes of S_0.

We will show that the local hypothesis (H3) is necessary and sufficient for S to be a strictly convex hypersurface in $\mathbb{R}^4$. See Figure 1 for a counter-example in $\mathbb{R}^2$.

We know that a hypersurface S, homeomorphic to S^3, divides $\mathbb{R}^4$ into two disjoint subsets: (i) B_S, which is closed and bounded ($\partial B_S = S \subset B_S$); and (ii) C_S, which is open and unbounded ($\partial C_S = S$). See Figure 2.

We set an orientation $N : S_0 \to NS_0$ to S_0 given by $N(p) = \frac{H_x(p)}{\|H_x(p)\|}$ and define an orthonormal moving frame to S_0 (i.e, regular orthonormal vector fields $X_i : S_0 \to TS_0$, $i = 1, 2, 3$ which spam TS_0) by the following way (See [3]):

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Now define the 4×4 matrices given by:

$$A_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} A_1 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} A_2 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} A_3 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{2}$$

For each $x \in S$ let

$$X_0(p) \stackrel{def}{=} N(p)$$

and

$$X_i(p) \stackrel{def}{=} A_i X_0(p) \tag{3}$$

for $i \in \{1, 2, 3\}$. Then we have $\langle X_i(p), X_j(p) \rangle = \delta_{ij}$, $0 \le i, j \le 3$ and, therefore, the vectors $X_1(p)$, $X_2(p)$ and $X_3(p)$ define an orthonormal basis for T_pS.

LEMMA 7 (Local representation of S_0). *For all $p \in S_0$, there exists a neighborhood U_p of p in S_0 and a rectangular coordinate system (x_1, x_2, x_3, x_4) such that U_p is the graph of a function $f^p : W \subset \mathbb{R}^3 \to \mathbb{R}$, $x_4 = f^p(x_1, x_2, x_3)$ where $f^p(0) = 0$, $f^p_x(0) = 0$, $f^p(x) > 0$ if $x \neq 0$ and the Hessian $f^p_{xx}(x)$ of f^p is positive-definite for all $x \in W$. (See Figure 3)*

Proof. The vectors $X_1(p), X_2(p), X_3(p)$ and $N(p)$ given by (3) define an orthonormal basis to $\mathbb{R}^4$. For all $z \in \mathbb{R}^4$ we have $z = p + x_1X_1(p) + x_2X_2(p) + x_3X_3(p) - x_4N(p)$. If $z \in S_0$, then $H(p + x_1X_1 + x_2X_2 + x_3X_3 - x_4N) = a$. As $\langle H_x(p), N(p) \rangle > 0$, then by the Implicit Function Theorem, there exists a function $f^p : W_0 \subset \mathbb{R}^3 \to \mathbb{R}$ defined in a neighborhood W_0 of 0 such that $x_4 = f^p(x_1, x_2, x_3)$ and $H(p + x_1X_2 + x_2X_2 + x_3X_3 - f^p(x_1, x_2, x_3)N) = a$. So, the graph of f^p represents S in a neighborhood of p in S_0. For $i \in \{1, 2, 3\}$ we have

$$\left\langle H_x(p), X_i(p) - f^p_{x_i}(0)N(p) \right\rangle = 0 \Longrightarrow f^p_{x_i}(0) = 0$$

and it is also easy to see that

$$f^p_{x_ix_j}(0) = \frac{\langle H_{xx}(p)X_i, X_j \rangle}{\|H_x(p)\|} \tag{4}$$

for $1 \le i, j \le 3$ and, therefore, the Hessian of f^p is positive-definite in a neighborhood $W \subset W_0$ of 0. So $f^p > 0$ in $W \backslash \{0\}$. ∎

We will need the following global Lemma

LEMMA 8. *For $p \in S_0$, let $O_p : \mathbb{R}^4 \to \mathbb{R}^4$ be the affine map given by*

$$O_p(x_1, x_2, x_3, x_4) = p + x_1 X_2(p) + x_2 X_2(p) + x_3 X_3(p) - x_4 N(p)$$

Let f^p be the function given by Lemma 7. Then the following holds:

$$\begin{aligned} x_4 = f^p(x_1, x_2, x_3) &\Rightarrow O_p(x_1, x_2, x_3, x_4) \in S \\ x_4 > f^p(x_1, x_2, x_3) &\Rightarrow O_p(x_1, x_2, x_3, x_4) \in B_S \\ x_4 < f^p(x_1, x_2, x_3) &\Rightarrow O_p(x_1, x_2, x_3, x_4) \in C_S \end{aligned} \tag{5}$$

for all (x_1, x_2, x_3, x_4) in a neighborhood of 0.

Proof. Take a large sphere B which contains S in its interior. As S has only one singularity and has positive curvature in all of its regular points, we can translate B to touch S in only one regular point $q \in S$ such that B and S are tangent at q. Then O_q must satisfies (5) in a neighborhood of 0. Since S_0 is connected then, by continuous dependence of f^p with p, O_p satisfies (5) for all $p \in S_0$. ▮

See Figure 4 for a counter-example of this Lemma for curves in $\mathbb{R}^2$ with more than one singularity.

LEMMA 9. *Let s be a line segment through $p \in S_0$ such that p is not in the boundary of s. Then $s \cap C_S \neq \emptyset$.*

Proof. Consider the function f^p given by Lemma 7. By Lemma 8, the set C_S satisfies $x_4 < f^p(x_1, x_2, x_3)$. In these coordinates, if there exists a point in s such that $x_4 > 0$, then, as s contains the origin, there exists a point in s such that $x_4 < 0$ and it follows that $s \cap C_S \neq \emptyset$. If points of s satisfy $x_4 = 0$, then for $b = (x_1, x_2, x_3, 0) \in s$, where $(x_1, x_2, x_3) \neq (0, 0, 0)$, we have $b \in C_S$. ▮

LEMMA 10. *The set B_S is convex in $\mathbb{R}^4$ and if $p \in S_0$, then $R_p \cap S = \{p\}$ where R_p is the hyperplane tangent to S_0 at p.*

Proof. To show that B_S is a convex set, it is enough to prove that given $x, y \in S$, $x \neq y$, then the line segment xy joining the points x and y is contained in B_S.

Let x and y be two distinct points in S such that $x \neq p_c$ and $y \neq p_c$. We know that the Hessian of H in x is definite, and this implies that S has positive curvature in x. Using Lemmas 7 and 8, we can represent S in a neighborhood U_x of x as a graph $x_4 = f^x(x_1, x_2, x_3)$, where f^x is a positive strictly convex function. The set B_S satisfies, in these coordinates, $x_4 \geq f^x(x_1, x_2, x_3)$. Therefore, if $w \in C_x$, then the line segment $xw \subset B_S$.

Let $\gamma_{xy} : [0,1] \to S$ be a continuous curve in S such that $\gamma_{xy}(0) = x$, $\gamma_{xy}(1) = y$ and $\gamma_{xy}(t) \neq p_c$ for all $t \in [0,1]$. The curve γ_{xy} can be chosen such that for all $t \in [0,1]$ the segment $x\gamma_{xy}(t)$ satisfies $x\gamma_{xy}(t) \cap \{p_c\} = \emptyset$. This is because the line r_x through x and p_c intersects S_0 in at most a discrete set accumulating in p_c and it is enough to require that γ_{xy} avoids this set.

If t is sufficiently small, then $\gamma_{xy}(t) \in U_x$ and so the segment $x\gamma_{xy}(t) \subset B_S$ and is transversal to S in x and in $\gamma_{xy}(t)$. Let $F \stackrel{def}{=} \{t \in (0,1) | x\gamma_{xy}(t) \cap C_S \neq \emptyset\}$. Suppose that $F \neq \emptyset$ and let $\widetilde{t} \stackrel{def}{=} \inf F > 0$. Then we have:

(i) $x\gamma_{xy}(\widetilde{t}) \cap C_S = \emptyset$ because, otherwise, by continuity of $x\gamma_{xy}(t)$ and openness of C_S, $x\gamma_{xy}(t) \cap C_S \neq \emptyset$ for $t < \widetilde{t}$, a contradiction. So $x\gamma_{xy}(\widetilde{t}) \subset B_S$.

(ii) The segment $x\gamma_{xy}(\widetilde{t})$ is transversal to S in x and in $\gamma_{xy}(\widetilde{t})$, because if it was tangent to S in any of these two points, there would be points of the segment $x\gamma_{xy}(\widetilde{t})$ in C_S, a contradiction to (i).

(iii) Suppose that there exists $w \in x\gamma_{xy}(\widetilde{t})$, $w \notin \{x, \gamma_{xy}(\widetilde{t})\}$ such that $w \in S$. But, by Lemma 9, the segment $x\gamma_{xy}(\widetilde{t})$ contains points of C_S, a contradiction with (i).

(iv) It follows that $x\gamma_{xy}(\widetilde{t}) \cap S = \{x, \gamma_{xy}(\widetilde{t})\}$ and by (ii), we have that for all t near $\widetilde{t}$, $x\gamma_{xy}(t)$ is also transversal to S in x and in $\gamma_{xy}(t)$. So, by continuity of $\gamma_{xy}(t)$, we have $x\gamma_{xy}(t) \subset B_S$ for all t near $\widetilde{t}$, contradicting the assumption on $\widetilde{t}$.

We conclude that $F = \emptyset$ and, by continuity of $x\gamma_{xy}(t)$ and compacity of B_S the segment $xy \subset B_S$.

If $y = p_c$, then let $\gamma_{xy} : [0,1] \to S$ be a continuous curve in S such that $\gamma_{xy}(0) = x$, $\gamma_{xy}(1) = p_c$ and $\gamma_{xy}(t) \neq p_c$ for all $t \in [0,1)$. The segment $x\gamma_{xy}(t) \in B_S$ for all $t \in (0,1)$ and, again by continuity of $x\gamma_{xy}(t)$ and compacity of B_S we conclude that the segment $xp_c \subset B_S$. This finishes the proof that B_S is a convex subset of $\mathbb{R}^4$.

Suppose that there exists a regular point $p \in S_0$ and a hyperplane R_p tangent to S_0 in p such that $R_p \cap S \supset \{p, z\}$, $z \neq p$. As B_S is convex, the segment $pz \subset B_S$. But $pz \subset R_p$ and therefore pz must intersect C_S near p, a contradiction. ∎

Remark 11. This result is also valid in $\mathbb{R}^n$, $n > 2$, with S having finitely many singularities. See Figures 1 and 4 for counter-examples in $\mathbb{R}^2$.

COROLLARY 12. *The regular hypersurface S_0 is strictly convex.*

Proof. We know that S_0 has contact of order 1 with any hyperplane R_p tangent to S_0 in $p \in S_0$. By Lemma 10 we have $R_p \cap S_0 = \{p\}$ and, therefore, R_p is a non-singular support hyperplane of S, concluding that S_0 is strictly convex. ▮

THEOREM 13. *Let S be a hypersurface satisfying the hypotheses (H1), (H2) and (H3). Then the non-regular hypersurface S is strictly convex.*

Proof. By Lemma 10, S is the boundary of the convex set B_S in $\mathbb{R}^4$ and all tangent hyperplanes of $S_0 = S\backslash\{p_c\}$ are non-singular support hyperplanes of S_0. ▮

4. POSITIVE-DEFINITE QUADRATIC FORMS IN $\mathbb{R}^4$

When a hypersurface S is given as the pre-image of a regular value of a $C^{k\geq 2}$ function $H : \mathbb{R}^4 \to \mathbb{R}$, its convexity depends on the Hessian of H. Precisely, a necessary condition for S to be strictly convex is that the Hessian $H_{xx}(p)$ is definite in T_pS for all $p \in S$.

Without loss of generality, we can assume that $H_{xx}(p)$ is positive-definite in T_pS, i.e., for all $v \in T_pS$, $v \neq 0$, $\langle H_{xx}(p)v, v\rangle > 0$. It may happen that $H_{xx}(p)$ is not definite in $T_p\mathbb{R}^4 \simeq \mathbb{R}^4$.

Let $L : \mathbb{R}^4 \to \mathbb{R}^4$ be a self-adjoint linear operator and $Q : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ be its associated quadratic form defined by $Q(v) = \langle Lv, v\rangle$. Let E be a 3-dimensional subspace of $\mathbb{R}^4$ and $\{X_i, i = 1, 2, 3\}$ a basis of E. We define the 3×3 symmetric matrix W by

$$W = (\langle LX_i, X_j\rangle)_{1\leq i,j\leq 3} \tag{6}$$

Then we have the following easy properties

Property 1. The quadratic form Q is positive-definite in E if and only if all the eigenvalues of W are positive.

Proof. See [13] for a proof. ▮

Property 2. Let $W = (w_{ij})_{1\leq i,j\leq 3}$ be a 3×3 symmetric matrix. Then all of its eigenvalues are positive if and only if the following conditions are satisfied

$$\begin{aligned} \det W &> 0 \\ trW &> 0 \\ D_W &> 0 \end{aligned}$$

where $D_W = w_{11}w_{22} - w_{12}^2 + w_{11}w_{33} - w_{13}^2 + w_{22}w_{33} - w_{23}^2$.

Proof. Its is imediate from its characteristic polynomial. ▮

4.1. Hamiltonians of type $\frac{p_x^2+p_y^2}{2} + V(x,y)$

Let H be an Hamiltonian function $H(x,y,p_x,p_y) = \frac{p_x^2+p_y^2}{2} + V(x,y)$, where $V : \mathbb{R}^2 \to \mathbb{R}$ is a $C^{k\geq 2}$ function, known as the potential function. Let $z = (x,y,p_x,p_y)$ be the coordinates of $\mathbb{R}^4$ and $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ the canonical projection on the plane (x,y). Suppose that $S \subset H^{-1}(E)$ is a compact hypersurface, homeomorphic to S^3, invariant by the Hamiltonian flow, with at most one singularity p_c which corresponds to an equilibrium point of the Hamiltonian vector field X_H, i.e., $\pi(p_c)$ is a critical point of $V(x,y)$. The set $B \stackrel{def}{=} \pi(S)$ is, therefore, homeomorphic to D^2, and its boundary is given by $\partial B = \{\pi(z), z \in S, z = (x,y,0,0)\}$ with a singularity at $\pi(p_c)$. The interior of B is given by $\overset{\circ}{B} \stackrel{def}{=} \{\pi(z), z \in S, p_x^2 + p_y^2 \neq 0\}$ and if $(x,y) \in \overset{\circ}{B}$ then $V(x,y) < E$. Let $S_0 \stackrel{def}{=} S\backslash\{p_c\}$. We have

$$\begin{aligned} H_x(z) &= (V_x, V_y, p_x, p_y) \\ H_{xx}(z) &= \begin{pmatrix} V_{xx} & V_{xy} & 0 & 0 \\ V_{xy} & V_{yy} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Let $z \in S_0$ be a point satisfying $p_x^2 + p_y^2 \neq 0$, i.e., $\pi(z) \in \overset{\circ}{B}$. A basis for $T_z S_0 = [H_x(z)]^{\perp}$ is given by

$$\begin{aligned} X_1 &= (0, 0, -p_y, p_x) \\ X_2 &= (p_x, p_y, -V_x, -V_y) \\ X_3 &= (p_y, -p_x, V_y, -V_x) \end{aligned}$$

because $X_i, i = 1, 2$ and 3, are L.I. and orthogonal to H_x. We want to verify local convexity of S_0 at z (or positive curvature of S_0 at z). Observe that since $H_{xx}(z)$ has two eigenvalues equal to 1, then $H_{xx}(z)$ is definite in $T_z S_0$ if and only if it is positive-definite in $T_z S_0$. The matrix W defined in (6) is given by

$$W_z = \begin{pmatrix} p_1^2 + p_2^2 & p_2 V_x - p_1 V_y & -p_2 V_y - p_1 V_x \\ p_2 V_x - p_1 V_y & f_1 & f_2 \\ -p2 V_y - p_1 V_x & f_3 & f_4 \end{pmatrix} \tag{7}$$

where $L = H_{xx}$ and f_1, f_2, f_3 and f_4 are given by

$$\begin{aligned} f_1 &= V_{xx} p_1^2 + 2 p_1 p_2 V_{xy} + V_{yy} p_2^2 + V_x^2 + V_y^2 \\ f_2 &= p_1 p_2 (V_{xx} - V_{yy}) + (p_2^2 - p_1^2) V_{xy}^2 \end{aligned}$$

$$\begin{aligned} f_3 &= p_1 p_2 (V_{xx} - V_{yy}) + (p_2^2 - p_1^2) V_{xy}^2 \\ f_4 &= V_{xx} p_2^2 + 2 p_1 p_2 V_{xy} + V_{yy} p_1^2 + V_x^2 + V_y^2 \end{aligned}$$

Then we have

$$\begin{aligned} \det W_z &= 4(E-V)^2 (2(E-V)(V_{xx}V_{yy} - V_{xy}^2) + \\ & \quad V_{xx}V_y^2 + V_{yy}V_x^2 - 2V_xV_yV_{xy}) \\ trW_z &= 2(E-V)(1 + V_{xx} + V_{yy}) + 2(V_x^2 + V_y^2) \\ D_{W_z} &= 4(E-V)^2(V_{xx} + V_{yy} + V_{xx}V_{yy}) + \\ & \quad 2(E-V)(1 + V_{xx} + V_{yy})(V_x^2 + V_y^2) + (V_x^2 + V_y^2)^2 \end{aligned} \tag{8}$$

Now we consider points $z \in S_0$ such that $p_x = p_y = 0$, i.e., $\pi(z) \in \partial B \backslash \pi(p_c)$. We can use the following vector as a basis for $T_z S_0$

$$\begin{aligned} X_1 &= (0, 0, 1, 0) \\ X_2 &= (0, 0, 0, 1) \\ X_3 &= (V_y, -V_x, 0, 0) \end{aligned}$$

and the matrix W_z defined in (6) is given by

$$W_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & V_{xx}V_y^2 + V_{yy}V_x^2 - 2V_xV_yV_{xy} \end{pmatrix}$$

So we have

$$\det W_z = V_{xx}V_y^2 + V_{yy}V_x^2 - 2V_xV_yV_{xy}$$

$$trW_z = 2 + V_{xx}V_y^2 + V_{yy}V_x^2 - 2V_xV_yV_{xy} = 2 + \det W \tag{9}$$

$$D_{W_z} = 1 + 2(V_{xx}V_y^2 + V_{yy}V_x^2 - 2V_xV_yV_{xy}) = 1 + 2\det W$$

PROPOSITION 14. *W_z is positive-definite for all $z \in S_0$ if and only if*

$$T_{W_z} \stackrel{def}{=} 2(E-V)(V_{xx}V_{yy} - V_{xy}^2) + V_{xx}V_y^2 + V_{yy}V_x^2 - 2V_xV_yV_{xy} > 0$$

for all $(x, y) \in B \backslash \pi(p_c)$.

Proof. Suppose that $T_{W_z} > 0$ for points in $B \backslash \pi(p_c)$. Then, by (8) and (9), $\det W_z > 0$ if $\pi(z) \in B \backslash \pi(p_c)$. This is a sufficient condition for W_z to be positive-definite for all $z \in S_0$ such that $\pi(z) \in \partial B \backslash \pi(p_c)$ because $V = E$ in ∂B.

If $z \in S_0$ and $\pi(z) \in \overset{\circ}{B}$ is sufficiently close to $\partial B \backslash \pi(p_c)$, then, by (8), we have $trW > 0$ and $D_W > 0$ because $V \overset{(x,y)\to\partial B}{\longrightarrow} E$ and $V_x^2 + V_y^2 > 0$ in $\partial B \backslash \pi(p_c)$. Then W is positive-definite in z. As $\overset{\circ}{B}$ is connected and $\det W_z > 0$ in $\overset{\circ}{B}$, then no eigenvalue of W_z can change its signal in $\overset{\circ}{B}$ and, therefore, they must be positive for all points in $\overset{\circ}{B}$. Then W_z is positive-definite for all $z \in S_0$ such that $\pi(z) \in \overset{\circ}{B}$. We conclude that W_z is positive-definite for all $z \in S_0$. The "only if" part is imediate. ▌

Proof. [Theorem 1] By Proposition 14, the inequality (1) is equivalent to positive curvature of regular points of S. If S is regular then a well-know result states that hypersurfaces diffeomorphic to S^3 with positive curvature are strictly convex (see [15]). If S has one singularity, then it satisfies the hypotheses (H1), (H2) and (H3). The Theorem 13 finishes the proof in this case. ▌

We now give a geometric interpretation for the condition (1). Let $R : B \to \mathbb{R}$ be a function given by $R(x,y) = \sqrt{E - V(x,y)}$. Then $R|_{\partial B} = 0$ and $R|_{\overset{\circ}{B}} > 0$. The curve ∂B is a Jordan curve which is $C^{k\geq 2}$ at all points but $\pi(p_c)$.

Proposition 15. *$T_{W_z} > 0$ for all $z \in S_0$ if and only if the curve $\partial B \backslash \pi(p_c)$ has non-zero curvature and the function R is C^2 strictly concave in $\overset{\circ}{B}$.*

Proof. The function R is C^2 strictly concave in $\overset{\circ}{B}$ if and only if the matrix

$$\tilde{W} \overset{def}{=} \begin{pmatrix} R_{xx} & R_{xy} \\ R_{xy} & R_{yy} \end{pmatrix}$$

is negative-definite. Then we have

$$\tilde{W} = -\frac{1}{4(E-V)^{\frac{3}{2}}} \begin{pmatrix} 2V_{xx}(E-V) + V_x^2 & 2V_{xy}(E-V) + V_xV_y \\ 2V_{xy}(E-V) + V_xV_y & 2V_{yy}(E-V) + V_y^2 \end{pmatrix}$$

and, therefore, $\tilde{W}$ is negative-definite if and only if $\det \tilde{W} > 0$ and $tr\ \tilde{W} < 0$, i.e.,

$$\begin{aligned} \det \tilde{W} &= \frac{2(E-V)(V_{xx}V_{yy} - V_{xy}^2) + V_{xx}V_y^2 + V_{yy}V_x^2 - 2V_xV_yV_{xy}}{16(E-V)^3} > 0 \\ -tr\ \tilde{W} &= \frac{2(E-V)(V_{xx} + V_{yy}) + V_x^2 + V_y^2}{4(E-V)^{\frac{3}{2}}} > 0 \end{aligned}$$

It follows that $\widetilde{W}$ is negative-definite if and only if det $\widetilde{W} > 0$ in $\overset{\circ}{B}$ because $-tr\ \widetilde{W} > 0$ for all points sufficiently close to $\partial B \backslash \pi(p_c)$. We conclude that $T_{W_z} > 0$ for all $z \in S_0$ such that $\pi(z) \in \overset{\circ}{B}$ if and only if R is C^2 strictly concave in $\overset{\circ}{B}$.

The points of the regular curve $\partial B \backslash \pi(p_c)$ satisfy the equation $V(x, y) = E$ and its curvature is given by

$$k(x, y) = \frac{\left|V_{xx}V_y^2 - 2V_{xy}V_xV_y + V_{yy}V_x^2\right|}{(V_x^2 + V_y^2)^{\frac{3}{2}}}$$

Therefore, $k(x, y)$ is positive if and only if $T_{W_z} > 0$ for all $z \in S_0$ such that $\pi(z) \in \partial B \backslash \pi(p_c)$. ▮

5. EXAMPLES

5.1. The Hénon-Heiles Hamiltonian

We now consider the Hamiltonian $H : \mathbb{R}^4 \to \mathbb{R}$ given by

$$H(x, y, p_x, p_y) = \frac{p_x^2 + p_y^2}{2} + V(x, y)$$

where $V(x, y) = \frac{x^2+y^2}{2} + bx^2y - \frac{y^3}{3}$ and b is a positive parameter. When $b = 1$, this is the Hénon-Heiles potential. It was first presented in [10] as a model for chaotic star motions in a galaxy with an axes of symmetry.

The point $(0, 1)$ is always a saddle of the potential function V and, for $p_x = p_y = 0$, it corresponds to an equilibrium of saddle-center type for the Hamiltonian flow associated to the function H. For a more detailed description of the Hamiltonian flow near a saddle-center, see [1], [12], [14], [8] and [9]. The saddle-center $p_c = (0, 1, 0, 0)$ is in the energy level $M_b \overset{def}{=} \{H = \frac{1}{6}\}$. The energy level M_b depends on the parameter b. Let $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ be the canonical projection given by $\pi(x, y, p_x, p_y) = (x, y)$. If $0 < b < 1$, then $\pi(M_b)$ has a component C_b which is homeomorphic to D^2 and contains $\pi(p_c)$. The boundary of C_b is a closed curve, which is regular except at $\pi(p_c)$. For all points $(x, y) \in C_b$, we have $-\frac{1}{2} \le y \le 1$ and $x^2 \le \frac{1-3y^2+2y^3}{3+6by}$. The points $(x, y) \in \partial C_b$ satisfy $V(x, y) = \frac{1}{6}$ (See Figure 5).

Let $S_b \overset{def}{=} \pi^{-1}(C) \cap M_b$. The hypersurface S_b is homeomorphic to S^3 and is regular except at p_c.

Lemma 16. *If* $(y, b) \in D \overset{def}{=} (-\frac{1}{2}, 0) \times (0, 1)$ *and* $8b^2y^3 + 12by^2 + 6y + 4b^2 - 3 > 0$ *then* $4by^2 + 3y + by + b > 0$.

Proof. We will show that $8by^2 + 6y + 2by + 2b > 0$ in D. We have

$$8by^2 + 6y > 3 - 4b^2 - 4by^2 - 8b^2y^3$$

Then

$$8by^2 + 6y + 2by + 2b > -4by^2 - 8b^2y^3 + 3 - 4b^2 + 2by + 2b$$

Now we have to show that

$$R(y,b) \stackrel{def}{=} -4by^2 - 8b^2y^3 + 3 - 4b^2 + 2by + 2b > 0$$

in D. In ∂D, we have $R(-\frac{1}{2}, b) = -3b^2 + 3 > 0$ (except if $b = 1$), $R(0,b) = 3 - 4b^2 + 2b = (1-b)(4b+3) + b > 0$, $R(y,0) = 3 > 0$ and $R(y,1) = -8y^3 - 4y^2 + 2y + 1$.

Moreover, $\frac{\partial R(y,1)}{\partial y} = -24y^2 - 8y + 2$ and it is easy to see that $\frac{\partial R(y,1)}{\partial y} > 0$ for $-\frac{1}{2} < y \le 0$. Then $R(y,1) > 0$ for all $-\frac{1}{2} < y \le 0$. We conclude that $R(y,b)$ is a positive function in ∂D except at the point $(-\frac{1}{2}, 1)$ where it is equal to zero. To show that $R > 0$ in D it is enough to prove that $R(y,b)$ has no critical points in D, i.e, the solutions of

$$(-8by - 24b^2y^2 + 2b, -4y^2 - 8b - 18by^3 + 2y + 2) = (0,0)$$

are not in D. But if $-8by - 24b^2y^2 + 2b = 0$, $b \neq 0$, then

$$y_{\pm} = \frac{-1 \pm \sqrt{1+3b}}{6b}$$

If $0 < b < 1$, then $y_+ > 0$ and $y_- < -\frac{1}{2}$ because

$$\begin{aligned} 9b(b-1) &< 0 \Leftrightarrow 9b^2 - 6b + 1 < 3b + 1 \Leftrightarrow (3b-1)^2 < 3b+1 \\ &\Rightarrow 3b - 1 < \sqrt{1+3b} \Rightarrow \frac{-1-\sqrt{1+3b}}{6b} < -\frac{1}{2} \end{aligned}$$

Therefore, there are no critical points of $R(y,b)$ in D concluding that $R(y,b) > 0$ in D. ∎

Proposition 17. *For all* $0 < b < 1$, S_b *is a strictly convex hypersurface in* $\mathbb{R}^4$.

Remark 18. This proposition is also valid for the regular energy levels inside S_b.

Proof. By Theorem 1, it is now sufficient to show that

$$2(V_{xx}V_{yy} - V_{xy}^2)(\frac{1}{6} - V) + V_x^2 V_{yy} + V_y^2 V_{xx} - 2V_x V_y V_{xy} > 0$$

for all $(x,y) \in C_b \backslash \pi(p_c)$. It is equivalent to show that

$$N(y,b) + F(y,b)x^2 + G(y,b)x^4 > 0 \qquad (10)$$

where $N(y,b) \overset{def}{=} (1+y)(1-y)^3(1+2by)$, $F(y,b) \overset{def}{=} 2b(y-1)(2by^2+3y+2by+2b)$ and $G(y,b) \overset{def}{=} 3b^2(1+2by)$. Let $B = (-\frac{1}{2},1) \times (0,1)$.

Solving the following equation for x^2

$$N(y,b) + F(y,b)x^2 + G(y,b)x^4 = 0$$

we have

$$x_\pm^2 = \frac{-K_1(y,b) \pm \sqrt{K_2(y,b)}}{b(3+6by)}$$

where

$$\begin{aligned} K_1(y,b) &= 3y^2 + 2by^3 - 3y - 2b \\ K_2(y,b) &= (1+2y)(1-y)^2(8b^2y^3 + 12by^2 + 6y + 4b^2 - 3) \end{aligned}$$

As $N(y,b) > 0$ and $G(y,b) > 0$ in B, we consider only the case $K_2(y,b) > 0$ ($x_\pm^2$ real) because otherwise the equation (10) is trivially true. In B, $K_2(y,b) > 0$ is equivalent to $8b^2y^3 + 12by^2 + 6y + 4b^2 - 3 > 0$.

Supposing that $K_2(y,b) > 0$, we will show now that $x^2 < x_-^2$ for all $(x,y) \in C_b \backslash \pi(p_c)$. To see this, it is enough to prove that $\frac{1-3y^2+2y^3}{3+6by} < x_-^2$ in B. And in B we have

$$\frac{1-3y^2+2y^3}{3+6by} < x_-^2 \Leftrightarrow \sqrt{(1+2y)(8b^2y^3+12by^2+6y+4b^2-3)} < 4by^2+3y+by+b \qquad (11)$$

and therefore by Lemma 16 we can square both sides of the last inequality of (11) to get

$$\frac{1-3y^2+2y^3}{3+6by} < x_-^2 \Leftrightarrow 3by + y + b + 1 > 0$$

and it is imediate to verify that $3by + y + b + 1 > 0$ in B. ▮

Remark 19. Considering the Hénon-Heiles case ($b = 1$) and using the same steps of the proof of Proposition 17, it is possible to prove that the component S_c diffeomorphic to S^3, which is in the energy level $\{H = c\}$

for $0 < c < 1/6$, is also strictly convex. It follows that in S_c there exists a periodic orbit P_c for the Hamiltonian flow which bounds a global surface of section of disk-type. It also implies the existence of infinitely many periodic orbits in S_c as there are more than 2 periodic orbits in S_c. See [11] and [4].

5.2. An integrable example

The Hamiltonian $H : \mathbb{R}^4 \to \mathbb{R}$ given by

$$H(x, y, p_x, p_y) = \frac{p_x^2 + p_y^2}{2} + V(x, y)$$

where $V(x, y) = \frac{x^2+ky^2}{2} + \frac{1}{2}(x^2 + y^2)^2$ is integrable where the the second integral F is given by the following function

$$F(x, y, p_x, p_y) = -\frac{(xp_y - p_x y)^2}{2(k-1)} + p_y^2 + ky^2 + \frac{1}{2}y^2(x^2 + y^2)$$

For $k > 0$, the Hamiltonian function is convex and provides an example of a system with two degrees of freedom with non-recurrent solutions. See [17] for more details.

We will consider only the case when the parameter k is negative. It follows that the origin is a saddle of the potential function V. The point $p_c \stackrel{def}{=} (0, 0, 0, 0)$ corresponds to a saddle-center equilibrium for the Hamiltonian flow associated to H and is in the energy level $M_k \stackrel{def}{=} \{H = 0\}$.

Let $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ be the canonical projection given by $\pi(x, y, p_x, p_y) = (x, y)$. Then $\pi(M_k)$ is symmetric related to the axes x and y. For $y \geq 0$, $\pi(M_k)$ has a component C_k which is homeomorphic to D^2 with a boundary ∂C_k which is regular except at the origin. The points of the boundary are given by $\partial C_k = \{(x, y)|y \geq 0, V(x, y) = 0\}$. Let $S_k \stackrel{def}{=} \pi^{-1}(C_k) \cap M_k$. The hypersurface S_k is homeomorphic to S^3 and regular except at p_c.

PROPOSITION 20. *For all $k < 0$, S_k is a strictly convex hypersurface in $\mathbb{R}^4$, homeomorphic to S^3, regular at all points except at p_c.*

Proof. Using Theorem 1, it is enough to show that

$$2(V_{xx}V_{yy} - V_{xy}^2)(E - V) + V_x^2 V_{yy} + V_y^2 V_{xx} - 2V_x V_y V_{xy} > 0$$

for all $(x, y) \in C_k \backslash \{(0, 0)\}$. It is equivalent to show that

$$K \stackrel{def}{=} 4(x^2 + y^2)^2 + 6(x^2 + ky^2) + 2(x^2 k + y^2) + 3k < 0$$

in $C_k \backslash \{(0, 0)\}$. We know that in $C_k \backslash \{(0, 0\}$, $V(x, y) \leq 0$, i.e.,

$$(x^2 + y^2)^2 + x^2 + ky^2 \leq 0$$

or

$$x^2 + y^2 + k \le -\frac{x^2}{y^2}(x^2 + y^2 + 1)$$

Then we have

$$\begin{aligned} K &= 4(x^2+y^2)^2 + 4(x^2+ky^2) + 2(x^2+ky^2) + 2(x^2k+y^2) + 3k \\ &\le 2k(x^2+y^2) + 2(x^2+y^2+k) + k \\ &< -\frac{2x^2}{y^2}(x^2+y^2+1) \le 0 \end{aligned}$$

We conclude that $K < 0$ in $C_k \backslash \{(0,0)\}$. ∎

Remark 21. Each connected component S_c of the energy level $\{H = c\}$, $c < 0$, diffeomorphic to S^3, is also strictly convex implying the existence of a periodic orbit P_c which bounds a global surface of section of disk-type. See [11].

ACKNOWLEDGMENT

The author gratefully thanks Clodoaldo Grotta Ragazzo for his support, suggestions and encouragement.

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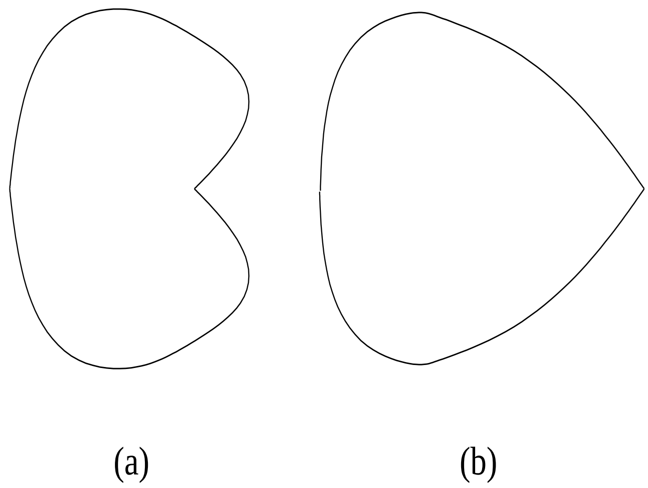

FIG. 1. A planar closed curve in $\mathbb{R}^2$ with positive curvature and one singularity may be the boundary of: (a) a non-convex domain and (b) a convex domain.

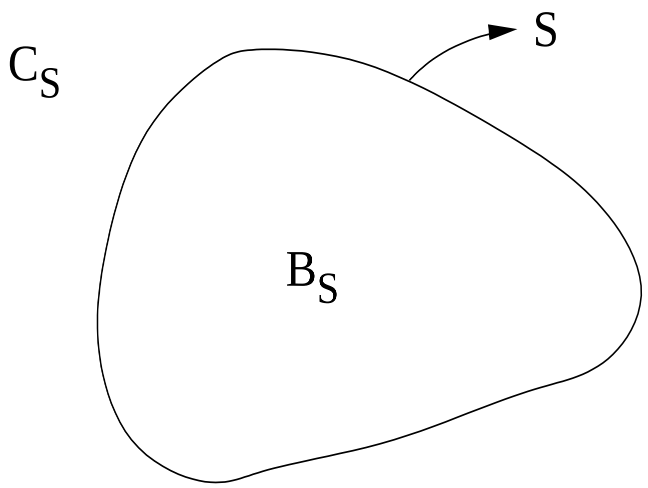

FIG. 2. Hypersurfaces in $\mathbb{R}^4$ homeomorphic to S^3 separate $\mathbb{R}^4$ in 2 sets: one bounded set B_S and one unbounded set C_S.

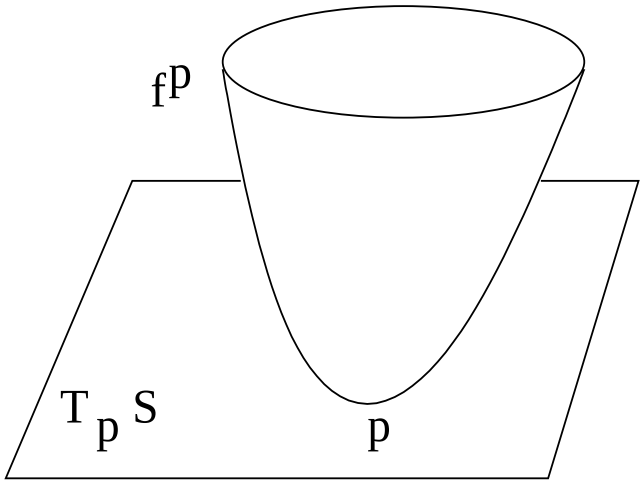

FIG. 3. The local representation f^p of S in a neighborhood of a point p with positive curvature is a strictly convex function.

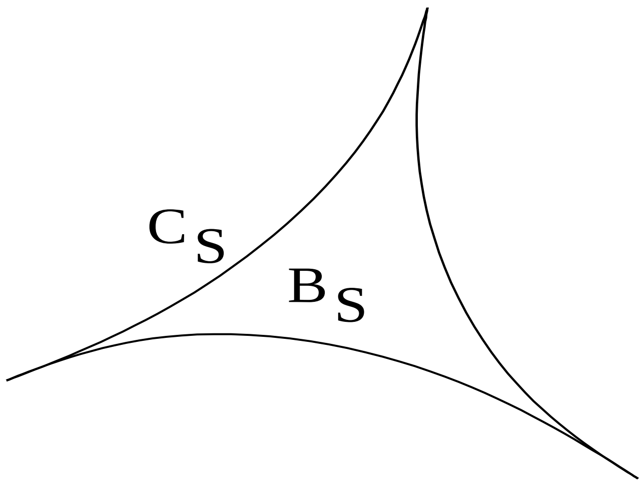

FIG. 4. The local representation f^p of a regular point p of a curve in $\mathbb{R}^2$ with three singularities may not satisfies the (analogous to $\mathbb{R}^2$) properties given by Lemma 8.

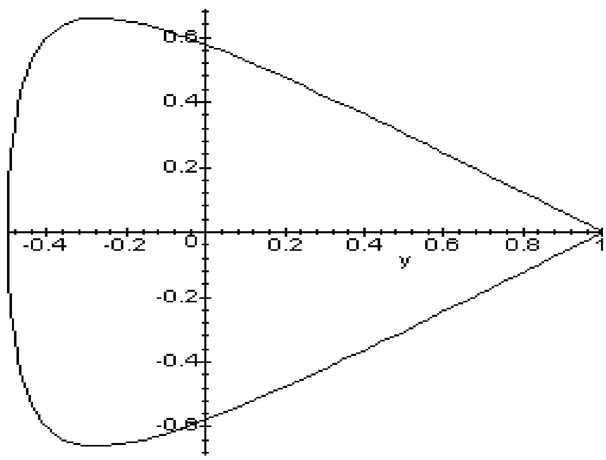

FIG. 5. The boundary of C_b is a simple closed curve, regular and with positive curvature in all points but $\pi(p_c)$.