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Plane Fields Related to Vector Fields on 3-Manifolds

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This paper is dedicated to Prof. Jorge Sotomayor.

This paper is a small collection of results about the topology of non-singular plane fields which are either transverse or tangent to nonsingular volume preserving vector fields on 3-manifolds. Emphasis is given to contact plane distributions and to restrictions of Hamiltonian vector fields to hypersurfaces in symplectic 4-manifolds.

Key Words: contact manifolds, Hamiltonian systems, trivializations, rotation numbers

1. INTRODUCTION

This paper is a collection of results about the geometry of certain plane distributions related to the dynamics of two degree of freedom Hamiltonian systems. The following example illustrates and motivates the problem. Let $(q_1, q_2, p_1, p_2) \stackrel{\text{def}}{=} x$ be Cartesian coordinates in $\mathbf{R}^4$ and $\omega = dq_i \wedge dp_i = d\lambda$ (the sum convention over repeated indices will be used throughout the paper) be the canonical symplectic form where $\lambda = (q_i dp_i - p_i dq_i)/2$. Let H be a real valued function in $\mathbf{R}^4$ and S be its level set $H = 1$ supposed regular. The Hamiltonian vector-field X_H associated to H is defined by the equation $dH = \omega(X_H, \cdot)$. Notice that if Y is any vector tangent to S then $dH(Y) = 0 = \omega(X_H, Y)$, namely X_H is contained in the kernel of $\omega|_S$ (the restriction of ω to S). The non-degeneracy of ω implies that $\ker \omega|_S$ is one dimensional. Therefore the direction of X_H is uniquely determined by the condition $X_H \in \ker \omega|_S = \ker d\lambda|_S$. The restriction of λ to S

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also defines a plane distribution on S, given by $\xi = \ker \lambda|_S$, with possible singularities at the origin of $\mathbf{R}^4$ or at points where S is tangent to $\ker \lambda$. So, the one form λ associates to each regular hypersurface S in $\mathbf{R}^4$ two distinct distributions: a direction field $\ker d\lambda|_S$ (which contains the original Hamiltonian vector filed) and a plane field $\ker \lambda$. If for all points of S this direction field is transversal to this plane distribution, then the plane distribution is necessarily regular and it is named "contact distribution". In this case $\lambda|_S$ is called contact form and the vector field X, uniquely determined by $d\lambda(X, \cdot) = 0$, $\lambda(X) = 1$, is called characteristic vector field or Reeb vector field. Notice that X_H and X possibly differ only by the time parameterization. This paper is about topological properties of this and related plane fields.

Contact distributions appear not only in the study of Hamiltonian systems. They are central objects in many different subjects in mathematics like symplectic topology, sub-Riemannian geometry, Cauchy-Riemann (CR) structures in complex geometry, non-holonomic mechanics, control theory, etc. In different areas distinct aspects of contact distributions, or even the same aspects, are studied using different methods. During some time I superficially explored many of these areas having as a main motivation future applications in the study of two degrees of freedom Hamiltonian systems. This paper is an overview of part of this investigation. Most of the material here presented is not new, except for some examples, counter-examples, and a result on canonical trivializations of certain plane fields presented in section 4.

This paper is subdivided as follow. Section 2 contains some important and well-known results on the topology of contact distributions on 3-manifolds. Section 3 is a study of contact distributions and related plane fields which appear on hypersurfaces in symplectic 4-manifolds (as in the example above). The plane fields in section 3 are transverse to their associated Hamiltonian vector fields. Section 4 is dedicated to the study of plane fields on hypersurfaces which are tangent to such Hamiltonian vector fields.

In this paper unless explicitly mentioned all functions, manifolds, distributions, etc, are supposed to be C^∞. The manifolds are supposed to be compact and boundary-less, that is closed, and three-dimensional. Of course many of the results presented below hold under more general hypotheses.

2. CONTACT STRUCTURES

As general references to this section see: [19] section 4, [11], [2] appendix 4, [6], and, specially, [30].

Definition 1. [Contact Distribution] Let ξ be a plane distribution on a three-dimensional manifold M. This distribution is called non-singular if each point $p \in M$ has a neighborhood U in which ξ is the kernel of a non vanishing one-form θ. It is called a contact distribution or contact structure if any local defining one-form θ satisfies $\theta \wedge d\theta \neq 0$ at every point in U. The pair (M, ξ) is called contact manifold.

Notice that a contact distribution satisfies a condition of maximal non-integrability $\theta \wedge d\theta \neq 0$ (Frobenius theorem) and it does not admit any local integral surface (see [2] appendix 4). The condition $\theta \wedge d\theta \neq 0$ does not depend on the local form used to define ξ. If θ and σ are two local non-vanishing forms such that $\xi = \ker \theta = \ker \sigma$ then there exists a non-vanishing real function f such that $\sigma = f\theta$, which implies that $\sigma \wedge d\sigma = f\theta \wedge d(f\theta) = f^2\theta \wedge d\theta$. This fact and the following argument imply that a contact structure determines an orientation on M. Let $U, V, \ldots$ be a finite open cover of M such that each of the sets $U, V, \ldots$ admits a one-form $\theta_U, \theta_V, \ldots$, respectively, which defines locally the contact structure. Then in the intersection of U and V these forms satisfy $\theta_U = f\theta_V$ for some non-vanishing function f. So, $\theta_U \wedge d\theta_U = f^2\theta_V \wedge d\theta_V$ and the local signed volume forms $\theta_U \wedge d\theta_U$ and $\theta_V \wedge d\theta_V$ have the same sign in $U \cap V$. Therefore, using a partition of unity these local volume forms can be glued to define a global signed volume form.

Definition 2. [Orientation] If M is a priori oriented and the sign of the above signed volume form associated to ξ is positive then the contact structure ξ is called positive, otherwise it is called negative.

Definition 3. [Co-orientation] A contact structure ξ is called co-orientable if there exists a global one-form θ, called contact form, such that $\xi = \ker \theta$. In this case ξ is also said transversally orientable, in the sense that there exists a globally defined nonsingular vector field transverse to ξ. Let the orientation of M be fixed. Then ξ is positive if $\theta \wedge d\theta > 0$ and negative if $\theta \wedge d\theta < 0$.

Let us see some examples of contact structures (see [2] appendix 4).

Example 4. [The one-jet space or the standard contact structure of $\mathbf{R}^3$.] Let $z = f(x)$ be a real-valued function on $\mathbf{R}$ and let $x \to (f'(x), z) \in \mathbf{R}^2$ be its one-jet extension. Let (x, y, z) be Cartesian coordinates on $\mathbf{R}^3$ and consider the one-form $\theta = dz - ydx$. Notice that $\theta \wedge d\theta = dx \wedge dy \wedge dz$, namely θ is a contact form and $\xi = \ker \theta$ is a co-oriented contact structure on $\mathbf{R}^3$ (with its usual orientation). A curve $\gamma : x \to (y, z)$ is the one-jet extension of some function $z = f(x)$ if, and only if, γ is an integral curve of ξ, that means γ is tangent to ξ at every point.

Example 5. [The space of oriented contact elements or oriented tangent lines (co-orientable).] Let Q be a two dimensional surface and TQ^* be its cotangent bundle. Each non-null covector p of T_qQ^* defines a line in T_qQ given by $\ker p$. Any other covector αp, with $\alpha \neq 0$, defines the same line. For each $q \in Q$ let S_qQ be the circle obtained from $T_qQ^* - \{0\}$ by its quotient under the equivalence relation $p \sim \alpha p$, $\alpha > 0$. The union of S_q for all $q \in Q$ defines SQ^*, a circle bundle over Q called the bundle of oriented lines of Q (or the oriented projectivization of TQ^* or the bundle of oriented contact elements). This bundle has a natural contact structure. Let $\pi : SQ^* \to Q$ be the bundle projection. Each point $z \in SQ^*$ is associated to an oriented line l_z in $T_{\pi z}Q$. Let $\xi_z \in T_zSQ^*$ be the pull-back of l_z under π, namely $Z \in \xi_z$ if $\pi_*Z \in l_z$, where π_* is the tangent map of π. The plane field ξ is a negative co-oriented contact distribution with respect to a natural orientation of SQ^* and with an associated contact form θ constructed in the following way. Let us set a Riemannian metric on Q. Then the bundle SQ^* can be identified with the unit circle bundle of TQ^*, that is SQ^* can be considered as a submanifold of TQ^* given by the set of covectors z of Q with $||z|| = 1$. Let θ be the canonical one-form (or Liouville form) on SQ^* which for $z \in SQ^*$ and $Z \in T_zSQ^*$ has value $\theta_z(Z) = z(\pi_*Z)$, namely $\theta = \pi^*z$. Clearly $\ker\theta = \xi$. In order to show that $\theta \wedge d\theta < 0$ let us choose a coordinate system (x^1, x^2) in a neighborhood U of $q \in Q$ such that: $q \to (0,0)$, $\{dx^1, dx^2\}$ is positively oriented and orthonormal at q. Let $\{\sigma^1, \sigma^2\}$ be an orthonormal reference coframe in U such that

$$\sigma^1 = dx^1 + a_1^1(x)dx^1 + a_2^1(x)dx^2, \qquad \sigma^2 = dx^2 + a_1^2(x)dx^1 + a_2^2(x)dx^2$$

with $a_j^i = \mathcal{O}(x)$. Then the function $(x,\phi) \to z = \cos(\phi)\sigma^1 + \sin(\phi)\sigma^2$, where $(x,\phi) \in V \subset \mathbf{R}^3$ and $z \in \pi^{-1}U$, defines a local parameterization of $\pi^{-1}U$. In these coordinates θ is also given by $\cos(\phi)\sigma^1 + \sin(\phi)\sigma^2$ (with the usual ambiguity of the notation), where σ^1, σ^2 are given above. Then

$$d\theta = -\sin(\phi)d\phi \wedge dx^1 + \cos(\phi)d\phi \wedge dx^2 + c\,dx^1 \wedge dx^2 + \mathcal{O}(x)$$

where c is some constant. Therefore at $x = (0,0)$ we obtain $\theta \wedge d\theta = -dx^1 \wedge dx^2 \wedge d\phi$.

Example 6. [The space of contact elements or tangent lines (non co-orientable).] Let SQ^* be the bundle of oriented lines defined above, identified with the unit circle bundle in TQ^*. Let $A : SQ^* \to SQ^*$ be the fiber preserving antipodal map $A(z) = -z$ and PQ^* be the manifold obtained from SQ^* under the identification of z with $A(z) = -z$. Notice that z and $-z$ have the same kernel, so they are associated to the same non-oriented line element in $T_{\pi(z)}Q$. The manifold PQ^* is called the bundle of tangent lines, or the bundle of contact elements, or the projectivized cotangent bundle. If θ is the contact one-form in SQ^* defined in example 5 then $A^*\theta_z(Z) = \theta_{A(z)}(A_*Z) = -z(\pi_*A_*Z) = -\theta_z(Z)$ where we used $\pi \circ A = \pi$. Therefore $A^*\theta = -\theta$ which implies that the contact structure $\xi = \ker\theta$ of SQ^* remains invariant under the action of A but not the contact form. So PQ^* enherits the contact structure of SQ^* but this contact structure is not co-orientable. More precisely, let $\rho : SQ^* \to PQ^*$ be the double cover projection associated to the quotient SQ^*/A. Let $z \in SQ^*$ be one of the preimages of $y \in PQ^*$ under ρ. Then in a neighborhood of z the map ρ is invertible and $\rho^{-1*}\theta = \mu$ defines a local contact form in a neighborhhod of y. If instead of z we have used $A(z)$ in this construction then we would obtain the local contact form $\rho^{-1*}\theta = -\mu$. The kernel of both forms generate the same contact distribution in PQ^* which we denote by η. Now suppose that there is a global contact form μ on PQ^* that generates η. Then $\rho^*\mu$ is a global contact form on SQ^* defining ξ and $\rho^*\mu$ has to be invariant under A because $A^*\rho^*\mu = (\rho \circ A)^*\mu = \rho^*\mu$. Moreover, $\rho^*\mu = f\theta$ with f everywhere different from zero, because all contact forms associated to ξ are of this form. Then $A^*\rho^*\mu = \rho^*\mu$ and $A^*\theta = -\theta$ imply that $f(z) = -f(-z)$ for all $z \in SQ^*$, which is impossible because f cannot be zero anywhere. Therefore we conclude that η is a non co-orientable contact structure on PQ^*.

After the definition of contact structure two natural questions are about the existence and classification of such structures. The following existence result give a definite answer to the first question.

Theorem 7 (Existence of contact forms). *Let M be any closed oriented 3-manifold. Then:*

a)M admits a contact structure (this was first proven by Martinet [25], see also [23] [24] [32]).

b)M has a contact structure in every homotopy class of non-singular plane distributions (a result essentially due to Lutz, see the previous references and [9]).

c)M has a parallelization by three contact forms θ^j, $j = 1, 2, 3$, and each contact distribution $\ker\theta^j$, considered as a plane bundle, is trivial [18].

d)M admits contact circles realizing any of the two orientations, where a contact circle is a pair of contact forms θ^1, θ^2 such that any linear combination $a_1\theta^1 + a_2\theta^2$ with constant coefficients a_1, a_2, $a_1^2 + a_2^2 = 1$, is also a contact form. [17].

The question about local classification of contact structures is completely solved by the following important theorem of Darboux (see [2] appendix 4).

Theorem 8 (Darboux). *There is a C^∞ coordinate system (x, y, z) in a sufficiently small neighborhood V of any point in a contact manifold such that in V the contact structure is given by the kernel of the standard contact form $dz - ydx$.*

So, from a topological point of view all contact structures look locally the same and their possible non-trivial topological properties must be global. A first result about global topology of contact structures is Gray stability theorem ([15], see also [26] chapter 3, [30]).

Theorem 9 (Gray). *Given a one-parameter family of contact forms θ_t (or in the non-co-orientable case contact structures ξ_t) on M there exist one-parameter families of diffeomorphisms ψ_t and real functions $f_t > 0$ such that $f_t\theta_t = \psi_t^*\theta_0$ (or $\xi_t = \psi_t^*\xi_0$).*

Therefore two contact structures that are homotopic through contact structures are isotopic. Important topological properties of contact manifolds are related to the way the contact structure intersect embedded surfaces. Recall that a plane distribution is integrable if and only if it is locally given by the kernel of a one-form θ that satisfies $\theta \wedge d\theta = 0$ (Frobenius theorem). So, a contact distribution does not admit any integral surface.

Definition 10. [Characteristic directions on a surface] Let ξ be a non-singular plane distribution on a manifold M and Q be a surface embedded in M. A point of intersection of the tangent plane field of Q and ξ is called regular if the intersection is tranversal and is called singular otherwise. The direction defined by the intersection at a regular point is called characteristic direction and the set of all characteristic directions plus the singular points is called characteristic field of Q.

Let M be a co-oriented contact manifold with contact form θ and X be the characteristic field of a surface Q embedded in M. A singular point of X is always isolated from closed orbits of X (it cannot be like a center of a Hamiltonian vector field in the plane). Indeed, let $D \subset Q$ be a disk containing a singular point p of X such that $\gamma = \partial D$ is a closed orbit of X. The integral of θ over X is zero because $X \subset \ker\theta$. Then, by Stokes theorem, the integral of $d\theta$ over D is also zero. But in a sufficiently small neighborhood of p the integral of $d\theta$ over D has to be non-null, because at the singular point p the tangent space of Q is generated by two vectors V_1, V_2 contained in the kernel of θ and if V_3 is a third linear independent vector then $\theta \wedge d\theta(V_1, V_2, V_3) \neq 0$ implies $d\theta(V_1, V_2) \neq 0$. So, γ cannot be too close to p. Moreover, for the integral of $d\theta$ over D to vanish it is necessary that the contact planes turn (or twist) sufficiently, with respect to the tangent planes of Q, in order to $i^*d\theta$ ($i : D \to M$ is the inclusion map) to change sign in D. It is quite interesting that in some sense this local property of non-existence of closed characteristic orbits near a singular point hold globally for certain types of contact structures.

Definition 11. [Overtwisted×Tight] A contact structure ξ in M is called overtwisted if there exists an embedded 2-disk in M such that ∂D is tangent to ξ (a curve tangent to ξ is called Legendrian curve) and ∂D does not contain singularities of the characteristic field, namely the tangent planes to D at ∂D are transverse to ξ. A contact structure ξ is called tight if it is not overtwisted.

Examples of overtwisted contact structures can be constructed as in the following (see [9], [5],[30]). Let us start with the standard contact structure in $\mathbf{R}^3$ (example 4) and change variables as $x = \sqrt{2}x'$, $y = \sqrt{2}y'$, $z = z'+x'y'$ to get (the prime will be omitted in the new variables) $dz + xdy - ydx$. This form written in cylindrical coordinates becomes $dz + r^2 d\phi$. Notice that along the rays $z = 0$, $\phi =$ constant the contact planes $dz + r^2 d\phi = 0$ rotate around the r-rays from angle 0 to $\pi/2$ as r goes from 0 to ∞. Note that no disk in the plane $z = 0$ centered at the origin has a boundary which is a Legendrian curve. The idea to turn the boundary of a disk like this into a Legendrian curve is to twist the contact planes of the previous structure along the r-rays. So, consider the one-form $\sigma = \cos r dz + r \sin r d\phi$. Note that $\sigma \wedge d\sigma > 0$ and that the planes $\sigma = 0$ turn infinitely many times around an r-axis as r-increases. The boundary of the disk $\{r \leq \pi, z = 0\}$ is a Legendrian curve but the disk is tangent to the contact structure at it. The disk $D = \{r \leq \pi, z = (\pi^2 - r^2)\}$ satisfies the hypotheses required in definition 11, so σ is an overtwisted structure in $\mathbf{R}^3$. It is not easy to show that a given contact structure is tight. A first proof that the standard contact structure in $\mathbf{R}^3$ is tight was given by Bennequin [5]. A main result on the classification of overtwisted contact structures on closed 3-manifolds is the following [9] (see also [11]).

Theorem 12 (Eliashberg). *Two overtwisted co-orientable contact structures are homotopic as contact structures (therefore isotopic by Gray theorem) if and only if they are homotopic as plane fields.*

Any contact structure can be made overtwisted through a surgery of the structure called Lutz twisting which does not change the homotopy type of the plane field (see [5], [9], [30]). The existence theorems 7 of Martinet and Lutz say that there exists a contact structure in every homotopy class of plane fields and therefore there exists an overtwisted co-oriented contact structure in every homotopy class of co-oriented plane fields (which are plane fields given by global non-vanishing one-forms). So, there are as many isotopy classes of overtwisted co-oriented contact forms as homotopy classes of co-oriented plane fields (see [16] for a characterization of these homotopy classes). A closed orientable 3-manifold M has a trivial cotangent bundle (theorem 7 c)). Choose a trivialization of the cotangent bundle of M and consider the unit sphere bundle associated to it. Then to each oriented plane in T_pM corresponds a unique covector in this sphere bundle. This implies that the space of co-oriented plane fields in M is in one-to-one correspondence with the space of mappings from M to S^2 and therefore, the homotopy classes of co-oriented planes in M are in one-to-one correspondence to the homotopy classes of mappings from M to S^2. For instance, for $M = \mathrm{S}^3$ these homotopy classes, denoted as $\pi_3(\mathrm{S}^2)$, are in one-to-one correspondence with the set $\mathbf{Z}$, which implies that there are infinitely many different homotopic classes of overtwisted co-orientable structures in S^3. Tight contact structures are much more rigid then the overtwisted. For example, in S^3 there is a unique tight contact structure up to isotopy [10], [11]. Moreover, there exists an oriented closed 3-manifold which does not admit any tight contact structure with oposite orientation [13]. More results about the topological classification of tight contact structures are given in [11].

3. FIELDS OF PLANES TRANSVERSE TO HAMILTONIAN VECTOR FIELDS

Let N be a 4-dimension manifold and ω be a symplectic form on N. The pair (N, ω) is called symplectic manifold. The main example is $\mathbf{R}^4$ with cartesian coordinates (q_1, q_2, p_1, p_2) and $\omega = dq_i \wedge dp_i$. A hypersurface M in N is a connected closed 3-manifold which can be described as the preimage of a regular value of some function $H : N \to \mathbf{R}$ called Hamiltonian function. A triple (N, ω, H) is called Hamiltonian system. Let i be the inclusion map of M into N and $i^*\omega$ be the pull-back of ω by i. The fact that ω is nondegenerate implies that the kernel of $i^*\omega$ is one-dimensional. The direction field given by the kernel of $i^*\omega$ is called the characteristic field of the hypersurface M contained (N, ω). Let (N, ω, H) be a Hamiltonian system. The vector field X_H in N given by $dH = \omega(X_H, \cdot)$ is called Hamiltonian vector field. Notice that the form $dH = \omega(X_H, \cdot)$ restricted to $M = H^{-1}(c)$ is null which implies that $X_H|_M$ is contained in the characteristic field of M. A Hamiltonian system (N, ω, H) induces not only a vector-field in $M = H^{-1}(c)$ but also a volume form, which is preserved by the flow of $X_H|_M$. This volume form is constructed in the following way. Let $\rho = (1/2)\omega \wedge \omega$ be a volume form in N and X be the Hamiltonian vector field associated to H. Since $L_X\omega = i_X d\omega + d i_X \omega = 0$ the flow of X preserves ρ. Let Y be any vector field defined in a neighborhood U of M such that Y is transverse to the level sets of H and $dH(Y) = -1$. For instance, if we set a Riemannian metric on U then Y can be chosen as $-\nabla H/||\nabla H||^2$. Let $\pi : TU \to TU$ be the projection onto the level sets of H given by $\pi(V) = V + Y dH(V)$ and σ be the three-form in U given by $\sigma(V_1, V_2, V_3) = (1/2)\omega \wedge \omega(Y, \pi V_1, \pi V_2, \pi V_3)$. Note that $\rho = \sigma \wedge dH$. To verify this identity it is enough to check that at each point of U the identity is true for a linearly independent quadruple of vectors (Y, V_1, V_2, V_3) where (V_1, V_2, V_3) are tangent to the level sets of H. Indeed, $\sigma \wedge dH(Y, V_1, V_2, V_3) = -dH(Y)\sigma(V_1, V_2, V_3) = \omega \wedge \omega(Y, V_1, V_2, V_3)/2$. Now, let $\mu = \sigma|_M$ be the three-form in M obtained from the restriction of σ to M. The form μ is also preserved by the flow of X because $0 = L_X(\omega \wedge \omega) = L_X\sigma \wedge dH + \sigma \wedge L_X dH = L_X\sigma \wedge dH$, which implies that $L_X\sigma|_M = 0$. The form μ does not depend on the vector field Y used in the costruction of σ. In fact, let $\tilde{Y}$ be a second vector field in U with the same properties and $\tilde{\sigma}$ be its associated three-form. If (V_1, V_2, V_3) are tangent to M then

$$(\mu - \tilde{\mu})(V_1, V_2, V_3) = (\sigma - \tilde{\sigma})(V_1, V_2, V_3) = \frac{1}{2}\omega \wedge \omega(Y - \tilde{Y}, V_1, V_2, V_3) = 0$$

because $dH(Y - \tilde{Y}) = 0$ implies that $Y - \tilde{Y} \in TM$. Notice that μ may not be the unique invariant volume form under the flow of X (excluding multiplication by a constant). If X has a nontrivial positive first integral $f : M \to \mathbf{R}$ then $f\mu$ is also invariant under the flow of X because $L_X(f\mu) = (L_X f)\mu + f L_X \mu = 0$. In this sense the uniqueness part of the following proposition, which is a consequence of the above argument, is quite interesting.

Proposition 13 (Invariant volume form). *Let (N, ω, H) be a Hamiltonian system of class C^k, for any $k = 1, \ldots, \infty$, analytic (this means that N, ω and H are C^k) and $M = H^{-1}(c)$ be a compact regular hypersurface. Given any vector field $Y : M \to TN$ of class C^1, transverse to M, and such that $dH(Y) = -1$, let μ be the three-form in M given by:*

$$\mu(V_1, V_2, V_3) = \frac{1}{2} i_Y(\omega \wedge \omega)|_M$$

Then:

a)μ is unique in the sense that it does not depend on the choice of the vector-field Y. This, in particular, implies that μ is as regular as the Hamiltonian system (N, ω, H).

b)μ is a volume form (if V_1, V_2, V_3 are positively oriented on M then Y, V_1, V_2, V_3 are positively oriented on N).

c)μ is invariant under the flow of the Hamiltonian vector of (N, ω, H) restricted to M.

Although the existence of an X-invariant volume form (usually called Liouville form) on M is stated in almost every text book on classical mechanics, in most of them the question of its uniqueness is not even mentioned. An exception is the book by Abraham and Marsden (see [1] section 3.4) where some type of uniqueness is cited. Notice that the regularity statement a) in the proposition is a consequence of the non-dependence of μ on the choice of Y, the explicit construction of σ, and the fact that Y can always be chosen as regular as (N, ω, H). In the real analytic case such result could not be obtained using a partition of unit argument like, for instance, the one used in [1]. There are certain vector fields Y transverse to M which play a crucial role in the construction of contact forms on M. This is the content of the next proposition which establishes a link between contact manifolds and hypersurfaces of symplectic manifolds. The statement below was adapted from [26] (section 3.4) where the reader also finds its proof.

Proposition 14 (Hypersurfaces of contact type). *Let (N, ω) be a symplectic manifold. A vector field Y in N is called a Liouville vector field if $L_Y\omega = \omega$. Let M be a compact hypersurface in N. Then the following are equivalent.*

a)There exists a contact form θ on M such that $d\theta = \omega|_M$.

b)There exists a Liouville vector field $Y : U \to TM$, defined in a neighborhood U of M, which is transverse to M.

If these conditions are satisfied then M is said of contact type. Moreover the contact form θ in a) can be chosen as $\theta = i_Y\omega|_M$ where Y is the vector-field in b).

The proof that b) implies a) is easy. If $\theta = i_Y\omega|_M$ then $d\theta = -i_Y d\omega + L_Y\omega = L_Y\omega = \omega$ and

$$\begin{aligned}\theta \wedge d\theta &= i_Y\omega \wedge di_Y\omega|_M = i_Y\omega \wedge (-i_Y d\omega + L_Y\omega)|_M \\ &= i_Y\omega \wedge \omega|_M = \frac{1}{2} i_Y(\omega \wedge \omega)|_M. \end{aligned} \tag{1}$$

implies that $\theta \wedge d\theta$ is a volume form on M.

Definition 15. [Reeb vector field] Let (M, θ) be a contact manifold. The vector field X in M uniquely determined by the conditions $i_X d\theta = 0$ and $\theta(X) = 1$ is called Reeb vector field.

Notice that if M is a hypersurface of contact type in (N, ω) then there exists infinitely many contact forms θ such that $\theta = d\omega$ (just take a particular θ and add a sufficiently small closed one-form to it). If Y is a Liouville vector field transverse to M then there exists a function H such that $H^{-1}(c) = M$ and $dH(Y) = -1$. Indeed, let h be any function defined in a neighborhood of M such that $h(x) = 0$ and $dh(x) \neq 0$ for $x \in M$. Then $H(x) = -h(x)/[dh(x)Y(x)]$ satisfies $dH(x)Y(x) = -1$ for $x \in M$. Let

$\theta = i_Y\omega|_M$ be the contact form given in proposition 14. The Hamiltonian vector field of (N, ω, H) restricted to M coincides with the Reeb vector field of θ because $i_X d\theta = i_X\omega = 0$ and $\theta(X) = \omega(Y, X) = -dH(Y) = 1$ (here is the reason for the minus sign appearing in the condition $dH(Y) = -1$ of proposition 13). Moreover, the volume form associated to θ satisfies equation (1) which implies that it is equal to the volume form μ defined in proposition 13. This shows that if M is a hypersurface of contact type in (N, ω) associated to the Liouville vector field Y then its Reeb vector field coincides with the Hamiltonian vector field of (N, ω, H) restricted to M, where $dH(Y)|_M = -1$, and its contact volume form coincides with the volume form given in proposition 13. So, a natural question is whether any hypersurface M is of contact type. An answer to it is provided by the following list of examples.

Example 16. [Star-shaped hypersurfaces in $\mathbf{R}^4$.] Consider $\mathbf{R}^4$ with its standard symplectic form ω. Let $(q_1, q_2, p_1, p_2) \stackrel{\text{def}}{=} x$ be cartesian coordinates in $\mathbf{R}^4$ and λ be the one-form $\lambda = (q_i dp_i - p_i dq_i)/2$ which satisfies $\omega = d\lambda$. If M is a star-shaped hypersurface with respect to the origin then $\lambda|_M$ is a contact form and M is of contact type. One way to verify this is to notice that $Y(x) = x/2$ (the radial vector field) satisfies $i_Y\omega = \lambda$ and $L_Y\omega = i_Y d\omega + d i_Y\omega = d\lambda = \omega$, which implies that Y is a Liouville vector-field transverse to M. Therefore, proposition 14 implies that $\lambda|_M = \theta = i_Y\omega$ is a contact form. The form θ generates a contact structure on M which is diffeomorphic to the "standard contact structure" of $\mathrm{S}^3 \subset \mathbf{R}^4$. This fact, which is interesting from a computational point of view, is discussed in the following (see [20] section 4). Let $h : \mathbf{R}^4 - \{0\} \to \mathbf{R}$ be the function that to each $x \in \mathbf{R}^4 - \{0\}$ associates a positive number $h(x)$ such that $xh(x) \in M$. If $x = s\tilde{x}$ for some positive number s then $xh(x) = \tilde{x}h(\tilde{x}) = sxh(sx)$ implies that h is homogeneous of degree -1. Now, if $H(x) = 1/h^2(x)$ then H is homogeneous of degree two, $M = H^{-1}(1)$, and $2H(x) = dH(x)x$ implies that $dH(x)Y(x) = 1$ for $x \in M$. Note that the Hamiltonian vector field of H is minus the Reeb vector field of θ. Let $\Phi : \mathrm{S}^3 \to M$ be the mapping $\Phi(x) = h(x)x$ where $||x|| = 1$. If $v \in T_x\mathrm{S}^3$ is written as a vector in $\mathbf{R}^4$ such that $(v, x) = 0$ then, using the explicit expression for λ in coordinates,

$$\Phi^*\theta v = \Phi^*\lambda|_M v = \lambda(xh(x))\Phi_* v = h(x)\lambda(x)[h(x)v + (dh(x)v)x] = h^2(x)\lambda(x)v$$

which implies that $\Phi^*\theta = h^2\lambda|_{\mathrm{S}^3} \stackrel{\text{def}}{=} h^2\theta_0$. The one form θ_0 is called the "standard contact form" of S^3 and it generates the same contact structure ξ as the form $\Phi^*\theta$. The Reeb vector field of $\Phi^*\theta$, which corresponds to the pull-back of the Hamiltonian vector field of $-H$ restricted to M, is transverse to the contact structure ξ. Therefore, any Hamiltonian vector field associated to a star-shaped hypersurface in $\mathbf{R}^4$ has a pull-back to S^3 which is transverse to the standard contact structure of S^3.

Definition 17. [Restricted contact type] A hypersurface M in a symplectic manifold (N, ω) is of restricted contact type if there exists a one-form λ in N such that $d\lambda = \omega$, and $\lambda|_M$ is a contact form on M.

If M is of restricted contact type then it is of contact type. All the star-shaped hypersurfaces of example 16 are of restricted contact type. Moreover, a hypersurface M in $(\mathbf{R}^4, dq_i \wedge dp_i)$, with trivial first de Rham cohomology $\mathrm{H}^1(M) = 0$, is of restricted contact type if, and only if, it is of contact type. This is not true if $\mathrm{H}^1(M) \neq 0$ as shown in the next example.

Example 18. [A 3-torus in $(\mathbf{R}^4, dq_i \wedge dp_i)$ which is of contact type but not of restricted contact type.] Consider the following Hamiltonian function in $(\mathbf{R}^4, dq_i \wedge dp_i)$:

$$H = \left\{2 - \frac{(q_1^2 + p_1^2)}{2}\right\}^2 + \left\{2 - \frac{(q_2^2 + p_2^2)}{2}\right\}^2$$

Any one-form λ in $\mathbf{R}^4$ such that $d\lambda = \omega$ must be of the form $\lambda = \lambda_0 + df$ where $\lambda_0 = (q_i dp_i - p_i dq_i)/2$ is the one-form used in example 16 and $f : \mathbf{R}^4 \to \mathbf{R}$. Let us define new coordinates in $\mathbf{R}^4 - \{q_1 = p_1 = 0\} \cup \{q_2 = p_2 = 0\}$

$$q_1 = \sqrt{2(I_1 + 2)} \sin\theta_1, \quad q_2 = \sqrt{2(I_2 + 2)} \sin\theta_2,$$

$$p_1 = \sqrt{2(I_1 + 2)} \cos\theta_1, \quad p_2 = \sqrt{2(I_2 + 2)} \cos\theta_2$$

where θ_i are angular coordinates and $I_i > -2$. In these new coordinates:

$$\lambda = \lambda_0 + df = -I_i d\theta_i - 2d\theta_1 - 2d\theta_2 + df, \qquad \omega = d\theta_i \wedge dI_i, \qquad H = I_1^2 + I_2^2$$

The hypersurface $M = H^{-1}(1)$ is a three-torus which can be parameterized by the three angles θ_1, θ_2, ϕ, where ϕ is given by

$$I_1 = \cos\phi, \qquad I_2 = \sin\phi$$

Note that a necessary condition for λ restricted to M to be a contact form is that $\lambda(X) \neq 0$ for any nonsingular vector field in M such that $d\lambda(X, \cdot) = \omega(X, \cdot) = 0$. In particular let X be the Hamiltonian vector field associated to H restricted to M

$$\dot\theta_1 = 2I_1 = 2\cos\phi, \quad \dot\theta_2 = 2I_2 = 2\sin\phi, \quad \dot I_1 = \dot I_2 = \dot\phi = 0.$$

Then

$$\lambda(X) = -2[1 + 2\sqrt{2}\sin\left(\phi + \frac{\pi}{4}\right)] + df(X)$$

The vector field X has two closed orbits $\gamma_1 = \{\phi = \pi/2, \theta_1 = 0, \theta_2 = 2t, t \in [0,\pi]\}$ and $\gamma_2 = \{\phi = \pi, \theta_1 = -2t, \theta_2 = 0, t \in [0,\pi]\}$ on which $\lambda|_{\gamma_1} = -6 + df(X)$ and $\lambda|_{\gamma_2} = 2 + df(X)$. This implies that for any choice of function f the integrals $\int_{\gamma_1} \lambda < 0$ and $\int_{\gamma_2} \lambda > 0$ which is impossible because $\lambda(X)$ must have the same sign all over M. Although M is not of restricted contact type it is of contact type. Let us add to $\lambda_0|_M$ the closed form $2d\theta_1 + 2d\theta_2$ to obtain the one-form in M

$$\theta = -I_i d\theta_i = -\cos\phi d\theta_1 - \sin\phi d\theta_2 \tag{2}$$

Note that $d\theta = \omega|_M$ and $\theta \wedge d\theta = -d\theta_1 \wedge d\theta_2 \wedge d\phi$ which shows that θ is a contact form and M is of contact type.

Example 19. [A hypersurface which is not of contact type.] The following example satisfies the conditions given in [8] (theorem 1) for a hypersurface to be not of contact type. As in example 18 consider a Hamiltonian function in $\mathbf{R}^4$ given by

$$H = h(I_1, I_2), \qquad \text{where} \quad I_1 = \frac{(q_1^2 + p_1^2)}{2}, \quad I_2 = \frac{(q_2^2 + p_2^2)}{2}$$

and

$$\begin{aligned} h(I_1, I_2) &= \phi(I_1 + I_2) + \phi(I_1 - I_2) + 0.01(I_1^2 - I_2^2)^2, \\ \text{where} \quad & \phi(z) = z^2(z^2 - 1)^2 \end{aligned} \tag{3}$$

Let M be the hypersurface in $\mathbf{R}^4$ that corresponds to the connected component of the level curve $h = 0.15$ shown in figure 1. The hypersurface M is algebraic and is diffeomorphic to S^3 (the level curve shown in figure 1 is diffeomorphic to $I_1 + I_2 = 1$ which is the standard sphere in S^3). Let a and c be the points on the plane $\{I_1, I_2\}$ shown in figure 1 which satisfy $\partial_{I_2} h(a) = \partial_{I_2} h(c) = 0$, $\partial_{I_1} h(a) < 0$, and $\partial_{I_1} h(c) > 0$. Using polar coordinates

$$q_1 = \sqrt{2I_1}\sin\theta_1, \quad q_2 = \sqrt{2I_2}\sin\theta_2, \quad p_1 = \sqrt{2I_1}\cos\theta_1, \quad p_2 = \sqrt{2I_2}\cos\theta_2,$$

with $I_i > 0$, we verify that there exists two periodic orbits γ_a and γ_c which satisfy $\gamma_a = \{(I_1, I_2) = a, \dot\theta_1 < 0, \dot\theta_2 = 0\}$ and $\gamma_c = \{(I_1, I_2) = c, \dot\theta_1 > 0, \dot\theta_2 = 0\}$. Now, since M is diffeomorphic to S^3, any one-form θ on M which verifies $d\theta = \omega|_M$ must also satisfy $\theta = \lambda_0|_M + df$ where f is a real valued function on M and $\lambda_0 = (q_i dp_i - p_i dq_i)/2 = -I_i d\theta_i$. This implies that $\int_{\gamma_a} \theta > 0$ and $\int_{\gamma_c} \theta < 0$, for any choice of f. So, by the same argument as in example 18, namely, that necessarily $\theta(X) \neq 0$ over M, we conclude that M is not of contact type.

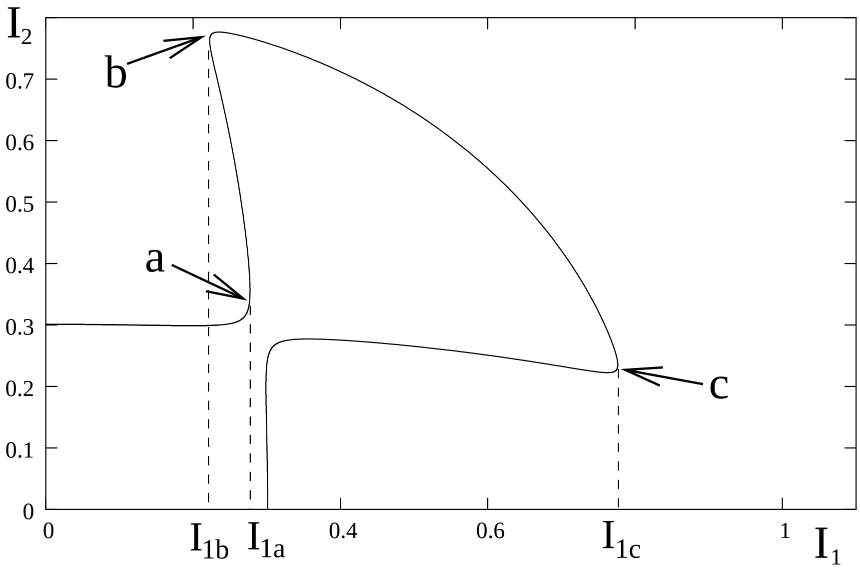

FIG. 1. Figure showing a connected component of the level curve $h = 0.15$ of function h defined in equation 3. The function h decreases as the level curve is crossed from the outside to the inside (by inside I mean the bounded region encircled by the level curve). The points a, b, and c are such that $\partial_{I_2}h(a) = \partial_{I_2}h(b) = \partial_{I_2}h(c) = 0$, $\partial_{I_1}h(a) < 0$, $\partial_{I_1}h(b) < 0$ and $\partial_{I_1}h(c) > 0$.

Example 20. [The unit sphere bundle of a Riemannian surface is of restricted contact type.]

Let Q be a surface with a given Riemannian metric and λ be the usual Liouville form defined on TQ^* (if σ is a one form on Q then $\pi^*\sigma = \lambda(\sigma)$ where $\pi : TQ^* \to Q$ is the canonical projection). Here we follow the notation of example 5. The form $\omega = -d\lambda$ is a symplectic form on TQ^*. If (q, pdq) are coordinates in TQ^* then $\lambda = pdq$ and $\omega = dq \wedge dp$. Let $SQ^* = \{z \in TQ^* : ||z|| = 1\}$ be the co-unit circle bundle of Q and θ be the one-form on SQ^*, defined in example 5, which satisfies $\theta = \lambda|_{SQ^*}$. It was shown in example 5 that θ is a contact form, so $(SQ^*, -\theta)$ is a submanifold of restricted contact type of (TQ^*, ω). If $H : TQ^* \to \mathbf{R}$ is the function $H(z) = ||z||^2/2$ then $SQ^* = H^{-1}(1/2)$. It is easy to show that the unit speed geodesic flow of Q when mapped to the cotangent bundle of Q (through the natural isomorphism $TQ \to TQ^*$, $v \to < v, \cdot >= z$), is the flow of the Hamiltonian vector field X of (TQ^*, ω, H) restricted to SQ^*. If $\gamma : [0, 1] \to Q$ is a unit speed geodesic and (q, pdq) are coordinates in TQ^* then $[0, 1] \to (q(t), p(t))$ is an integral curve of X where $q(t) = \gamma(t)$ and $p(t)dq =< \dot{\gamma}, \cdot >$. This implies that X restricted to SQ^* satisfies

$\theta(q,p)X = pdq(X) = <\dot{\gamma}, \dot{\gamma}> = 1$. So, the Reeb vector field of (SQ^*, θ) is the geodesic vector field X restricted to SQ^*.

Consider for instance the two-torus $\mathbf{R}^2/(2\pi\mathbf{Z}^2)$ with coordinates $(\theta_1, \theta_2)\text{mod}(2\pi, 2\pi)$ and Riemannian metric $d\theta_1^2 + d\theta_2^2$. In this case $\lambda = p_i d\theta_i$, $H = (p_1^2 + p_2^2)/2$, and $SQ^* = \{(\theta_1, \theta_2, p_1, p_2) : H = 1/2\}$. SQ^* is diffeomorphic to a three-torus and it can be parameterized by $(\theta_1, \theta_2, \phi)$ with $p_1 = \cos\phi$ and $p_2 = \sin\phi$. The contact form $\theta = \lambda|_{SQ^*} = \cos\phi d\theta_1 + \sin\phi d\theta_2$ is, except for a minus sign, the same as the one in equation (2). Notice that as contact manifolds the torus in this paragraph and that in example 18 are the same. Nevertheless the one appearing here is a submanifold of restricted contact type of (TQ^*, ω) while that in example 18 is not a submanifold of restricted contact type of $(\mathbf{R}^4, dq_i \wedge dp_i)$.

Example 19 shows that there is a hypersurface $M \simeq \mathrm{S}^3$ in $\mathbf{R}^4$ which do not admit a contact form θ such that $d\theta = \omega|_M$. In principle it could be possible that M would admit a contact form θ such that $d\theta = f\omega|_M$ where f would be a non-vanishing real function on M. This would imply that the Reeb vector field of θ would coincide with some Hamiltonian vector field on M. The next proposition shows that this is impossible.

Proposition 21. *The algebraic hypersurface $M \subset (\mathbf{R}^4, \omega)$ of example 19 does not admit any contact form θ such that $d\theta = f\omega$, where f is a real function on M which is different from zero everywhere.*

Proof. Suppose that there exists a form θ as in the proposition. Let X be its Reeb vector field and $(\theta_1, I_1, \theta_2, I_2)$ be the coordinates given in example 19. Since X must have the same direction of the Hamiltonian vector field of example 19 then X has three orbits (see figure 1) γ_a, γ_b, and γ_c which satisfy, possibly after multiplication of θ by -1,

$$\begin{aligned}
\gamma_a &= \{(I_1, I_2) = a, \dot{\theta}_1 < 0, \dot{\theta}_2 = 0\}, \\
\gamma_b &= \{(I_1, I_2) = b, \dot{\theta}_1 < 0, \dot{\theta}_2 = 0\}, \\
\gamma_c &= \{(I_1, I_2) = c, \dot{\theta}_1 > 0, \dot{\theta}_2 = 0\}.
\end{aligned}$$

As in example 19 we will show that two of the integrals $\int_{\gamma_a} \theta$, $\int_{\gamma_b} \theta$, and $\int_{\gamma_c} \theta$ have opposite signs, which is impossible since $\theta(X) = 1$.

The hypothesis $d\theta = f\omega|_M$ implies that $df \wedge \omega|_M = 0$ which when evaluated at three linear independent vectors (X, V_2, V_3) implies $df(X) = 0$. Let λ be the canonical one-form in $\mathbf{R}^4$, $\lambda = -I_j d\theta_j$, with $d\lambda = \omega$. Then from $d(df \wedge \lambda|_M) = 0$ results that there exists a one-form ν on $M \simeq \mathrm{S}^3$ such that $d\nu = df \wedge \lambda|_M$ which implies that $d(\theta - f\lambda|_M + \nu) = 0$. So, there exists a function g on M such that $\theta = f\lambda|_M - \nu + dg$. In the following the function g can be neglected because $\int_{\gamma_i}(\theta - dg) = \int_{\gamma_i} \theta$, for $i = a, b, c$. Let I_{1a}, I_{1b}, I_{1c} be the three values of I_1 shown in figure 1. Let $y_a : I_1 \to I_2$,

$I_1 \in [0, I_{1a}]$; $y_b : I_1 \to I_2$, $I_1 \in [I_{1a}, I_{1b}]$, and $y_c : I_1 \to I_2$, $I_1 \in [I_{1b}, I_{1c}]$, be the three branches of $I_1 \to I_2$ that solves $h(I_1, I_2) = 0.15$. Consider the three two-cells (with (∂_s, ∂_t) positively oriented):

$$\begin{aligned} C_a &= \{\theta_1 = t, I_1 = s, I_2 = y_a(s), \theta_2 = 0 : s \in [0_+, I_{1a}], t \in [0, 2\pi]\} \\ C_b &= \{\theta_1 = t, I_1 = s, I_2 = y_b(s), \theta_2 = 0 : s \in [I_{1a}, I_{1b}], t \in [0, 2\pi]\} \\ C_c &= \{\theta_1 = t, I_1 = s, I_2 = y_c(s), \theta_2 = 0 : s \in [I_{1b}, I_{1c}], t \in [0, 2\pi]\} \end{aligned}$$

where 0_+ in the definition of C_a means that C_a is the right limit, as $\epsilon \to 0$, of cells $C_{a\epsilon}$, with $s \in [\epsilon, I_{1a}]$, $\epsilon > 0$. Let us show that on the part P_a of M given by

$$P_a = \{(\theta_1, \theta_2, I_1 = s, I_2 = y_a(s)) : \theta_1 \in [0, 2\pi], \theta_2 \in [0, 2\pi], s \in [0_+, I_{1a}]\}$$

the function f depends only on I_1 and I_2. It will be denoted as $f = f(I_1, I_2) = f(s, y_a(s)) \stackrel{\text{def}}{=} f_a(s)$. The Hamiltonian vector field associated to h is integrable with first integrals I_1 and I_2. All of its orbits in P_a are contained in invariant tori $\{I_1 = s, I_2 = y_a(s)\}$ labeled by s and parameterized by (θ_1, θ_2). Every orbit on the torus s is dense if and only if the ratio

$$\frac{\dot\theta_1}{\dot\theta_2} = \frac{\partial_{I_1} h}{\partial_{I_2} h} = -\frac{dy_a}{ds}(s) = y_a'(s)$$

is irrational. But y_a is real analytic in $(0_+, I_{1a})$ and it is not a linear function, therefore y_a' is irrational almost everywhere on the interval $(0_+, I_{1a})$. Since f is constant along flow lines and the flow lines are dense in each "irrational torus", f does not depend on θ_1 and θ_2 on the set of irrational tori. Using that the irrational tori are dense in P_a we conclude that f does not depend on θ_1 and θ_2 at all. The same argument holds for f over analogous sets P_b and P_c. Then, $\partial(C_a + C_b + C_c) = \gamma_c$, Stokes theorem, and integration by parts, imply (the following notation will be used: $f_a(s) = f(s, y_a(s))$, $f_a'(s) = \partial_{I_1} f(s, y_a(s)) + \partial_{I_2} f(s, y_a(s)) y_a'(s)$, $\int_0^{I_{1a}} f_a(s) ds = A$, $f_b(s) = f(s, y_b(s))$, $\int_{I_{1a}}^{I_{1b}} f_b(s) ds = B$, etc)

$$\begin{aligned} \int_{\gamma_c} \nu &= \int_{C_a + C_b + C_c} d\nu = \int_{C_a + C_b + C_c} df \wedge \lambda|_M \\ &= \int_0^{I_{1a}} \int_0^{2\pi} f_a'(s)(-s) dt ds + \int_{I_{1a}}^{I_{1b}} \int_0^{2\pi} f_b'(s)(-s) dt ds \\ &\quad + \int_{I_{1b}}^{I_{1c}} \int_0^{2\pi} f_c'(s)(-s) dt ds \\ &= 2\pi \left\{ -I_{1a} f_a(I_{1a}) + \int_0^{I_{1a}} f_a(s) ds \right\} \end{aligned}$$

$$
\begin{aligned}
& +2\pi\left\{-I_{1b}f_b(I_{1b})+I_{1a}f_b(I_{1a})+\int_{I_{1a}}^{I_{1b}} f_b(s)ds\right\} \\
& +2\pi\left\{-I_{1c}f_c(I_{1c})+I_{1b}f_c(I_{1b})+\int_{I_{1b}}^{I_{1c}} f_c(s)ds\right\} \\
= \;& -2\pi I_{1c}f_c(I_{1c})+2\pi(A+B+C)
\end{aligned}
$$

where it was used that $f_a(I_{1a})=f_b(I_{1a})$ and $f_b(I_{1b})=f_c(I_{1b})$. Therefore, using $df(X)=0$, one gets

$$
\begin{aligned}
\int_{\gamma_c}\theta &= \int_{\gamma_c} f\lambda|_M - \int_{\gamma_c}\nu = f_c(I_{1c})\int_{\gamma_c}\lambda|_M - \int_{\gamma_c}\nu \\
&= -2\pi I_{1c}f_c(I_{1c}) - \int_{\gamma_c}\nu \\
&= -2\pi(A+B+C)
\end{aligned}
$$

The same reasoning and $\partial C_a=-\gamma_a$, $\partial(C_a+C_b)=-\gamma_b$ give

$$
\int_{\gamma_a}\theta = +2\pi A \qquad \text{and} \qquad \int_{\gamma_b}\theta = +2\pi(A+B)
$$

The inequalities $I_{1b}<I_{1a}<I_{1c}$ and the fact that f does not change sign imply that $AB<0$, $BC<0$, and $AC>0$. Now, if $\int_{\gamma_a}\theta\int_{\gamma_b}\theta\le 0$ then the proof is over. If $\int_{\gamma_a}\theta\int_{\gamma_b}\theta>0$ then $A(A+B)>0$ and

$$
\int_{\gamma_a}\theta\int_{\gamma_c}\theta = -4\pi^2A(A+B+C) = -4\pi^2[A(A+B)+AC]<0,
$$

which finishes the proof. ▮

Conversely to proposition 21, a Reeb vector field on a contact manifold (M,θ) is always the restriction of a Hamiltonian vector field. Indeed, let $N=M\times\mathbf{R}$ be a four-manifold and $\lambda=\mathrm{e}^a\theta$ be a one-form in N where a is a coordinate on the $\mathbf{R}$ factor of N. Then $\omega=d\lambda$ is a symplectic form and (N,ω) is a symplectic manifold. If $H:N\to\mathbf{R}$ given by $H=-a$ is a Hamiltonian function, then $M\times\{0\}=H^{-1}(0)$ and X is the Hamiltonian vector field of H restricted to $M\times\{0\}$. This shows that for closed three-manifolds the set of Reeb vector fields is a subset of the set of restrictions of Hamiltonian vector fields. Since restrictions of regular Hamiltonian vector fields to closed 3-manifolds are always volume preserving (Proposition 13) a natural question is: Is a volume preserving flow in a closed three-manifold always the restriction of a Hamiltonian vector field? The next proposition, which I learned in [19], answers this question.

Proposition 22. *A regular vector field X on a closed three-manifold M which preserves a volume form on M is always the restriction of the Hamiltonian vector field of a Hamiltonian system (N, ω, H) to $M = H^{-1}(0)$.*

Proof. Let Ω be a volume form in M which is invariant under the flow of X. Then $L_X \Omega = 0$ implies that $\mu = i_X \Omega$ is closed, $d\mu = L_X \Omega - i_X d\Omega = 0$. Let λ be a one-form in M such that $\lambda(X) = 1$ (put, for instance, a Riemannian metric in M such that $||X|| = 1$ and take λ as the projection onto X). Let $N = M \times (-\epsilon, \epsilon)$ be a four-manifold, $\epsilon > 0$, and consider the two-form $\omega = d(a\lambda) + \mu$ on N, where a is a coordinate on the factor $(-\epsilon, \epsilon)$ of N. If ϵ is sufficiently small then ω is non-degenerate. Therefore, for $\epsilon > 0$ small, (N, ω) is a symplectic manifold. Let $H : N \to \mathbf{R}$ be a Hamiltonian function given by $H = -a$. Then $H^{-1}(0) = M$ and X is the Hamiltonian vector field of (N, ω, H). ▌

Now, let us discuss the behavior of Reeb vector fields under reparameterization. The restriction of a Hamiltonian vector field to a hypersurface multiplied by any positive function is still the restriction of a Hamiltonian vector field, in general for a different Hamiltonian function. Let θ be a contact form. Then if ν is a sufficiently small closed one-form then $\sigma = \theta + \nu$ is also a contact form and $d\sigma = d\theta$ implies that the Reeb vector fields of σ and θ coincide up to multiplication by a positive function. Is it true that a Reeb vector field multiplied by any positive function is always a Reeb vector field of some contact form? Locally the answer to this question is yes. Globally it is no.

Proposition 23. *There exists a Reeb vector field X on a closed three-manifold M and a strictly positive function $h : M \to \mathbf{R}$ such that X/h is not the Reeb vector field of any contact form in M.*

Proof. Let Q be a surface obtained as the quotient of the Poincaré hyperbolic disc D by the discontinuos action of a discrete group Γ of isometries which fundamental domain is a regular octagon on D (see [22], section 5.4, in particular figure 5.4.3, or [4], chapter IV). Q is a compact surface of genus two with a Riemannian metric of constant negative curvature -1. It was shown in example 20 that the unit speed geodesic vector field mapped to $SQ^* = M$ by the natural isomorphism $v \to < v, \cdot >$ is the Reeb vector field X of a contact form θ. The unit cotangent bundle SD^* of D is trivial $SD^* \simeq D \times \mathrm{S}^1$ and can be covered by $D \times \mathbf{R}$ which is globally parameterized by (x, y, ϕ), where $(x, y) \in \mathbf{R}^2$, $x^2 + y^2 < 1$, are Cartesian coordinates on D and ϕ is a coordinate on $\mathbf{R}$. The horizontal diameter of D corresponds to two geodesics: $\gamma = \{x = 0, y = \overline{y}(t), \phi = 0 : t \in (-\infty, \infty)\}$ and $-\gamma = \{x = 0, y = \overline{y}(-t), \phi = \pi : t \in (-\infty, \infty)\}$, for some function $\overline{y} : \mathbf{R} \to \mathbf{R}$. The group of isometries Γ can be chosen such that γ and $-\gamma$ correspont to two closed geodesics of Q which will be denoted as γ_a and γ_b, respectively (see [4], in particular figures 12 and 19). Let $h : M \to \mathbf{R}$ be a strictly positive function and suppose that there exists a contact form σ on M such that $Y = X/h$ is its Reeb vector field. Then $\ker d\sigma = \ker d\theta$ which implies that there exists a non-vanishing function f such that $d\theta = f d\sigma$. As in the proof of proposition 21 f must satisfy $df(X) = 0$, namely f must be constant along the orbits of X. But, in this case X has a dense orbit in M (see [22], section 5.4) which implies that f is constant on the whole M. So, $d\theta = f d\sigma$ implies that $\theta = f\sigma + \nu$ where ν is a closed form. The normalizations $\theta(X) = 1$, $\sigma(Y) = 1$, and $Y = X/h$ imply $\nu(X) = 1 - fh$. Let $\overline{C}$ be the cell in SD^* given by $(s,t) \to (x = 0, y = \overline{y}(t), \phi = s) : s \in [0,\pi], t \in (-\infty,\infty)\}$. The quotient of $\overline{C}$ by the lift of the action of the group Γ to SD^*, $C = \overline{C}/\Gamma$, defines a cell in M which boundary is $\partial C = \gamma_a + \gamma_b$. This and $\nu(X) = 1 - fh$ imply that ($2l$ is the length of γ_a and γ_b)

$$\begin{aligned} 0 &= \int_C d\nu = \int_{\gamma_a} \nu + \int_{\gamma_b} \nu = \int_{-l}^{l} [1 - fh(\gamma_a(t))]dt + \int_{l}^{-l} [1 - fh(\gamma_b(t))]dt \\ &= -f \int_{-l}^{l} [h(\gamma_a(t)) - h(\gamma_b(-t))]dt. \end{aligned}$$

Since $f \neq 0$ this identity is verified only for very special functions h. If for instance h is constant equal to 1 over γ_a and constant equal to 2 over γ_b then one gets a contradiction. ▮

As it is clear from the discussion and examples above many hypersurfaces in a symplectic manifold carry a contact form with a Reeb vector field which has the direction of the characteristic field induced by the symplectic structure. So, all the topological information about the characteristic direction field is encoded in this contact form which additionally defines a contact structure on the hypersurface. Nevertheless, not all hypersurfaces carry such form. In the following, weaker structures that always exist on a hypersurface will be briefly discussed. In order to set a time parameterization for the characteristic vector field, let $M = H^{-1}(0)$ be a hypersurface in a Hamiltonian system (N, ω, H) and X be the Hamiltonian vector field restricted to M. Three structures are naturally induced on M by (N, ω, H): the closed two form $\omega|_M$, the vector field X, and the volume form Ω given in Proposition 13. On three-manifolds these three structures are not independent, given two of them the third is determined. In the following definition the two-form and the vector field were chosen as primary structures on M.

Definition 24. [Hamiltonian Structure] A Hamiltonian structure on a manifold M of odd dimension is a pair (ω, X) where ω is a closed two-form of maximal rank and X is a vector field such that $i_X\omega = 0$. The triple

(M, ω, X) will be called Hamiltonian manifold. The pair (ω, X) will be called a regular Hamiltonian structure if X does not have critical points.

Notice that ω is invariant under the flow of X, $L_X\omega = i_X d\omega + d i_X \omega = 0$. A volume form Ω invariant under the flow of X is intrinsically associated to a regular Hamiltonian structure. Let θ be any one-form on M such that $\theta(X) = 1$ and $\Omega = \theta \wedge \omega$. Then Ω does not depend on the choice of θ. If σ is another one form with $\sigma(X) = 1$ and (X, V_2, V_3) are three linear independent vectors at some point of M then $\sigma \wedge \omega(X, V_2, V_3) = \sigma(X)\omega(V_2, V_3) = \theta \wedge \omega(X, V_2, V_3)$. Moreover, the form Ω is invariant under the flow of X because $L_X\Omega = i_X d\Omega + d i_X \Omega = d\omega = 0$. In fact to each choice of one-form θ such that $\theta(X) = 1$ it is associated an "almost contact structure" on M.

Definition 25. [Almost contact structure] An almost contact structure on a three-manifold M is a pair (ω, θ) where ω is a two-form and θ is a one-form such that $\theta \wedge \omega \neq 0$. The triple (M, ω, θ) is called an almost contact manifold. (see [6], chapter 3)

Notice that in general the plane field defined by $\ker\theta$ is not invariant under the flow of X. Indeed, if there is a transversal plane field which is invariant under the flow of X then θ satisfies the following.

Proposition 26. *Let (M, ω, X) be a Hamiltonian manifold and ξ be a regular plane field on M which is transverse to X and is invariant under the flow of X. Let θ be a one-form on M such that $\xi = \ker\theta$ and $\theta(X) = 1$. Then $d\theta = f\omega$ where f is a real valued first integral of X, namely $df(X) = 0$. If f is strictly positive or negative then X is the Reeb vector field of θ. Moreover there exists Hamiltonian manifolds (M, ω, X) which do not admit any non singular plane field transverse to X which is invariant under the flow of X.*

Proof. Let V_2, V_3 be two vectors in $\ker\theta$. Then the hypothesis that $\ker\theta$ is invariant under the flow of X implies that $(L_X\theta)(V_j) = 0$, for $j = 1, 2$. But, $L_X\theta = i_X d\theta + d i_X \theta = i_X d\theta$ implies that $(L_X\theta)(X) = 0$. Therefore, $L_X\theta = 0$, θ is invariant under the flow of X, and $i_X d\theta = 0$. This last equation implies that $d\theta = f\omega$ for some function f and, as in the proof of proposition 21, $df(X) = 0$. If f is strictly positive or negative then $\theta \wedge d\theta \neq 0$ and θ is a contact form. Finally, $(M, \omega, X/h)$ is a Hamiltonian manifold for any strictly positive function h. Let (M, ω, X) and h be those considered in the proof of proposition 23. Then $df(X) = 0$ implies that f is constant. Let θ be such that $\theta(X/h) = 1$ and $\ker\theta$ be invariant under the flow of X/h. Then $d\theta = f\omega$ with f constant. If $f \neq 0$ then θ is a contact form and X/h is its Reeb vector field which contradits what was proved in proposition 23. If $f = 0$ then θ is a closed form and $\theta(X) = h$. Let C,

$\partial C = \gamma_a + \gamma_b$, be the cell defined in the proof of proposition 23. Then

$$\begin{aligned} 0 &= \int_C d\theta = \int_{\gamma_a} \theta + \int_{\gamma_b} \theta = \int_{-l}^{l} h(\gamma_a(t))dt + \int_{l}^{-l} h(\gamma_b(t))dt \\ &= -\int_{-l}^{l} [h(\gamma_a(t)) - h(\gamma_b(-t))]dt. \end{aligned}$$

As in the proof of proposition 23 if h is constant equal to 1 over γ_a and constant equal to 2 over γ_b then one gets a contradiction. ∎

To finish this section let us present an invariant of certain Hamiltonian manifolds (M, ω, X) introduced by V. I. Arnold [3]. This invariant is related to a phase space average of a sort of linking number ("asymptotic linking number") between pair of orbits, see [3] for details.

Definition 27. [Arnold invariant] Let (M, ω, X) be a closed Hamiltonian manifold such that ω is exact. Let λ be any one-form in M such that $d\lambda = \omega$. The Arnold invariant of (M, ω, X) is the real number:

$$I = \frac{\int_M \lambda \wedge \omega}{\int_M \Omega}$$

Notice that the form λ is determined up to the addition of a closed form ν. However, replacing λ by $\lambda+\nu$ does not change I because $\nu\wedge\omega = \nu\wedge d\lambda = -d(\nu \wedge \lambda)$ and Stokes theorem imply that $\int_M \nu \wedge \omega = 0$. The hypothesis ω is closed is always verified if M is a hypersurface in an exact symplectic manifold, namely a manifold where the symplectic form is the derivative of a one-form as, for instance, the cotangent bundle of a surface with its canonical symplectic form (example 20). If X is the Reeb vector field of a contact form λ then $I = 1$.

4. FIELDS OF PLANES TANGENT TO HAMILTONIAN VECTOR FIELDS

In the last section we mostly studied properties of fields of planes transverse to a given Hamiltonian vector-field. Special emphasis was given to contact distributions which are transverse to and invariant under the flow of their Reeb vector fields. In this section we will study non singular fields of planes that are tangent to Hamiltonian vector fields. The following interesting example was taken from [7].

Example 28. [Hypersurfaces in $\mathbf{R}^4$.] Let I and J be the 2×2 matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and A_0, A_1, A_2, A_3 be the 4×4 matrices

$$A_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad A_1 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{4}$$

The matrices A_i, $i = 1,2,3$, are anti-symmetric and multiply like the unit quaternions: $A_1A_1 = A_2A_2 = A_3A_3 = -A_0$, $A_1A_2 = -A_2A_1 = A_3$,$A_2A_3 = -A_3A_2 = A_1$, $A_3A_1 = -A_1A_3 = A_2$. Let $\omega = dq_i \wedge dp_i$ be the standard symplectic form in $\mathbf{R}^4$, with its usual Euclidean structure $(\cdot,\cdot)$ and coordinates $(q_1, q_2, p_1, p_2) = x$. Let H be a Hamiltonian function in $\mathbf{R}^4$ and $M = H^{-1}(0)$ be a regular hypersurface. If $V_0(x) = -\mathrm{grad}H(x)/||\mathrm{grad}H(x)||$ denotes the normal field to M then $V_i = A_iV_0$, $i = 1,2,3$, form an orthonormal frame on M. The vector field V_3 is the Hamiltonian vector field of H normalized to have Euclidean norm one. Therefore the plane field $\{V_1, V_2\}$ is transverse to V_3 and the plane fields $\{V_2, V_3\}$ and $\{V_1, V_3\}$ are tangent to V_3.

Plane fields tangent to vector fields naturally appear in several questions related to Hamiltonian dynamics. In the following we present some of them. If a regular vector field X on M admits a nontrivial global first integral $f : M \to \mathbf{R}$ then $\ker df$ defines a field of planes (with singularities) tangent to X. In this direction the following proposition holds (compare to proposition 26).

Proposition 29 ([12] lemma 2.2.2 and [30] lemma 0.4.3). *Let X be a vector field on a 3-manifold M and ξ be a C^1 regular plane field tangent to X. Suppose the flow of X preserves ξ. Then ξ is integrable, namely it is the tangent plane field of a foliation of M.*

Proof. Let θ be a local one-form such that $\xi = \ker\theta$. The invariance of ξ implies $L_X\theta = h\theta$, for some real valued function h. Then

$$i_X\theta \wedge d\theta = (i_X\theta)d\theta - \theta \wedge i_X d\theta = -\theta \wedge L_X\theta = -h\theta \wedge \theta = 0$$

Therefore $\theta \wedge d\theta = 0$ and by Frobenius theorem ξ is integrable. ▮

For instance, an Anosov flow preserves two plane fields tangent to its vector field. These plane fields integrate to give the stable and unstable foliations of the flow [22]. Anosov flows are also related to other interesting tangent plane fields. Suppose that a vector field X of a Hamiltonian manifold (M, ω, X) admits a pair of tangent plane fields given by $\ker\theta^1$ and $\ker\theta^2$ (in example 28 $\theta^1 = (\cdot, V_1)$ and $\theta^2 = (\cdot, V_2)$). A theorem of Mitsumatsu [29] (proposition 3) and Eliashberg and Thurston [12] (proposition 2.2.6) plus conservation of volume by the flow of X imply that if θ_1 and θ_2 are contact forms with opposite orientations (for instance $\theta_1 \wedge d\theta_1 > 0$ and $\theta_2 \wedge d\theta_2 < 0$, see definitions 2 and 3) then X is an Anosov vector field. Conversely, if X is an Anosov vector field then there exists a pair of plane fields tangent to X which define a pair of contact structures with opposite orientations ([29] proposition 2 and [12] proposition 2.2.4).

Another subject where tangent plane fields appear is the following. The existence of two plane fields (always assumed to be regular) tangent to a vector field X is equivalent to the trivialization of any plane bundle transverse to X. The existence of this trivialization plays a central role in the next definition.

Definition 30. [Rotation number] Let (M, ω, X) be a Hamiltonian manifold. Given a one form θ with $\theta(X) = 1$, suppose that the field of planes $\xi = \ker \theta$ admits a trivialization $\{V_1, V_2\}$ such that $\{V_1, V_2, X\}$ are positively oriented with respect to the orientation of M. Let $P : TM \to \xi$ be the projection onto ξ given by $PY = Y - \theta(Y)X$ and $\phi_t : M \to M$ be the flow of X. Given a vector $Y \in \xi_x$ let $a(t, x, Y)$ be the angle of rotation of $P\phi_{t*}Y$ with respect to $V_1(\phi_t(x))$, measured continuously from $t = 0$. If the limit

$$\lim_{t\to\infty} \frac{1}{t} a(t, x, Y)$$

exists then it is independent of the vector Y and it is called the rotation number $r(x)$ of x.

It is possible to prove (see [31] and [14] for details) that for almost all x, with respect to the measure defined by the volume form Ω, the limit above exists and the function $x \to r(x)$ is integrable. The integral of r over M is called "Ruelle invariant" [14]. It measures the average angle of rotation of vectors transverse to X under the action of the tangent map of the flow. The dependence of this number on the choice of the trivialization is discussed in [14]. Also in this reference, the Arnold invariant, the Ruelle invariant, and the so called "Calabi invariant", are simultaneously discussed and compared.

Cotangent bundles are the most important symplectic manifolds of classical mechanics. The next example is a generalization of example 28 to cotangent bundles.

Example 31. [Cotangent bundles of oriented Riemannian surfaces.] Let Q be an oriented Riemannian surface with Riemannian metric g. Let ∇ and $J : TQ \to TQ$ be the Levi-Civita connection and the complex structure associated to g, respectively (J rotates vectors counter-clockwise by $\pi/2$ with respect to a fixed orientation). The complex structure $J^* : TQ^* \to TQ^*$ is defined by $\theta \to \theta \circ J$. Notice that if $\{V_1, V_2\}$ is an oriented orthonormal frame at T_xQ and $\{\theta_1, \theta_2\}$ is the dual frame at T_xQ^* then $JV_1 = V_2$

and $J^*\theta_1 = -\theta_2$, so J^* rotates covectors clockwise. Let $\pi : TQ^* \to Q$ be the cotangent bundle projection and $\pi_* : TTQ^* \to TQ$ its tangent map. The vector bundle $P : TTQ^* \to TQ^*$ has a natural "vertical sub-bundle" $Y \subset TTQ^*$ defined by the kernel of π_*, namely $v \in T_\theta TQ^*$ is in Y_θ if and only if $\pi_* v = 0$. Now, the Riemannian metric will be used to define a complementary "horizontal bundle" to Y. A differentiable curve $t \to \delta(t) \in TQ^*$ is called adapted to a vector $v \in T_\theta TQ^*$ if $\delta(0) = \theta$ and $\dot{\delta}(0) = v$. Note that the curve $t \to \delta(t) \in TQ^*$ defines a covector field along its projection $\gamma(t) = \pi\delta(t)$ (if $\gamma(t) = b$ is a single point for all t then $t \to \delta(t) \in T_b Q^*$ is a curve of covectors over b). The covariant derivative of a covector field δ along a curve $t \to \gamma(t)$ in Q is defined by $D^*\delta/dt = (D\delta^\sharp/dt)^\flat$, where D/dt is the usual covariant derivative along curves (see [27] chapter 2) and $\sharp : TQ^* \to TQ$ and $\flat : TQ \to TQ^*$ are given by $g(\theta^\sharp, \cdot) = \theta$ and $X^\flat = g(X, \cdot)$, respectively. Let $K : TTQ^* \to TQ^*$ be a bundle map defined as: given $v \in T_\theta TQ^*$ let δ be a curve adapted to it and $K(v) = (D^*\delta/dt)_{t=0}$. Note that $\pi\theta = \pi K(v)$, so K maps the fiber over θ to the fiber over $\pi\theta$. Moreover, K projects V_θ isomorphically on TQ^* since for a vertical vector $v \in T_\theta TQ^*$ an adapted curve δ can be chosen such that $\pi\delta(t) = \pi\theta$ and in this case the covariant derivative D^*/dt acts just as the ordinary derivative. The sub-bundle $W \subset TTQ^*$ defined by the kernel of K is called horizontal sub-bundle. Thus $TTQ^* = W \oplus Y$ where $P|_W \to TQ^*$ and $P|_Y \to TQ^*$ are bundles over TQ^* and $\oplus$ denote the Whitney sum of these bundles, namely the set of all pairs $(w, y) \in W \times Y$ such that $P|_W w = P|_Y y$. The bundle map $\pi_* \oplus K : TTQ^* = W \oplus Y \to TQ \oplus TQ^*$, given by $\pi_* \oplus K(w, y) = (\pi_* w, Ky) \in TQ \oplus TQ^*$, is a fiber isomorphism and maps the base manifold TQ^* of TTQ^* onto the base manifold Q of $TQ \oplus TQ^*$, if $P(v) = P(w, y) = \theta$ and $\pi\theta = b$, then $\pi_* \oplus K(v) \in T_b Q \oplus T_b Q^*$. So, $TTQ^* = W \oplus Y$ is isomorphic to the pull back $\pi^*(TQ \oplus TQ^*)$ of $TQ \oplus TQ^*$ by π (if $\pi : E \to B$, is a vector bundle and $f : M \to B$ then the pull back of E by f is the set of all pairs $(v, x) \in E \times M$ such that $\pi(v) = f(x)$, the projection $f^*E \to M$ is $(v, x) \to x$, see [28] section 3 for details). There are several canonical structures defined on $TQ \oplus TQ^*$: a Riemannian metric $< \cdot, \cdot >_\sim$ given by $< (u_1, u_2), (v_1, v_2) >_\sim = g(u_1, v_1) + g(u_2^\sharp, v_2^\sharp)$, a symplectic structure $\tilde{\omega}$ given by $\tilde{\omega}((u_1, u_2), (v_1, v_2)) = v_2(u_1) - u_2(v_1)$, and a complex structure $\tilde{J}_1$ given by $\tilde{J}_1(u_1, u_2) = (Ju_1, J^*u_2)$. These structures canonically define two more complex structures on $TQ \oplus TQ^*$, denoted as $\tilde{J}_2$ and $\tilde{J}_3$, given by $< \tilde{J}_3 u, v >_\sim = \tilde{\omega}(u, v)$ and $\tilde{J}_2 = \tilde{J}_3 \circ \tilde{J}_1$. In order to check that $\tilde{J}_2 \circ \tilde{J}_2$ is minus the identity let b be a point in Q, $\{V_1, V_2\}$ be an orthonormal frame at $T_b Q$ and $\{\theta_1, \theta_2\}$ be the dual frame at TQ^*. Then $\{(V_1, 0), (V_2, 0), (0, \theta_1), (0, \theta_2)\}$ form an orthonormal frame at $(TQ \oplus TQ^*)_b$ and in this frame $\tilde{J}_3 = A_3$, $\tilde{J}_1 = A_1$, and $\tilde{J}_2 = A_2$ where A_1, A_2, and A_3 are the matrices given in equation 4. Now, let q be a coordinate system on a neighborhood $U \subset Q$ of b with $q(b) = 0$, such that at the point b: $\partial_{q1} = V_1$, $\partial_{q2} = V_2$ and $\nabla_{\partial_{qi}} \partial q_j = 0$, for $i, j = 1, 2$ (q can be chosen as Riemann normal coordinates centered at b). Then (q, pdq) are coordinates on TU^* and $(q, pdq, \dot{q}\partial_q, \dot{p}\partial_p)$ are coordinates on TTU^*. The special choice of coordinates q implies that $W(0,p) = \{(\dot{q}, \dot{p} = 0)\}$ and $Y(0,p) = \{(\dot{q} = 0, \dot{p})\}$. In these coordinates the Liouville form λ and the canonical symplectic form $\omega = -d\lambda$ on TQ^* (see example 20) are given by $\lambda = pdq$ and $\omega = -d\lambda = dq \wedge dp$. Let J_i be the pull back of $\tilde{J}_i$, $i = 1, 2, 3$, by $\pi_* \oplus K$, and $< \cdot, \cdot >$ and $\tilde{\tilde{\omega}}$ be the pull back of $< \cdot, \cdot >_\sim$ and $\tilde{\omega}$, respectively. The fact that $(\pi_* \oplus K)_{(0,p)}$ maps ∂q_1, ∂q_2, ∂p_1, and ∂p_2 to $(V_1, 0)$, $(V_2, 0)$, $(0, \theta_1)$, and $(0, \theta_2)$, respectively, implies that the matrix expressions of $\tilde{J}_i$, $< \cdot, \cdot >_\sim$, and $\tilde{\omega}$ coincides with those of J_i, $< \cdot, \cdot >$, and $\tilde{\tilde{\omega}}$, respectively, and moreover $\tilde{\tilde{\omega}} = \omega$. Finally, given a Hamiltonian function H on $(TQ^*, < \cdot, \cdot >)$ let $V_0(\theta) = -\text{grad}H(\theta)/||\text{grad}H(\theta)||$ be the normal vector to the regular level set $H^{-1}(0)$ at the point θ. Then the vectors $J_i V_0$, $i = 1, 2, 3$, form an orthonormal frame on $H^{-1}(0)$ and $J_3 V_0$ is the Hamiltonian vector field of H normalized to have norm one.

Definition 32. [Canonical trivialization] Let T^*Q be the cotangent bundle of an oriented Riemannian surface Q with metric g. Let $H : T^*Q \to \mathbf{R}$ be a Hamiltonian function, $M = H^{-1}(0)$ be a regular compact hypersurface, and X be the Hamiltonian vector field of (T^*Q, ω, H) restricted to M. Let $< \cdot, \cdot >$ be the metric on TT^*Q and J_1, J_2, and J_3 be the complex structures on TT^*Q, as defined in example 31. The trivialization $\{V_1, V_2, X\}$ of TM given by $V_1 = -J_1 \text{grad}H(x)$, $V_2 = -J_2 \text{grad}H(x)$, and $X = -J_3 \text{grad}H(x)$, for $x \in M$, will be called "the canonical trivialization of (M, g)". The one-form on M $\theta(\cdot) =< \cdot, X > /||X||^2$ and $\xi = \ker\theta$ will be called the canonical one form and the canonical transverse bundle of (M, g). Finally, $\{V_1, V_2\}$ will be called the canonical trivialization of ξ.

Proposition 33 (Homotopy invariance of canonical trivializations). *Let $M \subset T^*Q$ be a hypersurface as in definition 32. Let g_0 and g_1 be two Riemannian metrics on Q. Then the canonical trivializations of TM associated to g_0 and g_1 are smoothly homotopic through a family of canonical trivializations.*

Proof. The two Riemannian metrics g_0 and g_1 are homotopic through a smooth family of Riemannian metrics $g_t = tg_1 + (1-t)g_0$. Since all objects, like connections, complex structures J_t, etc, used in the definition of a canonical trivialization depend smoothly on the Riemannian metric, there exists a smooth family of trivializations of TM associated to g_t. ▮

Examples 28 and 31 show that many important Hamiltonian manifolds (M, ω, X) admit a trivialization of the transverse bundle $\xi = \ker\theta$ where $\theta(X) = 1$. In general the transverse bundle ξ is trivializable if and only if its "first Chern class" $c_1(\xi)$ is trivial (see [28]). The first Chern class admits a simple geometric description in this case. Let us fix a trivialization of $TM \to \{e_1, e_2, e_3\} \in \mathbf{R}^3$ and consider a Riemannian metric on M such that $\{e_1, e_2, e_3\}$ is an orthonormal frame. This trivialization can always be chosen such that $||X|| = 1$. So, X defines a Gauss map $X : M \to \mathrm{S}^2$ where S^2 is the unit ball in $\mathbf{R}^3$. Let a be the area form on S^2 divided by 4π (so that $\int_{\mathrm{S}^2} a = 1$). The pull back $\sigma = X^* a$ is closed, since a is closed, and the integral of σ over any two-cycle Σ is an integer, since $\int_\Sigma \sigma = \int_{X \circ \Sigma} a$. Therefore, σ determines an element $[\sigma]$ of $\mathrm{H}^2(M, \mathbf{Z})$. The class $[\sigma]$ is the first Chern class of the oriented bundle ξ ($\{V_1, V_2\} \in \xi$ is a positively oriented if $\{V_1, V_2, X\}$ is positively oriented with respect to $\{e_1, e_2, e_3\}$). See [28], specially appendix C, for details and [16], section 4, for a discussion of homotopy classes of oriented plane fields on three-manifolds. Notice that in order to show that a certain Hamiltonian manifold (M, ω, X) has a non-trivial transverse bundle ξ it is enough to find a cycle where the integral of σ does not vanish.

Example 34. [ξ is non-trivial and $[\omega] \neq 0$] Let S^2 be the unit sphere $\{x \in \mathbf{R}^3 : ||x|| = 1\}$, $\mathrm{S}^1 = \mathbf{R}/\mathbf{Z}$ and $M = \mathrm{S}^2 \times \mathrm{S}^1$. Consider the vector field X on M with trivial S^2-component and S^1-component $\dot{r} = 1$, $r \in \mathbf{R}$. Let ω be the closed two form on M which restricts to the usual area form on the factor S^2 and such that $\omega(X, \cdot) = 0$. Let $\psi : \mathrm{S}^2 \times \mathbf{R} \to \mathbf{R}^3$ be given by $(x, r) \to y = rx$ and $\theta_j = \psi^* dy_j$, $j = 1, 2, 3$, where dy_j are the usual cartesian coordinate forms of $\mathbf{R}^3$. Then $\{\theta_1, \theta_2, \theta_3\}$ restricted to $\mathrm{S}^2 \times \{r = 1\}$ provides a trivialization of $TM|_{\mathrm{S}^2 \times \{r=1\}}$ which can be easily extended to TM: $\theta_j(x, r) = \theta_j(x, 1)$. The Gauss map of X with respect to this trivialization is $\mathrm{S}^2 \times \mathrm{S}^1 \ni (x, r) \to (\theta_1(x), \theta_2(x), \theta_3(x)) \in \mathrm{S}^2$. Let $\Sigma = \mathrm{S}^2 \times \{r = 1\}$ be a two-cycle in M and σ be the form on M defined above, namely, the pull back of the area form of S^2 by the Gauss map. Then $\int_\Sigma \sigma = 1$ which implies that any transverse plane bundle to X is not trivializable.

In examples 28 and 31 the homologies $\mathrm{H}^2(M, \mathbf{Z})$ of both ω and σ were trivial (the last one because ξ was always trivial) and in example 34 the homologies of both ω and σ were non trivial. This would suggest that some relation $[\omega] = 0 \Rightarrow [\sigma] = 0$ or $[\omega] \neq 0 \Rightarrow [\sigma] \neq 0$ could hold independently of the vector field X. Both relations are false as it is shown in the examples below.

Example 35. [ξ is trivial and $[\omega] \neq 0$.] Let $M = \mathbf{R}^3/\mathbf{Z}^3$ be a three-torus (where $\mathbf{R}^3$ has Cartesian coordinates (x_1, x_2, x_3)), X be the vector field ∂_{x_1}, and $\omega = dx_2 \wedge dx_3$. Then $[\omega] \in \mathrm{H}^2(M, \mathbf{Z})$ is non trivial but the transverse bundle $\xi = \ker dx_1$ is trivial.

Example 36. ξ is non-trivial and $[\omega] = 0$.

Let $f : \mathbf{T}^2 \to \mathrm{S}^2$ be a map of degree one which, for instance, can be constructed in the following way. Let C_1 be the cylinder $\{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 = 1, |z| \leq 1\}$ and γ_+ and γ_- be the two circles $z = +1$ and $z = -1$ at the boundary of C_1, respectively. Let S^2 be given by $\{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\}$ and consider the map $\tilde{F} : C_1 \to \mathrm{S}^2$ which maps γ_+ onto $(0, 0, 1)$, γ_- onto $(0, 0, -1)$, and $(x, y, z) \in C_1$ with $|z| < 1$ to $(xs, ys, z) \in \mathrm{S}^2$ where $s = (1 - z^2)^{1/2}$. Now, let $\mathbf{T}^2$ be the torus obtained extending the cylinder C_1 to $|z| \leq 2$ and identifying the boundaries at $z = \pm 2$ and let F be the extension of $\tilde{F}$ given by: $(x, y, z) \to (0, sy, 2 - z)$, for $1 < z \leq 2$, with $s = \sqrt{1 - (2 - z)^2}$, and $(x, y, z) \to (0, sy, -2 - z)$, for $-2 \leq z < -1$, with $s = \sqrt{1 - (-2 - z)^2}$. Map F is continuous and have degree one. It can be approximated by a C^∞ map $f : \mathbf{T}^2 \to \mathrm{S}^2$ also with degree one. Let M be the three-torus $M = \mathbf{R}^3/\mathbf{Z}^3$, where $\mathbf{R}^3$ has Cartesian coordinates (x_1, x_2, x_3), and let Y be the vector field on M given by $Y(x, y, z) = f(x, y)$ where f is the function above with values in $\mathrm{S}^2 \subset \mathbf{R}^3$ and defined over the two torus $\mathbf{R}^2/\mathbf{Z}^2$. The form σ_0 associated to the Gauss map of Y is non-trivial, since its integral over the cycle $\Sigma = \{(x, y, z) : z = 0\}$ is one. Let θ_0 be a one-form on M such that $\theta_0(Y) = 1$ and let $\xi_0 = \ker \theta_0$ be a plane bundle transverse to Y. By theorem 7 b), M has a contact structure in the homotopy class of non-singular plane fields that contains ξ_0. Moreover, since ξ_0 is orientable the contact structure is co-orientable, see definition 3, and is given by the kernel of a globally defined contact form θ. Thus there is a homotopy of non-singular one-forms connecting θ_0 to θ. Let X be the Reeb vector field of θ. Then there is also a homotopy of non-singular vector-fields connecting Y and X (if $t \to \theta_t$ is the homotopy between θ_0 and θ, choose a Riemannian metric such that X is transverse to $\ker \theta$, define $\tilde{Y}_t$ as the vector orthogonal to $\ker \theta_t$ such that $\theta_t(\tilde{Y}_t) = 1$, and at the end make a homotopy of $\tilde{Y}_0$ to Y). Finally, consider the Hamiltonian manifold $(M, \omega = d\theta, X)$ and the form σ associated to the Gauss map of X. The homology class of $[\sigma] \in \mathrm{H}^2(M, \mathbf{Z})$ is non-trivial because the Gauss maps $\Sigma \simeq \mathbf{T}^2 \to \mathrm{S}^2$ associated to Y and X are homotopic, therefore they have the same degree which is equal to one.

Now, let us suppose that the Hamiltonian manifold (M, ω, X) has a trivializable transverse bundle $\xi = \ker \theta$, where $\theta(X) = 1$. There can exist more than one homotopy class of such trivializations. Indeed, let $\{V_1, V_2, X\}$ be a given trivialization. Then any other trivialization with the same orientation is determined by a choice of a vector field $\tilde{V}_1 = \cos(f)V_1 + \sin(f)V_2$, where $f : M \to \mathbf{R}/2\pi\mathbf{Z}$, and another one $\tilde{V}_2$ such that $\{\tilde{V}_1, \tilde{V}_2\}$ are linear independent and have the same orientation as $\{V_1, V_2\}$. So, the homotopy classes of possible oriented trivializations of ξ is the same as the homotopy classes of maps from M to S^1. The set of homotopy classes of all continuous maps from M to $\mathrm{S}^1 = \mathbf{R}/\mathbf{Z}$ endowed with the operation of addition given by $(f_1 + f_2)\text{mod}(2\pi\mathbf{Z})$ forms an Abelian group called Bruschlinsky group of M (or first cohomotopy group of M) which is denoted as $\pi^1(M)$ (see [21] chapter 2.7). Let $\phi \in \mathbf{R}/\mathbf{Z}$ be a coordinate system on S^1 and $d\phi$ be the generator of $\mathrm{H}^1(\mathrm{S}^1)$. To each map $f : M \to \mathrm{S}^1$ is associated the closed one form $f^*d\phi$ whose integral over any integer cycle is an integer. Moreover, the homology class of $f^*d\phi$ in $\mathrm{H}^1(M, \mathbf{Z})$ depends only on the homotopy class of f, so there is a natural function h^* from $\pi^1(M)$ to $\mathrm{H}^1(M, \mathbf{Z})$. It can be proved (see [21] chapter 2.7) that h^* is an isomorphism. Therefore there are as many homotopy classes of oriented trivializations of ξ as elements in $\mathrm{H}^1(M, \mathbf{Z})$.

The Ruelle number of (M, ω, X) depends only on the homotopy class of the trivialization of the transverse bundle ξ. An interesting discussion of this dependence is presented in [14], section 3.2. If M is a manifold such that $\mathrm{H}^2(M, \mathbf{Z}) = 0$ then the first Chern class of ξ is necessarily trivial, so ξ is trivializable. Moreover, by Poincaré duality $\mathrm{H}^1(M, \mathbf{Z})$ is also trivial and there is only a single homotopy class of trivializations of ξ. Therefore, for a Hamiltonian manifold (M, ω, X) with $\mathrm{H}^2(M, \mathbf{Z}) = 0$ the Ruelle number is uniquely defined. In this case, the Arnold number, which requires that ω is exact, is also defined. The main example of M with this property is the three sphere. The following theorem, which will be stated for future reference, is a collection of consequences of: some statements in example 31, definition 32, proposition 33, and the invariance of the rotation number (definition30) with respect to homotopic trivializations (see [14] page 1376). This theorem shows that for a Hamiltonian vector field in a cotangent bundle of a surface it is possible to uniquely define a rotation number for each periodic orbit and a Ruelle number to each regular level set M of H, regardless the complexity of $\mathrm{H}^2(M, \mathbf{Z})$. Notice that under the same hypotheses of the theorem the Arnold number is defined for all regular compact level sets of H.

THEOREM 37. *Let T^*Q be the cotangent bundle of an oriented paracompact surface Q and $\omega = -d\lambda$ be the canonical two-form on T^*Q. Let $H : T^*Q \to \mathbf{R}$ be a Hamiltonian function, $M = H^{-1}(0)$ be a regular compact hypersurface, and X be the Hamiltonian vector field of (T^*Q, ω, H) restricted to M. Choose any Riemannian metric g on Q and define its associated canonical trivialization given in definition* 32. *Then the following quantities, defined using this trivialization, do not dependend on the choice of g:*

a)The rotation number r_γ of a periodic orbit γ of X, which is given by $r_\gamma = r(x)$, for any $x \in \gamma$, where $r(x)$ is given in definition 30.

b)The Ruelle number of (M, ω, X) given by $\int_M r(x)\Omega(x)$ where Ω is the volume form of (M, ω, X).

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